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Uniform formulas for compound siphons, complementary siphons and characteristic vectors in deadlock prevention of flexible manufacturing systems

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Abstract Unmarked siphons in a Petri net modelling concurrent systems such as those in cloud computing induce deadlocks. The number of siphons grows exponentially with the size of a net. This problem can be relieved by computing compound (or strongly dependent) siphons based on basic siphons. A basic (resp. compound) siphon can be synthesized from an elementary (resp. compound called alternating) resource circuit. It however cannot be extended to cases where two elementary circuits intersect at a directed path rather than a single place (i.e., corresponding to a weakly dependent siphon). This paper develops a uniform formula not only for both cases but also valid for the complementary set of siphon and characteristic vectors. We further propose to generalize it to a compound siphon consisting of *n* basic siphons. This helps simplify the computation and the computer implementation to shorten the program size. Also, the formula is easier to be memorized without consulting the references due to the same underlying physics.

Keywords Petri nets · Deadlocks · Supervisory control · Flexible manufacturing systems (FMS) · Concurrent systems.

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Introduction

A flexible manufacturing system (FMS) is a computer controlled configuration where different operations can be executed. Generally it consists of various kinds of general-purpose workstations, a palletized and programmable material handling system, and other types of resources such as fixtures and buffers. To effectively operate an FMS and meet its production objectives, the use of limited resources among various competing jobs has to be carefully controlled or coordinated. Since various jobs are concurrently processed and these jobs have to share some common resources, deadlocks may occur in an FMS during its operation, which are undesirable phenomena in a highly automated FMS.

Petri nets have been a powerful tool to model and analyze the deadlock problem of FMS (Ezpeleta et al. 1995; Ferrarini et al. 1999; Jeng et al. 1999; Iordache et al. 2001; Zimmermann et al. 2001; Basile et al. 2004; Uzam et al. 2007; Hu et al. 2012a; Pla et al. 2012; Li et al. 2012).

First of all, Ezpeleta et al. pioneer a class of Petri nets (PN) called systems of simple sequential processes with resources (S^3PR) (Ezpeleta et al. 1995). Deadlocks can be avoided by adding a control place and associated arcs to each emptiable siphon S to prevent it from being unmarked. However, generally too many control places and arcs are required.

On the other hand, the iterative control method in Iordache et al. (2001) reduces the number of monitors by finding all emptiable siphons in each iteration step. The method becomes very difficult and remains complex even for a moderate-size model.

Additionally, Uzam and Zhou (2006; Uzam et al. 2007) classified a reachability graph into deadlock-zone (unsafe states) including deadlocks and critical bad markings that inevitably lead to deadlocks, and live-zone (safe states) representing the live markings of the reachability graph. By

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singling out a first-met bad marking from the reachability graph at each iteration, a control place is added via constructing a place invariant (PI) [based on the method in Yamalidou et al. (1996)] of a Petri net to prevent this bad marking from being reached. The number of monitors required is greatly reduced and the controlled model is much more (though not maximally) permissive. But the construction of RG suffers the state explosion problem and the method is not suitable for large nets. It also needs to solve linear programming problems with exponential complexity. Furthermore, some monitors are redundant.

For solving above linear programming problem, Huang and Pan (2010, 2011; Hu et al. 2012a; Pan et al. 2013) proposed the crucial marking transition separation instances to enhance the computational efficiency based on the theory of Regions.

Besides, Hu et al. (2011a, 2011b, 2012a; Hu and Li 2009, 2010), for the first time, investigates the deadlock resolution in the paradigm of Petri nets allowing assembly operations, multiple-type and multiple-quantity resource acquisition, and production ratio among jobs. They elegantly prove the separation of ratio and supervisory controls. A mathematical programming-based method is developed to synthesize liveness-enforcing supervisors. Their approach outperforms many traditional methods in both generality and efficiency.

Li and Zhou (2004) propose the concept of elementary siphons (generally much smaller than the set of all emptiable siphons in large Petri nets) to minimize the new addition of control places. They classify emptiable siphons into two kinds: elementary and dependent. By adding a control place for each elementary siphon S_e, all dependent siphons S are controlled too, thus reducing the number of monitors required rendering the approach suitable for large Petri nets. As a result, for complex systems, it is essential to apply the concept of elementary siphons to add monitors; the number of which is linear to the size of the nets modelling the systems. However, the number of dependent siphons is exponential to the size of the net, even though that of elementary siphons is linear.

We find all elementary siphons without the knowledge of all SMS. Furthermore, it is much more efficient to identify and compute the T-characteristic vectors from the structure. For instance, Chao (2010a,b) shows that in an S^3PR, an SMS can be synthesized from a strongly connected resource subnet and any strongly dependent siphon corresponds to a compound circuit where the intersection between any two elementary circuits is at most a resource place. Also any elementary siphon is a basic one constructed from an elementary circuit where all places are resources. Thus, the set of elementary siphons can be computed without the knowledge of all SMS.

Controllability means how a dependent siphon depends on its elementary siphons. The best controllability occurs when each elementary siphon is least restrictively controlled (when control depth variable $\xi = 1$), while not generating new siphons, and the dependent siphon is already controlled and needs no monitor (Liu et al. 2011b; Wang et al. 2012). This can be checked by the MLI (called *controllability*). If the MLI is not satisfied, then they perform a LIP test which is an NP-hard problem. This implies that the MLI test is only sufficient, but not necessary. Thus, the time to verify against the MLI for all dependent siphons is exponential as for previous approaches. It is essential to efficiently compute the dependent siphons and related variables. We propose a sufficient and necessary test in Chao (2010a) for adjusting control depth variables in an S^3PR to avoid the sufficient-only time-consuming LIP test (NP-complete problem) required previously for some cases.

If the above LIP test also fails, then some ξ must be greater than 1 (bless permissive). After assigning ξ for each elementary siphon, one checks again the MLI. Continue such adjustment until the MLI is satisfied. The larger the control depth variable, the fewer states the system will reach. The control policy for weakly dependent siphons (Liu et al. 2011a; Xiong et al. 2010) is rather conservative (Li and Zhou 2004) (such that fewer states are reached) due to some negative terms in the controllability [a marking linear inequality (MLI)].

For strongly dependent siphons, the complementary set $[S]$ of dependent siphon S equals the union of those of its component elementary siphons. The controllability or MLI is derived accordingly. However, it is unknown in Li and Zhou (2004) how $[S]$ relates to those of its component elementary siphons. As a result, the control policy for weakly dependent siphons is rather conservative (Li and Zhou 2004) since negative terms in the controllability are ignored.

Chao (2010a) develops a better MLI (Marking Linear Inequality) test by discovering how $[S]$ relates to those of its component elementary siphons. The resulting control policy for weakly dependent hence can reach more good states. It turns out that the complementary set $[S]$ and those of its component siphons follow the same relationship as that of *T-characteristic vectors*.

So far, none in the literature deal with how a compound siphon S relates to those of its component siphons. As a result, one has to synthesize compound siphons from compound resource circuits (i.e., containing only resource places) using our method in Chao (2006). Furthermore, we have developed theory (Chao 2007) to efficiently extract SMS incrementally rather than the traditional global approach. Only linear number of basic siphons needs to be searched. Adding and deleting common sets of places from existing ones (called composition method), one can derive the compound siphons with much reduced search time. It is easily subject to computer implementation in a very efficient way compared with all current techniques since all these steps can be expressed in terms of formulas. However, the relationship between a compound siphon S and its component siphons remains unknown. We will show that the relationship follows that of *T-characteristic vectors*.

We discover that different cases can be unified by the same physics resulting in a single formula to compute dependent siphons, their complementary set of places, and characteristic *T-vectors*. This helps to memorize the formula and simplify the implementation since the codes for different cases can be shortened with a single set of codes for the uniform formula. This relieves the problem of computing siphons (and related variables for controlling the siphons), the number of which grows exponentially with the size of a net. Furthermore, it is desirable to compute S, $[S]$, and η directly from the structures independent to whether S is a strongly or a weakly dependent siphon. We are the very first to uncover the uniform method to do so.

The rest of the paper is organized as follows. Section "Preliminaries" presents the preliminaries about Petri nets and S^{3}PR, respectively. Section "Motivation" motivates the reader by presenting some simple examples. The results are proved and generalized in "Theory" section. A well-known S^{3}PR example has been illustrated to show the advantages in "Computation by structures". Finally, section "Example" concludes the paper.

Preliminaries

Here only the definitions used in this paper are presented. The reader may refer to Murata (1989) and Chao (2006) for more Petri net details. For a Petri net (N, M_0), a non-empty subset S(resp. τ) of places is called a *siphon* (resp. *trap*) if $^{\bullet}S \subseteq S^{\bullet}$ (resp. $^{\bullet}\tau \subseteq \tau^{\bullet}$, i.e., every transition having an output (resp. input) place in S has an input (resp. output) place in S(resp. τ). If $M_0\,(S) = \sum_{p \in s} M_0\,(p) = 0$, S is called an *empty siphon* at M_0. A *minimal siphon* does not contain a siphon as a proper subset. It is called a *strict minimal siphon* (*SMS*), if it does not contain a trap.

A *P-vector* (place vector) is a column vector $Y : P \rightarrow Z$ indexed by P where Z is the set of integers. The *incidence matrix* of N is a matrix $[N] : P \times T \rightarrow Z$ indexed by P and T such that $[N] = [N]^{+} - [N]^{-}$ where $[N]^{+}(p, t) = F(t, p)$ and $[N]^{-}(p, t) = F(p, t)$. We denote column vectors where every entry equals 0 by $\mathbf{0}$. Y^T and $[N]^T$ are the transposed versions of a vector Y and a matrix $[N]$, respectively. Y is a *P-invariant* (place invariant) if and only if $Y \bullet \neq \mathbf{0}$ and $Y^T\,[N] = \mathbf{0}^T$ hold where '$\bullet$' means a vector or matrix multiplication. $||Y|| = p \in P | Y(p) \neq 0$ is the *support* of Y. A *minimal P-invariant* does not contain another *P-invariant* as a proper subset.

Definition 1 (Ezpeleta 1995) A System of Simple Sequential Process with Resources (S^3PR) is a Petri net $N = (P \cup p^0 \cup P_R, T, F)$ defined as the union of a set of nets $N_i = (P_i \cup \{p_i^0\} \cup P_{Ri}, T_i, F_i)$ sharing common places, where the following statements are true:

1. p_i^0 is called the process idle place of N_i. Elements in P_i and P_{Ri} are called activity and resource places, respectively. A resource place is called a resource for short in case of no confusion.
2. $P_{Ri} \neq \oslash$; $Pi \neq \oslash$; $p_i^0 \notin P_i$; $(P_i \cup \{p_i^0\}) \cap P_{Ri} = \oslash$; $\forall p \in P_i, \forall t \in {\bullet}p, \forall t' \in p{\bullet}, \exists r_p \in P_{Ri}, {\bullet}t \cap P_{Ri} = t'^{\bullet} \cap P_{Ri} = \{r_p\}$; $\forall r \in P_{Ri}, {}^{\bullet\bullet}r \cap P_i = r^{\bullet\bullet} P_i \neq \oslash$; $\forall r \in P_{Ri}, {}^{\bullet}r \cap r^{\bullet} = \oslash$; and ${\bullet\bullet}(p_i^0) \cap P_{Ri} = \oslash$.
3. N_i' is a strongly connected state machine, where $N_i' = (P_i \cup \{p_i^0\}, T_i, F_i)$ is the resulting net after the places in P_{Ri} and related arcs are removed from N_i.
4. Every circuit of N_i' contains place p_i^0.
5. Any two N_i' are composable when they share a set of common places. Every shared place must be a resource.
6. $H(r) = {}^{\bullet\bullet}r \cap P$ denotes the set of holders of r (operation places that use r). Any resource r is associated with a minimal P-invariant whose support is denoted by $\varrho_r = r \cup H(r)$.

An S^{3}PR is composed of some state machines (with choices) holding and releasing some common resources. Figure 1 shows an example of S^{3}PR. For a net system (N, M_0), a non-empty subset S (resp. τ) of places is called a *siphon* (resp. *trap*) if ${\bullet}S \subseteq S{\bullet}$ (resp. $\tau{\bullet} \subseteq {\bullet}\tau$), i.e., every transition having an output (resp. input) place in S has an input (resp. output) place in S(resp. τ). $R(S)$ is the set of resource places in S.

S is called an *empty siphon* at M_0 If $M_0\,(S) = \sum_{p \in s} M_0\,(p) = 0$. A *minimal siphon* does not contain a siphon as a proper subset. It is called an SMS (Strict Minimal Siphon), denoted by S, if it does not contain a trap.

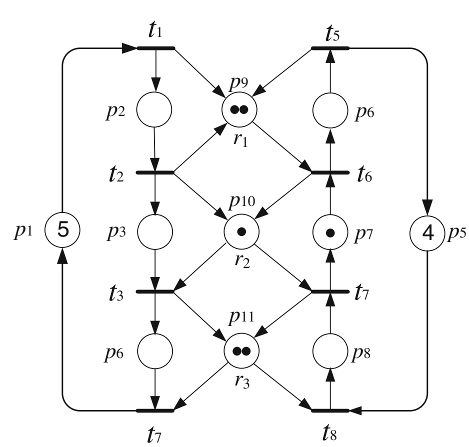

Fig. 1 An example S^{3}PR with strongly dependent siphon

If a siphon $S \subset ||Y||$, then $[S] = ||Y|| \backslash S$ is called the *complementary siphon* of S and $S \cup [S] = \cap_{r \in S} \varrho_r$ is the *support* of a P-*invariant*.

Tokens in a siphon S of an ordinary Petri net can either leak out to the complementary set $[S]$ of S or stay in S. Thus the sum of tokens in $S \cup [S]$ is a constant. $S \cup [S]$ forms the support of a minimal P-invariant.

Let $\Omega \subseteq P$ be a subset of places of N. P-*vector* λ_Ω is called the characteristic P-*vector* of Ω *iff* $\forall p \in \Omega$, $\lambda_\Omega(p) = 1$; otherwise $\lambda_\Omega(p) = 0$. η is called the characteristic T-*vector* of Ω, if $\eta^T = \lambda_\Omega^T \cdot [N]$, where $[N]$ is the incidence matrix. Physically, the firing of a transition t where $[\eta(t) > 0, \eta(t) = 0$, and $\eta(t) < 0]$ increases, maintains and decreases the number of tokens in S, respectively. Let $\eta_{S_\alpha}, \eta_{S_\beta}, \ldots$, and η_{S_γ}, $(\{\alpha, \beta, \ldots, \gamma\} \subseteq \{1, 2, \ldots, k\})$ be a linear independent maximal set of matrix $[\eta]$. Then $\Pi_E = \{S_\alpha, S_\beta, \ldots, S_\gamma\}$ is called a set of elementary siphons. $S \notin \Pi_E$ is called a strongly dependent siphon if $\eta_S = \sum_{S_i \in \Pi_E} a_i \eta_{S_i}$ where $a_i \geq 0$. $S \notin \Pi_E$ is called a weakly dependent siphon if $\exists$ non-empty $A, B \subset \Pi_E$, such that $A \cap B = \oslash$ and $\eta_S = \sum_{S_i \in A} a_i \eta_{S_i} - \sum_{S_i \in B} a_i \eta_{S_i}$ where $a_i > 0$.

Li and Zhou (2004) propose to find elementary siphons by constructing the characteristic P-vector (resp. T-vector) vector matrix $[\lambda]$ (resp. $[\eta]$) of the siphons in N followed by finding linearly independent vectors in $[\lambda]$ (resp. $[\eta]$). The siphons corresponding to these independent vectors are the elementary siphons in the net system. Note that the above calculation of linearly independent vectors do not assume N to be an S^3PR and are applicable to arbitrary nets.

Definition 2 (Chao 2006, 2010a,b) A sequence of nodes $x_1 x_2 \ldots x_n$ is called a path of N if $\forall i \in \{1, 2, \ldots, n-1\}$, $x_{i+1} \in x_i^{\bullet}$. An elementary path from x_1 and x_n is a path whose nodes are all different (except, perhaps, x_1 and x_n). A circuit is an elementary path with $x_1 = x_n$. A net N is strong connected if for every node pair (n_i, n_j), $n_i, n_j \in P \cup T$, there is a directed path from n_i to n_j.

The strongly connected circuit from which an SMS can be synthesized is called a core circuit. An elementary resource circuit is called a basic circuit, denoted by c_b. The siphon constructed from c_b is called a basic siphon. A compound circuit c is a circuit consisting of multiply interconnected elementary circuits $c_1, c_2, \ldots, c_n$ extending between Process 1 and 2. The SMS synthesized from compound circuit c using the Handle-Construction Procedure in Chao (2006) is called an n-compound siphon S. If for every pair of i, j, c_i, c_{i+1}, intersect at a resource place and r_i (resp. Path $[r_1 t_1 r_2 \ldots r_i t_i \ldots r_{k-1} t_{k-1} r_k]_1$), then $c = c_1 o c_2 o \ldots c_{n-1} o c_n$ (resp. $c_1 \oplus c_2 \oplus \ldots \oplus c_{n-1} \oplus c_n$). The corresponding synthesized siphon is denoted by $S = S_1 o S_2 o \ldots S_{n-1} o\ Sn$ (resp. $S = S_1 \oplus S_2 \oplus \ldots \oplus S_{n-1} \oplus S_n$).

Motivation

This section motivates the reader about the uniform formula described earlier. First we observe that (Table 1) for a strongly dependent 2-compound siphon,

$$\varsigma = \varsigma_1 + \varsigma_2 - \varsigma_{1,2} \tag{1}$$

where ς_{12} is the ς value for the siphon with $R(S) = R(S_1 \cap S_2)$, where $\varsigma = S, [S]$, and η, $R(S)$ is the set of resource places in S.

Next extend this equation to a weakly dependent 2-compound siphon (Table 2).

In Fig. 2, there are 3 elementary siphons $S_1 - S_3$ and 1 weakly dependent siphon S_4; their characteristic T-vectors η are shown in Table 2. In both cases, all compound siphons S, their $[S]$ and η share the same uniform formula.

Since the number of SMS grows exponentially with the size of a net, the time complexity of computing η for all compound siphons is exponential and quite time consuming. Thus, it is desired to compute compound siphons S, their complementary set of places $[S]$, and T-*characteristic*

Table 1 Types of siphons for the net in Fig. 1

SMS	[S]	η	Set of places	c
S_1	p_2, p_7	[-$t_1 + t_2 + t_6 - t_7$]	p_9, p_{10}, p_3, p_6	$c_1 = [p_9\, t_6\, p_{10}\, t_2\, p_9]$
S_2	p_3, p_8	[-$t_2 + t_3 + t_7 - t_8$]	p_{10}, p_{11}, p_4, p_7	$c_2 = [p_{10}\, t_7\, p_{11}\, t_3\, p_{10}]$
S_3	p_2, p_3, p_7, p_8	[-$t_1 + t_3 + t_6 - t_8$]	$p_9, p_{10}, p_6, p_{11}, p_4$	$c_3 = c_1 o\, c_2$

Table 2 Four SMS in Fig. 2 and their η. ($\eta_4 = \eta_1 + \eta_2 - \eta_3$)

SMS	[S]	η	Set of places	c
S_1	$p_2, p_3, p_8, p_9, p_{10}, p_{11}$	$[t_2 - t_4 + t_8 - t_9]$	$p_4, p_{12}, p_{13}, p_{14}, p_{15}$	$c_1 = [p_{15} t_2 p_{14} t_3 p_{13} t_8 p_{15}]$
S_2	$p_3, p_4, p_7, p_8, p_9, p_{10}$	$[t_1 - t_3 + t_7 - t_{10}]$	$p_5, p_{11}, p_{14}, p_{15}, p_{16}$	c_2=$[p_{14} t_4 p_{16} t_1 p_{15} t_2 p_{14}]$
S_3	p_3, p_8, p_9, p_{10}	$[t_2 - t_3 - t_4 + t_7]$	$p_4, p_{11}, p_{14}, p_{15}$	$c_3 = [p_{15} t_2 p_{14} t_6 p_{15}]$
S_4	$p_2, p_3, p_4, p_7, p_8, p_9, p_{10}, p_{11}$	$[t_1 + t_8 - t_9 - t_{10}]$	$p_5, p_{12}, p_{13}, p_{14}, p_{15}, p_{16}$	$c_4 = c_1 \oplus c_2$

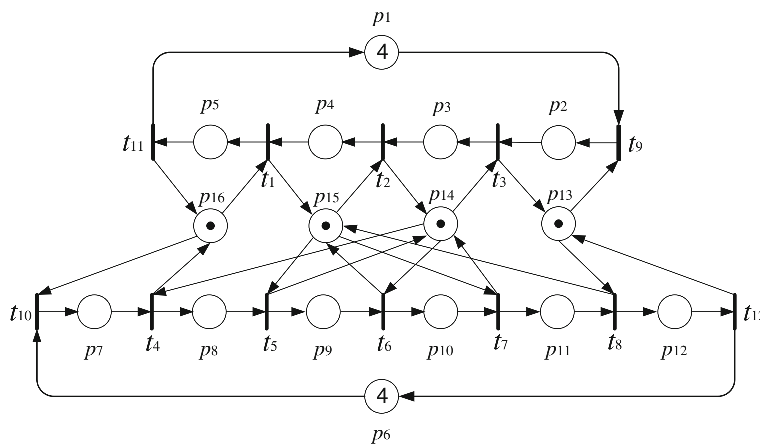

Fig. 2 Example weakly 2-compound siphon. ($\eta_0 = \eta_1 + \eta_2 - \eta_3$)

vectors η efficiently. To do so, we need to generalize Equation (1).

Theory

The following theorem provides the foundation for the uniform formula mentioned earlier.

Theorem 1 (Theorem 2 in *(Chao 2010a)):*
Let (N_0, M_0) be a net system and S_0 be a dependent SMS w.r.t. elementary siphons $S_1, S_2, \ldots, S_n, S_{n+1}, S_{n+2}, \ldots$, and S_{n+m} where $\eta_{S_0} = \sum_{i=1}^{n} \left(\boldsymbol{a_i}\eta_{S_i}\right) - \sum_{j=1}^{m} \left(\boldsymbol{b_{n+j}}\eta_{S_{n+j}}\right) = \boldsymbol{\sigma_a} - \boldsymbol{\sigma_b}$, $\boldsymbol{\sigma_a} = \sum_{i=1}^{n} \left(\boldsymbol{a_i}\eta_{S_i}\right)$, and $\boldsymbol{\sigma_b} = \sum_{j=1}^{m} \left(\boldsymbol{b_{n+j}}\eta_{S_{n+j}}\right)$.
Then

1. $\forall S \in S_0, S_1, S_2, \ldots, S_n, S_{n+1}, S_{n+2}, \ldots, S_{n+m}, \eta_S = -\eta[S]$ *(characteristic T-vector of the complementary set of siphon S equals the negative of that of S).*
2. $\lambda_{[S_0]} = \sum_{i=1}^{n} \left(a_i \lambda_{[S_i]}\right) - \sum_{j=1}^{m} \left(b_{n+j}\lambda_{[S_{n+j}]}\right)$, *where $a_i, b_j \in R$ (set of real numbers), $i \in, 2, \ldots, n$ and $j \in [1, 2, \ldots, m]$ (characteristic P-vectors of the complementary sets of siphon $S_0, S_1, S_2, \ldots, S_n, S_{n+1}, S_{n+2}, \ldots, S_{n+m}$ follow the same equation as that of the corresponding characteristic T-vectors).*
3. *The Marking Equality (ME) holds : $M([S_0]) = \sum_{i=1}^{n} (a_i M([S_i])) - \sum_{j=1}^{m} \left(b_{n+j} M\left(\left[S_{n+j}\right]\right)\right)$. $M \in R(N, M_0)$ (total tokens in the complementary sets of siphon $S_0, S_1, S_2, \ldots, S_n, S_{n+1}, S_{n+2}, \ldots, S_{n+m}$ follow the same equation as that of the corresponding characteristic T-vectors).*

Proof please refer to the proof of Theorem 2 in Chao (2010a). □

This *ME* (marking equality) says that the total number of tokens trapped in $[S_0]$ and $[S_i]$, follow the same linear algebraic relationship between η_{S0} and η_{Si}, $i = 1, 2, \ldots, n$, $n+1, \ldots, n+m$. This is because physically, -$\eta_{S(t)}$ is the number of tokens removed from S by firing t once.

Computation by structures

Recall that it is desirable to compute S, $[S]$, and η directly from the structures independent to whether S is a strongly or a weakly dependent siphon. We are the very first to uncover the uniform method to do so.

We first deal with strongly 2-compound siphons based on the following lemma.

Lemma 1 *Let $r \in P_R$, the minimal siphon containing r is $S = \varrho(r) = \{r\} \cup H(r)$ (also the support of a minimal P-invariant) with $[S] = \oslash$ and $\eta_S = 0$.*

Proof Obvious. □

Theorem 2 *For every compound circuit made of c_{b1} and c_{b2} in an S^3PR corresponding to an SMS S_0 such that $c_{b1} \cap c_{b2} = r$,*

1) *$\eta_0 = \eta_1 + \eta_2$, where $r \in P_R$ and η_0 is the η value for S_0*
2) *$\eta_0 = \eta_1 + \eta_2 - \eta_{12}$, where η_{12} is the characteristic T-vector of the minimal siphon containing r.*
3) *$[S_0] = [S_1] + [S_2] - [S_{12}]$.*
4) *$S_0 = S_1 + S_2 - S_{12}$.*
5) *$\varsigma = \varsigma_1 + \varsigma_2 - \varsigma_{1,2}$, where $\varsigma = S$, o $[S]$, o η.*

Proof 1. See the proof for Theorem 1 in Chao (2006).
2. It follows from Lemma 1 that $\eta_{12} = 0$. Thus, $\eta_0 = \eta_1 + \eta_2 {+} \eta_1 + \eta_2 - \eta_{12}$.
3. By Lemma 1, $[S_{12}] = \oslash$. By Corollary 3 in Li and Zhou (2004), $[S_0] = [S_1] + [S_2]$. Hence, $[S_0] = [S_1] + [S_2] - [S_{12}]$.
4. $(S_1 + [S_1]) + (S_2 + [S_2]) = \sum_{r \in R_1} \varrho(r) + \sum_{r \in R_2} \varrho(r) = \sum_{r \in R_3} \varrho(r) + \sum_{r \in R_1 \cap R_2} \varrho(r)$.

$\sum_{r \in R_3} \varrho(r) = s_1 + [s_1] + s_2 + [s_2] - \sum_{r \in R_1 \cap R_2} \varrho(r)\ .S_0 + [S_0] = S_0 + ([S_1] + [S_2] - [S_{12}]) = \sum_{r \in R_3} \varrho(r) = s_1 + [s_1] + s_2 + [s_2] - \sum_{r \in R_1 \cap R_2} \varrho(r)$.
$S_0 = S_1 + S_2 - (\sum_{r \in R_1 \cap R_2} \varrho(r) - [s_{12}]) = S_1 + S_2 - S_{12}$, where $S_{12} = \sum_{r \in R_1 \cap R_2} \varrho(r) - [s_{12}]$.

5. It follows from Parts 1–4 of this theorem.

□

This theorem proves Eq. (1) for a strongly 2-compound siphon. Let S_0 be a strongly dependent siphon, S_1, S_2, …, and S_n be elementary siphons, with $c_0 = c_1 o\ c_2 o \ldots o\ c_n$. c_0 (the core circuit from which to synthesize S_0) is a compound resource circuit containing $c_1, c_{2,\ldots,} c_n$ and the intersection between any two c_i and c_j, $i = j - 1 > 0$, is exactly a resource place, where c_i (i= 0, 1, 2,…, n) is the core circuit from which to synthesize S_i. We show in Theorem 2 of (Chao 2006) that $\eta_{S0} = \eta_{S1} + \eta_{S2} + \cdots + \eta_{Sn}$. Based on Theorem 3 the uniformity also holds for this case of strongly n-dependent siphons.

Thus, if S_0 is a WDS (weakly dependent siphon), the intersection between any two c_i and $c_{j,i} = j-1 > 0$ must contain more than one resource place.

In Fig. 2, $c_1 \cap c_2 = [p_{15} t_2 p_{14}]$ is not a single resource place, where $\eta_1(t_{15}) = \eta_2(t_{15}) = 1$. If $\Gamma = c_1 \cap c_2$ is not in a third basic circuit, we have $\eta_4(t_{15}) = \eta_1(t_{15}) + \eta_2(t_{15}) = 2$ against the fact in (Li and Zhou 2004) that any $\eta(t)$ must be one of 1, 0, and -1. Thus, Γ must be in a third basic circuit c_3 from which to synthesize S_3 so that $\eta_4(t_{15}) = \eta_1(t_{15}) + \eta_2(t_{15}) - \eta_3(t_{15}) = 1$.

Thus, if S_4 weakly depends on S_1 and S_2—synthesized from basic circuits c_1 and c_2, respectively, then there exists a third siphon S_3 such that $\eta_4 = \eta_1 + \eta_2 - \eta_3$.

Define $S_{1,2} = S_3$ and $S_0 = S_1 \oplus S_2$ since $R(S_1 \cap S_2) = R(S_3)$. $S_1 \oplus S_2$ is similar to S_1 o S_2 in terms of controllability to be shown later. $S_1 \oplus S_2$ is different than S_1o S_2 in that $R(S_1 \cap S_2)$ for the former contains more than one resource place while the latter contains only one resource place.

Definition 3 Let (N_0, M_0) be a net system and $S_0 = S_1 \oplus S_2$ denotes the fact that S_0 is a weakly dependent SMS w.r.t. elementary siphons S_1, S_2, and $S_{1,2} = S_3$ such that $\eta_0 = \eta_1 + \eta_2 - \eta_3$.

Theorem 3 *Let $S = S_1 \oplus S_2$. Then $\eta_0 = \eta_1 + \eta_2 - \eta_3$.*

Proof Let t be a transition in $\varrho \bullet \cup \bullet \varrho$, where $\varrho = R(c_0)$. There are two cases:

1) $c_1^a \cup c_{2^b}$is a single resource place. The case has been proved in Chao (2006).
2) $c_1^a \cup c_2^b$ contains more than a single resource place. It holds that $\eta_i(t) = \eta_0(t)$, and $\eta_j(t) = \eta_3(t)$, where $i, j \in \{1, 2\}, i \neq j \Rightarrow \eta_i(t) + \eta_j(t) - \eta_3(t) = \eta_i(t) = \eta_0(t)$. This theorem is thus proved.

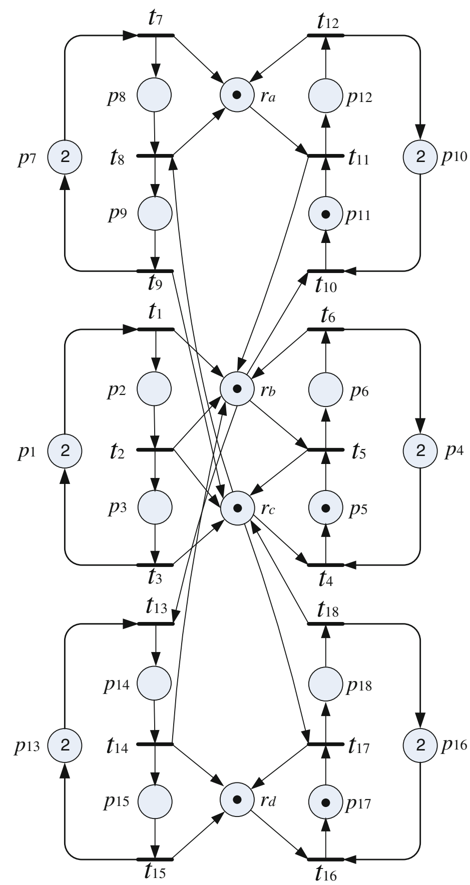

Fig. 3 Another example of weakly dependent siphons spanning more than two processes

□

We now propose the following:

Theorem 4 *Let $S_0 = S_1 \oplus S_2$ as defined in Definition 2, then*

1) $[S_0] = [S_1] \cup [S_2]$,
2) $[S_0] = [S_1] + [S_2] - [S_3]$,
3) $[S_1] \cap [S_2] = [S_{1,2}]$,
4) $S_0 = S_1 + S_2 - [S_{12}]$,
5) $\varsigma = \varsigma_1 + \varsigma_2 - \varsigma_{1,2}$, *where* $\varsigma = S$, *o* $[S]$, *o* η, *and*
6) $M([S_0]) = M([S_1]) + M([S_2]) - M([S_{1,2}])$.

Proof 1. $[S_0] = \{([S_1] \backslash [S_{1,2}]) \cup [S_{1,2}]\} \cup \{([S_2] \backslash [S_{1,2}]) \cup [S_{1,2}]\}$
$= [S_1] \cup [S_2]$, where $[S_i] = ([S_i] \backslash [S_{1,2}]) \cup [S_{1,2}]$.
Hence, $[S_0] = [S_1] \cup [S_2]$.
2. By Theorem 3, we have $\eta_0 = \eta_1 + \eta_2 - \eta_3$, which leads to $[S_0] = [S_1] + [S_2] - [S_3]$ by Corollary 1.
3. The fact that $[S_0] = [S_1] \cup [S_2]$ implies that $[S_0] = [S_1] + [S_2] - [S_1] \cap [S_2]$. Comparing this with Part 2 of this theorem leads to $[S_1] \cap [S_2] = [S_3] = [S_{1,2}]$ (By Def. 5, $S_{1,2} = S_3$).

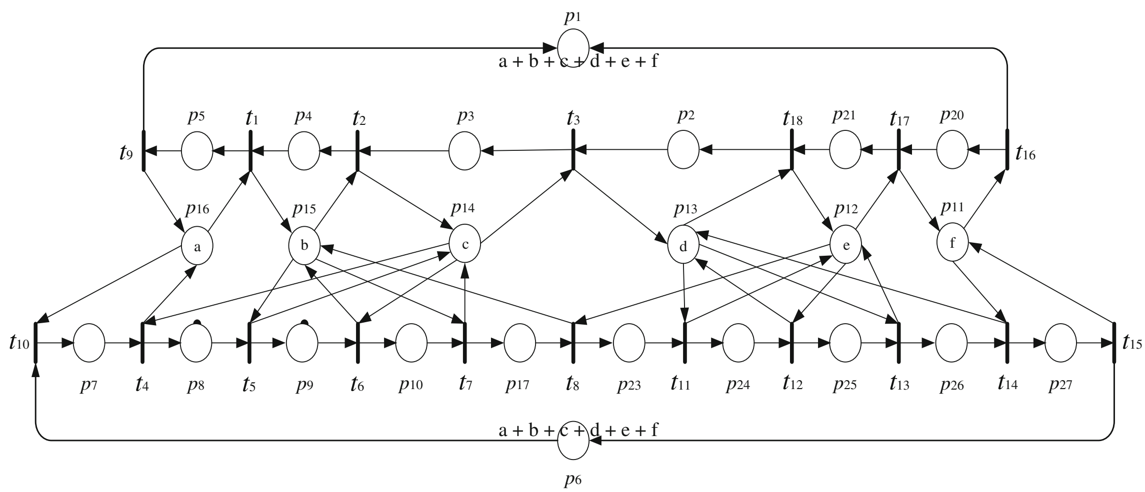

Fig. 4 Example weakly 3-compound siphon

Table 3 Eight SMS S, and core circuits in Fig. 4

S	Set of places	c
S_1	$p_5, p_{17}, p_{14}, p_{15}, p_{16}$	$c_1 = [p_{14}t_4p_{16}t_1p_{15}t_2p_{14}]$
S_2	$p_4, p_{26}, p_{12}, p_{13}, p_{14}, p_{15}$	$c_2 = [p_{15}t_2p_{14}t_3p_{13}t_{18}p_{12}t_8p_{15}]$
S_3	$p_2, p_{27}, p_{11}, p_{12}, p_{13}$	$c_3 = [p_{13}t_{18}p_{12}t_{17}p_{17}t_{14}p_{13}]$
S_4	$p_4, p_{17}, p_{14}, p_{15}$	$c_4 = [p_{15}t_2p_{14}t_6p_{15}]$
S_5	$p_2, p_{26}, p_{12}, p_{13}$	$c_5 = [p_{13}t_{18}p_{12}t_{12}p_{13}]$
S_6	$p_5, p_{27}, p_{11}, p_{12}, p_{13}, p_{14}, p_{15}, p_{16}$	$c_6 = c_1 \oplus c_2 \oplus c_3$
S_7	$p_5, p_{26}, p_{12}, p_{13}, p_{14}, p_{15}, p_{16}$	$c_7 = c_1 \oplus c_2$
S_8	$p_4, p_{27}, p_{11}, p_{12}, p_{13}, p_{14}, p_{15}$	$c_8 = c_2 \oplus c_3$

Table 4 Eight SMS S, $[S]$ and η in Fig. 4

S	[S]	η
S_1	$p_3, p_4, p_7, p_8, p_9, p_{10}$	$t_1 - t_3 + t_7 - t_{10}$
S_2	$p_2, p_3, p_8, p_9, p_{10}, p_{17}, p_{21}, p_{23}, p_{24}, p_{25}$	$t_2 - t_4 + t_{13} - t_{17}$
S_3	$p_{20}, p_{21}, p_{23}, p_{24}, p_{25}, p_{26}$	-$t_8 + t_{14} - t_{16} + t_{18}$
S_4	p_3, p_8, p_9, p_{10}	$t_2 - t_3 - t_4 + t_7$
S_5	$p_{21}, p_{23}, p_{24}, p_{25}$	-$t_8 + t_{13} - t_{17} + t_{18}$
S_6	$p_2, p_3, p_4, p_7, p_8, p_9, p_{10}, p_{17}, p_{20}, p_{21}, p_{23}, p_{24}, p_{25}, p_{26}$	$t_1 - t_{10} + t_{14} - t_{16}$
S_7	$p_2, p_3, p_4, p_7, p_8, p_9, p_{10}, p_{17}, p_{21}, p_{23}, p_{24}, p_{25}$	$t_1 - t_{10} + t_{13} - t_{17}$
S_8	$p_2, p_3, p_8, p_9, p_{10}, p_{17}, p_{20}, p_{21}, p_{23}, p_{24}, p_{25}, p_{26}$	$t_2 - t_4 + t_{14} - t_{16}$

4. The proof is the same as that of Part 4 of Theorem 1.
5. It follows from Theorem 3 and Parts 2 and 4 of this theorem.
6. From Part 3 of Theorem 1, we have $M([S_0]) = M([S_1]) + M([S_2]) - M([S_{1,2}])$.

□

Consider the S^{3}PR in Fig. 2, $[S_0]$= $p_2, p_3, p_4, p_7, p_8, p_9, p_{10}, p_{11}$. Based on Table 1, one can verify that 1) $[S_0] = [S_1] \cup [S_2]$ and 2) $[S_1] \cap [S_2] = [S_{1,2}] = S_3$. The same conclusion applies to the net in Fig. 3 where the transitions involved span more than two processes.

This theorem confirms the uniform computation for a weakly 2-compound siphon. The following theorem extends the uniform computation to a weakly *n-compound* siphon.

Theorem 5 *Let* $S_0 = S_1 \oplus S_2 \oplus \cdots \oplus \quad S_n$ *Then*

1) $\eta_0 = \eta_1 + \eta_2 + \cdots + \eta_n - \eta_{1,2} - \eta_{2,3} - \cdots - \eta_{n-1,n}$.
2) $[S_0] = [S_1] + [S_2] + \cdots + [S_n] - [S_{1,2}] - [S_{2,3}] - \cdots - [S_{n-1,n}]$.

Table 5 Eight SMS S and dependency in Fig. 4.

S	Set of places	Dependency
S_1	$p_5, p_{17}, p_{14}, p_{15}, p_{16}$	
S_2	$p_4, p_{26}, p_{12}, p_{13}, p_{14}, p_{15}$	
S_3	$p_2, p_{27}, p_{11}, p_{12}, p_{13}$	
S_4	$p_4, p_{17}, p_{14}, p_{15}$	
S_5	$p_2, p_{26}, p_{12}, p_{13}$	
S_6	$p_5, p_{27}, p_{11}, p_{12}, p_{13}, p_{14}, p_{15}, p_{16}$	$S_6 = S_1 \oplus S_2 \oplus S_3 = S_1 + S_2 + S_3 - S_{12} - S_{23}$
S_7	$p_5, p_{26}, p_{12}, p_{13}, p_{14}, p_{15}, p_{16}$	$S_7 = S_1 \oplus S_2 = S_1 + S_2 - S_{12}$
S_8	$p_4, p_{27}, p_{11}, p_{12}, p_{13}, p_{14}, p_{15}$	$S_8 = S_2 \oplus S_3 = S_2 + S_3 - S_{23}$

Table 6 Eight SMS $[S]$ and dependency in Fig. 4

[S]	Set of places	Dependency
$[S_1]$	$p_3, p_4, p_7, p_8, p_9, p_{10}$	
$[S_2]$	$p_2, p_3, p_8, p_9, p_{10}, p_{17}, p_{21}, p_{23}, p_{24}, p_{25}$	
$[S_3]$	$p_{20}, p_{21}, p_{23}, p_{24}, p_{25}, p_{26}$	
$[S_4]$	p_3, p_8, p_9, p_{10}	
$[S_5]$	$p_{21}, p_{23}, p_{24}, p_{25}$	
$[S_6]$	$p_2, p_3, p_4, p_7, p_8, p_9, p_{10}, p_{17}, p_{20}, p_{21}, p_{23}, p_{24}, p_{25}, p_{26}$	$[S_6] = [S_1] \oplus [S_2] \oplus [S_3] = [S_1] + [S_2] + [S_3] - [S_{12}] - [S_{23}]$
$[S_7]$	$p_2, p_3, p_4, p_7, p_8, p_9, p_{10}, p_{17}, p_{21}, p_{23}, p_{24}, p_{25}$	$[S_7] = [S_1] \oplus [S_2] = [S_1] + [S_2] - [S_{12}]$
$[S_8]$	$p_2, p_3, p_8, p_9, p_{10}, p_{17}, p_{20}, p_{21}, p_{23}, p_{24}, p_{25}, p_{26}$	$[S_8] = [S_2] \oplus [S_3] = [S_2] + [S_3] - [S_{23}]$

Table 7 Eight SMS η and dependency in Fig. 4

η	Characteristic T-vectors	Dependency
η_1	$t_1 - t_3 + t_7 - t_{10}$	
η_2	$t_2 - t_4 + t_{13} - t_{17}$	
η_3	$-t_8 + t_{14} - t_{16} + t_{18}$	
η_4	$t_2 - t_3 - t_4 + t_7$	
η_5	$-t_8 + t_{13} - t_{17} + t_{18}$	
η_6	$t_1 - t_{10} + t_{14} - t_{16}$	$\eta_6 = \eta_1 + \eta_2 + \eta_3 - \eta_{12} - \eta_{23}$
η_7	$t_1 - t_{10} + t_{13} - t_{17}$	$\eta_7 = \eta_1 + \eta_2 - \eta_{12}$
η_8	$t_2 - t_4 + t_{14} - t_{16}$	$\eta_8 = \eta_2 + \eta_3 - \eta_{23}$

3) $S_0 = S_1 + S_2 + \cdots + S_n - S_{1,2} - S_{2,3} - \cdots - S_{n-1,n}$.
4) $\varsigma = \varsigma_1 + \varsigma_2 + \cdots + \varsigma_n - \varsigma_{1,2} - \varsigma_{2,3} - \cdots - \varsigma_{n-1,n}$.

Proof 1. Prove by induction. The case of $n = 2$ holds by Theorem 5. Let $S_0 = S^* \oplus S_n$ where $S^* = S_1 \oplus S_2 \oplus \cdots \oplus S_{n-1}$ and the corresponding core circuit $c^* = c_1 \oplus c_2 \oplus \cdots \oplus c_{n-1}$. Assume it holds for the case of $n - 1$. Thus, $\eta^* = \eta_1 + \eta_2 + \cdots + \eta_{n-1} - \eta_{1,2} - \eta_{2,3} - \cdots - \eta_{n-2}, n-1$. $c_0 = c^* \oplus c_n$ is a compound circuit containing c^* and c_n. By Theorem 3, $\eta_0 = \eta^* + \eta_n - \eta_{*,n}$, where $\eta_{*,n}$ is the η value for the SMS S' with $R(S') = R(S^* \cap S_n)$ $= R(S_{n-1} \cap S_n) \Rightarrow \eta_0 = \eta^* + \eta_n - \eta_{n-1,n} \Rightarrow \eta_0$ $= \eta_1 + \eta_2 + \ldots + \eta_n - \eta_{1,2} - \eta_{2,3} - \ldots - \eta_{n-1,n}$.

2. Prove by induction. The case of $n = 2$ holds by Part 2 of Theorem 4. Let $S_0 = S^* \oplus S_n$ where $S^* = S_1 \oplus S_2 \oplus \cdots \oplus S_{n-1}$. Assume it holds for the case of n-1. Thus, $[S^*] = [S_1] + [S_2] + \cdots + [S_{n-1}] - [S_{2,3}] - \cdots - [S_{n-2,n-1}]$. $\exists p \in [S_0] \cap [S_n]$, $p \notin [S_i]$, $i \in 1, 2, \ldots, n-1 \Rightarrow [S_0]$ must contain a term $[S_n]$ since p appears exactly once in both $[S_0]$ and $[S_n]$. Places p' in $[S^*] \cap [S_n]$, however, appears twice in $[S^*] + [S_n]$. Thus, they must be deleted from $[S^*] + [S_n]$ to get $[S_0] = [S^*] + [S_n] - [S^*] \cap [S_n]$. $[S^*] \cap [S_n] = [S_{n-1}] \cap [S_n] = [S_{n-1,n}]$ (Part 3 of Theorem 4) $\Rightarrow [S_0] = [S^*] + [S_n] - [S^*] \cap [S_n] = [S_1] + [S_2] + \cdots + [S_{n-1}] - [S_{2,3}] - \cdots - [S_{n-2,n-1}] + [S_n] - [S_{n-1,n}] = [S_1] + [S_2] + \cdots + [S_n] - [S_{2,3}] - \cdots - [S_{n-1,n}]$.
3. The proof by induction is similar to that for Parts 1 and 2 of this theorem.
4. It follows Parts 1–3 of this theorem.

□

Thus, we prove the uniform formula for weakly *n-compound* siphons. It covers strongly dependent siphons as a special case where there are no negative terms in η_{s0}. Thus, the uniform formula holds irrespective to whether the compound siphon is strongly or weakly. This further enhances the uniformity of the formula.

One can employ Fig. 4 to demonstrate Theorem 5. As shown in Tables 3, 4, 5, 6 and 7, S, $[S]$ and η for weakly compound siphons, all share the same formula verifying Theorem 5.

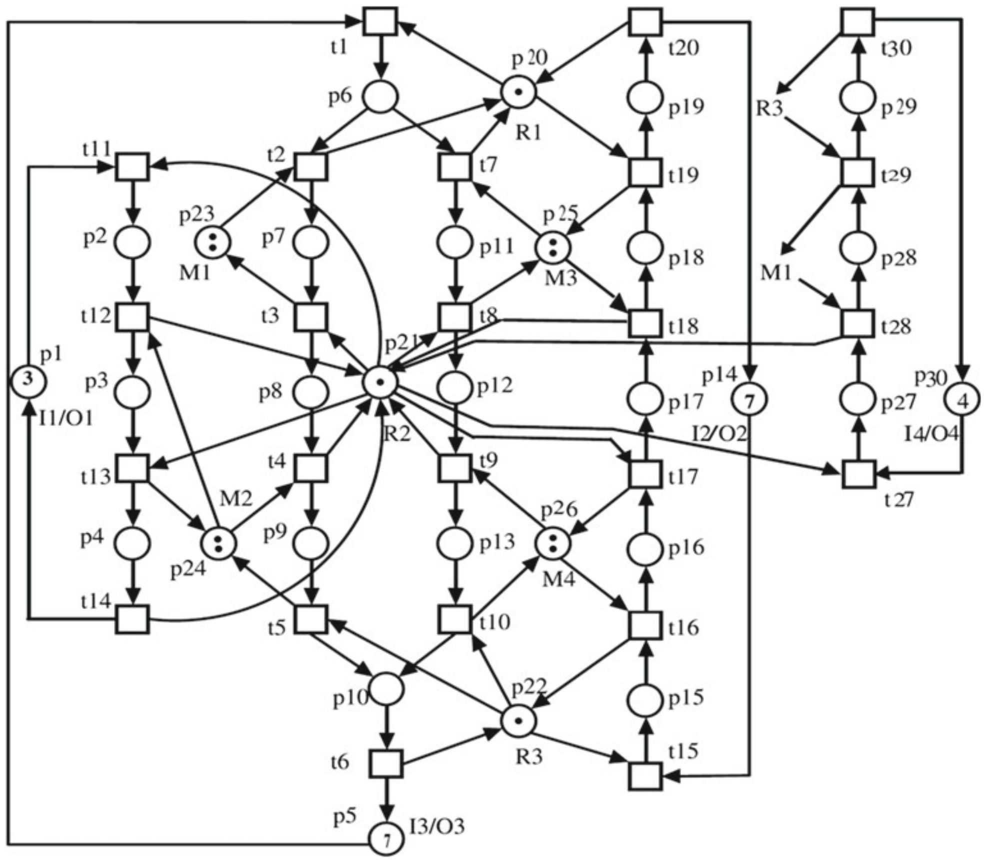

Fig. 5 A more complicated Petri net model of an FMS with weakly dependent siphons (Liu et al. 2011a)

Example

Figure 5 shows a classical example [adapted from Liu et al. (2011a)] with both strongly and weakly dependent siphons. The net system is an S^3PR and contains deadlocks. With the theory developed in this paper, it is easy (even manually without software help) to identify all basic and compound siphons. They happen to be elementary and dependent siphons, respectively. The uniform formulas further help compute all SMS, S, $[S]$, and η, from which one can construct the controlled model to be more permissive than that in Li and Zhou (2004) as shown in Liu et al. (2011a). There are eight elementary or basic siphons synthesized from 8 resource circuits using the handle-construction procedure in Chao (2006), 4 weakly and 21 strongly dependent siphons as shown in Tables 8 and 9 respectively. For example, S_3 is a strongly dependent SMS w.r.t. to S_4 and S_{18}, S_2 is a weakly dependent SMS w.r.t. to S_{17} and S_{20}, and S_{28} is another weakly dependent SMS w.r.t. to S_4 and S_{20}, respectively.

Table 8 Elementary or basic siphons, resource circuits, and η for the net in Fig. 5

Basic siphons	Places	c	η
S_1	$p_{10}, p_{16}, p_{22}, p_{26}$	$[p_{22}t_{10}p_{26}t_{16}p_{22}]$	$-t_9 + t_{10} - t_{15} + t_{16}$
S_4	$p_4, p_{10}, p_{17}, p_{21}, p_{22}, p_{24}, p_{26}$	$[p_{21}t_{17}p_{26}t_{16}p_{22}t_5p_{24}t_4p_{21}]$	$-t_3 + t_5 - t_{11} + t_{13} - t_8 + t_{10} - t_{15} + t_{17}$
S_{10}	$p_4, p_9, p_{12}, p_{17}, p_{21}, p_{24}$	$[p_{21}t_{13}p_{24}t_4p_{21}]$	$-t_3 + t_4 - t_{11} + t_{13}$
S_{16}	$p_2, p_4, p_8, p_{13}, p_{17}, p_{21}, p_{26}$	$[p_{21}t_{17}p_{26}t_9p_{21}]$	$-t_8 + t_9 - t_{16} + t_{17}$
S_{17}	$p_2, p_4, p_8, p_{12}, p_{19}, p_{20}, p_{21}, p_{23}, p_{25}$	$[p_{21}t_3p_{23}t_2p_{20}t_{19}p_{25}t_{18}p_{21}]$	$-t_1 + t_3 + t_8 - t_{17} + t_{19}$
S_{18}	$p_2, p_4, p_8, p_{12}, p_{18}, p_{21}, p_{25}$	$[p_{21}t_8p_{25}t_{18}p_{21}]$	$-t_7 + t_8 - t_{17} + t_{18}$
S_{19}	$p_2, p_4, p_8, p_{12}, p_{17}, p_{21}, p_{23}, p_{28}$	$[p_{21}t_3p_{23}t_{28}p_{21}]$	$-t_2 + t_3 - t_{27} + t_{28}$
S_{20}	$p_{10}, p_{21}, p_{22}, p_{23}, p_{26}, p_{29}$	$[p_{21}t_{17}p_{26}t_{16}p_{22}t_{29}p_{23}t_{28}p_{21}]$	$-t_2 + t_3 - t_{27} + t_{28}$

Table 9 Compound siphons, and their η relationship for the net in Fig. 5

Compound siphons	Places	η Relationship
S_2	$p_2, p_4, p_8, p_{10}, p_{19}, p_{20}, p_{21}, p_{22}, p_{23}, p_{24}, p_{25}, p_{26}, p_{29}$	$\eta_2 = \eta_{17} + \eta_{20} - \eta_{19}$
S_3	$p_4, p_{10}, p_{18}, p_{21}, p_{22}, p_{24}, p_{25}, p_{26}$	$\eta_3 = \eta_4 + \eta_{18}$
S_5	$p_4, p_9, p_{13}, p_{19}, p_{20}, p_{21}, p_{23}, p_{24}, p_{25}, p_{26}$	$\eta_5 = \eta_{10} + \eta_{16} + \eta_{17}$
S_6	$p_4, p_9, p_{13}, p_{18}, p_{21}, p_{24}, p_{25}, p_{26}$	$\eta_6 = \eta_{10} + \eta_{16} + \eta_{18}$
S_7	$p_4, p_9, p_{13}, p_{17}, p_{21}, p_{24}, p_{26}$	$\eta_7 = \eta_{10} + \eta_{16}$
S_8	$p_4, p_9, p_{12}, p_{19}, p_{20}, p_{21}, p_{23}, p_{24}, p_{25}$	$\eta_8 = \eta_{10} + \eta_{17}$
S_9	$p_4, p_9, p_{12}, p_{18}, p_{21}, p_{24}, p_{25}$	$\eta_9 = \eta_{10} + \eta_{18}$
S_{11}	$p_2, p_4, p_8, p_{10}, p_{19}, p_{20}, p_{21}, p_{22}, p_{23}, p_{25}, p_{26}$	$\eta_{11} = \eta_1 + \eta_{16} + \eta_{17}$
S_{12}	$p_2, p_4, p_8, p_{13}, p_{19}, p_{20}, p_{21}, p_{23}, p_{25}, p_{26}$	$\eta_{12} = \eta_{16} + \eta_{17}$
S_{13}	$p_2, p_4, p_8, p_{10}, p_{18}, p_{21}, p_{22}, p_{25}, p_{26}$	$\eta_{13} = \eta_1 + \eta_{16} + \eta_{18}$
S_{14}	$p_2, p_4, p_8, p_{13}, p_{18}, p_{21}, p_{25}, p_{26}$	$\eta_{14} = \eta_{16} + \eta_{18}$
S_{15}	$p_2, p_4, p_8, p_{10}, p_{17}, p_{21}, p_{22}, p_{26}$	$\eta_{15} = \eta_1 + \eta_{16}$
S_{21}	$p_2, p_4, p_8, p_{13}, p_{17}, p_{21}, p_{23}, p_{26}, p_{28}$	$\eta_{21} = \eta_{19} + \eta_{16}$
S_{22}	$p_4, p_9, p_{12}, p_{17}, p_{21}, p_{23}, p_{24}, p_{28}$	$\eta_{22} = \eta_{10} + \eta_{19}$
S_{23}	$p_2, p_4, p_8, p_{12}, p_{18}, p_{21}, p_{23}, p_{25}, p_{28}$	$\eta_{23} = \eta_{18} + \eta_{19}$
S_{24}	$p_2, p_4, p_8, p_{13}, p_{18}, p_{21}, p_{23}, p_{25}, p_{26}, p_{28}$	$\eta_{24} = \eta_{16} + \eta_{18} + \eta_{19}$
S_{25}	$p_4, p_9, p_{13}, p_{17}, p_{21}, p_{23}, p_{24}, p_{26}, p_{28}$	$\eta_{25} = \eta_{10} + \eta_{16} + \eta_{19}$
S_{26}	$p_4, p_9, p_{12}, p_{18}, p_{21}, p_{23}, p_{25}, p_{28}$	$\eta_{26} = \eta_{10} + \eta_{18} + \eta_{19}$
S_{27}	$p_4, p_9, p_{13}, p_{18}, p_{21}, p_{23}, p_{24}, p_{25}, p_{26}, p_{28}$	$\eta_{27} = \eta_{10} + \eta_{16} + \eta_{18} + \eta_{19}$
S_{28}	$p_4, p_{10}, p_{17}, p_{21}, p_{22}, p_{23}, p_{26}, p_{29}$	$\eta_{28} = \eta_4 + \eta_{20} - \eta_1 - \eta_{16}$
S_{29}	$p_4, p_{10}, p_{18}, p_{21}, p_{22}, p_{23}, p_{25}, p_{26}, p_{29}$	$\eta_{29} = \eta_4 + \eta_{18} + \eta_{20} - \eta_1 - \eta_{16}$
S_{30}	$p_4, p_{10}, p_{19}, p_{21}, p_{22}, p_{23}, p_{25}, p_{26}, p_{29}$	$\eta_{30} = \eta_4 + \eta_{17} + \eta_{20} - \eta_1 - \eta_{16} - \eta_{19}$

Note that $R(S_2) = R(S_{17} \cup S_{20}) = \{R_2, M_1\}$ contains two resource places, from which one can synthesize Basic Siphon S_{19}. Hence, $R(S_2) = S_{17} \oplus S_{20}$ is a weakly dependent siphon. Also $R(S_{28}) = R(S_4 \cup S_{20}) = \{R_2, R_3, M_4\}$ contains three resource places; one can synthesize two basic siphons S_{16} and S_1, accordingly. Thus, S_{12} in Theorem 4 is a 2-compound siphon S_{15} (shown in Table 9 depending on S_{16} and S_1) rather than a basic siphon.

Similarly, S_{29} is also a weakly dependent SMS w.r.t. to S_4 and S_{20}, respectively. But it strongly depends on S_{28} and S_{19}. Hence, $S_{29} = S_{28}$ o $S_{19} = (S_4 \oplus S_{20})$ o S_{19}, so is the structure of the core subnet $c_{29} = c_{28}$ o $c_{19} = (c_4 \oplus c_{20})$ o c_{19}. All S_{29}, $[S_{29}]$ and η_{29}can be computed uniformly, much more efficiently than the traditional way of finding out all SMS and extracting elementary siphons. Finally, $c_{29} = (c_4 \text{o} c_{17})$. For strongly dependent siphons, the same formula also holds except $S_{i,j} = [S_{i,j}] = \emptyset, \eta_{i,j} = 0, \forall i \neq j$.

Conclusion

In summary, we propose a new method to compute SMS (strict minimal siphons), and uniform formulas to compute SMS, their complementary sets and characteristic *T-vectors* for both strongly and weakly *n-compound* siphons based on the same underlying physics. We further propose to generalize it to a compound siphon consisting of *n* basic siphons. This helps to retain the formula in brain without consulting the references and simplify the computation plus its implementation to reduce the lines of codes. Future work can be directed to large S^3PR and more complicated systems.

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