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Valuation of quanto options in a Markovian regime-switching market: A Markov-modulated Gaussian HJM model

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ABSTRACT

We consider the valuation of European quanto call options in an incomplete market where the domestic and foreign forward interest rates are allowed to exhibit regime shifts under the Heath–Jarrow–Morton (HJM) framework, and the foreign price dynamics is exogenously driven by a regime switching jump-diffusion model with Markov-modulated Poisson processes. We derive closed-form solutions for four different types of quanto call options, which include: options struck in a foreign currency, a foreign equity call struck in domestic currency, a foreign equity call option with a guaranteed exchange rate, and an equity-linked foreign exchange-rate call.

1. Introduction

Recent works that consider alternative option pricing models have progressed in various directions: Zumbach (2012) explores the valuation of options when discrete-time ARCH processes

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drive the underlying asset prices; Xu et al. (2012) price vulnerable options under a continuous-time jump-diffusion setting; Hsu and Chen (2012) investigate the valuation of exchange-rate barrier options when interest rates are driven by a Lévy process. In view of incorporating the regime switching feature into the pricing model, Hamilton (1989) provides empirical evidence of business cycles under a regime-shift model of hidden Markov chain, Elliott et al. (2003) propose a regime-switching Brownian motion model with a Markov-modulated system that captures the volatilities-clustering feature. Simonato (2011), on the other hand, computes American option prices under a lognormal jump-diffusion setting based on a numerical approach.

In this research, the regime-switching feature is introduced two ways: one is based on the Markov-modulated HJM (MMHJM) model of Valchev (2004) for interest rates, and the other via the regime switching jump-diffusion model (RSJD) for foreign stock prices. Existing literature that incorporates either discrete or continuous regime-shifts in model parameters includes Bansal and Zhou (2002), who develop a term structure model in which the short rates and the market price of risk are subject to discrete-time regime shifts, and Zhu (2011), who shows that regime shifts are able to explain the predictability of excess returns.

When a financial market is incomplete, the pricing measure is not unique. In order to identify a risk-neutral measure for derivatives pricing under an incomplete market, Gerber and Shiu (1994) propose the Esscher transform approach that characterizes the risk-neutral measure by moment-generation functions, and Husmann and Todorova (2011) apply the equilibrium approach of Jarrow and Madan (1997) to the case of an incomplete lognormal market. In this paper, we adopt the regime-switching Esscher transform proposed by that Elliott et al. (2005) to identify the risk-neutral measure under which quanto call options can be priced.

Other approaches for options pricing that consider the presence of different sources of risk, such as liquidity and credit, are also worth noting. Sample works in this category include Ku et al. (2012), where the valuation framework of Leland (1985) in dealing with transaction costs is re-interpreted as an option-pricing problem under an incomplete-market setting when liquidity risk is present; Jarrow (2011), on the other hand, argues that asymmetric information structure in fact plays an important role in the determination of credit market equilibrium, and hence affects the capital structure of a firm.

Subsequent parts of this article are organized as follows: In Section 2, while the domestic and foreign forward interest rates are modeled by a MMHJM model, the exchange rate is assumed to follow a geometric Brownian motion, and the foreign stock prices are specified by a RSJD model. In Section 3, we use the regime-switching Esscher transform to construct a risk-neutral martingale measure. In Section 4, we derive closed-form solutions for four types of quanto options. The final section concludes our research findings.

2. Regime-switching model

In this section, we first specify the MMHJM model for the domestic and the foreign forward interest rates, and we also specify the RSJD model that the foreign stock prices are assumed to follow.

2.1. Specifications of Markov chains

According to Elliott et al. (2005),[4] we assume that the state space of Markov chain ξ is a set of two states: $I = \{e_1, e_2\} = \{(1,0),(0,1)\} \in \mathbb{R}^2$, which implies respectively, a boom or a recession (good or bad time) for the state of economy. A continuous-time Markov chain ξ has the transition matrix given as below:

$$\mathbf{P}(t) = \begin{bmatrix} p_{11}(t) & 1 - p_{11}(t) \\ 1 - p_{22}(t) & p_{22}(t) \end{bmatrix}. \tag{2.1}$$

[4] Elliott et al. (2005) also establish the occupation time of the moment generating function for Markov chain ξ.

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A Markov-modulated Poisson process (MMPP), $\Phi(t)$, represents a particular class of doubly-stochastic Poisson processes where the jump intensity is modulated by a Markov chain $\xi(t)$. In particular, we consider a set of nonnegative numbers $\{\lambda_1, \lambda_2\}$, where λ_i, $i = 1, 2$, denotes the intensity of the Poisson process when a Markov chain $\xi(t)$ is at state e_i at time t, i.e., $\lambda_i = (\lambda_1, \lambda_2) \cdot e_i$ where the dot $(\cdot)$ denotes the scalar product. $\xi(t)$ and $\Phi(t)$ can be defined by the joint probability, $\mathcal{P}_{ij}(n,t) = \mathbb{R}(\Phi(t) = n, \xi(0) = e_i,\ \xi(t) = e_j)$. The moment-generating function of the joint probability admits a unique solution $\mathcal{P}(u,t)$ such that, $\mathcal{P}(u,t) = \sum_{n=0}^{\infty} \mathcal{P}(n,t)u^n = \exp[(\mathbf{\Psi} - (1-u)\mathbf{\Lambda})t]$, where $\mathbf{\Lambda}$ represents the intensity matrix $\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$ (cf. Last and Brandt, 1995). The numerical method for computing $\mathcal{P}_{ij}(n,t)$ can be found in Abate and Whitt (1992).

2.2. Regime-switching HJM model for the forward interest rates

Let the regime-switching feature of interest rates be represented by a Markov chain ξ_f where the subscript f denotes for the forward interest rates with a two-state space $I_{\xi_f} = \{e_g, e_h\} = \{(1,0),(0,1)\}$. We use the following notations for the Markov-modulated parameters in the HJM model:

$$\alpha(t,T,\xi_f(t)) = (\alpha_1(t,T), \alpha_2(t,T)) \cdot \xi_f(t)$$

and

$$\mathbf{v}(t,T,\xi_f(t)) = ((\upsilon_{1,1}(t,T), \upsilon_{1,2}(t,T)) \cdot \xi_f(t),\ (\upsilon_{2,1}(t,T), \upsilon_{2,2}(t,T)) \cdot \xi_f(t),\ ((\upsilon_{3,1}(t,T),\ \upsilon_{3,2}(t,T))) \cdot \xi_f(t)), \tag{2.2}$$

where $(\upsilon_{m,1}(t, T), \upsilon_{m,2}(t, T))$, $m = 1, 2$, and 3, represent, respectively, the volatility structures of short-, mid- and long-term interest rates.

Following Valchev (2004), the dynamics of forward rates formulated by the MMHJM model under the physical measure $\mathbb{P}$ is given by:

$$df_k(t,\mathbf{T},\xi_f(t)) = \alpha_k(t,T,\xi_f(t)) + \mathbf{v}_k(t,T,\xi_f(t)) \cdot d\mathbf{W}_k(t), \tag{2.3}$$

where $\mathbf{W}_k(t) \in \mathbb{R}^n$ is a standard Brownian motion, $k \in \{D, F\}$ denotes, respectively, a domestic or a foreign country.

The domestic and the foreign money market accounts are given by:

$$\beta_k(t,\xi_f(t)) = \exp\left(\int_0^t r_k(u,\xi_f(u))\,du\right), \quad k \in \{D,F\}, \tag{2.4}$$

where $r_k(u, \xi(u)) = f_k(u, u, \xi(u))$ is the spot interest rate.

2.3. Regime-switching jump diffusion model for stock prices and BSM for spot FX rate

Let $S_F(\boldsymbol{t})$ be the price of a foreign stock under the RSJD model, and $X(t)$ the spot FX. The dynamics of the spot-FX rate under the Black–Scholes framework is given by[5]:

$$\frac{dX(t)}{X(t-)} = \mu_X\,dt + \boldsymbol{\sigma}_X \cdot d\mathbf{W}_D(t), \quad X(0) > 0, \tag{2.5}$$

under the physical measure $\mathbb{P}$. Hence the price dynamics of $S_F(t)$ under the RSJD model is found to be:

$$\frac{dS_F(t)}{S_F(t-)} = \mu_F\,dt + \boldsymbol{\sigma}_F \cdot d\mathbf{W}_F(t) + (\exp(Z_n) - 1)\,d\Phi_F(t;\xi_F), \quad S_F(0) > 0, \tag{2.6}$$

[5] We expect that the FX rate should be subject to a (much) weaker regime-switching (RS) impact of the domestic and the foreign interest rates (r_D and r_F). By interest rate parity $X(T)/X(t) = (1 + r_D)/(1 + r_F)$, the RS effects of the numerator r_D and the denominator r_F tend to offset with each other, and hence resulting in (much) weaker RS associated with the FX rate $X(T)$. In addition, by assuming away the RS of the FX rate, we avoid further mathematical complication, and thereby facilitating us to obtain a closed-form pricing model without losing its significance.

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where μ_l and $\boldsymbol{\sigma}_l$, $l \in \{F, X\}$, are constants, and $\mathbf{W}_k(t) \in \mathbb{R}^n$, where $k \in \{D, F\}$, are, respectively, the domestic or the foreign country. A foreign MMPP $\Phi_F(t;\, \xi_F)$ is used to model changes in the state of foreign economy. The jump term, $(\exp(Z_n) - 1)d\Phi_F(t;\, \xi_F)$, is a compound Poisson process. Z_n, where $n = 1, 2, 3\ldots$, represents a sequence of mutually independent jump sizes. The jump size variable Z_n has a normal distribution with mean μ_{FJ} and variance σ^2_{FJ}. All random variables are assumed to be mutually independent.

3. Risk-neutral Martingale measure via Esscher transform

In this section, a regime-switching Esscher transform is introduced and applied to the RSJD model such that the price dynamic process becomes a martingale.

3.1. Regime-switching Esscher measure

The filtrations of the foreign assets and the spot-FX rate are denoted, respectively, by $\mathcal{F}_t^{S_F}$ and $\mathcal{F}_t^{X}$. The filtration of hidden Markov chains $\xi_F \vee \xi_f$ is given by $\mathcal{F}_T^{\xi_f \vee \xi_F}$. The join filtration of the foreign assets (or stocks), the FX rate, the foreign forward interest rates, and the hidden Markov chain $\xi_F \vee \xi_f$ is denoted by a σ-algebra given by:

$$\mathcal{H}(t) = \sigma(\mathcal{F}_T^{\xi_f \vee \xi_F} \vee \mathcal{F}_t^{S_F} \vee \mathcal{F}_t^{X} \vee \mathcal{F}_t^{f_k}).$$

Two families of regime-switching parameters for the Esscher transform are denoted, respectively, by $\boldsymbol{\theta}^C(u, \xi_f(u))$ and $\theta^J(\xi_F(u))$ such that $\mathbb{Q}^{(\theta^C,\theta^J)} \sim \mathbb{P}$ on $\mathcal{H}(t)$, and are given as follows:

$$\boldsymbol{\theta}^C(u, \xi_f(u)) = \left(\left(\boldsymbol{\theta}^C_{1,1}(u), \boldsymbol{\theta}^C_{1,2}(u)\right) \cdot \xi_f(u),\ \left(\boldsymbol{\theta}^C_{2,1}(u), \boldsymbol{\theta}^C_{2,2}(u)\right) \cdot \xi_f(u),\ \left(\boldsymbol{\theta}^C_{3,1}(u), \boldsymbol{\theta}^C_{3,2}(u)\right) \cdot \xi_f(u)\right),$$

where $\theta^J(\xi_F(u)) = (\theta^J_1, \theta^J_2) \cdot \xi_F(u) \quad \forall t \leqslant u \leqslant T \leqslant \overline{T}$.

The foreign regime-switching Esscher transform under the RSJD model takes the following definition: (cf. Bo et al., 2010)

$$\left.\frac{d_{\mathbb{Q}}^{F,(\theta^C,\theta^J)}}{d\mathbb{P}}\right|_{\mathcal{H}(t)} = \frac{\exp\left(\int_t^T \boldsymbol{\theta}^C(u, \xi_f(u)) \cdot d\mathbf{W}_F(u)\right)}{E\left[\exp\left(\int_t^T \boldsymbol{\theta}^C(u, \xi_f(u)) \cdot d\mathbf{W}_F(u)\right)\middle|\mathcal{H}(t)\right]} \frac{\exp\left(\int_t^T \boldsymbol{\theta}^J(\xi_F(u)) Z_{u-}\, d\Phi_F(u;\xi_F)\right)}{E\left[\exp\left(\int_t^T \boldsymbol{\theta}^J(\xi_F(u)) Z_{u-}\, d\Phi_F(u;\xi_F)\right)\middle|\mathcal{H}(t)\right]} \tag{3.1}$$

The martingale condition for the foreign stock under the foreign martingale measure $\mathbb{Q}^{F,(\theta^C,\theta^J)}$, where the superscript "F" indicates the foreign measure, is given as follows:

Theorem 1.

$$\begin{aligned} &-\int_u^T \alpha_F(u, s, \xi_f(u))ds + \frac{1}{2}\left\|-\mathbf{V}_F(u, T, \xi_f(u)) + \boldsymbol{\sigma}_F + \boldsymbol{\theta}^C(u, \xi_f(u))\right\|^2 \\ &\quad = \frac{1}{2}\|\boldsymbol{\sigma}_F\|^2 + \frac{1}{2}\|\boldsymbol{\theta}^C(u, \xi_f(u))\|^2 - \mu_F \quad \text{and} \\ &\phi_F(\theta^J(\xi_F(u)) + 1) - \phi_F(\theta^J(\xi_F(u))) = 0, \end{aligned} \tag{3.2}$$

where $\mathbf{V}_F(u, T, \xi_f(u)) = \int_u^T \boldsymbol{\sigma}_F(u, s, \xi_f(u))ds, \quad 0 \leqslant u \leqslant s \leqslant T \leqslant \overline{T}$, $\|\cdot\|$ *denotes the norm in* $\mathbb{R}^n$, *and* $\phi_F(u) = E[\exp(uZ_n)] = \exp(u\mu_{F,J} + \frac{1}{2}u^2\sigma_{FJ})$ *is the moment-generating function of the jump size variable.*

Proof (*See Appendix A*). □

Let $S_F^{Q,F}(T)$ be the time T foreign-stock process discounted by the foreign money market account (β_F in (2.4)) under the foreign martingale measure $\mathbb{Q}^{F,(\theta^{C*},\theta^{J*})}$, and the superscript "$Q, F$" denotes for the foreign risk-neutral measure.

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The process $S_F^{Q,F}(T)$ can be represented as follows:

$$S_F^{Q,F}(T) = S_F^{Q,F}(t)\exp\left(\int_t^T r_F(u,\xi_f(u))\,du - \frac{1}{2}\int_t^T \|\boldsymbol{\sigma}_F\|^2\,du + \int_t^T \boldsymbol{\sigma}_F\cdot d\mathbf{W}_F^{Q,F}(u)\right) \cdot \exp\left(\int_t^T Z_{u-}^{Q,F}\,d\Phi_F^{Q,F}(u;\xi_F)\right), \tag{3.3}$$

where $\|\cdot\|$ denotes the Euclidean norm, $\mathbf{W}_F^{Q,F}(u)$ is standard Brownian motion and $Z_{u-}^{Q,F}$ is normally distributed with mean $-\frac{1}{2}\sigma_{F,J}^2$ and variance $\sigma_{F,J}^2$. $\Phi_F^{Q,F}(u;\xi_F)$ therefore admits an new intensity matrix $\Lambda_F^{Q,F}$ given by:

$$\Lambda_F^{Q,F} = \begin{bmatrix} \lambda_{F,1}\phi(\theta^{J*}) & 0 \\ 0 & \lambda_{F,2}\phi(\theta^{J*}) \end{bmatrix} = \Lambda_F \exp\left(-\frac{\mu_{F,J}^2}{2\sigma_{F,J}^2} + \frac{\sigma_{F,J}^2}{8}\right).$$

In addition, under the risk-neutral measure $\mathbb{Q}^{F,(\theta^{C*},\theta^{J*})}$, the interest rate in $S_F^{Q,F}(T)$ is modulated by the Markov chain ξ_f, while the jump intensities of the $S_F^{Q,F}(T)$ are modulated by the Markov chain ξ_F. Existing empirical studies, such as Bansal and Zhou (2002) and Estrella and Hardouvelis (1991), suggest that interest rates are not only intimately related to business cycles, but also act as economic leading indicators. In this research we see that the stock-price dynamics are influenced by the effects of regime-switching interest rates via the Markov chain ξ_f associated with the yield curve. This feature is incorporated into the stock-price dynamics given in (3.3) to capture the changes in economic cycles. This formulation is also supported by Harvey (1989) who shows that the bond market reveals more information about future economic growth than the stock market.

4. Valuation of European quanto options

Without loss of generality, in the following we shall omit the superscript notation under the RSJD model for the risk-neutral measure.

Some model parameters can be expressed explicitly in terms of the occupation times of the state of Markov chain ξ_f over the option duration $[t, T]$.

That is:

$$\int_t^T V_{i,m}(u,T,\xi_f(u))\,du = \int_t^T V_{i,m}(u,T)\cdot\xi_f(u)\,du = \sum_{h=1}^{2}\int_t^T V_{i,m,h}(u,T)\varphi(\delta_h)d\delta_h(t,T),$$

where $i \in \{D,F\}$, m = 1, 2, 3, and $\delta_h(t,T)$, h = 1, 2, denotes the two-state occupation time, and $\varphi(\delta_1,\delta_2)$ denotes the joint probability distribution for the occupation times $(\delta_1(t,T), \delta_2(t,T))$, which can be determined by its corresponding moment-generating function given in Elliott et al. (2005). Therefore, the quanto call options under study $C_w^{RS}(t;\xi_f,n)$, w = 1,2,3,4, depend on the occupation times and jump number (n).

For simplicity, the price notation $C_w^{RS}(t;\delta_1,\delta_2,n|\mathcal{H}(t))$ is replaced by the notation $C_w^{RS}(t;\xi_f,n)$ in the pricing models given below. Then, with regime-switching of forward interest rates under the MMHJM model, a European quanto call option can be expressed in terms of the occupation times given by Theorem 2:

Theorem 2.

$$C_w(t) = \sum_{n=0}^{\infty}\sum_{i}^{2}\pi_i^{\xi_F}Q_{ij}^*(T-t,n)\int_t^T\int_t^T C_w^{RS}(t;\delta_1,\delta_2,n)\varphi(\delta_1,\delta_2)\,d\delta_1\,d\delta_2,$$

where $\pi_i^{\xi_F}$ is the stationary state of Markov chain ξ_F used as initial value.

Let T be an expiry date of an option and K_k, $k \in \{D,F\}$, be an exercise price. Four types of quanto call options $C_w^{RS}(t;\xi_f,n)$, w = 1,2,3,4, are considered and given by the following four corollaries.

As was observed by Reiner (1992), an investor often hedges against the currency exposure of his/her foreign-stock investments by a large variety of options. In this following we provide closed-form solutions for four different types of quanto options that suffice such hedging needs.

Corollary 1 (*Options struck in a foreign currency*). *The domestic currency-denominated terminal payoff of a foreign-equity call option stuck in foreign currency is given by*:

$$C_1(T) = X(T)(S_F(T) - K_F)^+,$$

where the terminal payoff of a foreign-stock call option is converted into domestic currency at the spot exchange rate at expiry.

With the risk of jump and regime-switching interest rates, the arbitrage-free price of this European quanto call option at time t is equal to

$$C_1^{RS}(t;\xi_f,n) = X(t)[S_F(t)\Upsilon(d_{1,1}) - K_F B_F(t,T,\xi_f(t))\Upsilon(d_{1,2})], \tag{4.1}$$

where $\Upsilon(\cdot)$ is a cumulative normal distribution function,

$$d_{1,1} = \frac{\ln\left(\frac{S_F(t)}{K_F B_F(t,T,\xi_f(t))}\right) + \frac{1}{2}\left(\int_t^T \|\zeta_1(u,T,\xi_f(u))\|^2\,du + n\sigma_{FJ}^2\right)}{\sqrt{\int_t^T \|\zeta_1(u,T,\xi_f(u))\|^2\,du + n\sigma_{FJ}^2}}$$

$$d_{1,2} = d_{1,1} - \sqrt{\int_t^T \|\zeta_1(u,T,\xi_f(u))\|^2 + n\sigma_{F,J}^2},\ \ \zeta_1(u,T,\xi_f(u)) = \boldsymbol{\sigma}_F + \mathbf{V}_F(u,T,\xi_f(u)),$$

and K_F is the strike price in foreign currency (Proof: See Appendix B, Case 1).

Corollary 2 (*A foreign equity call struck in domestic currency*). *An investor wishes to receive a positive payoff from a foreign equity market, but would like the underlying foreign stock to be denominated in domestic currency at expiry. The payoff of this type of European quanto call options at expiry T is given by:*

$$C_2(T) = (X(T)S_F(T) - K_D)^+.$$

Then, with the risk of jump and regime-switching interest rates, the arbitrage-free price of this type of European quanto-call options at time t is equal to

$$C_2^{RS}(t;\xi_f,n) = X(t)S_F(t)\Upsilon(d_{2,1}) - K_D B_D(t,T,\xi_f(t))\Upsilon(d_{2,2}) \tag{4.2}$$

where

$$d_{2,1} = \frac{\ln\left(\frac{X(t)S_F(t)}{K_D B_D(t,T,\xi_f(t))}\right) + \frac{1}{2}\left(\int_t^T \|\zeta_2(u,T,\xi_f(u))\|^2\,du + n\sigma_{FJ}^2\right)}{\sqrt{\int_t^T \|\zeta_2(u,T,\xi_f(u))\|^2\,du + n\sigma_{FJ}^2}}$$

$$d_{2,2} = d_{2,1} - \sqrt{\int_t^T \|\zeta_2(u,T,\xi_f(u))\|^2 + n\sigma_{F,J}^2},\ \ \zeta_2(u,T,\xi_f(u)) = \boldsymbol{\sigma}_F + \boldsymbol{\sigma}_X + \mathbf{V}_D(u,T,\xi_f(u))$$

and K_D is the strike price in domestic currency (Proof: See Appendix B, Case 2).

Corollary 3 (*A foreign equity call option with a guaranteed exchange rate*). *An investor wishes to capture a positive payoff on his foreign equity investment, but also desires to eliminate all exchange-rate risk by denominating the foreign payoff in domestic currency.*

The payoff of this type of options at expiry T is given by:

$$C_3(T) = \chi(S_F(T) - K_F)^+,$$

where χ is the pre-specified exchange rate with which the option's payoff is converted into domestic currency.

With the risk of jump and regime-switching interest rates, the arbitrage-free price of the call option at time t is equal to

$$C_3^{RS}(t;\xi_f,n)=\chi B_D(t,T,\xi_f(t))\cdot\left(\frac{S_F(t)}{B_F(t,T,\xi_f(t))}\exp\left(\int_t^T\zeta_4(u,T,\xi_f(t))\cdot(-\mathbf{V}_F(u,T,\xi_f(u))-\boldsymbol{\sigma}_F)du\right)\Upsilon(d_{3,1})-K_F\Upsilon(d_{3,2})\right) \tag{4.3}$$

where

$$d_{3,1}=\frac{\left(\ln\left(\frac{S_F(t)}{K_FB_F(t,T,\xi_f(t))}\right)-\int_t^T\zeta_4(u,T,\boldsymbol{\xi}_f(u))\cdot(\mathbf{V}^F(u,T,\boldsymbol{\xi}_f(t))+\boldsymbol{\sigma}_F)\,du\right)}{\sqrt{\int_t^T\|\zeta_3(u,T,\xi_f(u))\|^2\,du+n\sigma_{F,J}^2}}+\frac{\frac{1}{2}\left(\int_t^T\|\zeta_3(u,T,\xi_f(u))\|^2\,du+n\sigma_{F,J}^2\right)}{\sqrt{\int_t^T\|\zeta_3(u,T,\xi_f(u))\|^2\,du+n\sigma_{F,J}^2}},\ d_{3,2}=d_{3,1}-\sqrt{\int_t^T\|\zeta_3(u,T,\xi_f(u))\|^2\,du+n\sigma_{F,J}^2},$$

and

$$\zeta_4(u,T,\xi_f(u))=\boldsymbol{\sigma}_X-\mathbf{V}_F(u,T,\xi_f(u))+\mathbf{V}_D(u,T,\xi_f(u))$$

$\zeta_3(u, T, \xi_f(u)) = \boldsymbol{\sigma}_F + \mathbf{V}_F(u, T, \xi_f(u))$ (Proof: See Appendix B, Case 3).

Corollary 4 (*An equity-linked foreign exchange-rate call*). *Finally, an investor wants to hold a foreign stock whose payoff depends on the payoff of a foreign-exchange call option. The final payoff of this type of quanto options is given by:*

$$C_4(T)=S_F(T)(X(T)-K_D)^+.$$

With the risk of jump and regime-switching interest rates, the arbitrage-free price of this quanto call option at time t is equal to

$$C_4^{RS}(t;\xi_f,n)=S_F(t)\cdot\left[X(t)\Upsilon(d_{4,1})-\frac{B_D(t,T,\xi_f)}{B_F(t,T,\xi_f)}K_D\exp\left(\int_t^T-(\boldsymbol{\sigma}_F+\mathbf{V}_F(u,T,\xi_f(u)))\cdot\zeta_4(u,T,\xi_f(u))dt\right)\Upsilon(d_{4,2})\right], \tag{4.4}$$

where

$$d_{4,1}=\frac{\ln\left(\frac{X(t)}{K_D}\frac{B_F(t,T,\xi_f)}{B_D(t,T,\xi_f)}\right)+\int_t^T(\mathbf{V}_F(u,T,\xi_f(u))+\boldsymbol{\sigma}_F)\cdot\zeta_4(u,T,\xi_f(u))\,du}{\sqrt{\int_t^T\|\zeta_4(u,T,\xi_f(u))\|^2\,du}}+\frac{\frac{1}{2}\left(\int_t^T\|\zeta_4(u,T,\xi_f(u))\|^2\,du\right)}{\sqrt{\int_t^T\|\zeta_4(u,T,\xi_f(u))\|^2\,du}},$$

$$d_{4,2}=d_{4,1}-\sqrt{\int_t^T\|\zeta_4(u,T,\xi_f(u))\|^2\,du}$$

(Proof: See Appendix B, Case4).

5. Conclusions

Through a regime-switching Brownian motion, we introduce regime shifts in the dynamics of zero-coupon yields under a Markov-modulated HJM (MMHJM) market. In addition, the price dynamics of foreign stocks are also allowed to exhibit regime shifts via the regime-switching jump diffusion (RSJD) setting. We adopt the regime-switching Esscher transform proposed by Elliott et al. (2005) to construct the risk-neutral measure under which the prices of quanto call options can be found. Our research findings include explicit closed-form solutions for four different types of European quanto call options.

Appendix A. Appendix

$$E^{F(\theta^{C*},\theta^{J*})}[S_F(t)|\mathcal{H}(u)] = E\left[S_F(t)\frac{d\mathbb{Q}^{F(\theta^{C*},\theta^{J*})}}{d\mathbb{P}}\bigg|\mathcal{H}(u)\right]$$
$$= E\left[\frac{1}{e^{\int_u^t r^F(s,\xi_f(s))\,ds}}S_F(u)\exp\left(\int_u^t \mu_F\,ds - \frac{1}{2}\int_u^t \|\boldsymbol{\sigma}_F\|^2\,ds + \int_u^t \boldsymbol{\sigma}_F\cdot d\mathbf{W}_F(s) + \int_u^t Z_{s-}\,d\Phi_F(s;\xi_F)\right)\right.$$
$$\left.\times\frac{\exp\left(\int_u^t \boldsymbol{\theta}^C(s,\xi_f(s))\,d\mathbf{W}_F(s) + \int_u^t \theta^J(\xi_F(s))Z_{s-}\,d\Phi_F(s;\xi_F)\right)}{E\left[\exp\left(\int_u^t \boldsymbol{\theta}^C(s,\xi_f(s))\cdot d\mathbf{W}_F(s) + \int_u^t \theta^J(\xi_F(s))Z_{s-}\,d\Phi_F(s;\xi_F)\right)\Big|\mathcal{H}(t)\right]}\bigg|\mathcal{H}(u)\right] = S_F(u)$$

where $0 \leqslant u \leqslant t \leqslant \overline{T}$.

The martingale condition under the risk-natural measure $\mathbb{Q}^{D(\theta^{F,C*},\theta^{F,J*})}$ is given as follows:

$$E\left[\exp\left(\int_0^t -r^F(s,\xi_f(s))ds + \int_0^t [\boldsymbol{\sigma}_S + \boldsymbol{\theta}^C(s,\xi_F(s))]\cdot dW^D(s)\right)\bigg|\mathcal{H}(0,t)\right]$$
$$= \exp\left(\frac{1}{2}\int_0^t \|\boldsymbol{\sigma}_S\|^2\,ds + \frac{1}{2}\int_0^t \|\boldsymbol{\theta}^C(s,\xi_f(s))\|^2\,ds - \int_0^t \mu_{F,J}\,ds\right) \quad \text{and} \quad \theta^{J*} = \frac{-\mu_{F,J}}{\sigma_{F,J}^2} - \frac{1}{2}. \tag{A.1}$$

It implies Theorem 1.

A.1. Change of parameters under new measure

Considering Brownian motion only, we reduce the Esscher transform to Girsanov Theorem.

$$dW^{Q,F}(t) = -\boldsymbol{\theta}^{C*}\,dt + dW_F(t)$$

where $W^{Q,F}(t)$ is Brownian motion under the new measure $\mathbb{Q}^{F,(\theta^{C*},\theta^{J*})}$.

Next, we consider the jump term under the measure $\mathbb{Q}^{F,(\theta^{C*},\theta^{J*})}$.

$$\frac{d\mathbb{Q}^{F,\theta^{J*}}}{d\mathbb{P}}\bigg|_{\sigma(Y=1)} = \frac{\exp\left(\int_u^t \theta^{J*}\mathbf{Z}_{S-}\,d\Phi(s)\right)}{E[\exp\left(\int_u^t \theta^{J*}\mathbf{Z}_{S-}\,d\Phi(s)\right)\big|\mathcal{H}(u)]} = \frac{\exp\left(\theta^{J*}\sum_{n=1}^{N_1(t-u)}Z_n\right)}{E[\exp\left(\theta^{J*}\sum_{n=1}^{N_1(t-u)}Z_n\right)\big|\mathcal{H}(u)]}.$$

Let $u = 0$

Case (1): $\xi_F(t) = 1$

$$\mathbb{Q}^{F,\theta^{J*}}(Z_1\in dz_1,\ldots,Z_m\in dz_m,\ N_1(t)=m) = \frac{\exp(\theta^{J*}\sum_{n=1}^m Z_n)}{E\left[\exp\left(\theta^{J*}\sum_{n=1}^m Z_n\right)\big|\sigma(\mathcal{F}_0^S\vee\mathcal{F}_T^{\xi_f})\right]}\cdot\mathbb{P}(Z_1\in dz_1,\ldots,Z_m$$
$$\in dz_m,\ N_1(t)=m) = \frac{\exp(\theta^{J*}Z_1)}{E[\exp(\theta^{J*}Z_n)]}\mathbb{P}(Z_1\in dz_1)\cdot\cdots\cdot\frac{\exp(\theta^{J*}Z_n)}{E[\exp(\theta^{J*}Z_n)]}\mathbb{P}(Z_m$$
$$\in dz_m)\times\frac{E[\exp(\theta^{J*}Z_n)]^m}{E[\exp(\theta^{J*}um_{n=1}^m Z_n)\big|\sigma(\mathcal{F}_0^S\vee\mathcal{F}_T^{\xi_f})]}\mathbb{P}(N_1(t)=m).$$

We apply Girsanov Theorem to the term $\frac{\exp(\theta^{J*}Z_1)}{E[\exp(\theta^{J*}Z_n)]}\mathbb{P}(Z_1\in dz_1)$.

Then,

$$\frac{E[\exp(\theta_1^{J*}Z_n)]^m}{E\left[\exp\left(\theta_1^{J*}\sum_{n=1}^{N_1(\tau)}Z_n\right)\big|\sigma(\xi_F(t)=1)\right]}\mathbb{P}(N_1(t)=m)$$
$$= \frac{1}{E\left[\exp\left(\theta_1^{J*}\sum_{n=1}^{N_1(t)}Z_n\right)\big|\sigma(\xi_F(t)=1)\right]}\exp(-\lambda_{F,1}\phi(\theta_1^{J*})t)\frac{(\lambda_{F,1}\phi(\theta_1^{J*})t)^m}{m!}\exp(-\lambda_{F,1}t)\times\exp(\lambda_{F,1}\phi(\theta_1^{J*})t)$$
$$= \exp(-\lambda_{F,1}\phi(\theta_1^{J*})t)\frac{(\lambda_{F,1}\phi(\theta_1^{J*})t)^m}{m!}.$$

Thus, it remains a Poisson density with $\lambda_{F,1}^{Q,F} = \lambda_{F,1}\phi(\theta_1^{J*})$ under the measure $\mathbb{Q}^{F,(\theta^{C*},\theta^{J*})}$.

Case (2): $\xi(t) = 2$, the driving process is similar. Hence, we have a new jump intensity matrix $\Lambda_F^{Q,F} = \begin{bmatrix} \lambda_{F,1}^{Q,F} & 0 \\ 0 & \lambda_{F,2}^{Q,F} \end{bmatrix}$ of the MMPP. □

Appendix B. Appendix

Lemma B. *The asset forward prices* $S_F^T(t)$ *under the foreign forward martingale measure* $\mathbb{Q}^{T*,F}$ *can be written as follows*:

$$\begin{aligned}\frac{dS_F^T(t)}{S_F^T(t-)} &= (\boldsymbol{\sigma}_F + \mathbf{V}_F(t,T,\xi_f(t)))\cdot\mathbf{V}_F(t,T,\xi_f(t))\,dt + (\boldsymbol{\sigma}_F + \mathbf{V}_F(t,T,\xi_f(t)))\cdot d\mathbf{W}_F^{Q,F} + (\exp(Z_n^{Q,F}) \\ &\quad - 1)\,d\Phi_F^{Q,F}(t;\xi_F) \\ &= \zeta_F(t,T,\xi_f(t))\cdot d\mathbf{W}_F^T + (\exp(Z_n^{Q,k}) - 1)\,d\Phi_F^{Q,F}(t;\xi_F),\end{aligned} \tag{B.1}$$

where $\zeta_F(t,T,\xi_f(t)) = \boldsymbol{\sigma}_F + \mathbf{V}_F(t,T,\xi_f(t))$.

B.1. Case1: Options struck in a foreign currency

The arbitrage-free price of a European call option at time t equals

$$C_1(t,\xi_f,n) = E^{\mathbb{Q}^D}\left(\exp\left(-\int_t^T r_D(u,\xi_f(u))\,du\right)X(T)(S_F(T)-K_F)^+\middle|\mathcal{H}(t)\right).$$

Proof.

$$\begin{aligned}C_1(t,\xi_f,n) &= E^{\mathbb{Q}^D}\left(\exp\left(-\int_t^T r_D(u,\xi_f(u))\,du\right)X(T)(S_F(T)-K_F)^+\middle|\mathcal{H}(t)\right) \\ &= X(t)E^D\left(\exp\left(\int_t^T(-r_F(u,\xi_f(u)))\,du - \frac{1}{2}\int_t^T\|\boldsymbol{\sigma}_X\|^2\,du + \int_t^T\boldsymbol{\sigma}_X\cdot d\mathbf{W}_D(u)\right)(S_F(T)-K_F)^+\middle|\mathcal{H}(t)\right) \\ &= X(t)E^{\mathbb{D}^{F,(\theta^{C*},\theta^{J*})}}\left(\exp\left(\int_t^T(-r_F(u,\xi_f(u)))\,du\right)(S_F(T)-K_F)^+\middle|\mathcal{H}(t)\right),\end{aligned}$$

$$\begin{aligned}\frac{C_1(t,n,\xi_f)}{B_F(t,T,\xi_f(t))} &= X(t)E^{\mathbb{Q}^{F,(\theta^{F,C*},\theta^{F,J*})}}\left(\frac{\beta_F(t,\xi_f(t))}{\beta_F(T,\xi_f(T))}\frac{B_F(T,T,\xi_f(T))}{B_F(t,T,\xi_f(t))}(S_F(T)-K_F)^+\middle|\mathcal{H}(t)\right) \\ &= X(t)E^{\mathbb{Q}^{T,F,(\theta^{C*},\theta^{J*})}}((S_F(T)-K_F)^+|\mathcal{H}(t)) \\ &= X(t)E^{\mathbb{Q}^{T,F,(\theta^{F,C*},\theta^{F,J*})}}\left(\left(\frac{S_F(T)}{B_F(T,T,\xi_f(T))}-K_F\right)^+\middle|\mathcal{H}(t)\right).\end{aligned}$$

By using (B.1), we know $S_F^T(t)$ under the foreign forward martingale measure $\mathbb{Q}^{T*,F}$.

Hence, we obtain (4.1) □

B.2. Case 2: A foreign equity call stuck in domestic currency

The arbitrage-free price of a European call option at time t equals

$$C_2(t,\xi_f,n) = E^{\mathbb{Q}^D}\left(\exp\left(-\int_t^T r_D(u,\xi_f(u))\,du\right)(X(T)S_F(T)-K_D)^+\middle|\mathcal{H}(t)\right).$$

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Proof. Furthermore, the price of a foreign asset denominated in units of domestic currency is denoted $S_{F*,X*}(t) \equiv X(t)S_F^*(t)$ and given by

$$S_{F*,X*}(T) = S_{F*,X*}(t)\exp\left(\int_t^T (r_D(u,\xi_f(u)) - \boldsymbol{\sigma}_F\cdot\boldsymbol{\sigma}_X)\,du - \frac{1}{2}\int_t^T \|\boldsymbol{\sigma}_X\|^2 du - \frac{1}{2}\int_t^T \|\boldsymbol{\sigma}_F\|^2 du\right)$$
$$\times \exp\left(\int_t^T (\boldsymbol{\sigma}_X + \boldsymbol{\sigma}_F)\,dW_D^*(u)\right)\exp\left(\int_t^T Z_{u-}^{Q,F}\,d\Phi_F^{Q,F}(u;\xi_F)\right).$$

Their dynamics returns are given by:

$$\frac{dS_{F*,X*}(t)}{S_{F*,X*}(t-)} = r_D(t,\xi_f(t))\,dt + (\boldsymbol{\sigma}_X + \boldsymbol{\sigma}_F)\cdot d\mathbf{W}_D^*(t) + (\exp(Z_{t-}^{Q,F}) - 1)\,d\Phi_F^{Q,F}(t;\xi_F) \tag{B.2}$$

$$\frac{dB_D(t,T,\xi_f(t))}{B_D(t,T,\xi_f(t))} = r_D(t,\xi_f(t))\,dt - \mathbf{V}_D(t,T,\xi_f(t))\cdot d\mathbf{W}_D^*(t) \tag{B.3}$$

By using (B.2) and (B.3), we find the returns dynamics $S_{F*,X*}(t)$ under forward measure

$$\frac{dS_{F*,X*}(t)/B^D(t,T,\xi_f(t))}{S_{F*,X*}(t)/B^D(t,T,\xi_f(t))} = (\mathbf{V}_D(t,T,\xi_f(t)) + \boldsymbol{\sigma}_X + \boldsymbol{\sigma}_F)\cdot d\mathbf{W}_D^{T*}(t) + (\exp(Z_{t-}^{Q,F}) - 1)\,d\Phi_F^{Q,F}(t;\xi_F).$$

Hence $\frac{C_2(t,\xi_f,n)}{B_D(t,T,\xi_f(t))} = E^{\mathbb{Q}^D}\left(\frac{\beta_D(t,\xi_f(t))}{\beta_D(T,\xi_f(T))}\frac{B_D(T,T,\xi_f(T))}{B_D(t,T,\xi_f(t))}(X(T)S_F(T) - K_D)^+\right)$ becomes $\frac{C_2(t,\xi_f,n)}{B^D(t,T,\xi_f(t))} = E^{\mathbb{Q}^{T,D*}}((X(T)S^F(T) - K^D)^+)$. Then, we can obtain (4.2). □

B.3. Case3: A guaranteed-exchange rate foreign equity call option

The arbitrage-free price of a European call option at time t equals

$$C_3(t,\xi_f,n) = \chi E^{\mathbb{Q}^D}\left(\exp\left(-\int_t^T r_D(u,\xi_f(u))\,du\right)(S_F(T) - K_F)^+\Big|\mathcal{H}(t)\right).$$

Proof.

$$C_3(t,\xi_f,n) = \chi E^{\mathbb{Q}^D}\left(\exp\left(-\int_t^T r_D(u,\xi_f(u))\,du\right)(S_F(T) - K_F)^+\Big|\mathcal{H}(t)\right)$$

$$\frac{C_3(t,\xi_f,n)}{B_D(t,T,\xi_f(t))} = \chi E^{\mathbb{Q}^D}\left(\frac{\beta_D(t,\xi_f(t))}{\beta_D(T,\xi_f(T))}\frac{B_D(T,T,\xi_f(T))}{B_D(t,T,\xi_f(t))}(S_F(T) - K_F)^+\Big|\mathcal{H}(t)\right)$$
$$= \chi E^{\mathbb{Q}^{T,D*}}((S_F(T) - K_F)^+|\mathcal{H}(t)) = \chi E^{\mathbb{Q}^{T,D*}}\left(\left(\frac{S_F(T)}{B_F(T,T,\xi_f(T))} - K_F\right)^+\Big|\mathcal{H}(t)\right)$$

Clearly, the forward asset prices $S_F^T(t)$ and the forward FX rate process $X^T(T)$ are martingale processes under the forward martingale measures $\mathbb{Q}^{T*,F}$. We find it useful to express the price of the underlying foreign stock under the domestic measure $\mathbb{Q}^{D(\theta^{C*},\theta^{J*})}$ called $S_F^*(t)$. Hence, the dynamics of the foreign asset returns can be rewritten as given below:

$$\frac{dS_F^*(t)}{S_F^*(t-)} = r_F(t,\xi_f(t))\,dt + \boldsymbol{\sigma}_F\cdot d\mathbf{W}_F^{Q,F}(t) + \left(\exp(Z_n^{Q,F}) - 1\right)d\Phi_F^{Q,F}(t;\xi_F)$$
$$= r_F(t,\xi_f(t))\,dt - \boldsymbol{\sigma}_F\boldsymbol{\sigma}_X\,dt + \boldsymbol{\sigma}_F\cdot d\mathbf{W}_D^{Q,X}(t) + \left(\exp(Z_n^{Q,F}) - 1\right)d\Phi_F^{Q,F}(t;\xi_F),$$

where $\mathbf{W}_D^{Q,X}(t)$ is domestic Brownian motion of exchange rate, $Z_n^{Q,F} \sim \mathcal{N}\left(-\frac{1}{2}\sigma_{F,J}^2, \sigma_{F,J}^2\right)$.

Using Ito's lemma, we find the dynamics of the foreign forward price under the domestic martingale

$$\frac{dF^{Q,D}(t)}{F^{Q,D}(t)} = \frac{d(S_F^{QDF*}(t)/B_F^{QD}(t,T,\xi_f(t)))}{(S_F^{QD}(t)/B_F^{QD}(t,T,\xi_f(t)))}$$
$$= (\boldsymbol{\sigma}_F + \mathbf{V}_F(t,T,\xi_f(t)))\cdot d\mathbf{W}_D^{Q,D}(t) + (\mathbf{V}_F(t,T,\xi_f(t)) - \boldsymbol{\sigma}_X)\cdot(\boldsymbol{\sigma}_F + \mathbf{V}_F(t,T,\xi_f(t)))dt + (\exp(Z_{t-}^{Q,F}) - 1)\,d\Phi_F^{Q,F}(t;\xi_F)$$

Consequently, under the domestic forward measure, we

$$d\mathbf{W}_D^T(t) = d\mathbf{W}_D^{Q,D}(t) + \mathbf{V}_D(t,T,\xi_f(t))\,dt$$

Hence,

$$\frac{dF_D^{T*}(t)}{F_D^{T*}(t)} = (\boldsymbol{\sigma}_F + \mathbf{V}_F(t,T,\xi_f(t)))\cdot d\mathbf{W}_D^T(t) + (\boldsymbol{\sigma}_F + \mathbf{V}_F(t,T,\xi_f(t)))\cdot(-\boldsymbol{\sigma}_X + \mathbf{V}_F(t,T,\xi_f(t)) - \mathbf{V}_D(t,T,\xi_f(t)))\,dt + (\exp(Z_{t-}^{Q,F}) - 1)\,d\Phi_F^{Q,F}(t;\xi_F),$$

$$\frac{dF_D^{T*}(t)}{F_D^{T*}(t)} = (\boldsymbol{\sigma}_F + \mathbf{V}_F(t,T,\xi_f(t)))\cdot d\mathbf{W}_D^T - (\boldsymbol{\sigma}_F + \mathbf{V}_F(t,T,\xi_f(t)))\cdot\zeta_4(t,T)dt + (\exp(Z_{t-}^{Q,F}) - 1)\,d\Phi_F^{Q,F}(t;\xi_F),$$

where $\zeta_4(u,T,\xi_f(u)) = \boldsymbol{\sigma}_X - \mathbf{V}_F(u,T,\xi_f(u)) + \mathbf{V}_D(u,T,\xi_f(u))$

$$C_3(T,\xi_f,n) = \chi B_D(t,T,\xi_f(t))E^{\mathbb{Q}^{T,D*}}\left((F_D^{T*}(T) - K^F)^+\middle|\mathcal{H}(t)\right)$$

$$C_3(t,\xi_f) = \chi B_D(t,T,\xi_f(t))\left(\frac{S_F(t)}{B_F(t,T,\xi_f(t))}\exp\left(\int_t^T \zeta_4(t,T,\xi_f(t))\cdot(-\mathbf{V}_F(t,T,\xi_f(t)) - \boldsymbol{\sigma}_F)\,du\right)\Upsilon(d_{3,1}) - K_F\Upsilon(d_{3,2})\right).$$

Then, we obtain (4.3) □

B.4. Case 4: An equity-linked foreign exchange call

The arbitrage-free price of a European call option at time t equals

$$C_4(t,\xi_f,n) = E^{\mathbb{Q}^D}\left(\exp\left(-\int_t^T r_D(u,\xi_f(u))\,du\right)S_F(T)(X(T) - K_D)^+\middle|\mathcal{H}(t)\right)$$

Proof.

$$\frac{C_4(t,\xi_f,n)}{B_D(t,T,\xi_f(t))} = E^{\mathbb{Q}^D}\left(\frac{\beta_D(t,\xi_f(t))}{\beta_D(T,\xi_f(T))}\frac{B_D(T,T,\xi_f(T))}{B_D(t,T,\xi_f(t))}S_F(T)(X(T) - K_D)^+\middle|\mathcal{H}(t)\right)\frac{C_4(t,\xi_f,n)}{B_D(t,T,\xi_f(t))}$$
$$= E^{\mathbb{Q}^{T,D*}}(S_F(T)(X(T) - K_D)^+|\mathcal{H}(t)), \tag{B.4}$$

Consider foreign stock and bond under domestic measure

$$S_F^{Q,D}(T) = S_F^{Q,D}(t)\exp\left(\int_t^T (r_F(u,\xi_f(u)) - \boldsymbol{\sigma}_F\cdot\boldsymbol{\sigma}_X)\,du - \frac{1}{2}\int_t^T \|\boldsymbol{\sigma}_F\|^2\,du + \int_t^T \boldsymbol{\sigma}_F\cdot dW_D^{Q,D}(u) + \int_t^T Z_{u-}^{Q,F}\,d\Phi_F^{Q,F}(u;\xi_F)\right)\frac{dB_F(t,T.\xi_f(t))}{B_F(t,T,\xi_f(t))}$$
$$= [r_F(t,\xi_f(t)) + \boldsymbol{\sigma}_X\cdot\mathbf{V}_F(t,T,\xi_f(t))]\,dt - \mathbf{V}_F(t,T,\xi_f(t))\cdot d\mathbf{W}_D^{Q,D}(t)$$

Here, the dynamics $S_F^{Q,D}$ under forward measure associated with $\frac{S_F^{Q,D}}{B_F}$ as follows:

$$\frac{dG_D^{T*}(t)}{G_D^{T*}(t)} = (\boldsymbol{\sigma}_F + \mathbf{V}_F(t,T,\xi_f(t)))d\mathbf{W}_D^T(t) - (\boldsymbol{\sigma}_F + \mathbf{V}_F(t,T,\xi_f(t)))\cdot\boldsymbol{\zeta}_4(t,T,X(t))\,dt + (\exp(Z_{t-}^{Q,F}) - 1)\,d\Phi_F^{Q,F}(t,\xi_F).$$

Hence the (B.4) becomes

$$\frac{C_4(t,\xi_f,n)}{B_D(t,T)}\frac{1}{G_D^{T*}(t)} = E^{\mathbb{Q}^{T*,D*}}\left(\frac{G_D^{T*}(T)}{G_D^{T*}(t)}(X(T) - K_D)^+\middle|\mathcal{H}(t)\right).$$

Under the measure $\mathbb{Q}^{T*,F}$, we have the following exchange rate

$$\frac{dX^T(t)}{X^T(t)} = \zeta_4(t,T,\xi_f(t))dW_D^T(t),$$

where $\zeta_4(t,T,\xi_f(t)) = \boldsymbol{\sigma}_X - \mathbf{V}_F(t,T,\xi_f(t)) + \mathbf{V}_D(t,T,\xi_f(t))$ under the measure associated with $G_D^{T*}(T)$, the dynamics of exchange rate is given blow:

$$\frac{dX^T(t)}{X^T(t)} = \zeta_4(t,T,\xi_f(t))(d\mathbf{W}_F^T + (\boldsymbol{\sigma}_F + \mathbf{V}_F(t,T,\xi_f(t)))\,dt)$$

Hence,

$$\frac{C_4(t)}{B_D(t,T,\xi_f(t))}\frac{1}{G(t)} = \exp\left(\int_t^T -(\boldsymbol{\sigma}_F + \mathbf{V}_F(u,T,\xi_f(u)))\cdot\zeta_4(u,T,\xi_f(u))\,du\right) \cdot E^{\mathbb{Q}^{T*,F}}\left((X(T)-K_D)^+ \middle| \mathcal{H}(t)\right)$$

Hence the call as follows:

$$C_4(,t\xi_f,n) = B_D(t,T,\xi_f(t))\frac{S_F^{QD}(t)}{B_F^{QD}(t,T,\xi_f(t))} \cdot E^{\mathbb{Q}^{T*,F}}\left(\left(X(T) - K_D\exp\left(\int_t^T -(\boldsymbol{\sigma}_F + \mathbf{V}_F(u,T,\xi_f(u)))\cdot\zeta_4(u,T,\xi_f(u))\,du\right)\right)^+ \middle| \mathcal{H}(t)\right)$$

$$C_4(t,\xi_f,n) = S_F(t)\left(X(t)\Upsilon(d_{4,1}) - \frac{B_D(t,T,\xi_f(t))}{B_F(t,T,\xi_f(t))}K_D\exp\left(\int_t^T -(\boldsymbol{\sigma}_F + \mathbf{V}_F(u,T,\xi_f(u)))\zeta_4(u,T,\xi_f(u))\,du\right)\Upsilon(d_{4,2})\right)$$

Then, we obtain (4.4) □

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