A tale of two regimes: Theory and empirical evidence for a Markov-modulated jump diffusion model of equity returns and derivative pricing implications

Charles Chang\textsuperscript{a,b,1}, Cheng-Der Fuh\textsuperscript{c}, Shih-Kuei Lin\textsuperscript{d,*}

\textsuperscript{a}Shanghai Advanced Institute of Finance, China
\textsuperscript{b}Chinese University of Hong Kong, Hong Kong
\textsuperscript{c}Graduate Institute of Statistics, National Central University, Taiwan
\textsuperscript{d}Department of Money and Banking, Risk and Insurance Research Center (RIRC), National Chengchi University, Taiwan

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\textbf{A B S T R A C T}
We provide closed-form solutions for a continuous time, Markov-modulated jump diffusion model in a general equilibrium framework for options prices under a variety of jump diffusion specifications. We further demonstrate that the two-state model provides the leptokurtic return features, volatility smile, and volatility clustering observed empirically for the Dow Jones Industrial Average (DJIA) and its component stocks. Using 10 years of stock return data, we confirm the existence of jump intensity switching and clustering, illustrate transition probabilities, and verify superior empirical fit over competing Poisson-style models.

\textsuperscript{*} Corresponding author. Address: NO.64, Sec.2, ZhiNan Rd.,Wenshan District, Taipei City 11605, Taiwan (R.O.C).
E-mail addresses: charleschang@saif.sjtu.edu.cn (C. Chang), square@nccu.edu.tw (S.-K. Lin).
\textsuperscript{1} Address: 211 Huaihai W. Rd., Ste. 603, Shanghai, China. Tel.: +86 21 6293 3102.
Some models, such as those of Pan (2002), Bakshi et al. (1997), Eraker (2004), and Bates (2000) consider stochastic volatility with jump risks.

\textbf{1. Introduction}
Leptokurtic and asymmetric stock return distributions, the volatility smile, and volatility clustering are critical empirical observations that have made the consistent theoretical description of asset prices elusive, especially in the case of options. The literature provides two general modeling directions to capture these features: the stochastic volatility class of models, such as Hull and White (1987), Duan (1995), Heston (1993), Heston and Nandi (2000), and Stein and Stein (1991), and the jump diffusion class of models, such as those of Björk et al. (1997) and Glasserman and Kou (2003).\textsuperscript{2} They capture leptokurtosis and the volatility smile but generally do not explain volatility clustering because increments are assumed to be independent in both the diffusion and the jump.

In this paper, we propose a Markov-modulated jump diffusion model (MMJM), in which the jump frequency of the underlying asset changes over time according to the state of the economy, which is governed by a continuous Markov chain. The jump behavior, in turn, affects option and stock prices. In the simplest case where there are only two states, we illustrate that periods of high (low) jump arrivals tend to be followed by periods of continued high (low) jump arrivals, which results in jump clustering.\textsuperscript{3} By investigating the Dow Jones Industrial Average (DJIA), we document this phenomenon empirically and attribute it to extended periods of sustained structural abnormality such as the credit crisis or the dotcom bubble. Whereas PJMs only capture abnormal price movements resulting from information events or surprises, MMJMs add color...
by allowing for extended periods of abnormality that may result in periods with frequent jumps in which prices may react with heightened sensitivity to the arrival of information, depending on their states. This clustering of jumps results in volatility clustering during the high-jump-frequency state. We empirically document the existence of these excited or quiet states and demonstrate the superior empirical fit of our model over PJMs.

Our model is not the first of the MMJMs. This class of models may introduce variation into the asset pricing process through the drift, Normal diffusion, or abnormal variation (jump) attributes. As previously mentioned, because it is Markov-modulated, each variation “holds” for a period of time, which is an important distinction from the more general class of stochastic volatility models. Elliott et al. (2007) and Bo et al. (2010) investigate general Markov-modulated jump diffusion models in which the market interest rate, jump frequency, drift and volatility of the underlying asset price vary over time, governed by a continuous Markov chain; the latter focus on currency options. These models are developed in the discrete time context, do not provide closed-form solutions, and hence do not address empirical fit.

Elliott and Siu (2011) incorporate structural changes in economic conditions, such as financial crises, into the description of price dynamics, and Chen (2010) uses an MMJM to indicate how business cycles impact prices. These demonstrate the intuitive attractiveness of this class of models because they naturally fit the state-dependent nature of MMJM, emphasizing the characteristic that the economy stays in each state for an extended period rather than varying at all points. Catastrophe- or weather-based assets such as insurance or certain futures may be valued in this way. Similarly, MMJMs may also be used to model periods of high asset co-integration for collapse models. In general, however, these MMJMs address the continuous component of volatility, allowing for mean-reverting qualities of volatility. To our knowledge, ours is the first to isolate the impact of jump behavior and the jump clustering phenomenon.

In this paper, we choose to model stochastic volatility as a MMJM for two main reasons. First, the literature, along with the findings of this study, documents strong empirical evidence of jump behavior that is not generated in Poisson-type stochastic volatility models, including Ball and Torous (1983, 1985), Beckers (1981), Bates (1991), and Eraker et al. (2003). Second, empirical observations and anecdotal evidence show that jumps in equity markets are not independent but seem to come in bursts, with certain periods being more prone to jumps than others; i.e., we empirically observe jump clustering. The internet bubble period and recent financial crisis are two examples of such jump-sensitive periods, representing boom and bust periods, respectively. Björk et al. (1997) and Glasserman and Kou (2003) each study jump

### Table 1

Evidence from Dow Jones index and return. Table 1 plots the dynamics of the Dow Jones Industrial Average index and its return from 1999/1/4 to 2008/12/31. In Panel A, we show the Dow Jones Industrial Average, and Panel B we graph returns of the DJIA. Horizontal bands about 0% indicate a ±3% band, where returns outside of the band may be seen as jumps. Periods with few such jumps are considered quiet, those with more jumps are considered excited.

**Panel A: The dynamic process of Dow Jones Industrial Average Index**

![Graph of Dow Jones Industrial Average Index](image1)

**Panel B: Extreme returns on the Dow Jones index**

![Graph of Extreme returns on the Dow Jones index](image2)
dependence. The former tests arbitrage bounds and completeness characteristics whereas the latter examines interest rate derivatives when dependence in jump sizes and jump times are allowed. Ding et al. (2009) looks specifically at a birth process as a model of dependence, but none of these papers directly test the empirical features and fit of their respective models.4

A third, more mundane motivation for this modeling environment is that it allows us to derive closed-form options prices for a variety of jump specifications that capture all three of the aforementioned empirical characteristics. In contrast, most stochastic volatility models require simulated options prices and volatility parameters.5 Applying Lucas’s general equilibrium framework, we provide closed-form formulas for options and options-on-futures prices using the moment-generating function of the Markov-modulated Poisson process. When the jump size follows a specific distribution, such as a lognormal distribution, we devise explicit analytic formulas for equilibrium prices. This analytical tractability is another advantage of our modeling technique.

To test our model’s empirical characteristics, we present a two-state, Markov-modulated process to describe information arrivals; that is, when the economy switches between two states of jump intensity. Using 10 years of stock return data, we confirm the existence of switches in jump intensity and show that modeling the dynamic nature of jump dependence effectively captures overall changing volatility, supporting the use of this class of models to demonstrate stochastic volatility. Importantly, we are further able to calculate transition probabilities between states. Table 1 plots the returns of the Dow Jones Industrial Average index from 1999/1/4 to 2008/12/31. In Panel A, we show the level of the DJIA, and beneath it in panel B, we show instances of large returns (jumps). Points outside the middle band represent returns with magnitudes in excess of 3%. We denote periods with frequent (infrequent) jumps to be the “excited” (“quiet”) state of the Markov process. From 1999 to 2000, the market is quiet but then switches to an excited state for the three years during the internet bubble. Then, from 2003 to 2007, jump frequencies return to a quiet state until the financial crisis. The MMJM specification captures the empirical observation of jumps as well as the dynamics of time-varying jump intensity.

This also explains why our model generates volatility clustering and jump dependence, characteristics that we empirically observe in this study for both the Dow Jones Industrial Average as a whole and each of its constituent stocks. PJs such as Kou (2002), which uses double exponential jump sizes, are generally able to describe leptokurtosis and the volatility smile but not volatility clustering.

Allowing for different arrival rates for jumps aligns intuition with empirical stimuli such as the triggering of the recent financial crisis (exogenous) and quiet markets, with switches in these states caused by exogenous shocks to the system.

The remainder of this paper is organized as follows. Section 2 introduces the structure of the model, presents necessary assumptions in a general equilibrium framework, and provides a closed-form solution for option prices under lognormal jump size. Section 3 presents an expectation maximization algorithm used to parameterize and test the empirical fit of the Black–Scholes, PJM, and MMJM for the DJIA and its component stocks. Section 4 numerically demonstrates the leptokurtic, asymmetric, implied volatility surface (smile) and the volatility clustering features of the MMJM. Section 5 concludes.

2. The general equilibrium framework and option pricing formula

2.1. The general equilibrium framework for the MMJM

Consider the general equilibrium framework of Lucas (1978) in a frictionless market, where there is a representative consumer in a rational expectations economy that maximizes an objective function of the following form: max E[\int_0^T U(c(t), t) dt]. E is the unconditional expectation operator, and U(c(t), t) is a continuously differentiable, strictly concave utility function that strictly increases in consumption c(t). For the sake of simplicity, we consider the power utility function as follows:

\[ U(c, t) = \left\{ \begin{array}{ll} e^{-\sigma} & \text{if } 0 < \alpha < 1, \\ e^{-\sigma \log c} & \text{if } \alpha = 0 \end{array} \right. \]

where \( \theta \) is a positive discount rate and \( \alpha \) is the risk aversion parameter. We assume predictable, locally bounded, self-financing feasible trading strategies under non-negative wealth constraints at all times.6

Assume that an exogenous Markovian endowment process \( \delta(t) \) is available to the investor. As demonstrated by Stokey and Lucas (1989), the equilibrium price of the security \( p(t) \) must satisfy the Euler equation, \( p(t) = \frac{E[U(c(t+1), t+1)|t]}{e^{-\rho(t+1)}} \), for all \( T \in [t, T_0] \), where \( U_i \) is the partial derivative of \( U \) with respect to \( c \) and \( T_0 \) denotes a finite liquidation date for the security. \( \delta(t) \) denotes filtration at time \( t \), which provides the information we will have in the future.

The investor optimally consumes the endowment \( \delta(t) \), i.e., \( c(t) = \delta(t) \), for all \( t > 0 \) so that the equilibrium price becomes the following:

\[ p(t) = \frac{e^{-\sigma(T)} E[U(c(T+1), T+1)|t]}{e^{-\sigma(T)} E[U(c(T+1), T+1)|t]} \]

First, let us investigate the Markov-modulated Poisson process. Let \( \Phi(t) \) denote a particular class of doubly stochastic Poisson processes for which the underlying state is governed by a homogeneous Markov chain. In particular, consider a series of nonnegative numbers \( \{\lambda_1, \lambda_2, \ldots, \lambda_k\} \), where \( \lambda_i \) denotes the intensity of the Poisson process if the underlying Markov chain \( q(t) \) is at state \( i \) at time \( t \). Then, \( \{q(t), [P_i; t \in \mathbb{X}] \} \) is a Markov jump process on the state space \( \mathbb{Q} = \{1, \ldots, I\} \) with a transition rate of \( \Psi(i) \).

Define \( \Psi := (\Psi(i,j)) \) and \( P(n,t) := (P_{ij}(n,t)) \), and denote \( A \) as an \( I \times I \) diagonal matrix with diagonal elements \( \lambda_i \). For \( 0 \leq z \leq 1 \), define \( P_z(t) := \sum_{n=0}^{\infty} P(n,t) z^n \) with \( P(0,0) = (1_{I \times I} \lambda_z) \), where \( D_z = 1 \), if \( i = j \) and \( 0 \) otherwise. Hence, \( P_z(0,0) = (D_z) \). Using Kolmogorov’s forward equation, the derivative of \( P_z(t) \) becomes \( \frac{d}{dt} P_z(t) = \sum_{n=0}^{\infty} P(n,t + 1) - \lambda_z \). Its unique solution is \( P_z(t) = e^{(1-z) \lambda_z t} \), where \( e^{A} := \sum_{n=0}^{\infty} \frac{A^n}{n!} \) for any \( (I \times I) \)-matrix \( A \) and \( A^0 := (I) \). Applying the Laplace inverse transformation, we may find the joint distribution of \( X \) and \( \Phi \) at time \( t \) to be given by \( P_{ij}(n, t) := P(q(t) = j, \Phi(t) = n) = P(q(t) = j, \Phi(t) = n) \) at time \( t \) with initial \( q(0) = i \), where \( n \) is a positive integer. Then, we may conduct a numerical inversion using the method proposed by Abate and Whitt (1992), which presents a Fourier-series method for numerically inverting the probability generating function.

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4 Chan (2003, 2004) study a correlated bi-variate Poisson model, with autoregressive jumps in the latter case, and find empirical evidence of both independent and correlated jumps but test only foreign exchange.

5 Heston and Nandi (2000) provide a closed-form option pricing formula in which variance follows a GARCH process.

6 These are common assumptions where “predictability” constrains agents to choose portfolios at any time \( t \) based only on information available before \( t \), “locally bounded” implies finite cumulative mean and variance, and “self-financing” means that portfolio wealth at time \( t \) is equal to initial wealth plus trading gains net of consumption. Dybvig and Huang (1988), Naik and Lee (1990), and Kou (2002) show that non-negativity eliminates arbitrage opportunities.
Next, consider the dynamic processes of the endowment and asset price following a Markov-modulated jump diffusion. Under the physical measure $\mathbb{P}$, the endowment follows:

$$
\frac{d\hat{Y}(t)}{\hat{Y}(t)} = \mu_1(t)dt + \sigma_1dW_1(t) + \left(\sum_{i=1}^{\Phi_1(t)} \hat{Y}_n \right),
$$

(1.3)

where the drift $\mu_1(t)$ represents the instantaneous return of $\hat{Y}(t)$ at time $t$, the volatility $\sigma_1$ of the stock price is assumed to be constant, $W_1(t)$ denotes a one-dimensional standard Wiener process under the physical measure $\mathbb{P}$, $\Phi_1(t)$ is a Markov-modulated Poisson process with finite state $X$, and $\hat{Y}_n$ is an independent sequence of jump sizes when the jump event occurs. The resulting sample path for the endowment process will be continuous except at finite points in time where jumps occur. Jump frequency depends on the state of the economy.

Then, consider the underlying asset price $S(t)$, which also follows a Markov-modulated jump diffusion process as follows:

$$
\frac{dS(t)}{S(t)} = \mu_0(t)dt + \sigma_0dW_0(t) + \left(\sum_{i=1}^{\Phi_0(t)} \hat{Y}_n \right),
$$

(1.4)

where $W_0(t)$ denotes a Brownian motion independent of $W_1(t)$, $\rho$ represents the constant correlation coefficient of the underlying asset and the endowment, and $\hat{Y}_n$ is a sequence of jump sizes related to that of the endowment through a power function such that $\hat{Y}_n = Y_n^b$, where $b \in (-\infty, \infty)$ is an arbitrary constant. Note that the same Markov-modulated Poisson process, $\Phi_0(t)$ affects both the endowment and asset price processes.

We assume a convenient bound for the summation of jump sizes, such that $\sum_{i=1}^{\Phi(t)} = \frac{1}{(\sum_{i=1}^{\Phi(t)} \hat{Y}_n)} < \infty$, and assume that $\sum_{i=1}^{\Phi(t)} < \infty$, $\mathbb{E}\left[\frac{1}{(\sum_{i=1}^{\Phi(t)} \hat{Y}_n)}\right] = \sum_{i=1}^{\Phi(t)} \frac{1}{(\sum_{i=1}^{\Phi(t)} \hat{Y}_n)}$. The above assumption guarantees that the term $\frac{1}{(\sum_{i=1}^{\Phi(t)} \hat{Y}_n)}$ is finite for the endowment and asset prices and for the Markov-modulated Poisson process under the first derivative of the jump size distribution with the lognormal jump size setting applied by Merton (1976), which satisfies this notion of finiteness.

Finally, we assume that the discount rate $\theta$ should be sufficiently large such that $t > (1 - a) \mu_1(t) + 0.5 \sigma_1^2 \mathbb{E}[Y_n^b]$, where $\mathbb{E}[Y_n^b]$ is the expectation of the jump size $Y_n^b$. This assumption guarantees that the term structure of the interest rate is determined. Additionally, the interest rate is endogenously determined.

With loss of generality, let $B(T)$ be the price of a zero-coupon bond with maturity date $T$, and assume that the interest rate $r(t) = \lim_{t \to T} \frac{\ln(B(t))}{t}$ is a deterministic function of $t$. Therefore, $B(T) = e^{-\int_0^T r(t) dt}$, establishing a relation between the asset return and the interest rate.

With this model setup, we present the following theorem, whose full proof can be found in Appendix A:

**Theorem 1.** The Markov-modulated jump diffusion model describes the equilibrium requirement (1.2) for the zero-coupon bond and the asset price if and only if $\mu_0(t) = r(t) + \sigma_1 \rho (1 - a) - \eta(t) = \theta + (1 - a) \mu_1(t) + \frac{1}{2} \sigma_1^2 \mathbb{E}[Y_n^b]$, where $\eta(t) = \mathbb{E}\left[\frac{\ln(B(t))}{t}\right]$ is the dividend yield expectation for the Markov-modulated jump diffusion model.

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7. This setting extends Naik and Lee (1990) and Koo (2002), which assume that $\Phi_1(t)$ is a Poisson process and $\mu_1(t)$ is a constant.

8. Note that when $a_1 = 2a_2 = \ldots = 2a_k = 1$, then $2a_k = 1$, because $\eta(t) = \mathbb{E}\left[\ln\left(\mathbb{E}\left[\frac{\ln(B(t))}{T}\right]\right)\right]$, this assumption reduces to $\theta = 1 - a \mu_1(t)$, the parallel assumption used in Koo (2002).

---

9. Note that if $a_1 = 2a_2 = \ldots = 2a_k = 1$, or $a_k = 0$ for $k \in X$ and $a_k \to \infty$ where $t \in X \times \mathbb{E}$, then $\eta(t) = \mathbb{E}\left[\frac{\ln(B(t))}{t}\right]$, which is a constant, and we arrive at Koo’s (2002) findings. As a result, we can consider the Poisson class to be a special case of our more general model.
The proof of Theorem 2 is provided in Appendix B. This is again a more general result that nests traditional Poisson models. If \( b \to 0 \) or \( \bar{y} \to 0 \) with a probability of 1, then the pricing formulas reduce to the corresponding Black–Scholes formulas. Similarly, if \( \bar{y}_1 = \bar{y}_2 = \cdots = \bar{y}_k = \bar{y} \) or \( \bar{y}_k \to 0 \) for \( k \in X \) and \( \bar{y}_k \to \infty \) for \( i \in X \) and \( i \neq k \), the pricing formulas reduce to the Merton (1976) PJM results, where \( N(t) \) is a Poisson process with a jump rate of \( \lambda (\frac{c(t)}{y(t)} + 1) \) or \( \lambda (\frac{c(t)}{y(t)} - 1) \), respectively. If \( \lambda = 0 \), they further reduce to the Black–Scholes formulas.

Consider a lognormal distribution for jump size. Let \( \sigma^2 \) denote the variance of the logarithm of \( Y_n \) and \( \mu_y \) the mean of the logarithm of \( Y_n \). The closed-form solution of a European call option is given by \( M_{C,y}(0) = \sum_{m=0}^{\infty} \left( C(S,0), K, T, \frac{1}{\sqrt{2\pi}} \right) \right) dt \), \( \sigma(m) \sum_{n=1}^{\infty} \left( \pi_i Q_n(m,T) \right) \), where \( \mu_y \) and \( \gamma \) are m jumps, and \( \gamma = \mu_y + \frac{1}{2} \sigma^2 \). The variance of the asset price is \( \sigma^2(m) = \sigma^2 + m \gamma^2 / T \) and \( Q_n(m,T) \) is the new transition probability of jumps from state \( i \) at time 0 to state \( j \) at time \( T \), where \( \eta(t) = d \log \left( \sum_{n=0}^{\infty} \sum_{i=1}^{\infty} \left( \zeta + 1 \right)^i \pi_i P_i(n,t)/dt \right) \) is a predictable process. The price of a European call on a futures contract is \( M_{C,y}(0) = \sum_{m=0}^{\infty} \left( C(F(0,T)), K, T, \frac{1}{\sqrt{2\pi}} \right) \right) dt \), \( \sigma(m) \sum_{n=1}^{\infty} \left( \pi_i Q_n(m,T) \right) \).

Note that, as before, if \( \zeta \to 0 \) or \( \mu_y \to 0 \) and \( \sigma^2 \to 0 \), the pricing formulas reduce to the Black–Scholes results, and if the Markov-modulated Poisson process reduces to a standard Poisson, the pricing formula also reduces to Merton’s results. The joint probability of the state and jumps may be computed using Laplace inverse transformation or numerical inversion, and the closed-form formulas for options or options on futures can be evaluated. Indeed, we calculate closed-form solutions in a general equilibrium framework for options and options on futures prices under a variety of jump diffusion specifications and also demonstrate the existence of jumps and the three previously mentioned critical empirical characteristics: leptokurtic and asymmetric features, volatility smile and surface, and volatility clustering. All subsequent tests assume a lognormal distribution for jump size; results for a full default are quantitatively identical and available upon request.

3. Parameterization and empirical fit

To test these theoretical findings, we collect daily data for the DJIA as well as its constituent stocks from Datastream from January 1999 through the end of December 2008. We simplify our analysis by utilizing our model using a two-state MJM. Summary statistics are presented in Table 2 and show the number of days for each year in which DJIA returns exceeded 2%, 3%, and ±5%. For example, looking at the ±3% threshold, there are no days with the mean of the loga-

<table>
<thead>
<tr>
<th>Year</th>
<th>1999</th>
<th>2000</th>
<th>2001</th>
<th>2002</th>
<th>2003</th>
<th>2004</th>
<th>2005</th>
<th>2006</th>
<th>2007</th>
<th>2008</th>
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<td>252</td>
<td>252</td>
<td>252</td>
<td>252</td>
<td>252</td>
<td>250</td>
<td>253</td>
</tr>
<tr>
<td>Days exceeding ±2%</td>
<td>16</td>
<td>29</td>
<td>24</td>
<td>52</td>
<td>15</td>
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<tr>
<td>Days exceeding ±3%</td>
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<td>15</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>1</td>
<td>39</td>
</tr>
<tr>
<td>Days exceeding ±5%</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>13</td>
</tr>
<tr>
<td>Average return × 10⁻⁴</td>
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<td>1.77</td>
<td>4.60</td>
<td>–8</td>
<td>8.18</td>
<td>5.03</td>
<td>3.06</td>
<td>4.58</td>
<td>2.30</td>
<td>–0.2</td>
</tr>
</tbody>
</table>

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cause it provides an attractive computational alternative. This involves a two-step process. The first step is the E-step or expectation step. In this step, we compute the \( Q \) function, that is, the expectation of the log complete-data likelihood function \( \log \Pr(R(t), \tilde{q}_t | \Theta^{(k-1)}) \), for the hidden variables \( \tilde{q}_t \) given returns \( R \) and parameters \( \Theta^{(k-1)} \). The general \( Q \) function is \( Q(\Theta, \Theta^{(k-1)}) = E[\log \Pr(R, \tilde{q}_t | \Theta) | R, \Theta^{(k-1)}] \). Adapted to this MMJM, we have the following:

\[
\begin{align*}
Q(\Theta, \Theta^{(k-1)}) &= \sum_{t=1}^{2} \sum_{i=1}^{T} \log p(q_{t,i} | \tilde{R}, \Theta^{(k-1)}) \\
&\quad + \sum_{i=1}^{T} \sum_{t=2}^{T} \log p(q_{t,i} | \tilde{R}, \Theta^{(k-1)}) \\
&\quad + \sum_{i=1}^{T} \sum_{t=1}^{T} \log q_t(R(t), m; \mu, \sigma, \lambda_i, \mu_i, \sigma_i) \\
&\quad \times \Pr(q_{t,i} = j, n_t = m | \tilde{R}, \Theta^{(k-1)}).
\end{align*}
\]

(2.3)

This \( Q \) function can be decomposed into three terms: the first is dependent on the initial probability, the second is dependent on the transition probabilities, and the third is dependent on the parameters of the Normal density given returns and parameters from the \((k-1)\)th iteration.

The second step is the M-step or maximization step. In this step, we maximize the \( Q \) function to update the parameter set as \( \Theta^{(k)} = \arg \max Q(\Theta, \Theta^{(k-1)}) \). By the iteration and recursive computation of two steps, the parameters converge the \( Q \) function to the local maximum in the incomplete-data likelihood function.

### 3.2. Empirical fit compared to Poisson jump diffusion and Black-Scholes model

In Table 3, we present the model fit results for the DJIA and for each of the component stocks. "PJM" denotes the fit results for of the Poisson jump diffusion model, and "MMJM" denotes the fit results derived from the Markov-modulated jump model with two states. For example, in the row labeled DJIA, the mean and standard deviation of the log return for the PJM are 2.75E–04 and 0.0078, respectively. The additional jump components prescribe a drift component of \(-1.09E \text{–} 03\) and a volatility component of 0.0186. Of particular interest is the jump rate, which is found to be 0.2684, ostensibly an “average” jump rate. In contrast, when we examine MMJM, we find that the jump rate in the quiet state is 0.05 whereas in the excited state is 1.78; that is, there are distinctly different jump rates in the different states.

In the final column of Table 3, we test whether the data fit the PJM better than BSM (Black Scholes model), and whether MMJM outperforms PJM. Specifically, we apply the following likelihood ratio test (LRT):

\[
L_{\text{LRT}} = -2 \log \frac{L_0(\Theta)}{L_d(\Theta)} \sim \chi^2_{2(n-1)}.
\]

where \( L_0(\Theta) \) is the likelihood function under the null hypothesis, \( L_d(\Theta) \) is the likelihood function under the alternate, and \( d \) denotes the degree of the parameters between the \( H_0 \) and \( H_1 \) constraints. If \( \Lambda > \chi^2_{2(n-1)} \), \( H_0 \) is rejected. The respective null hypotheses are that the PJM and MMJM do not hold. The difference in the number of
parameters for PJM (BSM) vs. MMJM (PJM) is 4 (3) and defines the value of $d$. At a confidence level of $1 - \alpha = 95\%$, the critical value for the former test is $Z_{0.05}^2 = 1.96$. For the DJIA, the test yields a likelihood ratio $A_2$ of 380.1(>9.49), implying that the MMJM significantly dominates the PJM (by the likelihood ratio denoted $A_2$ in Table 3). Similarly, the PJM significantly dominates the BSM (by the likelihood ratio denoted $A_1$ in Table 3). These conclusions are likewise true for each of the 30 constituent stocks. In every test, the likelihood ratios are consistently in the hundreds.

Given the relatively inflexible nature of the BSM and PJM, it may not be surprising that they underperform MMJM. Stochastic volatility models introduce flexibility through the variance term(s), essentially allowing for infinite variance regimes as well as mean reversions in variance. Our MMJM assumes a constant continuous variance but, importantly, can address volatility and jump clustering behavior as well as the dynamics of jump probability. Table 4 Panel A presents time series representations of the DJIA and jump characteristics. We see that the DJIA has experienced both bull and bear markets over the test period, with varying levels of volatility. This varying volatility is captured by the MMJM by fluctuating periods of high and low jump intensity. To characterize each period as high- or low-intensity, we use the Baum–Welch algorithm (see Rabiner, 1989 and Bilmes, 1998), a forward/backward procedure that finds the joint probability of seeing a partial return sequence before time $t$ and ending up at a particular state at time $t$, while simultaneously finding the conditional probability of the partial sequence after time $t$ given that we started at a particular state at time $t$. Essentially, at each point in time, it determines the state by looking at both the preceding and subsequent sequences of returns. Using this procedure, we find that the first 1000 days of the period seem to be predominantly characterized by a high jump intensity (the excited state) whereas the next 1000 are characterized by a low one (the probability of the quiet state is nearly 1 for the entire period). The test period ends with a relatively consistent period of high intensity, corresponding to the financial crises of 2007 and 2008.

These conclusions are true not only for the DJIA but for each constituent stock, without exception, as illustrated in Panel B. The median of the 30 constituent stocks and jump states are graphically illustrated. Again, the second 1000 days (the period from 2003 to 2006) represent the quiet state whereas the first and last periods both exhibit generally high jump intensity, with the trend in the first 1000 day being less obvious when looking at the cross-section of constituent stocks than when looking at the DJIA. That notwithstanding, in general, we find that returns do appear to follow a MMJM. Indeed, as shown below, we find that jump clustering is a potential, even probable, channel for volatility clustering.

### 3.3. Empirical fit compared to stochastic volatility models

Hull and White (1987) propose stochastic volatility through the variance term of the Geometric Brownian motion to price options.

**Table 4**

Time Series and Jump States for the DJIA and Component Stocks. Panel A presents the daily time series for the DJIA, with the low jump intensity state probability overlayed. Described in the right-side axis and dotted graph is the probability of being in the quiet state, when the probably is 0, the market is in state 2 (the excited state). Panel B presents the same for the median level and state probabilities of the 30 component stocks.

**Panel A: DJIA Index and Jump State**

<table>
<thead>
<tr>
<th>Dow Jones Index</th>
<th>Probability of quiet state</th>
</tr>
</thead>
<tbody>
<tr>
<td>14,000</td>
<td>0.1</td>
</tr>
<tr>
<td>12,000</td>
<td>0.2</td>
</tr>
<tr>
<td>10,000</td>
<td>0.3</td>
</tr>
<tr>
<td>8,000</td>
<td>0.4</td>
</tr>
<tr>
<td>6,000</td>
<td>0.5</td>
</tr>
</tbody>
</table>

1999/1/5 2000/12/27 2002/12/27 2004/12/22 2006/12/15 2008/12/11

**Panel B: Component Stocks Median Level and Jump State**

<table>
<thead>
<tr>
<th>Median of component stocks</th>
<th>Probability of quiet state</th>
</tr>
</thead>
<tbody>
<tr>
<td>90</td>
<td>0.1</td>
</tr>
<tr>
<td>80</td>
<td>0.2</td>
</tr>
<tr>
<td>70</td>
<td>0.3</td>
</tr>
<tr>
<td>60</td>
<td>0.4</td>
</tr>
<tr>
<td>50</td>
<td>0.5</td>
</tr>
</tbody>
</table>

1999/1/5 2000/12/27 2002/12/27 2004/12/22 2006/12/15 2008/12/11
and standard deviation $\sigma_v(t)$. The variance of variance $\mathbb{V}_v(t)$ is the Brownian motion under the physical measure, and $\mathbb{W}_v(t)$ is the Brownian motion under the $\mathbb{P}$ measure. The correlation $\rho$ of the stock and variance processes is \( \rho \). Corr$(\mathbb{W}_v(t), \mathbb{dV}_v(t)) = \rho$.

Bates (2005) prices American options under stochastic volatility and jump diffusion processes with systematic jumps and volatility risk, and Bates (2000) uses S&P 500 fudures to investigate the empirical fit of options prices in this framework as follows:

\[
\frac{dS(t)}{S(t)} = \mu_d dt + \sqrt{V(t)} \mathbb{dW}_Y(t)
\]

\[
\frac{dV(t)}{\sqrt{V(t)}} = \kappa_V (\theta_V - V(t)) dt + \sigma_V V(t) \mathbb{dW}_Y(t),
\]

where $\mu_d$ denotes the instantaneous return of the stock at time $t$, $\kappa_V$ is the speed of mean reversion, $\theta_V$ is the unconditional variance, $\sigma_V$ determines the variance of variance, $\mathbb{W}_Y(t)$ is the Brownian motion under the physical measure, and $\mathbb{W}_Y(t)$ is the Brownian motion under the $\mathbb{P}$ measure. The correlation $\rho$ of the stock and variance processes is $\rho$.

The evidence of jump behavior is particularly interesting. Whereas SV models may account for volatility clustering, they cannot address jump clustering, in which periods of high (low) jump arrival rates are followed by periods of continued high (low) jump arrival rates. This phenomenon, which was observed anecdotally during the internet bubble and financial crisis, is a chief motivation of MMJsMs, something we demonstrate empirically in the following sections.

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**Table 5**

Empirical parameters and fit of SVM vs. SVJM. This table presents parameter and fit results for the SVM and SVJM models, for the DJIA and each constituent stock. `SVM` denotes estimates of the model in Eq. (2.5), and `SVJM` denotes those of the jump model in Eq. (2.6). In the row labeled `DJIA` for `SVM`, the mean of the log return is 4.03E $-$ 04. The speed of mean reversion is 0.3; unconditional variance is 2.71 $-$ 4E; and the standard deviation of the variance is 1.68 $-$ 02. In the final two columns of Table 5, we compare the empirical fit of the models, first testing SVJM vs. SVM and then SVJM vs. PJM. Similarly, we apply the likelihood ratio test $A_3$ with the null hypothesis that returns follow the SVM (the alternative hypothesis is that they follow SVJM). The likelihood ratio test $A_3$ has a null hypothesis that returns follow PJM (the alternative hypothesis is that they follow SVJM). We find that SVJM significantly outperforms SVM and PJM for the DJIA index and all 30 constituent stocks. Put simply, there are both stochastic volatility and jump features in these asset returns.

<table>
<thead>
<tr>
<th>SVM parameters</th>
<th>SVJM parameters</th>
<th>LRT parameters</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mu_d \times 10^{-4}$</td>
<td>$\mu_d \times 10^{-4}$</td>
<td>$\mu_d \times 10^{-4}$</td>
</tr>
<tr>
<td>DJIA</td>
<td>4.03</td>
<td>1.38</td>
</tr>
<tr>
<td>AA</td>
<td>1.54</td>
<td>2.31</td>
</tr>
<tr>
<td>XLY</td>
<td>0.34</td>
<td>2.63</td>
</tr>
<tr>
<td>AIG</td>
<td>-2.06</td>
<td>2.38</td>
</tr>
<tr>
<td>BA</td>
<td>6.05</td>
<td>3.85</td>
</tr>
<tr>
<td>CAT</td>
<td>3.93</td>
<td>2.01</td>
</tr>
<tr>
<td>C</td>
<td>1.84</td>
<td>2.12</td>
</tr>
<tr>
<td>CO</td>
<td>1.25</td>
<td>2.27</td>
</tr>
<tr>
<td>DIS</td>
<td>1.09</td>
<td>2.01</td>
</tr>
<tr>
<td>DD</td>
<td>-4.39</td>
<td>2.01</td>
</tr>
<tr>
<td>XOM</td>
<td>12.11</td>
<td>1.14</td>
</tr>
<tr>
<td>GE</td>
<td>-2.37</td>
<td>2.01</td>
</tr>
<tr>
<td>GM</td>
<td>-0.79</td>
<td>2.01</td>
</tr>
<tr>
<td>HPQ</td>
<td>5.00</td>
<td>2.01</td>
</tr>
<tr>
<td>HD</td>
<td>1.17</td>
<td>2.01</td>
</tr>
<tr>
<td>HON</td>
<td>4.01</td>
<td>2.01</td>
</tr>
<tr>
<td>INTC</td>
<td>3.76</td>
<td>2.01</td>
</tr>
<tr>
<td>IBM</td>
<td>2.24</td>
<td>2.01</td>
</tr>
<tr>
<td>JPM</td>
<td>-0.32</td>
<td>2.01</td>
</tr>
<tr>
<td>JNJ</td>
<td>2.35</td>
<td>2.01</td>
</tr>
<tr>
<td>MCD</td>
<td>3.03</td>
<td>2.01</td>
</tr>
<tr>
<td>MRK</td>
<td>2.68</td>
<td>2.01</td>
</tr>
<tr>
<td>MSFT</td>
<td>-3.73</td>
<td>2.01</td>
</tr>
<tr>
<td>MMM</td>
<td>0.84</td>
<td>2.01</td>
</tr>
<tr>
<td>MO</td>
<td>7.00</td>
<td>2.01</td>
</tr>
<tr>
<td>PFE</td>
<td>-3.04</td>
<td>2.01</td>
</tr>
<tr>
<td>PG</td>
<td>-3.04</td>
<td>2.01</td>
</tr>
<tr>
<td>T</td>
<td>5.29</td>
<td>2.01</td>
</tr>
<tr>
<td>UTX</td>
<td>0.32</td>
<td>2.01</td>
</tr>
<tr>
<td>VME</td>
<td>0.00</td>
<td>2.01</td>
</tr>
</tbody>
</table>
Table 6

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Default calibration</th>
<th>New price (base = 12.604)</th>
<th>Difference</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_1$</td>
<td>1.00</td>
<td>12.5998</td>
<td>-0.0042</td>
</tr>
<tr>
<td>$a_2$</td>
<td>1.00</td>
<td>12.6072</td>
<td>0.0032</td>
</tr>
<tr>
<td>$\lambda_1$</td>
<td>5.00</td>
<td>12.6220</td>
<td>0.0180</td>
</tr>
<tr>
<td>$\lambda_2$</td>
<td>1.00</td>
<td>12.6044</td>
<td>0.0004</td>
</tr>
<tr>
<td>$-\ln B (t, T)$</td>
<td>0.02</td>
<td>12.6721</td>
<td>0.0681</td>
</tr>
<tr>
<td>$\sigma$</td>
<td>0.20</td>
<td>13.0057</td>
<td>0.4017</td>
</tr>
<tr>
<td>$\mu_T$</td>
<td>0.02</td>
<td>12.6230</td>
<td>0.0190</td>
</tr>
<tr>
<td>$\sigma_T$</td>
<td>0.02</td>
<td>12.6168</td>
<td>0.0128</td>
</tr>
<tr>
<td>$T$</td>
<td>0.50</td>
<td>12.8726</td>
<td>0.2686</td>
</tr>
</tbody>
</table>

3.4. Sensitivity analysis

Option prices derived through MMJM depend on several parameters. It is helpful, then, to test the impact of variations in each of these parameters on option estimates. Table 6 reports these sensitivities. In each case, the parameter is perturbed by a factor of 1.1 with all other parameters being set to default values. The resulting price change is noted as well as the difference from the default specification.

Price is most sensitive to changes in volatility $\sigma$, although this parameter is likely to be relatively straightforward and fixed for a given period in most applications. The first four parameters are those that are most specific to MMJM, though small perturbations in these variables do not affect option prices very much. An increase in parameter $a_1$ means that the Markov chain will exit state 1, the excited state, more rapidly. As a result, there is an overall reduction in the jump rate such that the option value is reduced. The opposite is true for $a_2$. A higher value means the chain exits the dormant state more rapidly, resulting in increased jump rates and option prices. Parameters $\lambda_1$ and $\lambda_2$ are the jump rates associated with each state in which $\lambda_1$ exceeds $\lambda_2$. An increase in either of these leads to a higher overall jump rate and higher option prices. The remaining general options pricing variables behave as expected, but we note that the four Markov-specific parameters have, in fact, the smallest impact on pricing, indicating that the pricing model is relatively stable with regard to the estimation of these parameters.

4. Key model features

Here, we demonstrate the leptokurtic, asymmetric, volatility surface/smile, and, most importantly, volatility clustering features of MMJM with two states. Although all three features are empirically well-founded, the majority of related models exhibit only the first two characteristics. The two-state MMJM generates all three features due to the state-based nature of the jump diffusion. We provide more details in the following.

4.1. Leptokurtic and asymmetric features

A litany of work has suggested that log returns exhibit leptokurtic and asymmetric features. In the context of the MMJM with two states, mean, variance, skewness, and kurtosis may be calculated as presented in Appendix C. According to the estimated results, these four summary values are presented in Table 7 for each stock.

4.2. Volatility smile and surface

The observation that implied volatility is not constant for different levels of moneyness has been well-documented in a variety of empirical and theoretical settings. The ability to allow for this so-called “smile” feature is often considered a critical objective for asset pricing models. We test the two-state Markov-modulated jump diffusion model empirically using DJIA index options and illustrate its characteristics using data from September 30, 2008 with maturity on August 30, 2009. Using our closed-form solution for options prices, presented as Theorem 2 of MMJM, and assuming lognormal jump size, we calculate call values and implied volatility for a variety of strike prices. The results are exhibited in Table 9.

As seen in Panel A, implied volatility is not constant but instead generates a smile-like curvature, consistently with empirical findings. Similarly, by varying both strike price and time to maturity, we generate the volatility surface presented in Panel B. Again, we find that volatility implied by MMJM decreases with moneyness and does so at a decreasing rate. The curvature is most severe for the closest maturity and becomes flatter as the maturity lengthens. We should emphasize this serves only to illustrate that MMJM can generate smile-shaped implied volatility curves. We do not test whether these values are empirically accurate; we leave this subject for future studies.

4.3. Volatility clustering

The volatility clustering phenomenon, documented as early as Mandelbrot (1963), essentially describes that periods of high (low) volatility are generally followed by periods of continued high (low) volatility. Cont (2005) investigates several economic mechanisms that have been proposed to explain this clustering, which generally centers around the behavioral tendencies of market participants or the news arrival process. Our model follows Cont (2005) in that we propose a mechanism of heterogeneous arrivals of information in the form of a MMJM.

Consider the case of a two-state MMJM and Eq. (2.1). Let...
The autocorrelation of volatility, consider squared returns. Let \( \mu_x = 0 \) and \( \mu_Y = 0 \). The autocorrelation of the squared return is the following:

\[
\rho_k = \frac{(p_{11} + p_{22} - 1)^2 (\lambda_1 - \lambda_2)^2 \sigma_{x Y}^4 (1 - p_{11})(1 - p_{22})}{(1 - p_{11})(\lambda_1 \sigma_Y^2 + (1 - p_{22})(2 - p_{11} - p_{22})), (1.31)}
\]

When jump rates are equal \( (\lambda_1 = \lambda_2) \), \( p_{11} + p_{22} = 1 \), or the mean of the jump size is zero \( (\mu_x = 0) \), return is uncorrelated.

To investigate the autocorrelation of volatility, consider squared returns. Let \( \mu_x = 0 \) and \( \mu_Y = 0 \). The autocorrelation of the squared return is the following:

\[
\rho_k = \frac{(p_{11} + p_{22} - 1)^2 (\lambda_1 - \lambda_2)^2 \sigma_{x Y}^4 (1 - p_{11})(1 - p_{22})}{(1 - p_{11})(\lambda_1 \sigma_Y^2 + (1 - p_{22})(2 - p_{11} - p_{22})), (1.31)}
\]

Where jump rates are equal \( (\lambda_1 = \lambda_2) \), \( p_{11} + p_{22} = 1 \), or the mean of the jump size is zero \( (\mu_x = 0) \), return is uncorrelated.
two are close to independent and the likelihoods of staying in a state or switching are approximately the same. If \( \rho_y \) is equal to 0, the model reduces to the Normal diffusion model because \( \rho_y = 0 \). The autocorrelation is again zero, and there is no volatility clustering. In contrast, as \( p_{11} + p_{22} \) increases and approaches 2, the autocorrelation of volatility (that is, volatility clustering) is most significant. Therefore, although our model nests other important classes of models that do not exhibit volatility clustering, it additionally captures this phenomenon when the sum of \( p_{11} \) and \( p_{22} \) is close to 2.

Most importantly, if \( k_1 \) is equal to \( k_2 \), the model reduces to PJM and the autocorrelation of the squared return is zero. Again, there is no volatility clustering. Therefore, it is precisely the existence of two distinct states with two different jump rates that generates and that captures clustering.

Table 10 presents empirical estimates for \( p_{11} \) and \( p_{22} \) for the DJIA and all of its constituent stocks. The sum of these two values is nearly always within a few hundredths of 2. Indeed, the sums range from 1.9341 to 1.9983. Hence, volatility clustering is substantial. Panel A illustrates these values. In Panel B, we see that the model prescribes and the data indicate negligible autocorrelation in log returns. However, in Panel D, we see that MMJM predicts a substantially positive autocorrelation in the squared log return that decays as lags lengthen. This trend is also exhibited in the data, with autocorrelations ranging between 0.2 and 0.4 for the first 10 lags and steadily declining as lag length increases. The model captures not only the existence of volatility clustering but seems to generally describe the magnitude and decay of the clustering as well. Again, clustering in our framework arises from there being different jump rates for different states of the economy (in our case, two). That is, volatility clustering arises from jump clustering.

We draw similar graphs for the DJIA constituent stocks, but, in the interest of space, we do not present them here. Qualitatively identical results are found and are available upon request.
5. Conclusion

In this study, we propose a Markov-modulated jump diffusion model in which jump frequency switches over time according to the state of the economy. The model captures the leptokurtic, asymmetric, volatility smile, and volatility clustering features and demonstrates the dynamic nature of jump arrival rates in the financial market. We derive a closed-form formula for a European call option in a general equilibrium framework and provide the sensitivity of price to parameter estimates, finding that price is relatively stable with respect to the Markov-specific parameters. Using this closed form, we then apply a discrete time analog to a two-state Markov-modulated jump model via an expectation and maximization (EM) algorithm to estimate parameters and demonstrate the aforementioned features using 10 years of data for the Dow Jones Industrial Average and its constituent stocks.

Whereas general stochastic volatility models may capture observed leptokurtosis, volatility smile, and volatility clustering, we
show that even SV models indicate the existence of jumps. However, existing SV models cannot determine the role of jumps in clustering. MMJM addresses this concern, empirically showing that jump clustering is significant in the stock market. These results indicate the need for additional extension work to incorporate both varying continuous volatility and jump clustering to address the behavior of volatility and jump risk in the financial market.

Appendix A. General equilibrium for Markov modulated jump diffusion models

Based on the bond price, we can find the relation between the interest rate and the endowment and the new pricing probability measure. By using the Girsanov theorem for the Markov-modulated jump diffusion model (see Björk et al., 1997) we obtain, under $P'$, $W^i_t := W^i_t - \sigma^i(a - 1)t$ as a Brownian motion. Further, under $P'$ the new Markov-modulated Poisson process with the invariant transition rate and the new jump frequencies $\{\lambda_i^j, \lambda_i^j, \ldots, \lambda_i^j\}$ where $\lambda_i^j := \lambda_i^{\text{jump}} - \lambda_i^{\text{jump}}$. The new jump size, $Y^i_n$, has the probability density $f_Y^i(y) = \left(\frac{1}{\lambda_i^{\text{jump}}}\right) y_i^{1-\lambda_i^{\text{jump}}}(y_i^{1-\lambda_i^{\text{jump}}})$. Therefore, the dynamics of $S(t)$ are given by

$$dS(t) = (\mu(t) + \sigma(t)\rho(a - 1))dt + \sigma(dW^i_t + \sqrt{1 - \rho^2}dW^j_t)$$

(A.1)

where $\eta(t) = \frac{d\mu(t)}{d\eta(\mu)}(\sum_{n=1}^{\infty} Y^i_n - 1)$. Then, under the new Markov-modulated jump diffusion model, the dynamic process of $S(t)$ is

$$dS(t) = (\mu(t) + \sigma(t)\rho(a - 1) - 1 - \omega(t))dt + \sigma(dW^i_t + \sqrt{1 - \rho^2}dW^j_t)$$

(A.2)

If $S(t)$ satisfies (A.2) in the equilibrium setting $r(t) = \theta + (1 - \theta)\mu_t(t) - \frac{1}{2}\sigma^2(1 - \theta)(2 - a - \eta(t)) > 0$, we must have $\mu(t) + \sigma(t)\rho(a - 1) + \eta(t) = r(t)$. On the other hand, if (A.2) is satisfied under the measure $P'$, then the dynamics of $S(t)$ are given by

$$dS(t) = r(t)dt - \eta(t)dt + \sigma(dW^i_t + \sqrt{1 - \rho^2}dW^j_t)$$

(A.3)

from which Eq. (1.5) follows.

Appendix B. Option pricing formulas

Proof of Theorem 2.

(1) Under the conditions of $q(0) = i$ and $q(T) = j$ and given the conditional jump times $\Phi(T) = m$, $\Phi_0 = \prod_{m=1}^{\infty} Y^i_n$, and $Z$ is the standard normal distribution, we can rewrite Eq. (1.5) as

$$S(T) = S(0) \exp \left( \int_0^T (r(t) - \eta(t) - 1/2\sigma^2)dt + \sigma \sqrt{T}Z \right) \Phi_0$$

with transition probability $Q_i(m, T)$. Hence, under the rational expectations setting, the equilibrium price of the call option in the Markov-modulated jump diffusion model is

$$M^i(0) = \sum_{m=0}^{\infty} \left( E \left( C(S(0)|L(T)|\Phi_0, T, m, \int_0^T r(t)dt, \sigma^i, \Phi^i(T) = m \right) \prod_{i=1}^{m} \sum_{j=1}^{\infty} \pi_{i,j} \right)$$

where

$$C(S(0)|L(T)|\Phi_0, T, m, \int_0^T r(t)dt, \sigma^i, \Phi^i(T) = m)$$

is the option price in the Black–Scholes formula with the stock price $S(0)|L(T)|\Phi_0$, the strike price $K$, the maturity day $T$, the deterministic interest rate $r(t)$, and the volatility of the stock price $\sigma^i$. Define $E^i$ as the expectation operator under the distribution $P^i$, and

$$d^i(\pm) = \frac{\log(E(S(0)|L(T)|\Phi_0, K, \int_0^T r(t)dt))}{\sigma^i \sqrt{T}}$$

(2) Using Eq. (2.8) and the definition of the futures, we obtain

$$E^i(B(0,T)|T'|\Phi^i(T')) = \sum_{m=0}^{\infty} E \left( C(F(0,T)|L(T)|\Phi_0, K, T, \int_0^T r(t)dt, \sigma^i, \Phi^i(T) = m \right) \prod_{i=1}^{m} \sum_{j=1}^{\infty} \pi_{i,j}$$

where $d^i(\pm) = \frac{\log(E(F(0,T)|L(T)|\Phi_0, K, \int_0^T r(t)dt))}{\sigma^i \sqrt{T}}$.

Appendix C. The mean, variance, skewness, and kurtosis of the Markov-modulated jump diffusion model

The mean, variance, skewness, and kurtosis may be calculated as follows (details available upon request):

Mean $\mu = \mu, E(\Phi)$.

(C.1)

Variance $\sigma^2 = \sigma^2 + \delta^i E(\Phi) + \mu^i_2 E(\Phi^2) - E(\Phi^2)$.

(C.2)

Skewness $3(\mu_i \delta^2 + 2\mu^i_2)E(\Phi^2) - E(\Phi^2) + 3\mu^i_2 E(\Phi^2) - E(\Phi^2)$.

(C.3)

Kurtosis $\left( \frac{\sigma^2 + \delta^2 E(\Phi) + \mu^i_2 E(\Phi^2) - E(\Phi^2)}{\sigma^2 + \delta^2 E(\Phi) + \mu^i_2 E(\Phi^2) - E(\Phi^2)} \right)^2$

(C.4)

where

$$E(\Phi) = \sum_{i=1}^{2} P(q_i = i) \lambda_i,$$

$$E(\Phi^2) = \sum_{i=1}^{2} \pi_i \lambda_i + \sum_{i=1}^{2} \pi_i \lambda_i^2,$$

$$E(\Phi^3) = \sum_{i=1}^{2} \pi_i \lambda_i + 3 \sum_{i=1}^{2} \pi_i \lambda_i^2 + \sum_{i=1}^{2} \pi_i \lambda_i^3,$$

$$E(\Phi^4) = \sum_{i=1}^{2} \pi_i \lambda_i + 7 \sum_{i=1}^{2} \pi_i \lambda_i^2 + 6 \sum_{i=1}^{2} \pi_i \lambda_i^3 + \sum_{i=1}^{2} \pi_i \lambda_i^4.$$
Appendix D. Volatility clustering in the Markov-modulated jump diffusion models

Let $\lambda_t \neq \lambda$, $\mu_t \neq 0$, and $p_{11} + p_{22} \neq 0$. The autocorrelation function of the return is

$$
\rho_k = \frac{\text{cov}(R(t), R(t-k))}{\sqrt{\text{var}(R(t))}\sqrt{\text{var}(R(t-k))}}
$$

$$
= \frac{\text{cov}(\hat{z}(t)\hat{Y}(t), \hat{z}(t-k)\hat{Y}(t-k))}{\sqrt{\text{var}(R(t))}\sqrt{\text{var}(R(t-k))}}.
$$

To evaluate (D.1), we first compute $\text{Var}(R(t)) = \text{Var}(\mu + \sigma Z(t) + \hat{z}(t)\hat{Y}(t))$. Note that

$$
\text{Var}(\mu + \sigma Z(t) + \hat{z}(t)\hat{Y}(t)) = \sigma^2 + \frac{1}{2(p_{11} - p_{22})^2} \lambda_1 \sigma_1^2 + \frac{1}{2(p_{11} - p_{22})^2} \lambda_2 \sigma_2^2 + \frac{1}{2(p_{11} - p_{22})^2} (\lambda_1 - \lambda_2)^2 \sigma_1^2.
$$

Next, we compute $\text{Cov}(\hat{z}(t)\hat{Y}(t), \hat{z}(t-k)\hat{Y}(t-k))$, which equals

$$
E \left( \sum_{n=1}^{N_1} \sum_{i=1}^{N_2} \hat{y}_n, \sum_{n=1}^{N_1} \sum_{i=1}^{N_2} \hat{y}_n \right) = \frac{1}{2(p_{11} - p_{22})^2} \sum_{n=1}^{N_1} \sum_{i=1}^{N_2} \hat{y}_n
$$

(D.3)

Let $\nu_1, \nu_2, \nu_3$ be the eigenvalues of the transition matrix (2.2), a simple calculation yields

$$
\nu_1 \nu_2 \nu_3 = \frac{1}{2(p_{11} - p_{22})^2}
$$

(D.4)

Putting equations (D.2) and (D.4) into equation (D.1), we obtain

$$
\rho_k = \frac{\text{cov}(R(t), R(t-k))}{\sqrt{\text{var}(R(t))}\sqrt{\text{var}(R(t-k))}}
$$

$$
= \frac{(p_{11} + p_{22} - 1)(1 - p_{11})(1 - p_{22})(\lambda_1 - \lambda_2)^2 \sigma_1^2}{(2(p_{11} - p_{22})^2)A}.
$$

Further, we let $\mu \approx 0$, then the autocorrelation of the squared return will be

$$
\rho_k = \frac{\text{cov}(R^2(t), R^2(t-k))}{\sqrt{\text{var}(R^2(t))}\sqrt{\text{var}(R^2(t-k))}}
$$

$$
= \frac{(p_{11} + p_{22} - 1)(1 - p_{11})(1 - p_{22})(\lambda_1 - \lambda_2)^2 \sigma_1^2}{(2(p_{11} - p_{22})^2)A^2}
$$

where

$$
A = \frac{2\sigma^4 + 4\sigma^2 \sigma_1^2 \sigma_1^2 (1-p_{11})(1-p_{22}) (\lambda_1-\lambda_2)^2 + \sigma_1^2 (1-p_{11})(1-p_{22})^2 \sigma_1^2 (\lambda_1-\lambda_2)^2}{(2(p_{11} - p_{22})^2)^2}.
$$

References


