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VALUATION OF QUANTO INTEREST RATE DERIVATIVES IN A CROSS-CURRENCY LIBOR MARKET MODEL

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ABSTRACT

This article is to provide the analytical valuation formulae of quanto interest rate derivatives based on a cross-currency LIBOR market model. The dynamics of forward LIBOR rates is a multi-factor model which incorporates the domestic and foreign interest rates and the exchange rate processes in a cross-currency environment. Under the framework, the pricing formulae of quanto interest rate derivatives are easy to implement in practice and model parameters can be acquired easily from the market quantities. The empirical results are shown to be sufficiently accurate and robust as compared to Monte Carlo simulation.

Key words and phrases: Cross-currency LIBOR market model, exotic quanto swap, quanto cap, quanto floor, quanto swap.
JEL classification: G13.

1. Introduction

In this paper we consider the valuation of cross-currency interest rate derivatives based on a *cross-currency LIBOR market model* (CLMM, hereafter) introduced by Wu and Chen (2007). The settings of the CLMM are in a cross-currency environment where the domestic and foreign interest rates and the exchange rate processes are incorporated in the model.

Cross-currency derivatives have been growing in increasing trade volume and become prevalent during the past decade. The major impetus to the growth of this market is due to a close relation between global interest rates and increasing volatility in financial markets. The growth of these derivatives provides investors to participate in foreign equity, money or bond markets for the purpose of speculation or hedging. The main rationale for entering into such contracts is to take benefits from interest rate differentials between countries, while avoiding direct exposure in currency risk.

A *quanto swap* (QS, hereafter), also known as a *differential swap*, is a variation of an interest rate swap which differs in that one payment of two swap legs is associated with a foreign interest rate. An investor who enters into a swap contract makes payments denominated in a domestic LIBOR rate in domestic currency. In return he receives payments based on the foreign LIBOR rate denominated in domestic currency. Both payments and receipts of a QS are made on a floating rate basis with reset in-advance and paid in-arrears features.

A *quanto cap* (QC, hereafter) is an interest rate option or a series of options whose payoffs are based upon a foreign LIBOR rate in excess of an absolute strike rate (cap rate). The payoffs are also denominated in domestic currency with a paid-in-arrears feature. A QC, like an ordinary cap, can be used to limit the up-side interest rate risk. Analogously, a quanto floor (QF, hereafter) is an insurance against a decline in interest rate.

An *exotic quanto swap* (EQS, hereafter) is a variant of a QS with a feature that the foreign payoff can be designed for customization in over-the-counter markets. In this paper, we shall introduce a prevalent case and demonstrate the separable structure of the product. In addition, the pricing formula and the hedging strategy of an EQS will be derived via other quanto derivatives in this paper.

Though there are some earlier researches on the issue of QSs, but the valuation of QCs, QFs and EQSs is relatively few in the literature, especially in the framework of the CLMM. Litzenberger (1992) firstly discussed a QS. Jamshidian (1993) used a replication method to derive the valuation formula of a QS, but it is difficult for further analysis. Turnbull (1993) introduced the dynamics of domestic and foreign forward interest rates under the Amin and Jarrow (1991, AJ) framework and priced

QSs with principals denominated in domestic and foreign currency. Wei (1994) derived the pricing formula of a QS based on a mean-reverting Ornstein-Uhlenbeck process for domestic and the foreign spot interest rates and a lognormal process for the exchange rate. Brace and Musiela (1997, BM) calculated the prices of at-the-money European options in the domestic and cross-currency economy in the Heath, Jarrow, and Morton (1992, HJM) framework. Chang, Chung, and Yu (2002, CCY) followed the framework adopted by Wei and established the formulae of a QS with principal denominated in a domestic, foreign or third-country currency. Brigo and Mercurio (2006, BM) provided the pricing formulae of a QC as well as a QF through lognormal martingales under the domestic forward martingale measure.

The pricing framework in this article is based under the CLMM which is extended from Brace, Gatarek and Musiela (1997, BGM). The BGM model, also known as the *LIBOR market model* (LMM, hereafter), is a continuous-time model of simple forward LIBOR rates which are market observable quantities. They are suitable for pricing interest rate derivatives with an feature that is reset in-advance and paid in-arrears.

In comparision with the aforementioned studies, the valuation formulae of QSs, QCs/QFs, and EQSs derived in this article have several advantages. First, previous pricing formulae have the difficulty of transforming traded quantities into model parameters, and hence not easy to implement in practice. Unlike the instantaneous short rate models, the forward LIBOR rates are market observable, thereby circumventing the problem transforming market data into model parameters. In Section 4, we will use market data from the U.S. and U.K. to calibrate model parameters and demonstrate numerical results accurate enough for practitioners. Second, the pricing formulae are shown to be analytically tractable, thereby leading to pricing efficiency with precision for practical implementation. In contrast, the model developed in this paper has an advantage to avoid complicated settings inherited in other abstract interest rate models and time-consuming Monte Carlo simulation and other numerical analysis, which leads to preventing the products' issuers and dealers from using them for application. Third, by avoiding the explosion problem in the HJM framework, forward LIBOR rates have a lognormal volatility structure which avoids the forward rates from becoming negative with positive probability.

This article is organized as follows. In Section 2, we will demonstrate an arbitrage-free CLMM and model settings. Section 3 develops the valuation formulae of quanto interest rate derivatives and identifies the relationship among them. Section 4 provides a calibration procedure for practical implementation and numerical examples to show how the models actually work. The conclusion is made in the last section.

2. The Model

In this section we briefly introduce the arbitrage-free CLMM and list the notations used in our model. Assume that trading takes place continuously in time over an interval $[0, \mathbf{T}]$, $0 < \mathbf{T} < \infty$. The uncertainty is described by the filtered probability space $\left(\Omega, F, Q, \mathcal{F}_{t\in[0,T]}\right)$, where the filtration is generated by independent standard Brownian motions $W(t) = (W_1(t), W_2(t), \ldots, W_n(t))$. Q represents the domestic spot martingale probability measure. There are some domestic and foreign assets in this economy. Hereafter, the subscript $k \in \{d, f\}$ represents the kth country's asset with 'd' for domestic and 'f' for foreign.

$f_k(t,T)$ = the kth country's forward interest rate contracted at time t for instantaneous borrowing and lending at time T with $0 \leq t \leq T \leq \mathbf{T}$.

$L_k(t,T)$ = the kth country's forward LIBOR rate contracted at time t and effective at time T for a simple compounded period $[T, T+\delta]$ with $0 \leq t \leq T \leq \mathbf{T}$.

$P_k(t,T)$ = the time t price of the kth country's zero-coupon bond paying one currency unit at maturity T.

$r_k(t,T)$ = the kth country's risk-free short rate at time t.

$\beta_k(t)$ = the kth country's money market account at time t with $\beta_k(t) = \exp\left[\int_0^t r_k(s)ds\right]$ and initial value $\beta_k(0) = 1$.

$X(t)$ = the spot exchange rate at $t \in [0, \mathbf{T}]$ for one unit of foreign currency expressed in terms of units of domestic currency.

The CLMM is developed from the AJ model which incoporated the dynamics of the domestic and foreign assets in the economy under the domestic martingale measure Q. The risk-neutral probability measure Q is induced by the domestic money market account $\beta_d(t)$ and the domestic and foreign forward interest rate $f_k(t,T)$ and the exchange rates $X(t)$ are denominated in units of $\beta_d(t)$ in the model. The drift and volatility terms of the dynamics of assets are determined by the arbitrage-free relationship. The following proposition briefly describes the pricing framework.

Proposition 1. The forward interest rates, zero-coupon bonds and exchange rate dynamics under the measure Q

For any time $T \in [0, \mathbf{T}]$, the dynamics of the forward interest rates $f_k(t,T)$, the zero-coupon bond $P_k(t,T)$ and the exchange rate $X(t)$ in the CLMM under the domestic martingale measure Q are given as follows.

$$\begin{aligned} df_d(t,T) &= \eta_d(t,T) \cdot \sigma_d(t,T)d(t) + \eta_d(t,T) \cdot dW(t), \\ df_f(t,T) &= \eta_f(t,T) \cdot (\sigma_f(t,T) - \sigma_X(t))d(t) + \eta_f(t,T) \cdot dW(t), \\ \frac{dP_d(t,T)}{P_d(t,T)} &= r_d(t)d(t) - \sigma_d(t,T) \cdot dW(t), \\ \frac{dP_f(t,T)}{P_f(t,T)} &= (r_f(t) + \sigma_X(t) \cdot \sigma_f(t,T))d(t) - \sigma_f(t,T) \cdot dW(t), \\ \frac{dX(t)}{X(t)} &= (r_d(t) - r_f(t))d(t) + \sigma_X(t) \cdot dW(t), \end{aligned}$$

where $t \in [0, T]$. $\eta_k(t,T)$ and $\sigma_k(t,T)$ denote, respectively, the volatility processes of the kth country's forward interest rate $f_k(t,T)$ and the zero-coupon bond $P_k(t,T)$. $\sigma_X(t)$ stands for the volatility process of the spot exchange rate $X(t)$.

The dynamics of the forward LIBOR rates $L_k(t,T)$ in the CLMM under the domestic martingale measure Q are not yet specified. However, the forward LIBOR rate $L_k(t,T)$ is related to the zero-coupon bond $P_k(t,T)$ and forward interest rate $f_k(t,T)$. By fixing some accrual periods δ, the forward LIBOR rate is defined by

$$1 + \delta L_k(t,T) = \frac{P_k(t,T)}{P_k(t,T+\delta)} = \exp\left(\int_T^{T+\delta} f_k(t,u)du\right). \tag{1}$$

where δ is the length of a compounding period $[T, T+\delta]$.

We assume that $L_k(t,T)$ has a lognormal volatility structure and its dynamic process is given by

$$dL_k(t,T) = \mu_k(t,T)d(t) + L_k(t,T)\gamma_k(t,T) \cdot dW(t), \tag{2}$$

where $\gamma_k(t,T)$ represents the volatility processes of the kth country's forward LIBOR rate $L_k(t,T)$ and $\mu_k(t,T)$ is some drift function.

To determine the drift terms of the forward LIBOR rates under the CLMM, we will use the arbitrage-free relationship between the drift and the volatility terms in Proposition 1. The derivation is shown as follows.

Let us define that $Z(t) = \int_T^{T+\delta} f_f(t,u)du$ and $L_f(t,T) = (1/\delta)(\exp(Z(t)) - 1)$. After some calculations, the dynamics of $L_f(t,T)$ and $Z(t)$ are shown below.

$$dL_f(t,T) = \frac{1}{\delta}\exp\left(\int_T^{T+\delta} f_f(t,u)du\right)\left(dZ(t) + \frac{1}{2}dZ(t)dZ(t)\right) \tag{3}$$

and

$$\begin{aligned} dZ(t) = &\left(\frac{1}{2}\|\sigma_f(t,T+\delta)\|^2 - \frac{1}{2}\|\sigma_f(t,T)\|^2 - \sigma_X(t)\cdot(\sigma_f(t,T+\delta) - \sigma_f(t,T))\right)d(t) \\ &+ (\sigma_f(t,T+\delta) - \sigma_f(t,T))\,dW(t). \end{aligned} \tag{4}$$

Substituting equations (1) and (4) into (3), the foreign forward LIBOR rate dynamic becomes

$$\begin{aligned} dL_f(t,T) = &\frac{1}{\delta}(1 + \delta L_f(t,T))(\sigma_f(t,T+\delta) - \sigma_f(t,T))\cdot(\sigma_f(t,T+\delta) - \sigma_X(t))\,d(t) \\ &+ \frac{1}{\delta}(1 + \delta L_f(t,T))(\sigma_f(t,T+\delta) - \sigma_f(t,T))\cdot dW(t). \end{aligned} \tag{5}$$

With the assumption that the foreign forward LIBOR rate's volatility structure is lognormal, we can compare the the volatility term in (2) and (5) and obtain the following relationship.

$$\frac{1}{\delta}(1 + \delta L_f(t,T))(\sigma_f(t,T+\delta) - \sigma_f(t,T)) = L_f(t,T)\gamma_f(t,T). \tag{6}$$

Finally, by substituting (6) into the drift term in (5), the foreign forward LIBOR rate dynamics in the CLMM under the domestic martingale measure Q can be expressed as follows.

$$\frac{dL_f(t,T)}{L_f(t,T)} = \gamma_f(t,T)\cdot(\sigma_f(t,T+\delta) - \sigma_X(t))\,d(t) + \gamma_f(t,T)\cdot dW(t). \tag{7}$$

In the same way, the domestic forward LIBOR rate process can be derived as follows.

$$\frac{dL_d(t,T)}{L_d(t,T)} = \gamma_d(t,T)\cdot\sigma_d(t,T+\delta)d(t) + \gamma_d(t,T)\cdot dW(t). \tag{8}$$

There are two main terms to be determined in the forward LIBOR rate dynamics. First, the zero-coupon bond volatility process $\{\sigma_k(t,T)\}$ is stochastic rather than deterministic, which makes the dynamics (7) and (8) difficult to solve for the forward rate $L_k(t,T)$. Instead, according to the recurrence relation between $\sigma_k(t,T+\delta)$ and $\gamma_k(t,T)$, $\sigma_k(t,T)$ can be approximated by $\overline{\sigma}_k^\tau(t,T)$ with a fixed initial time τ, which is defined by

$$\overline{\sigma}_k^\tau(t,T) = \sum_{i=1}^{[\delta^{-1}(T-t)]} \frac{\delta L_k(\tau, T-i\delta,\delta)}{1+\delta L_k(\tau,T-i\delta,\delta)} \gamma_k(t,T-i\delta), \tag{9}$$

where $t \in [\tau, T-\delta]$, and $T-\delta > 0$. It means that the calendar time of the process $\{L_k(t,T-i\delta,\delta)\}_{t\in[\tau,T-i\delta]}$ is frozen at its initial time τ. By substituting (4) into the drift and volatility terms in (1) and (2), the resulting equations become solvable and the distribution of the forward LIBOR rate is approximated lognormally distributed.

Second, the payments and receipts of quanto interest rate derivatives are made on a floating rate basis with reset in-advance and paid in-arrears features. It is relatively easier to price under the domestic forward martingale measure $Q^{T+\delta}$ than under the domestic martingale measure Q. The forward martingale measure $Q^{T+\delta}$ is induced by the domestic zero-coupon bond $P_d(t,T+\delta)$ as the numeraire. By using the changing-numeraire mechanism, the drift and volatility terms of the assets dynamics under the forward martingale measure $Q^{T+\delta}$ can be specified. The result is shown in Proposition 2.

Proposition 2. The forward LIBOR rates dynamics under the measure $Q^{T+\delta}$
For any time $T \in [0, \mathbf{T}]$, the dynamics of the forward LIBOR rates $L_k(t,T)$ in the CLMM under the forward martingale measure $Q^{T+\delta}$ are given as follows.

$$\frac{dL_d(t,T)}{L_d(t,T)} = \gamma_d(t,T)\cdot dW(t),$$
$$\frac{dL_f(t,T)}{L_f(t,T)} = \gamma_f(t,T)\cdot\left(\overline{\sigma}_f^\tau(t,T+\delta) - \overline{\sigma}_d^\tau(t,T+\delta) - \sigma_X(t)\right) d(t) + \gamma_f(t,T)\cdot dW(t),$$

where $t \in [0,T]$ and $\overline{\sigma}_k^\tau(t,T)$ is defined in (9).

In Proposition 2, the drift term of the forward LIBOR rate $L_k(t,T)$ is specified by the approximated zero-coupon bond volatility $\overline{\sigma}_k^\tau(t,T)$ in (9). The volatility term remain unchanged under the forward martingale measure $Q^{T+\delta}$. By avoiding the stochastic characteristic, the distribution of the forward LIBOR rate $L_k(t,T)$ is lognormally distributed.

The CLMM has the following merits. Unlike the abstract rates modeled in the short rate and forward rate models, the LIBOR rates in the CLMM are market-observable. Moreover, the cap pricing formula is consistent with the market formula, namely the Black formula. Therefore, the volatility of the forward LIBOR rates $\gamma_k(t,T)$ and the approximated bond volatility $\overline{\sigma}_k^{\tau}(t,T)$ are easy to be calibrated under the CLMM. Specifically, they can be extracted via the pricing model from quoted prices in financial markets, such as caps and floors. Once the parameters are calibrated, the pricing formulae under the CLMM are analytically tractable for practical application. In addition, Rogers (1996) indicated that the Gaussian models, such as Hull and White (1990) and Gaussian HJM, may distort the pricing of derivatives under certain circumstances due to possible negative interest rates. However, the forward LIBOR rate in the CLMM are log-normally distributed, thereby making the LIBOR rates to be positive and preventing the negative rate problem arising from using the Gaussian interest rate models. As a result, the CLMM is very general and useful for pricing many cross-currency interest rate derivatives, such as QSs, QCs/QFs, and EQSs. In the next section, variants of the cross-currency interest rate derivatives are priced as examples.

3. Valuation of Quanto Interest Rate Derivatives

In this section, we employ the CLMM derived in the previous section to price the interest rate-based derivatives in the cross-currency economy. We firstly derive the formula of a QS under this framework. A QC and a QF can be derived in a similar way. Finally, we demonstrate the relationship among an EQS and other quanto derivatives and derive related pricing formulae.

T_j = the reset date, $j = 0, 1, \cdots, n-1$.

T_{j+1} = the payment date, $j = 0, 1, \cdots, n-1$.

τ = the current time and the cash flow time is defined by $0 \le \tau \le T_0 < T_1 < \cdots < T_n \le \mathbf{T}$.

$\mathbf{T}_R$ = the set of reset dates T_j and $j = 0, 1, \cdots, n-1$.

$\mathbf{T}_P$ = the set of payment dates T_{j+1} and $j = 0, 1, \cdots, n-1$.

δ = the year fraction between time interval $[T_{j-1}, T_j]$, $j = 1, 2, \cdots, n$.

N_d = notional principal denominated in domestic currency.

$\Phi(\cdot)$ = the standard normal distribution.

3.1 Quanto swaps

A QS is similar to an interest rate swap in a single currency. It is a contract in which two counterparties agree to exchange domestic floating-rate payments for foreign floating-rate payments. Both notional principals are denominated in domestic currency. The impetus for investors to enter such a contract is to take advantage of interest differentials between two countries. We define a QS as follows.

Definition 1. Quanto Swaps
A QS is a contract swapping the payments at future times $\mathbf{T}_P$ *with reset dates* $\mathbf{T}_R$ *in notional principal* N_d*. From the perspective of a counterparty who pays the domestic floating rate and receives the foreign floating rate payment, the cash flow stream at time* T_j *is given as follow:*

$$N_d L_f(T_{j-1}, T_{j-1})\delta - N_d \left(L_d(T_{j-1}, T_{j-1}) + R \right) \delta, \tag{10}$$

where $L_f(T_{j-1}, T_{j-1})$ *and* $L_d(T_{j-1}, T_{j-1})$ *denotes, respectively, the foreign and domestic LIBOR rates for the period* $[T_{j-1}, T_j]$*, observed at time* T_{j-1}*.* R *is a spread in basis points and may be positive or negative.*

Based on the CLMM framework, we can derive the pricing formula of a QS as defined in Definition 1. The result is presented in the following theorem.

Theorem 1. The Pricing Formula of a Quanto Swap
Under the CLMM, the no-arbitrage value of a QS with a spread R *at current time* τ *is given by*

$$QS(\tau, K) = N_d \delta \sum_{j=1}^{n} P_d(\tau, T_j) \left\{ L_f(\tau, T_{j-1}) \rho(\tau, T_{j-1}) - L_d(\tau, T_{j-1}) - R \right\}, \tag{11}$$

where

$$\rho(\tau, T_{j-1}) = \exp\left(\int_{\tau}^{T_{j-1}} \mu_f(t, T_{j-1}) d(t) \right),$$
$$\mu_f(t, T_{j-1}) = \gamma_f(t, T_{j-1}) \cdot (\overline{\sigma}_f^{\tau}(t, T_j) - \overline{\sigma}_d^{\tau}(t, T_j) - \sigma_X(t)).$$

Proof. See Appendix A. □

In Theorem 1, there are four main points to emphasize. First, when pricing cross-currency linked derivatives, we have to adjust the drift of the foreign asset by a covariance term under the domestic martingale measure Q. The covariance term denotes the correlation between the exchange rate and the foreign asset price. In comparison with the drift terms in Proposition 2, the dynamic of the foreign forward LIBOR rate $L_f(t,T)$ is adjusted by a covariance term $\gamma_f(t,T)\cdot\sigma_X(t)$. The intuition of the adjustment is when we price foreign assets in units of domestic currency, we are using a fixed exchange rate for conversion. The covariance term captures future exchange rate fluctuations. After taking exponent and integrating with time interval $[\tau,T_{j-1}]$, the covariance term becomes

$$qa(\tau,T_{j-1})=\exp\left(\int_{\tau}^{T_{j-1}}(\gamma_f(t,T_{j-1})\cdot\sigma_X(t))d(t)\right).$$

It is so called *quanto adjustment*. From the domestic investors' perspective under the domestic martingale measure, the floating payment with the foreign interest rate $L_f(\tau,T_{j-1})$ is adjusted by $qa(\tau,T_{j-1})$.

Second, the spread R is specified to compensate for the differentials between domestic and the foreign interest rates. The fair quanto swap rate at contract initiation is the rate to make the contract a zero-sum game. In other words, R is the solution to the zero-price of the QS at start. We easily obtain

$$R=\frac{\sum_{j=1}^{n}P_d(\tau,T_j)\left\{L_f(\tau,T_{j-1})\rho(\tau,T_{j-1})-L_d(\tau,T_{j-1})\right\}}{\sum_{j=1}^{n}P_d(\tau,T_j)}. \tag{12}$$

Thirdly, the hedging strategy could be constructed by rewriting (10) in an alternative expression. Using the relationship between the forward LIBOR rate and the zero-coupon bonds as shown in (1), the no-arbitrage price in (11) can be expressed as follows.

$$QS(\tau,K)=N_d\sum_{j=1}^{n}\left\{DF(\tau,T_j)\left(P_f(\tau,T_{j-1})-P_f(\tau,T_j)\right)-P_d(\tau,T_{j-1})+(1-\delta R)P_d(\tau,T_j)\right\},$$

where

$$DF(\tau,T_j)=\rho(\tau,T_j)\frac{P_d(\tau,T_j)}{P_f(\tau,T_j)}. \tag{13}$$

$DF(\tau, T_j)$ denotes the discount factor to reveal the expected cash flow of the foreign leg under the domestic martingale measure. Specifically, the reciprocal of the foreign bond, $\frac{1}{P_f(\tau,T_j)}$, represents the expected terminal payoff of the foreign leg at time T_j for one unit of foreign currency invested at τ. This expected foreign cash flow is discounted by the domestic bond $P_d(\tau, T_j)$ and adjusted by $\rho(\tau, T_j)$ which is effected by quanto adjustment $qa(\tau, T_{j-1})$.

Hedging a QS is relatively simple due to the linearity of the cash flow of two legs. Via equation (13), we can construct a hedging portfolio, H, with the domestic and the foreign zero-coupon bonds $P_k(t, T)$ as given below.

$$H = N_d \sum_{j=1}^{n} \left\{ \Delta_{1j} P_f(\tau, T_{j-1}) + \Delta_{2j} P_f(\tau, T_j) + \Delta_{3j} P_d(\tau, T_{j-1}) + \Delta_{4j} P_d(\tau, T_j) \right\}.$$

The hedging ratios are

$$\Delta_{1j} = DF(\tau, T_j), \quad \Delta_{2j} = -DF(\tau, T_j), \quad \Delta_{3j} = -1, \quad \Delta_{4j} = 1 - \delta R.$$

The hedging strategy H shows the proper units of the zero-coupon bonds that must be held to replicate a QS synthetically. Specifically, it contains holding long Δ_{1j} units of the foreign bond with maturity at T_{j-1} and Δ_{4j} units of the domestic bond with maturity at T_j, meanwhile selling short Δ_{2j} units of the foreign bond with maturity at T_j and Δ_{3j} units of the domestic bond with maturity at T_{j-1}.

Fourthly, we compare our CLMM pricing formula with related literature as follows. Turnbull (1993) described the dynamics of domestic and foreign forward interest rates under the HJM framework. Wei (1994) and CCY (2002) derived the pricing formulas under the mean-reverting Ornstein-Uhlenbeck process of the short rate. In contrast, our pricing formula in the CLMM framework is clear and full of economic intuition. The CLMM is obviously term-structure consistent, and the model parameters can be easily calibrated from market quantities. In addition, it avoids the problem of negative interest rates inherited in the previous models. Therefore, our pricing formula should be more accurate, feasible and tractable in practice.

3.2 Quanto caps/floors

A QC/QF is similar to a cap/floor in a single currency. The major difference is that the forward LIBOR rate is denominated in foreign currency rather than in domestic currency. The payoff is in arrears as the case of a normal cap, but is denominated in domestic currency. A QC/QF can be decomposed additively. Indeed, its discount payoff is a sum of quanto caplets/floorlets. We define a QC/QF as follows.

Definition 2. Quanto Caps/Floors
A QC/QF is a series of options with maturities at future times $\mathbf{T}_P$ *with reset dates* $\mathbf{T}_R$ *in notional principal* N_d*. The owner of a QC/QF has a right whether to exercise the options. The cash flow stream at time* T_j *is given as follows.*

$$N_d \left(\phi L_f(T_{j-1}, T_{j-1}) - \phi K\right)^+ \delta,$$

where $L_f(T_{j-1}, T_{j-1})$ *denotes the foreign LIBOR rate for the period* $[T_{j-1}, T_j]$ *observed at time* T_{j-1}*.* K *is a strike level of interest rate in basis points.* $\phi = 1$ *denotes a QC and* $\phi = -1$ *for a QF.*

According to the Definition 2, the pricing formula of a QC in the following theorem are presented in Theorem 2.

Theorem 2. The Pricing Formula of a Quanto Cap
Under the CLMM, the no-arbitrage value of a QC with strike level of interest rate K *at current time* τ *is given by*

$$QC(\tau, K) = N_d \delta \sum_{j=1}^{n} P_d(\tau, T_j) \left\{ L_f(\tau, T_{j-1}) \rho(\tau, T_{j-1}) \Phi\left(d_1(\tau, T_{j-1})\right) - K \Phi\left(d_2(\tau, T_{j-1})\right) \right\}, \tag{14}$$

where $\rho(\tau, T_{j-1})$ *are defined in Theorem 1, and*

$$\nu_f(\tau, T_{j-1}) = \int_{\tau}^{T_{j-1}} \|\gamma_f(t, T_{j-1})\|^2 d(t),$$
$$d_1(\tau, T_{j-1}) = \frac{\ln(L_f(\tau, T_{j-1})\rho(\tau, T_{j-1})/K) + \frac{1}{2}\nu_f(\tau, T_{j-1})}{\sqrt{\nu_f(\tau, T_{j-1})}},$$
$$d_2(\tau, T_{j-1}) = d_1(\tau, T_{j-1}) - \sqrt{\nu_f(\tau, T_{j-1})}.$$

Proof. See Appendix B. □

Several interesting points in Theorem 2 are worth noting as follows. Like a QS, the covariance term $\rho(\tau, T_{j-1})$ of domestic and foreign interest rates and the exchange rate also has an impact on the pricing result. The derived pricing formula of a QC under the CLMM is a Black type model, which is an analytical solution and easy to be implemented in practice. In addition, the volatility of the forward LIBOR rates $\gamma_k(t,T)$ and the approximated bond volatility $\overline{\sigma}_k^{\tau}(t,T)$ are easy to be calibrated via the pricing model. That is, they can be extracted via the model from quoted prices of caps in financial market.

Second, a hedging strategy is as in the case of a QS. The pricing formula is rewritten as an alternative expression in terms of domestic and foreign zero-coupon bonds, and a quanto adjustment term. This is shown below.

$$\begin{aligned} &QC(\tau, K) \\ &= N_d \sum_{j=1}^{n} \{DF(\tau, T_j)\left(P_f(\tau, T_{j-1}) - P_f(\tau, T_j)\right)\Phi\left(d_1(\tau, T_{j-1})\right) - \delta K P_d(\tau, T_j)\Phi\left(d_2(\tau, T_{j-1})\right)\}, \end{aligned}$$

where $DF(\tau, T_j)$ is defined in (13).

Equation (14) provides a way to construct a hedging strategy. In contrast to a QS, hedging a QC is not straightforward due to the nonlinearity of its cash flows. However, we can construct a delta-neutral hedging portfolio, H, in terms of the domestic and the foreign bonds $P_k(t,T)$. This is given below.

$$H = N_d \sum_{j=1}^{n} \left\{\Delta_{1j} P_f(\tau, T_{j-1}) + \Delta_{2j} P_f(\tau, T_j) + \Delta_{3j} P_d(\tau, T_j)\right\}.$$

The hedging ratios are

$$\begin{aligned} \Delta_{1j} &= DF(\tau, T_j)\Phi\left(d_1(\tau, T_{j-1})\right), \\ \Delta_{2j} &= -DF(\tau, T_j)\Phi\left(d_2(\tau, T_{j-1})\right), \\ \Delta_{3j} &= -\delta K \Phi\left(d_2(\tau, T_{j-1})\right). \end{aligned}$$

This hedging strategy shows the proper units of the zero-coupon bonds that must be held in a replicated portfolio of a QC similar to the case of a QS presented in equation (13).

Thirdly, the derived pricing formulae are suffieiently accurate and robust and easy to implement in comparision with other models. BM (1997) derived the price of an exotic cap on a basket of LIBOR rate in a cross-currency economy within the Gaussian HJM framework. However, it is well-known that the instantaneous forward rates are not observable in the market and the parameters in the HJM model are difficult to calibrate. In addition, the Gaussian HJM forward rates could result negative with a positive probability. BM (2006) derived the pricing formula of a QC through lognormal martingales under the domestic forward martingale measure. The drift and volatility structure of the dynamics are determined by the forward LIBOR rate and the forward exchange rate. Comparatively, we employ the approximation techniques to obtain lognormal volatilities and access the compact formula without loss of the accuracy. The numerical analyses are demonstrated in the following section. These approximation techniques are first presented by BGM (1997) and then adopted by Brace and Womersley (2000).

Analogously, a quanto floor is an insurance against declining interest rate. The pricing process of a QF is similar to a QC. We present it in Theorem 3.

Theorem 3. The Pricing Formula of a Quanto Floor
Under the CLMM, the no-arbitrage value of a QF with strike level of interest rate K at current time τ is given by

$$QF(\tau) = N_d \delta \sum_{j=1}^{n} P_d(\tau, T_j) \left\{ K\Phi\left(-d_2(\tau, T_{j-1})\right) - \L_f(\tau, T_{j-1})\rho(\tau, T_{j-1})\Phi\left(-d_1(\tau, T_{j-1})\right) \right\}.$$

3.3 Exotic quanto swaps

The CLMM can be used to price more complicated products in the interest rate markets. By structuring the components of QSs, QCs and QFs, we show that various customized products could be designed for the special purpose. An EQS is a variant of a QS with an additional feature in that instead of the foreign payoff being directly based on the foreign LIBOR rate, the payment in the foreign leg has a different payoff structure in different ranges of the foreign LIBOR rate. We define an EQS as follows.

Definition 3. Exotic Quanto Swaps

An EQS is a contract swapping the payment at future times $\mathbf{T}_P$ with reset dates $\mathbf{T}_R$ in notional principal N_d. From the perspective of a counterparty who pays the domestic floating rate and receives the foreign floating payment, the cash flow stream at time T_j is given as follows:

$$N_d L_f^*(T_{j-1}, T_{j-1})\delta - N_d \left(L_d(T_{j-1}, T_{j-1}) + R\right)\delta, \tag{15}$$

where

$$L_f^*(T_{j-1}, T_{j-1}) = \begin{cases} L_f(T_{j-1}, T_{j-1}) & L_f(T_{j-1}, T_{j-1}) \leq R_d, \\ R_d & R_d \leq L_f(T_{j-1}, T_{j-1}) \leq R_m, \\ R_u - L_f(T_{j-1}, T_{j-1}) & R_m \leq L_f(T_{j-1}, T_{j-1}) \leq R_u, \\ 0 & \text{otherwise.} \end{cases}$$

$L_f^*(T_{j-1}, T_{j-1})$ denotes a reference rate determined by the foreign LIBOR rate. R is a spread in basis points and may be positive or negative. R_d, R_m and R_u denote three different levels of interest rate, where $R_u = R_d + R_m$.

As given in Definition 3, the payment of the foreign leg in a common case is increasing in the low level of the interest rate, fixed in the medium range, and declining as the interest rate getting higher. It makes a trapezoid figure between the payoff and the foreign LIBOR rate. An EQS are usually adopted to exploit the differential in the global interest rate market. For example, corporate borrowers with debt referred to the domestic LIBOR rate may take advantage of the inverse floating-rate debt based on the foreign LIBOR rate when the level of interest rates is rising higher gradually. Bond managers can exploit the spread to enhance a portfolio's yield by receiving the foreign payment in a low-level environment fo the foreign rate.

In a similar way, we can derive the pricing formula of an EQS based on the CLMM. The result is presented directly without proof in the following theorem.[1]

Theorem 4. The Pricing Formula of Exotic Quanto Swaps

Under the CLMM, the no-arbitrage value of an EQS with spread of interest rate R at

[1] We omit derivation for the snake of parsimony and available upon request.

current time τ is given by

$$
\begin{aligned}
EQS(\tau, R) = N_d \delta \sum_{j=1}^{n} P_d(\tau, T_j)\{ & L_f(\tau, T_{j-1})\rho(\tau, T_{j-1})\Phi\left(-d_1(\tau, T_{j-1}, R_d)\right) \\
& + R_d[\Phi\left(d_2(\tau, T_{j-1}, R_d)\right) - \Phi\left(d_2(\tau, T_{j-1}, R_m)\right)] \\
& + R_u[\Phi\left(d_2(\tau, T_{j-1}, R_m)\right) - \Phi\left(d_2(\tau, T_{j-1}, R_u)\right)] \\
& - L_f(\tau, T_{j-1})\rho(\tau, T_{j-1})[\Phi(d_1(\tau, T_{j-1}, R_m)) - \Phi(d_1(\tau, T_{j-1}, R_u))] \\
& - L_d(\tau, T_{j-1}) - R\},
\end{aligned} \tag{16}
$$

where

$$
\begin{aligned}
d_1(\tau, T_{j-1}, *) &= \frac{\ln(L_f(\tau, T_{j-1})\rho(\tau, T_{j-1})/*) + \frac{1}{2}\nu_f(\tau, T_{j-1})}{\sqrt{\nu_f(\tau, T_{j-1})}}, \\
d_2(\tau, T_{j-1}, *) &= d_1(\tau, T_{j-1}, *) - \sqrt{\nu_f(\tau, T_{j-1})},
\end{aligned}
$$

and $\rho(\tau, T_{j-1})$ and $\nu_f(\tau, T_{j-1})$ are defined in Theorem 2.

The intuition and model parameters of the pricing formula of an EQS in (16) can be explained accordingly in a similar way as those given in Theorem 1–2. We next focus on a replicating and hedging strategy of an EQS.

Analogously, we can construct a delta-neutral hedging portfolio for an EQS in terms of the domestic and foreign bonds $P_k(t, T)$. Hedging ratios can be identified after some rearrangement from the relationship between the forward LIBOR rates and the domestic and foreign zero-coupon bonds. It is shown as follows.

$$
H = N_d \sum_{j=1}^{n} \{\Delta_{1j} P_f(\tau, T_{j-1}) + \Delta_{2j} P_f(\tau, T_j) + \Delta_{3j} P_d(\tau, T_{j-1}) + \Delta_{4j} P_d(\tau, T_j)\} \tag{17}
$$

where the hedging ratios are given by

$$
\begin{aligned}
\Delta_{1j} &= DF(\tau, T_j)\left\{\Phi\left(-d_1(\tau, T_{j-1}, R_d)\right) - \Phi\left(d_1(\tau, T_{j-1}, R_m)\right) - \Phi\left(d_1(\tau, T_{j-1}, R_u)\right)\right\}, \\
\Delta_{2j} &= -DF(\tau, T_j)\left\{\Phi\left(-d_1(\tau, T_{j-1}, R_d)\right) - \Phi\left(d_1(\tau, T_{j-1}, R_m)\right) - \Phi\left(d_1(\tau, T_{j-1}, R_u)\right)\right\}, \\
\Delta_{3j} &= -1, \\
\Delta_{4j} &= \delta R_d\{\Phi(d_2(\tau, T_{j-1}, R_d)) - \Phi(d_2(\tau, T_{j-1}, R_m))\} \\
&\quad + \delta R_u\{\Phi(d_2(\tau, T_{j-1}, R_m)) - \Phi(d_2(\tau, T_{j-1}, R_u))\} + (1 - \delta R).
\end{aligned}
$$

The hedging strategy as shown in (17) can be explained in a similar way as given in the previous sections. This portfolio contains four zero-coupon bonds with the corresponding hedging ratios.

In addition, an EQS can be synthetically replicated by prevailing market quantities QSs, QCs and QFs according to the cash flow stream in Definition 3. The no-arbitrage condition tells us that the price of an EQS is the linear combination of such quanto interest rate derivatives. The relationship of these quanto products is presented in the following theorem.

Theorem 5. The Relationship of Quanto Derivatives
An EQS can be synthetically replicated by a linear combination of QSs, QCs and QFs. Under the CLMM, the no-arbitrage value at time τ of an EQS is given by

$$\begin{aligned} EQS(\tau, R) &= QS(\tau, R) - QC(\tau, R_d) - QC(\tau, R_m) + QC(\tau, R_u) \\ &= -(L_d(\tau, T_{j-1}) + R)P_d(\tau, T_j)\delta - QF(\tau, R_d) - QF(\tau, R_m) + QF(\tau, R_u). \end{aligned} \tag{18}$$

Proof. See Appendix C. □

The pricing formula of an EQS in (18) implies that hedging an EQS is straightforward due to the linearity of cash flow components that are composed of basic quanto derivatives QSs, QCs and QFs. If this relation among the component derivatives is in disequilibrium, an arbitrage opportunity arises for taking a profit.

4. Calibration and Numerical Analysis

This section provides a calibration procedure and numerical examples for practical implementation.

4.1 Calibration procedure

The calibration of an interest rate model is one of the significant parts of its implementation. Given observable forward LIBOR rates and market-quoted volatilities of caplets, the pricing parameters in the pricing framework are virtually obtainable and immediate. The calibration methodology presented by Rebonato (1999) is employed in our paper. We perform a simultaneous calibration of the CLMM to the instantaneous total volatilities and correlation surface of the underlying forward LIBOR rates and the exchange rate. To illustrate the procedure, we assume there are n assets in a cross-currency economy. The assets include $(n-1)/2$ forward LIBOR rates of the k-th country and the exchange rate. The number of random shocks of the term structure of the forward rates, denoted by m, is determined based on a compromise between the simplicity and the accuracy in practical application. Here, we employ three stylized factors ($m = 3$), such as level, slope and curvature of the term structure of interest rates.

The steps of calibration are given as follows. First, the instantaneous volatility of the forward LIBOR rate $\gamma_k(t, T)$ is chosen such that the caplet price is correctly recovered. In other words, the market-quoted caplet volatility for T years from current time τ, denoted by $\sigma_{imp}(T)$, is given by

$$\int_{\tau}^{T+\delta} \|\gamma_k(t, T+\delta)\|^2 d(t) = \sigma_{imp}^2(T)T.$$

Next, we assume that the current market term structure is recovered by a piecewise-constant instantaneous total volatility depending only on the duration of the underlying rates. The time-dependent volatility of each period for each rate can be determined by the stripping method presented in Rebonato (1999). To describe the volatility structure engaged in the CLMM, the element S_i is defined by

$$S(i) = \int_{Ti-1}^{Ti} \|\gamma_k(t, T_i)\|^2 d(t),$$

where the subscript i labels the ith forward LIBOR rate. The instantaneous volatility is converted into a time-dependent volatility structure and $S(i)$ is solved by the recurrence relation as shown below

$$\sum_{k=1}^{i} S(i) = \sigma_{imp}^2(T)T.$$

The volatility structure is reported below.

$$S \equiv \begin{pmatrix} S_1 & & & & \\ S_2 & S_1 & & & \\ S_3 & S_2 & S_1 & & \\ \vdots & \vdots & \vdots & \ddots & \\ S_n & S_{n-1} & S_{n-2} & \dots & S_1 \end{pmatrix}.$$

Besides, the instantaneous volatility of the exchange rate can be extracted from the on-the-run currency options prices in practice or calculated directly from historical data of the underlying exchange rate. For simplicity, we assume that the term structure of volatility is flat.

Second, the correlation structure of the underlying rates in Proposition 2 is given by

$$dW(t)dW(t) = \Sigma d(t),$$

where Σ is a correlation matrix which represents the market correlation of the underlying rates. Σ is an n-rank, positive-definite, and symmetric matrix and can be written as

$$\Sigma = AA' = P\Lambda P,$$

where P is a real orthogonal matrix and Λ is a diagonal matrix. A is defined by $A \equiv P\Lambda^{1/2}$. Next, the full-rank correlation matrix Σ is approximated by selecting a suitable correlation matrix Σ^B which can be decomposed by a m-rank matrix B.

$$BdZ(t)(BdZ(t))' = BB'd(t) = \Sigma^B d(t).$$

The appropriate choice of the matrix B can be obtained by using by Rebonato (1999) and shown as follows

$$b_{i,k} = \begin{cases} \cos\theta_{i,k} \prod_{j=1}^{k-1} \sin\theta_{i,j} \,, & \text{if } k = 1, 2, ..., m-1, \\ \prod_{j=1}^{k-1} \sin\theta_{i,j} \,, & \text{if } k = m. \end{cases}$$

Then a suitable matrix $\hat{B}$ is obtained by finding the solution $\hat{\theta}$ of the following optimization problem

$$\min_{\theta} \sum_{i,j=1}^{n} |\Gamma_{i,j}^{B} - \Gamma_{i,j}|^2.$$

Hence, the approximated correlation matrix $\Sigma^B = \hat{B}\hat{B}'$ mimics the correlation of the market Σ.

4.2 Numerical analysis

In this subsection, we provide numerical examples to illustrate the application of pricing formulae of QSs, QCs and EQSs. In comparision with previous researches, the model parameters in our model are calibrated from the market observable quantities to examine the model efficiency and accuracy.

The accuracy of the pricing formulae will be compared to Monte Carlo simulation to examine the robustness of the pricing formulae. We consider short and long maturities for these quanto interest rate derivatives with different levels of spread of interest rate R and strike rate K to examine the robustness of the pricing models. In addition, three calendar dates are set to the current time τ to include the different shapes and levels of the term structures of forward LIBOR rates for two countries.

The market data associated with the forward LIBOR rates, the exchange rate and the cap volatilities are obtained between 2008/01/01 and 2010/12/31 from a domestic country (U.S.) and foreign country (U.K.). [2] To consider different market situations, the current time τ is set at three various dates including 2008/01/01, 2009/01/01, and 2010/01/01. The term structures of the forward LIBOR rates of the two countries are shown in Table 5 of Appendix E and the volatilities structures implied in cap markets are shown in Table 4.

In Table 1–3, the pricing results of QSs, QCs and EQSs are separately demonstrated. The current time τ is date T_0 ($\tau = T_0$). The maturity of these quanto interest rate derivatives are considered respectively for one, three, and five years. ($T_{n-1} = 1, 3, 5$)

[2] In fact, we have examined more data from 2005/01/03 to 2007/12/31. The pricing results are shown robust and precision in comparision with Monte Carlo simulation. The output is available upon request.

The year fraction δ of the time interval $[T_{j-1}, T_j]$ is half a year ($\delta = 1/2$). Three different spreads R are examined for sensitivity analysis of QSs and EQSs while three different strike rates K are examined for QCs. The pricing results are reported based on three calendar dates.

The pricing results of Theorem 1–4 given in Table 1–3 are listed with the results of Monte Carlo simulation and standard errors. The results show that the derived formulae are sufficiently accurate in comparision with Monte Carlo simulation for the short and long maturities. At different levels of spread or strike, the pricing models perform robustly in the empirical study. In addition, the pricing results also exhibit satisfactorily in different scenarios of the term structures of interest rates. The overall results show that the derived pricing formulae for quanto interest rate derivatives are good substitutes for simulation and exhibit a time-efficiency characteristic, thereby avoiding the problem of excessive time-consuming simulation.

5. Conclusion

We have derived the analytical valuation formulae for quanto interest rate derivatives that are frequently traded in financial markets. The pricing model is a multi-factor framework that includes the dynamics of domestic and foreign LIBOR rates and the foreign exchange rate. The valuation formulae are easy for practical implementation and the model parameters can be easily acquired from market quantities. In addition, the derived formulae are robust, analytically tractable and sufficiently accurate and thereby suitable for pricing QSs, QCs and EQSs in practice.

Table 1 The value of QSs

	R		1-year		3-year		5-year
2008/01/01	−2.00%	AF	0.04832	AF	0.09350	AF	0.12049
		MC	0.04836	MC	0.09348	MC	0.12073
		SE	3.14×10^{-5}	SE	9.19×10^{-5}	SE	1.40×10^{-4}
	0.00%	AF	0.01928	AF	0.02834	AF	0.02200
		MC	0.01929	MC	0.02833	MC	0.02219
		SE	3.15×10^{-5}	SE	9.19×10^{-5}	SE	1.40×10^{-4}
	2.00%	AF	−0.00976	AF	−0.03681	AF	−0.07650
		MC	−0.00977	MC	−0.03676	MC	−0.07658
		SE	3.14×10^{-5}	SE	9.18×10^{-5}	SE	1.41×10^{-4}
			(1.320%)		(0.870%)		(0.440%)
2009/01/01	−2.00%	AF	0.03963	AF	0.06927	AF	0.08066
		MC	0.03965	MC	0.06898	MC	0.08080
		SE	5.67×10^{-5}	SE	2.01×10^{-4}	SE	3.45×10^{-4}
	0.00%	AF	0.00995	AF	0.00136	AF	−0.02359
		MC	0.00991	MC	0.00160	MC	−0.02378
		SE	5.73×10^{-5}	SE	2.07×10^{-4}	SE	3.49×10^{-4}
	2.00%	AF	−0.01973	AF	−0.06655	AF	−0.12784
		MC	−0.01981	MC	−0.06700	MC	−0.12794
		SE	5.65×10^{-5}	SE	1.98×10^{-4}	SE	3.48×10^{-4}
			(0.670%)		(0.040%)		(−0.450%)
2010/01/01	−2.00%	AF	0.03590	AF	0.06640	AF	0.07200
		MC	0.03596	MC	0.06612	MC	0.07222
		SE	8.07×10^{-5}	SE	4.15×10^{-4}	SE	7.68×10^{-4}
	0.00%	AF	0.00612	AF	−0.00135	AF	−0.03087
		MC	0.00617	MC	−0.00106	MC	−0.03065
		SE	8.23×10^{-5}	SE	4.05×10^{-4}	SE	7.86×10^{-4}
	2.00%	AF	−0.02366	AF	−0.06909	AF	−0.13374
		MC	−0.02367	MC	−0.06978	MC	−0.13365
		SE	8.15×10^{-5}	SE	4.48×10^{-4}	SE	7.64×10^{-4}
			(0.410%)		(−0.040%)		(−0.600%)

[a] The current time τ is set at three dates including 2008/01/01, 2009/01/01, and 2010/01/01. The maturities of QSs are examined for one, three, and five years. ($T_{n-1} = 1, 3, 5$) The year fraction δ is half a year ($\delta = 1/2$).

[b] R denotes the level of spread in basis points and may be positive and negative by Definition 1. We examined three different levels of spread ($R = -2.00\%$, 0.00%, 2.00%). AF stands for the pricing result using analytic formula in equation (11) of Theorem 1. MC and SE represent, respectively, the numerical result using Monte Carlo simulation with 50,000 random paths and standard errors.

[c] The number in the parenthesis denotes the level of swap rates in equation (12). The swap rate makes the contract a zero-sum game at the beginning.

Table 2 The value of QCs

	K		1-year		3-year		5-year
		AF	0.06144	AF	0.11956	AF	0.16517
	1.00%	MC	0.06140	MC	0.11965	MC	0.16508
		SE	2.06×10^{-5}	SE	5.09×10^{-5}	SE	7.03×10^{-5}
		AF	0.03240	AF	0.05533	AF	0.07059
2008/01/01	3.00%	MC	0.03240	MC	0.05530	MC	0.07064
		SE	2.04×10^{-5}	SE	4.83×10^{-5}	SE	6.43×10^{-5}
		AF	0.00734	AF	0.01088	AF	0.01353
	5.00%	MC	0.00734	MC	0.01087	MC	0.01353
		SE	1.05×10^{-5}	SE	2.41×10^{-5}	SE	3.15×10^{-5}
		AF	0.01665	AF	0.03573	AF	0.04844
	1.00%	MC	0.01662	MC	0.03573	MC	0.04840
		SE	4.02×10^{-5}	SE	1.19×10^{-4}	SE	1.56×10^{-4}
		AF	0.00166	AF	0.01059	AF	0.01681
2009/01/01	3.00%	MC	0.00167	MC	0.01066	MC	0.01671
		SE	2.74×10^{-5}	SE	9.80×10^{-5}	SE	1.30×10^{-4}
		AF	0.00059	AF	0.00574	AF	0.00957
	5.00%	MC	0.00060	MC	0.00584	MC	0.00949
		SE	1.91×10^{-5}	SE	8.37×10^{-5}	SE	1.11×10^{-4}
		AF	0.00812	AF	0.04414	AF	0.07442
	1.00%	MC	0.00809	MC	0.04436	MC	0.07429
		SE	3.83×10^{-5}	SE	1.51×10^{-4}	SE	2.15×10^{-4}
		AF	0.00177	AF	0.02002	AF	0.03723
2010/01/01	3.00%	MC	0.00173	MC	0.02005	MC	0.03706
		SE	2.42×10^{-5}	SE	1.27×10^{-4}	SE	1.94×10^{-4}
		AF	0.00057	AF	0.01120	AF	0.02248
	5.00%	MC	0.00056	MC	0.01116	MC	0.02231
		SE	1.60×10^{-5}	SE	1.10×10^{-4}	SE	1.69×10^{-4}

[a] The current time τ is set at three dates including 2008/01/01, 2009/01/01, and 2010/01/01. The maturities of QCs are examined for one, three, and five years. ($T_{n-1} = 1, 3, 5$) The year fraction δ is half a year ($\delta = 1/2$).

[b] K denotes the strike level of interest rate in basis points by Definition 2. We examined three different levels of strike (K = 1.00%, 3.00%, 5.00%). AF stands for the pricing result using analytic formula in equation (14) of Theorem 2. MC and SE represent, respectively, the numerical result using Monte Carlo simulation with 50,000 random paths and standard errors.

Table 3 The value of EQSs

	R		1-year		3-year		5-year
		AF	-0.01621	AF	-0.02014	AF	-0.02802
	-2.00%	MC	-0.01619	MC	-0.02006	MC	-0.02825
		SE	2.93×10^{-5}	SE	8.35×10^{-5}	SE	1.28×10^{-4}
		AF	-0.04525	AF	-0.08529	AF	-0.12652
2008/01/01	0.00%	MC	-0.04519	MC	-0.08520	MC	-0.12669
		SE	2.92×10^{-5}	SE	8.40×10^{-5}	SE	1.28×10^{-4}
		AF	-0.07430	AF	-0.15045	AF	-0.22501
	2.00%	MC	-0.07429	MC	-0.15054	MC	-0.22497
		SE	2.93×10^{-5}	SE	8.34×10^{-5}	SE	1.27×10^{-4}
		AF	0.03099	AF	0.04549	AF	0.04671
	-2.00%	MC	0.03094	MC	0.04533	MC	0.04640
		SE	3.99×10^{-5}	SE	1.75×10^{-4}	SE	3.18×10^{-4}
		AF	0.00131	AF	-0.02242	AF	-0.05754
2009/01/01	0.00%	MC	0.00134	MC	-0.02241	MC	-0.05736
		SE	4.08×10^{-5}	SE	1.79×10^{-4}	SE	3.18×10^{-4}
		AF	-0.02837	AF	-0.09033	AF	-0.16179
	2.00%	MC	-0.02837	MC	-0.09037	MC	-0.16135
		SE	4.03×10^{-5}	SE	1.62×10^{-4}	SE	2.97×10^{-4}
		AF	0.03171	AF	0.03180	AF	0.01078
	-2.00%	MC	0.03184	MC	0.03149	MC	0.01048
		SE	7.35×10^{-5}	SE	3.95×10^{-4}	SE	7.48×10^{-4}
		AF	0.00193	AF	-0.03595	AF	-0.09209
2010/01/01	0.00%	MC	0.00193	MC	-0.03598	MC	-0.09168
		SE	7.21×10^{-5}	SE	3.86×10^{-4}	SE	7.03×10^{-4}
		AF	-0.02786	AF	-0.10369	AF	-0.19495
	2.00%	MC	-0.02787	MC	-0.10315	MC	-0.19450
		SE	7.56×10^{-5}	SE	3.92×10^{-4}	SE	6.96×10^{-4}

[a] The current time is set at three dates including 2008/01/01, 2009/01/01, and 2010/01/01. The maturities of EQSs are examined for one, three, and five years ($T_{n-1} = 1, 3, 5$). The year fraction is half a year ($\delta = 1/2$). Three different levels of interest rates are set at $R_d = 2.00\%$, $R_m = 4.00\%$, $R_u = 6.00\%$.

[b] R denotes the level of spread in basis points and may be positive and negative by Definition 3. We examined three different levels of spread ($R = -2.00\%$, 0.00%, 2.00%). AF stands for the pricing result using analytic formula in equation (16) of Theorem 4. MC and SE represent, respectively, the numerical result using Monte Carlo simulation with 50,000 random paths and standard errors.

Appendix

A. Proof of Theorem 1

Denote the forward martingale measure as Q^{T_j} with respect to the numeraire by the domestic zero-coupon bond $P_d(t,T_j)$. According to Proposition 2, the dynamics of the forward LIBOR rates $L_k(t,T_{j-1})$ under the domestic martingale measure Q are given as follows.

$$\frac{dL_d(t,T_{j-1})}{L_d(t,T_{j-1})} = \gamma_d(t,T_{j-1})\cdot dW(t), \tag{A.1}$$

$$\frac{dL_f(t,T_{j-1})}{L_f(t,T_{j-1})} = \gamma_f(t,T_{j-1})\cdot\left(\overline{\sigma}^{\tau}_f(t,T_j) - \overline{\sigma}^{\tau}_d(t,T_j) - \sigma_X(t)\right)d(t) + \gamma_f(t,T_{j-1})\cdot dW(t), \tag{A.2}$$

where $L_d(t,T_{j-1})$ is a martingale under Q^{T_j}.

The no-arbitrage price of a QS with a change of measure from the measure Q^T to the forward martingale measure Q^{T_j} is given by

$$\begin{aligned} QS(\tau,R) &= \mathrm{E}^Q\left\{\sum_{j=1}^{n}\frac{\beta_d(\tau)}{\beta_d(T_j)}N_d\left(L_f(T_{j-1},T_{j-1})\delta - (L_d(T_{j-1},T_{j-1})+R)\delta\right)|F_\tau\right\} \\ &= N_d\delta\sum_{j=1}^{n}P_d(\tau,T_j)\mathrm{E}^{Q^{T_j}}\left\{L_f(T_{j-1},T_{j-1}) - L_d(T_{j-1},T_{j-1}) - R|F_\tau\right\}. \end{aligned} \tag{A.3}$$

The stochastic integral form of the foreign LIBOR rate $L_k(t,T_{j-1})$ over time interval $[\tau,T_{j-1}]$ under the measure Q^{T_j} can be evaluated directly from (A.1) and (A.2) and shown as follows.

$$L_d(T_{j-1},T_{j-1}) = L_d(\tau,T_{j-1})\exp\{-\frac{1}{2}\nu_d(\tau,T_{j-1}) + Z_d(\tau,T_{j-1})\}, \tag{A.4}$$

$$L_f(T_{j-1},T_{j-1}) = L_f(\tau,T_{j-1})\rho(\tau,T_{j-1})\exp\{-\frac{1}{2}\nu_f(\tau,T_{j-1}) + Z_f(\tau,T_{j-1})\}, \tag{A.5}$$

where

$$\begin{aligned} \nu_k(\tau,T_{j-1}) &= \int_\tau^{T_{j-1}}\|\gamma_k(t,T_{j-1})\|^2 d(t), \\ \rho(\tau,T_{j-1}) &= \exp\left(\int_\tau^{T_{j-1}}\mu_f(t,T_{j-1})d(t)\right), \\ \mu_f(t,T_{j-1}) &= \gamma_f(t,T_{j-1})\cdot(\overline{\sigma}^{\tau}_f(t,T_j) - \overline{\sigma}^{\tau}_d(t,T_j) - \sigma_X(t)), \end{aligned}$$

and

$$Z_k(\tau, T_{j-1}) = \int_{\tau}^{T_{j-1}} \gamma_k(t, T_{j-1}) \cdot dW(t)$$

is a normal variate with variance $\nu_k(\tau, T)$, $k \in \{d, f\}$.

Hence, the pricing formula of a QS can be derived by substituting (A.4) and (A.5) into (A.3) and given as follows.

$$QS(\tau, R) = N_d \delta \sum_{j=1}^{n} P_d(\tau, T_j) \left\{ Ł_f(\tau, T_{j-1}) \rho(\tau, T_{j-1}) - L_d(\tau, T_{j-1}) - R \right\}.$$

B. Proof of Theorem 2

The no-arbitrage price of a QC is derived via a change of measure from the domestic martingale measure Q into the forward martingale measure Q^{T_j} and given as follows.

$$\begin{aligned} QC(\tau, K) &= \mathrm{E}^{Q} \left\{ \sum_{j=1}^{n} \frac{\beta_d(\tau)}{\beta_d(T_j)} N_d (L_f(T_{j-1}, T_{j-1}) - K)^{+} \delta \middle| F_\tau \right\} \\ &= N_d \delta \sum_{j=1}^{n} P_d(\tau, T_j) \mathrm{E}^{Q^{T_j}} \left\{ (L_f(T_{j-1}, T_{j-1}) - K)^{+} \middle| F_\tau \right\} \\ &= N_d \delta \sum_{j=1}^{n} P_d(\tau, T_j) \left\{ \mathrm{E}^{Q^{T_j}} \left(L_f(T_{j-1}, T_{j-1}) I_{\{A\}} | F_\tau \right) - \mathrm{E}^{Q^{T_j}} (I_{\{A\}} | F_\tau) \right\}, \end{aligned} \tag{B.1}$$

where $A = \{L_f(T_{j-1}, T_{j-1}) \geq K\}$. $I_{\{\cdot\}}$ denotes an indicator function.

The second expectation in (B.1) can be derived in a similar way as given in Appendix A. The first expectation is somewhat complicated to compute under the measure Q^{T_j}. However, the pricing result can be derived by using the changing-numeraire mechanism to obtain a new martingale measure. The new martingale measure, denoted by Q^{R_j}, is defined by the Radon-Nikodym derivative ζ_t via Girsanov's theorem. That is

$$\zeta_t := \frac{dQ^{T_j}}{dQ^{R_j}} = \{-\frac{1}{2} \nu_f(\tau, T_{j-1}) + Z_f(\tau, T_{j-1})\},$$

where $\nu_f(\tau, T_{j-1})$ and $Z_f(\tau, T_{j-1})$ are defined in Appendix A. The first expectation in (B.1) is then evaluated under the new probability measure Q^{R_j} and rearranged as follows.

$$QC(\tau,K)=N_d\delta\sum_{j=1}^{n}P_d(\tau,T_j)\left\{L_f(\tau,T_{j-1})\rho(\tau,T_{j-1})\mathrm{E}^{Q^{R_j}}(I_E|F_\tau)-\mathrm{E}^{Q^{T_j}}(I_E|F_\tau)\right\}. \tag{B.2}$$

where $\rho(\tau,T_{j-1})$ is defined in Appendix A. Again, the stochastic integral form of the foreign LIBOR rate $L_f(t,T_{j-1})$ over time interval $[\tau,T_{j-1}]$ under the measure Q^{R_j} is given by

$$L_f(T_{j-1},T_{j-1})=L_f(\tau,T_{j-1})\rho(\tau,T_{j-1})\exp\left\{\frac{1}{2}\nu_f(\tau,T_{j-1})+Z_f(\tau,T_{j-1})\right\}. \tag{B.3}$$

Substituting (B.3) into (B.2), the pricing formula of a QC is therefore derived as follows.

$$QC(\tau,K)=N_d\delta\sum_{j=1}^{n}P_d(\tau,T_j)\{L_f(\tau,T_{j-1})\rho(\tau,T_{j-1})\Phi(d_1(\tau,T_{j-1}))-K\Phi(d_2(\tau,T_{j-1}))\},$$

where

$$d_1(\tau,T_{j-1})=\frac{\ln(L_f(\tau,T_{j-1})\rho(\tau,T_{j-1})/K)+\frac{1}{2}\nu_f(\tau,T_{j-1})}{\sqrt{\nu_f(\tau,T_{j-1})}},$$
$$d_2(\tau,T_{j-1})=d_1(\tau,T_{j-1})-\sqrt{\nu_f(\tau,T_{j-1})}.$$

C. Proof of Theorem 5

Using simple set calucation, we can rewrite the reference rate as follows.

$$\begin{aligned}L_f^*(T_{j-1},T_{j-1})&=L_f(T_{j-1},T_{j-1})I_{\{A_{0d}\}}+R_dI_{\{A_{dm}\}}+(R_u-L_f(T_{j-1},T_{j-1}))I_{\{A_{mu}\}}\\&=L_f(T_{j-1},T_{j-1})-(L_f(T_{j-1},T_{j-1})-R_d)I_{\{A_d\}}\\&\quad-(L_f(T_{j-1},T_{j-1})-R_m)I_{\{A_m\}}+(L_f(T_{j-1},T_{j-1})-R_u)I_{\{A_u\}},\end{aligned} \tag{C.1}$$

where

$$A_d=\{R_d\le L_f(T_{j-1},T_{j-1})\},$$
$$A_m=\{R_m\le L_f(T_{j-1},T_{j-1})\},$$
$$A_u=\{R_u\le L_f(T_{j-1},T_{j-1})\}$$

are the subsets in probability space Ω and $I_{\{\cdot\}}$ denotes an indicator function.

By substituting (C.1) into (15), we can find out the relation among EQSs, QSs, and QCs as follows.

$$N_d L_f(T_{j-1},T_{j-1})\delta - N_d\left(L_f(T_{j-1},T_{j-1}) - R_d\right)^+\delta - N_d\left(L_f(T_{j-1},T_{j-1}) - R_m\right)^+\delta$$
$$+N_d\left(L_f(T_{j-1},T_{j-1}) - R_u\right)^+\delta - N_d\left(L_f(T_{j-1},T_{j-1}) + R\right)\delta.$$

Thus, the payoff can be decomposed into a QS and three QCs with different strikes. The no-arbitrage price at time τ of an EQS is therefore given by

$$\begin{aligned} EQS(\tau,R) &= QS(\tau,R) - QC(\tau,R_d) - QC(\tau,R_m) + QC(\tau,R_u) \\ &= -(L_d(\tau,T_{j-1}) + R)P_d(\tau,T_j)\delta - QF(\tau,R_d) - QF(\tau,R_m) + QF(\tau,R_u). \end{aligned} \tag{C.2}$$

The second equality in (C.2) can be derived directly from put-call parity of quanto interest rate derivatives.

$$QC(\tau,R) = QF(\tau,R) + L_f(\tau,T_{j-1}) - R\cdot P_d(\tau,T_j).$$

D. The Market Data

The Market data are drawn from the DataStream database. We list the cap volatilities and the initial forward LIBOR rates from U.S. and U.K. in the following tables. The data is used to compute the QSs, QCs/QFs, and EQSs in Section 4.2.

Table 4 Cap Volatilities Quoted in the U.S. and U.K. Market

	U.S.			U.K.		
year	2008/1/1	2009/1/1	2010/1/1	2008/1/1	2009/1/1	2010/1/1
1	27.26	79.02	105.80	17.33	81.42	68.39
2	31.07	69.62	76.74	18.35	59.03	59.03
3	29.55	59.68	59.95	17.71	45.67	48.41
4	27.75	52.63	49.90	16.76	38.36	41.44
5	26.24	47.93	43.35	15.95	33.32	36.26

* The quoted volatilities of the caps in the U.S. and U.K. market over the past three years are shown in this table.

Table 5 Initial Forward LIBOR Rates

	U.S.			U.K.		
year	2008/1/1	2009/1/1	2010/1/1	2008/1/1	2009/1/1	2010/1/1
0	4.561	1.260	0.328	6.121	2.982	0.841
0.5	3.575	1.252	0.974	5.054	1.883	1.461
1.0	3.540	1.486	1.790	4.641	2.099	2.435
1.5	3.218	1.729	2.521	4.489	3.327	3.203
2.0	3.815	2.072	3.038	4.598	3.352	3.706
2.5	3.868	2.382	3.481	4.408	3.031	4.026
3.0	3.991	2.510	3.728	4.424	3.200	4.121
3.5	4.075	2.617	4.052	4.299	3.380	4.201
4.0	4.092	2.664	4.184	4.199	3.328	4.174
4.5	4.176	2.686	4.279	4.089	3.275	4.202
5.0	4.127	2.677	4.311	4.015	3.261	4.138

* The forward LIBOR rates in the U.S. and U.K. market over the past two years are shown in this table. The rates are obtained from the associated bond prices derived from the zero curves.

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