# Solving $C_{k} / C_{m} / 1 / N$ queues by using characteristic roots in matrix analytic methods 

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#### Abstract

In this paper, we study a $C_{k} / C_{m} / 1 / N$ open queueing system with finite capacity. We investigate the property which shows that a product of the Laplace Stieltjes Transforms of interarrival and service times distributions satisfies an equation of a simple form. According to this equation, we present that the stationary probabilities on the unboundary states can be written as a linear combination of vector product-forms. Each component of these products is expressed in terms of roots of an associated characteristic polynomial. As a result, we carry out an algorithm for solving stationary probabilities in $C_{k} / C_{m} / 1 / N$ systems, which is independent of $N$, hence greatly reducing the computational complexity. © 2006 Elsevier Inc. All rights reserved.


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## 1. Introduction

We consider a $C_{k} / C_{m} / 1 / N$ open queueing model which restrains a finite number of customers in the system, assuming that both interarrival and service times are Coxian distributions with $k$ and $m$ phases respectively. There is only one server in the system. Customers are served under the First-come First-served (FCFS) discipline. In this paper, our goal is to explore an invariant solution structure arising in the model where a vector expression of stationary probability distributions is apparent in system states by using the matrix analytic method.

Matrix analytic methods have proved useful in the study of Markov chains of phase-type distributions, providing a solution tool which is relevant for queueing theory and computing probability distributions in many applications. A fundamental idea is to substitute independent identical distributions by conditional independence and conditional distributions given a Markov process. In a specific example, it can be viewed as a systematic approach to generalization in modelling of bursty traffic in telecommunication networks. Le Boudec [1] obtained an expression of the stationary probabilities of $P H / P H / 1$ by using this approach. He showed that

[^0]all the eigenvectors used in the expression of the stationary probabilities of $\mathrm{PH} / \mathrm{PH} / 1$ system are Kronecker products and gave a formula for computing eigenvalues by constructing an associated characteristic polynomial. Luh [2] used a similar approach to derive stationary probabilities in terms of linear combinations of product-forms in studying a system of two stations in tandem. In this paper, we shall prove a general result of existence of vector product-forms of $C_{k} / C_{m} / 1 / N$ which was provided neither in [1] nor in [2].

This method requires finding all singularities of a given matrix function in the unit disk and then using them to obtain a set of linear equations in the finite number of unknown boundary probabilities. The remaining probabilities and other measures of interest are then computed from the boundary probabilities. Based on our approach, unboundary state probabilities can be written as a linear combination of vector product-forms. Moreover, the solution procedure is independent of $N$, explicitly reducing the computational complexity of solving a complicated problem in quasi-birth-death (QBD) processes. We adopt a popular factorization procedure in algorithm for solving stationary probabilities and calculating the performance of $C_{k} / C_{m} / 1 / N$ system.

The remainder of this paper is organized as follows. In Section 2, we show that in analysis how the detailed state transitions are defined. We prove that the Laplace Stieltjes Transforms (LST) of interarrival and service times distributions may satisfy an equation derived from unboundary states of the state balance equations. The proof includes derivation of a characteristic polynomial and solutions of unboundary state probabilities. In Section 3, we present that unboundary state probabilities can be written in vector product-forms and show that the stationary probabilities on the unboundary states can be written as a linear combination of vector product-forms. For solving complicated numerical problems, we use the least square approach for solving stationary probabilities and calculating the performance of a $C_{k} / C_{m} / 1 / N$ system. We give an example to illustrate our approach in Section 4. In Section 5, we draw some conclusions and make recommendations for further investigation.

## 2. Analysis of state balance equations

### 2.1. Interarrival and service times

We consider a $C_{k} / C_{m} / 1 / N$ open queueing system containing a finite number of customers which is denoted by $N$. There is one server and service discipline is First-come-First-served. We assume that both interarrival and service times are Coxian as defined in the following. Assume that both interarrival and service times are of phase type with the parameters $k$ and $m$ for the number of phases respectively. The length of phase $j$ is exponentially distributed with a given rate $\lambda_{j}$. After phase $j, j=1,2, \ldots, k$, the interarrival time comes to an end with probability $p_{j}$, and it enters the next phase with probability $1-p_{j}$. Obviously, set $p_{k}=1$ for the last phase. The LST of the probability distribution function (DF) of the interarrival times is

$$
f_{1}^{*}(x)=\sum_{j=1}^{k} \frac{p_{j} \lambda_{j}}{x+\lambda_{j}} \prod_{r=1}^{j-1} \frac{\left(1-p_{r}\right) \lambda_{r}}{x+\lambda_{r}},
$$

where we define $\prod_{r=1}^{0}(\cdot)=1$. A similar characterization of distribution is defined for the service time, except that the symbols $\mu_{j}$ and $q_{j}, j=1,2, \ldots, m$ take the place of $\lambda_{j}$ and $p_{j} . f_{2}^{*}(x)$ denotes the LST of the service times with $q_{m}=1$. Denote by $\rho$ the utilization factor in system which is written as a fraction of derivatives of $f_{i}^{*}(x)$ at $x=0$, namely,

$$
\rho=\frac{\dot{f}_{2}^{*}(0)}{\dot{f}_{1}^{*}(0)} .
$$

The system state is denoted as ( $n, r, i$ ), where $n$ is the number of customers in service and in the waiting room, $0 \leqslant n \leqslant N$, and $r$ (resp. $i$ ) is the phase of the interarrival time (resp. the service time), $1 \leqslant r \leqslant k$ ( $1 \leqslant$ $i \leqslant m$ ). If $n=0$, a state is denoted by $(0, r, 0)$ since the server is idle. The states with $1 \leqslant n<N$ are called unboundary states. The states with $n=0$ and $n=N$ are called boundary states. The probabilities of state $(n, r, i)$ are denoted as $y_{n, r, i}$.

### 2.2. The state balance equations

By considering all possible state transitions in the system, we write the following state balance equations of the system according to its system states:

1. $n=0, r=1, \ldots, k, i=0$

$$
y_{0, r, 0}\left\{\lambda_{r}\right\}=\sum_{j=1}^{m} \mu_{j} q_{j} y_{1, r, j}+\lambda_{r-1}\left(1-p_{r-1}\right) y_{0, r-1,0},
$$

2. $1 \leqslant n<N$
(a) $r=1, i=1$

$$
\begin{equation*}
y_{n, 1,1}\left\{\lambda_{1}+\mu_{1}\right\}=\sum_{l=1}^{k} \lambda_{l} p_{l} y_{n-1, l, 1}+\sum_{j=1}^{m} \mu_{j} q_{j} y_{n+1,1, j} . \tag{2.1}
\end{equation*}
$$

(b) $r=1, i=2, \ldots, m$

$$
\begin{equation*}
y_{n, 1, i}\left\{\lambda_{1}+\mu_{i}\right\}=\sum_{l=1}^{k} \lambda_{l} p_{l} y_{n-1, l, i}+\mu_{i-1}\left(1-q_{i-1}\right) y_{n, 1, i-1} . \tag{2.2}
\end{equation*}
$$

(c) $r=2, \ldots, k, i=1$

$$
\begin{equation*}
y_{n, r, 1}\left\{\lambda_{r}+\mu_{1}\right\}=\lambda_{r-1}\left(1-p_{r-1}\right) y_{n, r-1,1}+\sum_{j=1}^{m} \mu_{j} q_{j} y_{n+1, r, j} \tag{2.3}
\end{equation*}
$$

(d) $r=2, \ldots, k, i=2, \ldots, m$

$$
\begin{equation*}
y_{n, r, i}\left\{\lambda_{r}+\mu_{i}\right\}=\lambda_{r-1}\left(1-p_{r-1}\right) y_{n, r-1, i}+\mu_{i-1}\left(1-q_{i-1}\right) y_{n, r, i-1} . \tag{2.4}
\end{equation*}
$$

3. $n=N$
(a) $r=1, i=1, \ldots, m$

$$
y_{N, 1, i}\left\{\left(1-p_{1}\right) \lambda_{1}+\mu_{i}\right\}=\sum_{l=1}^{k} \lambda_{l} p_{l} y_{N-1, l, i}+\mu_{i-1}\left(1-q_{i-1}\right) y_{N, 1, i-1} .
$$

(b) $r=2, \ldots, k-1, i=1, \ldots, m$

$$
y_{N, r, i}\left\{\left(1-p_{r}\right) \lambda_{r}+\mu_{i}\right\}=\lambda_{r-1}\left(1-p_{r-1}\right) y_{N, r-1, i}+\mu_{i-1}\left(1-q_{i-1}\right) y_{N, r, i-1} .
$$

(c) $r=k, i=1, \ldots, m$

$$
y_{N, k, i}\left\{\mu_{i}\right\}=\lambda_{k-1}\left(1-p_{k-1}\right) y_{N, k-1, i}+\mu_{i-1}\left(1-q_{i-1}\right) y_{N, k, i-1} .
$$

We define the values of $y_{0,0,0}, y_{0,0, j}$ and $y_{0, r, i}$ to be 0 for $i=2, \ldots, m$ as well as $y_{n, r, i}=0, n>N$, and $y_{n, r, 0}=0$, $n>0$. In addition, we set $\lambda_{0} \triangleq 0$, and $y_{0, r, 1} \triangleq y_{0, r, 0}$.

### 2.2.1. Separation of variables technique

Initially, we consider the equations of unboundary states from (2.1)-(2.4). We use a familiar technique from the theory of partial differential equations, the separation of variables technique [3]. We assume that the unboundary state probabilities are of the form:

$$
y_{n, r, i}=D_{r} R_{i} w^{n} \quad 1 \leqslant n<N .
$$

We need to determine $D_{r}, R_{i}$, and $w$. There will be several values for $w$, which lead to different $D_{r}$ and $R_{i}$. These values can be substituted into (2.1)-(2.4)

For $1 \leqslant n<N$,
(2a) $r=1, i=1$

$$
\left\{\lambda_{1}+\mu_{1}\right\} D_{1} R_{1} w^{n}=\sum_{l=1}^{k} \lambda_{l} p_{l} D_{l} R_{1} w^{n-1}+\sum_{j=1}^{m} \mu_{j} q_{j} D_{1} R_{j} w^{n+1} .
$$

(2b) $r=1, i=2, \ldots, m$

$$
\begin{equation*}
\left\{\lambda_{1}+\mu_{i}\right\} D_{1} R_{i} w^{n}=\sum_{l=1}^{k} \lambda_{l} p_{l} D_{1} R_{i} w^{n-1}+\mu_{i-1}\left(1-q_{i-1}\right) D_{1} R_{i-1} w^{n} . \tag{2.5}
\end{equation*}
$$

(2c) $r=2, \ldots, k, i=1$

$$
\begin{equation*}
\left\{\lambda_{r}+\mu_{1}\right\} D_{r} R_{1} w^{n}=\lambda_{r-1}\left(1-p_{r-1}\right) D_{r-1} R_{1} w^{n}+\sum_{j=1}^{m} \mu_{j} q_{j} D_{r} R_{j} w^{n+1} \tag{2.6}
\end{equation*}
$$

(2d) $r=2, \ldots, k, i=2, \ldots, m$

$$
\begin{equation*}
\left\{\lambda_{r}+\mu_{i}\right\} D_{r} R_{i} w^{n}=\lambda_{r-1}\left(1-p_{r-1}\right) D_{r-1} R_{i} w^{n}+\mu_{i-1}\left(1-q_{i-1}\right) D_{r} R_{i-1} w^{n} . \tag{2.7}
\end{equation*}
$$

Therefore, there exists a constant $x$, which is independent of $r$ and $i$, such that

$$
\begin{equation*}
\frac{\lambda_{r} D_{r}-\lambda_{r-1}\left(1-p_{r-1}\right) D_{r-1}}{D_{r}}=-x=\frac{\mu_{i-1}\left(1-q_{i-1}\right) R_{i-1}-\mu_{i} R_{i}}{R_{i}} \tag{2.8}
\end{equation*}
$$

By induction of (2.8) on $r$, we find that

$$
D_{r}=D_{1} \prod_{l=1}^{r-1} \frac{\left(1-p_{l}\right) \lambda_{l}}{x+\lambda_{l+1}}=D_{1} u(x, r), \quad r=1, \ldots, k
$$

where $u(x, r) \triangleq \prod_{l=1}^{r-1} \frac{\left(1-p_{l}\right) \lambda_{l}}{x+\lambda_{l+1}}$, and

$$
R_{i}=R_{1} \prod_{j=1}^{i-1} \frac{\left(1-q_{j}\right) \mu_{j}}{\mu_{j+1}-x}=R_{i, s} v(x, i), \quad i=1, \ldots, m
$$

where $v(x, i) \triangleq \prod_{j=1}^{i-1} \frac{\left(1-q_{j}\right) \mu_{j}}{\mu_{j+1}-x}$.
Lemma 1. The relation between $x$ and $w$ is given by

$$
w=\sum_{r=1}^{k} \frac{p_{r} \lambda_{r}}{x+\lambda_{r}} \prod_{l=1}^{r-1} \frac{\left(1-p_{l}\right) \lambda_{l}}{x+\lambda_{l}}=f_{1}^{*}(x) .
$$

Proof. Multiplying (2.5) by $D_{r} / D_{1}$, we have that

$$
\begin{equation*}
\mu_{i-1}\left(1-q_{i-1}\right) D_{r} R_{i-1} w^{n}=\lambda_{1} D_{r} R_{i} w^{n}+\mu_{i} D_{r} R_{i} w^{n}-\sum_{l=1}^{k} \lambda_{l} p_{l} \frac{D_{l}}{D_{1}} D_{r} R_{i} w^{n-1} . \tag{2.9}
\end{equation*}
$$

Substituting (2.9) to (2.7), and dividing by $R_{i} w^{n-1} D_{r}$, it can be written as

$$
\left(\frac{\lambda_{r} D_{r}-\lambda_{r-1}\left(1-p_{r-1}\right) D_{r-1}}{D_{r}}-\lambda_{1}\right) w=-\sum_{l=1}^{k} \lambda_{l} p_{l} \frac{D_{l}}{D_{1}} .
$$

From (2.8), we have

$$
\left(-x-\lambda_{1}\right) w=-\sum_{l=1}^{k} \lambda_{l} p_{l} \frac{D_{l}}{D_{1}} .
$$

Dividing by $\left(-x-\lambda_{1}\right)$, we obtain

$$
w=\sum_{l=1}^{k} \frac{\lambda_{l} p_{l}}{x+\lambda_{1}} \frac{D_{l}}{D_{1}}=\sum_{r=1}^{k} \frac{\lambda_{r} p_{r}}{x+\lambda_{r}} \frac{x+\lambda_{r}}{x+\lambda_{1}} \prod_{l=1}^{r-1} \frac{\left(1-p_{l}\right) \lambda_{l}}{x+\lambda_{l+1}}=\sum_{r=1}^{k} \frac{\lambda_{r} p_{r}}{x+\lambda_{r}} \prod_{l=1}^{r-1} \frac{\left(1-p_{l}\right) \lambda_{l}}{x+\lambda_{l}} \frac{x+\lambda_{r}}{x+\lambda_{r}}=f_{1}^{*}(x) .
$$

Theorem 1. If $f_{1}^{*}(x)$ and $f_{2}^{*}(x)$ are LST of DF of the interarrival times and the service times, then we have the equation:

$$
f_{1}^{*}(x) f_{2}^{*}(-x)=1
$$

Proof. By having (2.7) divided by $w^{n}$, we give

$$
\begin{equation*}
\mu_{i-1}\left(1-q_{i-1}\right) D_{r} R_{i-1}-\mu_{i} D_{r} R_{i}=\lambda_{r} D_{r} R_{i}-\lambda_{r-1}\left(1-p_{r-1}\right) D_{r-1} R_{i}, \quad i=2, \ldots, m . \tag{2.10}
\end{equation*}
$$

Again, by making (2.6) divided by $w^{n}$, it yields

$$
\begin{equation*}
\sum_{j=1}^{m} \mu_{j} q_{j} D_{r} R_{j} w-\mu_{1} D_{r} R_{1}=\lambda_{r} D_{r} R_{1}-\lambda_{r-1}\left(1-p_{r-1}\right) D_{r-1} R_{1} . \tag{2.11}
\end{equation*}
$$

Dividing (2.10) by $R_{i}$, we obtain

$$
\begin{equation*}
\mu_{i-1}\left(1-q_{i-1}\right) D_{r} \frac{R_{i-1}}{R_{i}}-\mu_{i} D_{r}=\lambda_{r} D_{r}-\lambda_{r-1}\left(1-p_{r-1}\right) D_{r-1} . \tag{2.12}
\end{equation*}
$$

Dividing (2.11) by $R_{1}$, we find

$$
\begin{equation*}
\sum_{j=1}^{m} \mu_{j} q_{j} D_{r} \frac{R_{j}}{R_{1}} w-\mu_{1} D_{r}=\lambda_{r} D_{r}-\lambda_{r-1}\left(1-p_{r-1}\right) D_{r-1} \tag{2.13}
\end{equation*}
$$

Combining (2.12) and (2.13), we find that

$$
\begin{equation*}
\mu_{i-1}\left(1-q_{i-1}\right) D_{r} \frac{R_{i-1}}{R_{i}}-\mu_{i} D_{r}=\sum_{j=1}^{m} \mu_{j} q_{j} D_{r} \frac{R_{j}}{R_{1}} w-\mu_{1} D_{r} . \tag{2.14}
\end{equation*}
$$

Rearranging (2.14) and dividing it by $D_{r}$, we have

$$
\mu_{i-1}\left(1-q_{i-1}\right) R_{i-1} \frac{1}{R_{i}}-\mu_{i}+\mu_{1}=\sum_{j=1}^{m} \mu_{j} q_{j} \frac{R_{j}}{R_{1}} w .
$$

Therefore it produces

$$
\frac{\mu_{i-1}\left(1-q_{i-1}\right) R_{i-1}-\mu_{i} R_{i}}{R_{i}}+\mu_{1}=\sum_{j=1}^{m} \mu_{j} q_{j} R_{j} \frac{1}{R_{1}} w .
$$

From (2.8), we have

$$
1=w \frac{\sum_{j=1}^{m} \mu_{j} q_{i} R_{j}}{\left(-x+\mu_{1}\right) R_{1}},
$$

which implies

$$
1=w \sum_{i=1}^{m} \frac{\mu_{i} q_{i}}{-x+\mu_{i}} \prod_{j=1}^{i-1} \frac{\left(1-q_{j}\right) \mu_{j}}{-x+\mu_{j}},
$$

and

$$
f_{1}^{*}(x) f_{2}^{*}(-x)=1
$$

Now, we need to determine the number of roots that satisfy the equation in Theorem 1. Here, we state the main theorem.

Theorem 2. The equation $f_{1}^{*}(x) f_{2}^{*}(-x)=1$ has troots except $x=0$. If $\rho<1$, t equals $m$ and the equation has $m$ roots with positive real parts. If $\rho>1$, t equals $k$ and the equation has $k$ roots with negative real parts. If $\rho=1$, it is indeterminate.

In order to prove the theorem, we first state Rouchés theorem:
Rouche's theorem: Let C be a "scroc", and let $\phi(z)$ be a function which is meromorphic in C" where there is neither zeros nor poles on C. Suppose that $\phi(x)$ can be written as the sum of two function meromorphic in $C^{*}, \phi(z)=\varphi(z)+\psi(z)$, such that $\varphi(z) \neq 0$. Further, suppose that $|\varphi(z)|>|\psi(z)|$ on $C$. Then the change in the argument of $\phi(z)$ when $z$ describes $C$ is the same as the change in the argument of $\varphi(z)$, and the difference between the number of zeros, $Z$, and the number of pole, $P$, is the same for both functions: $Z_{\phi}-P_{\phi}=Z_{\varphi}-P_{\varphi}$.

## Proof of Theorem 2

Case 1: To show that if $\rho<1, t$ equals $m$ and the equation has $m$ solutions with positive real parts.Let $\varphi(x)=1, \psi(x)=f_{1}^{*}(x) f_{2}^{*}(-x), \phi(x)=1-\psi(x)=1-f_{1}^{*}(x) f_{2}^{*}(-x)$, and define $\Omega_{\epsilon, r}$ as the path in the $x$-plane as shown in Fig. 1. We restrict ourselves to $\epsilon$ small enough and $r$ large enough for all roots of $f_{1}^{*}(x) f_{2}^{*}(-x)=1$ with positive real part of a line segment inside $\Omega_{\epsilon, r}$. If we can prove that $|\psi(z)|<1$ on $\Omega_{\epsilon, r}$, then we have $Z_{\phi}-P_{\phi}=0$, equivalently $Z_{\phi}=P_{\phi}$. Because $\phi(x)$ has $m$ poles with positive real parts, it implies that it also has $m$ zeros with positive real parts.

1. $|x|=r, r \rightarrow \infty$.

Then $\lim _{|x| \rightarrow \infty} f_{1}^{*}(x)=0$ and $\lim _{|x| \rightarrow \infty} f_{2}^{*}(-x)=0$. Thus, for large enough values of $r$, we have

$$
\begin{equation*}
|\psi(z)|=\left|f_{1}^{*}(x) f_{2}^{*}(-x)\right|<1 . \tag{2.15}
\end{equation*}
$$

2. $\operatorname{Re}\{x\}=\epsilon$.

Consider $f_{1}(x)$ and $f_{2}(-x)$, i.e., the probability density functions of the interarrival and service times. If $\operatorname{Re}\{x\}<\mu=\min \left\{\mu_{1}, \mu_{2}, \ldots, \mu_{m}\right\}$, then we have
$f_{2}^{*}(-x)=\int_{0}^{+\infty} \mathrm{e}^{x t} f_{2}(t) \mathrm{d} t<+\infty$.
Moreover, we have that

$$
\left|f_{1}^{*}(x)\right|=\left|\int_{0}^{+\infty} \mathrm{e}^{-x t} f_{1}(t) \mathrm{d} t\right| \leqslant \int_{0}^{+\infty}\left|\mathrm{e}^{-x t}\right| f_{1}(t) \mathrm{d} t=\int_{0}^{+\infty} \mathrm{e}^{-s t} f_{1}(t) \mathrm{d} t, \quad \text { let } \operatorname{Re}\{x\}=s=f_{1}^{*}(s),
$$



Fig. 1. $\rho<1$.
and

$$
\left|f_{2}^{*}(-x)\right| \leqslant\left|f_{2}^{*}(-s)\right| .
$$

Hence, it implies that

$$
\begin{equation*}
\left|f_{1}^{*}(x) f_{2}^{*}(-x)\right| \leqslant f_{1}^{*}(s) f_{2}^{*}(-s) \tag{2.16}
\end{equation*}
$$

By assumption of the utilization factor $\rho<1$, we have $\dot{f}_{1}^{*}(0)-\dot{f}_{2}^{*}(0)<0$. It is easy to show that the derivative at $s=0$ of $f_{1}^{*}(s) f_{2}^{*}(-s)$ is given by $\dot{f}_{1}^{*}(0)-f_{2}^{*}(0)$, whose value is less than 0 . Because $\psi(0)=1$ and $\psi^{\prime}(0)<0, \psi(x)$ is a decreasing function and its value is less than 1 when $x \rightarrow 0^{+}$. Therefore, for $\epsilon$ small enough and $0<s<\epsilon$, we have $0<f_{1}^{*}(s) f_{2}^{*}(-s)<1$. By (2.16), we have
$\operatorname{Re}\{x\}=\epsilon \rightarrow|\psi(z)|=\left|f_{1}^{*}(x) f_{2}^{*}(-x)\right|<1$
By (2.15) and (2.17), we have prove that $|\psi(z)|<1$ on $\Omega_{\epsilon, r}$. By Rouche's theorem, it has $m$ zeros with positive real parts since $\phi(x)$ has $m$ poles with positive real parts. It completes the first part of the proof.
Case 2: To show that if $\rho>1, t$ equals $k$ and the equation has $t$ solutions with negative real parts.Let $\varphi(x)=1$, $\psi(x)=f_{1}^{*}(x) f_{2}^{*}(-x), \phi(x)=1-\psi(x)=1-f_{1}^{*}(x) f_{2}^{*}(-x)$, and define $\Omega_{-\epsilon, r}$ as the path in the $x$-plane as shown in Fig. 2. We restrict ourselves to $\epsilon$ small enough and $r$ large enough for all roots of $f_{1}^{*}(x) f_{2}^{*}(-x)=1$ with negative real part of a line segment inside $\Omega_{-\epsilon, r}$. If we can prove that $|\psi(z)|<1$ on $\Omega_{-\epsilon, r}$, then we have $Z_{\phi}-P_{\phi}=0$, equivalently, $Z_{\phi}=P_{\phi}$. Because $\phi(x)$ has $k$ poles with negative real parts, it implies that it also has $k$ zeros with negative parts.

1. $|x|=r, r \rightarrow \infty$.

Then $\lim _{|x| \rightarrow \infty} f_{1}^{*}(x)=0$ and $\lim _{|x| \rightarrow \infty} f_{2}^{*}(-x)=0$. Thus, for large enough values of $r$, we have

$$
\begin{equation*}
|\psi(z)|=\left|f_{1}^{*}(x) f_{2}^{*}(-x)\right|<1 \tag{2.18}
\end{equation*}
$$

2. $\operatorname{Re}\{x\}=\epsilon$.

We have that with a similar argument in case 1

$$
\begin{equation*}
\left|f_{1}^{*}(x) f_{2}^{*}(-x)\right| \leqslant f_{1}^{*}(s) f_{2}^{*}(-s) \tag{2.19}
\end{equation*}
$$

Now, by the assumption of the utilization factor $\rho>1$, we have $\dot{f}_{1}^{*}(0)-\dot{f}_{2}^{*}(0)>0$. It is easy to show that the derivative at $s=0$ of $f_{1}^{*}(s) f_{2}^{*}(-s)$ is given by $\dot{f}_{1}^{*}(0)-\dot{f}_{2}^{*}(0)>0$. Because $\psi(0)=1$ and $\psi^{\prime}(0)>0, \psi(x)$ is increasing function and less than 1 when $x \rightarrow 0^{-}$. Therefore, for $-\epsilon$ small enough and $-\epsilon<s<0$, we have $0<f_{1}^{*}(s) f_{2}^{*}(-s)<1$. By (2.19), we have

$$
\begin{equation*}
\operatorname{Re}\{x\}=-\epsilon \rightarrow|\psi(z)|=\left|f_{1}^{*}(x) f_{2}^{*}(-x)\right|<1 \tag{2.20}
\end{equation*}
$$



Fig. 2. $\rho>1$.

By (2.18) and (2.20), we prove that $|\psi(z)|<1$ on $\Omega_{-\epsilon, r}$. By Rouche's theorem and $\phi(x)$ has $k$ poles with negative real parts, it also has $k$ zeros with negative real parts. However, when $\rho=1$, the zeros may located at either the positive side or the negative side of the plane.

Note that $k$ is the number of phases of the arrival process and $m$ is the number of phases of the service time. For the finite capacity system, a steady state exists if the utilization factor $\rho>1$. It shows that the solution will take the positive halfplanes in this case. However, if $\rho=1$, it may take either side of the plane because the solution alternates.

## 3. A model with matrix forms

### 3.1. Assumptions and problem description

Let $P H\left(\boldsymbol{\beta}_{i}, \mathbf{S}_{i}\right)$ be a Coxian distribution $i=1$ and 2 with respect to interarrival time and service time distributions. They may be presented by an initial probability vector $\left(\boldsymbol{\beta}_{i}, 0\right)$ and transition rate matrix

$$
\left[\begin{array}{cc}
\mathbf{S}_{i} & \gamma_{i}^{\prime} \\
\mathbf{0} & 0
\end{array}\right] .
$$

Denote by $\mathbf{e}^{\prime}$ a column vector of all entries equal to 1 with an appropriate dimension of its multiplier. $\boldsymbol{\gamma}_{i}^{\prime}$ is a column vector such that $\mathbf{S}_{i} \mathbf{e}^{\prime}+\gamma_{i}^{\prime}=\mathbf{0}$. Note that the representation $\left(\boldsymbol{\beta}_{i}, \mathbf{S}_{i}\right)$ fully characterizes a phase-type distribution (see [4]).

By a matrix representation, it is well known that $f_{1}^{*}(x)$ may be written as

$$
f_{1}^{*}(x)=\boldsymbol{\beta}_{1}\left(x \mathbf{I}_{1}-\mathbf{S}_{1}\right)^{-1} \boldsymbol{\gamma}_{1}^{\prime},
$$

where $\mathbf{I}_{i}, i=1,2$, denote identity matrices with dimension of $k$ and $m$ respectively. Similarly, $f_{2}^{*}(x)$ is written as

$$
f_{2}^{*}(x)=\boldsymbol{\beta}_{2}\left(x \mathbf{I}_{2}-\mathbf{S}_{2}\right)^{-1} \boldsymbol{\gamma}_{2}^{\prime} .
$$

Following a similar discussion in Liefvoort [5], we define

$$
\mathbf{E}=\left(\begin{array}{cc}
\gamma_{1}^{\prime} \boldsymbol{\beta}_{2} & \mathbf{S}_{1} \\
\mathbf{S}_{2} & \gamma_{2}^{\prime} \boldsymbol{\beta}_{1}
\end{array}\right) .
$$

Assumption 1. $\boldsymbol{\gamma}_{i}^{\prime} \boldsymbol{\beta}_{i}+\mathbf{S}_{i}$ is irreducible, or equivalently $-\boldsymbol{\beta}_{i} \mathbf{S}_{i}^{-1}>0, i=1,2$.
Assumption 2. All eigenvalues of $\mathbf{E}$ are simple.
It was proved that a zero of $f_{1}^{*}(x) f_{2}^{*}(-x)=1$ is also an eigenvalue of $\mathbf{E}$ in [5]. Those solutions are in term related to the construction of vector product-forms in stationary distributions of states in system. These assumptions are necessary to construct the product-form solutions in the model.

### 3.2. Matrix of transition rates

We arrange the states $(n, r, i)$ in lexicographic order and partition of the state space according to the number of customers, $n$. For a fixed $n$ the state can be lexicographically ordered in accord with phase $r$ and $i$. Let $\mathscr{L}_{n}=\{(n, r, i) \mid 1 \leqslant r \leqslant k, 1 \leqslant i \leqslant m\}, n=0,1,2, \ldots, N . \mathscr{L}_{n}$ is defined as sets of unboundary states, $1 \leqslant n<$ $N . \mathscr{L}_{0}$ and $\mathscr{L}_{N}$ are defined as sets of boundary states. Denote by $\mathbf{y}$ the row-vector of stationary probability partitioned corresponding to $\mathscr{L}_{n}$ as: $\mathbf{y}=\left(\mathbf{y}_{0}, \mathbf{y}_{1}, \ldots, \mathbf{y}_{N}\right)$ where $\mathbf{y}_{n}$ is a row-vector indicating $n$ customers in
system. Define $\mathbf{Q}$ the transition rate matrix of the chain according to the arrangement of $\mathscr{L}_{n}$. Then $\mathbf{Q}$ is of the block-tridiagonal form and written as

$$
\mathbf{Q}=\begin{gathered}
\\
\mathcal{L}_{0} \\
\mathcal{L}_{1} \\
\mathcal{L}_{2} \\
\vdots \\
\mathcal{L}_{N-2} \\
\mathcal{L}_{N-1} \\
\mathcal{L}_{N}
\end{gathered}\left[\begin{array}{ccccccc}
\mathcal{L}_{0} & \mathcal{L}_{1} & \mathcal{L}_{2} & \cdots & \mathcal{L}_{N-2} & \mathcal{L}_{N-1} & \mathcal{L}_{N} \\
\mathbf{B}_{0} & \mathbf{A}_{0} & & & & & \\
\mathbf{C}_{0} & \mathbf{B} & \mathbf{A} & & & & \\
& \mathbf{C} & \mathbf{B} & \mathbf{A} & & & \\
& & \ddots & \ddots & \ddots & & \\
& & & \mathbf{C} & \mathbf{B} & \mathbf{A} & \\
& & & & \mathbf{C} & \mathbf{B} & \mathbf{A} \\
& & & & & \mathbf{C} & \mathbf{B}_{1}
\end{array}\right] .
$$

The submatrices could be written as Kronecker product and Kronecker sum whose operations were defined in Bellman [6] and denoted by $\otimes$ and $\oplus$, respectively. They are expressed by

$$
\begin{array}{lll}
\mathbf{A}_{0}=\gamma_{1}^{\prime} \boldsymbol{\beta}_{1} \otimes \boldsymbol{\beta}_{2}, & \mathbf{B}_{0}=\mathbf{S}_{1}, & \mathbf{C}_{0}=\mathbf{I}_{1} \otimes \boldsymbol{\gamma}_{2}^{\prime}, \\
\mathbf{A}=\gamma_{1}^{\prime} \boldsymbol{\beta}_{1} \otimes \mathbf{I}_{2}, & \mathbf{B}=\mathbf{S}_{1} \oplus \mathbf{S}_{2}, & \mathbf{C}=\mathbf{I}_{1} \otimes \boldsymbol{\gamma}_{2}^{\prime} \boldsymbol{\beta}_{2}, \\
& \mathbf{B}_{1}=\left(\mathbf{S}_{1}+\mathbf{R}_{1}\right) \oplus \mathbf{S}_{2}, & \tag{3.1}
\end{array}
$$

where $\mathbf{R}_{1}$ is a matrix with $\operatorname{diag}\left[\gamma_{1}, \gamma_{2}, \gamma_{3}, \ldots, \gamma_{k}\right]$ and $\gamma_{j}, j=1,2, \ldots, k$, are elements of $-\mathbf{S}_{1} \mathbf{e}^{\prime}$.
For the state balance equations $\mathbf{y Q}=\mathbf{0}$ and the normalization condition $\mathbf{y e}^{\prime}=1$, we give the following equations:

$$
\begin{align*}
& \mathbf{y}_{0} \mathbf{B}_{0}+\mathbf{y}_{1} \mathbf{C}_{0}=\mathbf{0},  \tag{3.2}\\
& \mathbf{y}_{0} \mathbf{A}_{0}+\mathbf{y}_{1} \mathbf{B}+\mathbf{y}_{2} \mathbf{C}=\mathbf{0},  \tag{3.3}\\
& \mathbf{y}_{n-1} \mathbf{A}+\mathbf{y}_{n} \mathbf{B}+\mathbf{y}_{n+1} \mathbf{C}=\mathbf{0}, \quad 2 \leqslant n \leqslant N-1,  \tag{3.4}\\
& \mathbf{y}_{N-1} \mathbf{A}+\mathbf{y}_{N} \mathbf{B}_{1}=\mathbf{0}  \tag{3.5}\\
& \mathbf{y e}^{\prime}=1 \tag{3.6}
\end{align*}
$$

It is easy to rewrite the balance equations by substituting equations in (3.1) into (3.2)-(3.5). It yields

$$
\begin{align*}
& \mathbf{y}_{0} \mathbf{S}_{1}+\mathbf{y}_{1}\left(\mathbf{I}_{1} \otimes \boldsymbol{\gamma}_{2}\right)=\mathbf{0},  \tag{3.7}\\
& \mathbf{y}_{0}\left(\gamma_{1} \boldsymbol{\beta}_{1} \otimes \boldsymbol{\beta}_{2}\right)+\mathbf{y}_{1}\left(\mathbf{S}_{1} \oplus \mathbf{S}_{2}\right)+\mathbf{y}_{2}\left(\mathbf{I}_{1} \otimes \boldsymbol{\gamma}_{2} \boldsymbol{\beta}_{2}\right)=\mathbf{0},  \tag{3.8}\\
& \mathbf{y}_{n-1}\left(\gamma_{1} \boldsymbol{\beta}_{1} \otimes \mathbf{I}_{2}\right)+\mathbf{y}_{n}\left(\mathbf{S}_{1} \oplus \mathbf{S}_{2}\right)+\mathbf{y}_{n+1}\left(\mathbf{I}_{1} \otimes \boldsymbol{\gamma}_{2} \boldsymbol{\beta}_{2}\right)=\mathbf{0}, \quad 2 \leqslant n \leqslant N-1,  \tag{3.9}\\
& \mathbf{y}_{N-1}\left(\gamma_{1} \boldsymbol{\beta}_{1} \otimes \mathbf{I}_{2}\right)+\mathbf{y}_{N}\left(\left(\mathbf{S}_{1}+\mathbf{R}_{1}\right) \oplus \mathbf{S}_{2}\right)=\mathbf{0} . \tag{3.10}
\end{align*}
$$

### 3.3. Vector product-form solutions

In this section, our intention is to construct a solution basis that will solve the system of general equation (3.9) for $2 \leqslant n \leqslant N-1$. With this purpose, we define a vector product-form solution.

According to Assumption 2 and Theorem 2, let $x_{s}$ be a simple root of $f_{1}^{*}(x) f_{2}^{*}(-x)=1, s=1, \ldots, t$ and set $w_{s}=f_{1}^{*}\left(x_{s}\right)$, for $w_{s} \neq 0$. Given $x_{s}$, we define $\mathbf{u}_{s}$ and $\mathbf{v}_{s}$ as follows:

$$
\begin{align*}
\mathbf{u}_{s} & =a_{1}(s) \boldsymbol{\beta}_{1}\left(\mathbf{S}_{1}-x_{s} \mathbf{I}_{1}\right)^{-1}  \tag{3.11}\\
\mathbf{v}_{s} & =a_{2}(s) \boldsymbol{\beta}_{2}\left(\mathbf{S}_{2}+x_{s} \mathbf{I}_{2}\right)^{-1} . \tag{3.12}
\end{align*}
$$

where $a_{1}(s), a_{2}(s)$ are constants such that $\mathbf{u}_{s} \mathbf{e}^{\prime}=\mathbf{v}_{s} \mathbf{e}^{\prime}=1$. Simply, set

$$
a_{1}(s)=\frac{x_{s}}{w_{s}-1}, \quad a_{2}(s)=\frac{x_{s} w_{s}}{w_{s}-1} \quad \text { for } w_{s} \neq 1
$$

We explain it in the following. For any given $s$, rewriting (3.11) and multiplying it by $\mathbf{e}^{\prime}$, we have

$$
\mathbf{u}_{s} \mathbf{S}_{\mathbf{l}} \mathbf{e}^{\prime}=a_{1}(s) \boldsymbol{\beta}_{1} \mathbf{e}^{\prime}+x_{s} \mathbf{u}_{s} \mathbf{e}^{\prime} \Rightarrow a_{1}(s) f_{1}^{*}\left(x_{s}\right)=a_{1}(s)+x_{s},
$$

implying $a_{1}(s)=x_{s} /\left(w_{s}-1\right)$.
From (3.11) and (3.12), we can easily derive

$$
\mathbf{u}_{s} \mathbf{S}_{1} \mathbf{e}^{\prime}=a_{1}(s) f_{1}^{*}\left(x_{s}\right)=a_{1}(s) w_{s}, \quad \mathbf{v}_{s} \mathbf{S}_{2} \mathbf{e}^{\prime}=a_{2}(s) f_{2}^{*}\left(-x_{s}\right)=\frac{a_{2}(s)}{w_{s}} .
$$

Since there are $t$ solutions, we define $s=1,2, \ldots, t$,

$$
\begin{equation*}
\mathbf{w}_{s, n}=w_{s}^{n}\left(\mathbf{u}_{s} \otimes \mathbf{v}_{s}\right), \quad 2 \leqslant n \leqslant N-1, \tag{3.13}
\end{equation*}
$$

where $\mathbf{u}_{s} \in \mathbb{C}^{k}, \mathbf{v}_{s} \in \mathbb{C}^{m}$, and $w \in \mathbb{C}$.
Lemma 2. Given $x_{s}, \mathbf{w}_{s, n}$ satisfies Eq. (3.9).
Proof. Inserting (3.13) into (3.9) divided by $w_{s}^{n}$, it becomes

$$
\begin{aligned}
& \left(\mathbf{u}_{s} \otimes \mathbf{v}_{s}\right)\left(\mathbf{S}_{1} \oplus \mathbf{S}_{2}\right)-\left\{\frac{1}{w_{s}}\left(\mathbf{u}_{s} \otimes \mathbf{v}_{s}\right)\left(\mathbf{S}_{1} \mathbf{e}^{\prime} \boldsymbol{\beta}_{1} \otimes \mathbf{I}_{2}\right)+w_{s}\left(\mathbf{u}_{s} \otimes \mathbf{v}_{s}\right)\left(\mathbf{I}_{1} \otimes \mathbf{S}_{2} \mathbf{e}^{\prime} \boldsymbol{\beta}_{2}\right)\right\} \\
& \quad=\mathbf{u}_{s} \mathbf{S}_{1} \otimes \mathbf{v}_{s}+\mathbf{u}_{s} \otimes \mathbf{v}_{s} \mathbf{S}_{2}-\left\{\frac{1}{w_{s}}\left(\mathbf{u}_{s} \otimes \mathbf{v}_{s}\right)\left(\mathbf{S}_{1} \mathbf{e}^{\prime} \boldsymbol{\beta}_{1} \otimes \mathbf{I}_{2}\right)+w_{s}\left(\mathbf{u}_{s} \otimes \mathbf{v}_{s}\right)\left(\mathbf{I}_{1} \otimes \mathbf{S}_{2} \mathbf{e}^{\prime} \boldsymbol{\beta}_{2}\right)\right\} \\
& =a_{1}(s) \boldsymbol{\beta}_{1} \otimes \mathbf{v}_{s}+\mathbf{u}_{s} \otimes a_{2}(s) \boldsymbol{\beta}_{2}-\left\{\frac{1}{w_{s}}\left(\mathbf{u}_{s} \otimes \mathbf{v}_{s}\right)\left(\mathbf{S}_{1} \mathbf{e}^{\prime} \boldsymbol{\beta}_{1} \otimes \mathbf{I}_{2}\right)+w_{s}\left(\mathbf{u}_{s} \otimes \mathbf{v}_{s}\right)\left(\mathbf{I}_{1} \otimes \mathbf{S}_{2} \mathrm{e}^{\prime} \boldsymbol{\beta}_{2}\right)\right\} \\
& =a_{1}(s) \boldsymbol{\beta}_{1} \otimes \mathbf{v}_{s}+a_{2}(s) \mathbf{u}_{s} \otimes \boldsymbol{\beta}_{2}-\left\{\frac{1}{w_{s}}\left(a_{1}(s) w_{s} \boldsymbol{\beta}_{1} \otimes \mathbf{v}_{s}\right)+w_{s}\left(\mathbf{u}_{s} \otimes\left(\frac{a_{2}(s)}{w_{s}} \boldsymbol{\beta}_{2}\right)\right\} .\right.
\end{aligned}
$$

Hence it balances Eq. (3.9).
Now any linear combination of $\mathbf{w}_{s, n}, 2 \leqslant n \leqslant N-1$ obviously satisfies the balance equation (3.9). Let

$$
\begin{equation*}
\mathbf{y}_{n}=\sum_{s=1}^{t} c_{s} \mathbf{w}_{s, n}, \quad c_{s} \in \mathbb{C} \tag{3.14}
\end{equation*}
$$

where $c_{s}$ is the coefficients with respect to $\mathbf{w}_{s, n}$. Since the system is stable, at least one of the coefficient $c_{s}$ must be nonzero. Thus, for an appropriate choice of $c_{s}, \mathbf{y}_{n}$ presents for unboundary state probabilities.

## 4. The numerical method

### 4.1. Boundary state probabilities

In this section, we are going to present a numerical method to solve boundary stationary probabilities, i.e., $\mathbf{y}_{0}, \mathbf{y}_{N}$ as well as $\mathbf{y}_{1}$. Note $\mathbf{y}_{0}$ is a $k$-dimensional vector, $\mathbf{y}_{1}$ and $\mathbf{y}_{N}$ are $k m$-dimensional vectors.

Let $\mathbf{c}=\left(c_{1}, c_{2}, \ldots, c_{t}\right)$ where $t$ depends on the condition of $\rho$. If $\rho=1$, we may choose $t=\max \{k, m\}$. It is easy to check that $\mathbf{S}_{1}$ and $\mathbf{B}_{1}$ are invertible. Let $\mathbf{y}_{0}=-\mathbf{y}_{1} \mathbf{C}_{0} \mathbf{S}_{1}^{-1}$ and $\mathbf{y}_{N}=-\mathbf{y}_{N-1} \mathbf{A} \mathbf{B}_{1}^{-1}$ derived directly from (3.2) and (3.5). By adopting (3.14) and rewriting (3.3) and (3.4), we have

$$
\begin{align*}
& \mathbf{y}_{0} \mathbf{A}_{0}+\mathbf{y}_{1} \mathbf{B}-\mathbf{c} \mathbf{G}_{1}=\mathbf{0}  \tag{4.1}\\
& \mathbf{y}_{N-2} \mathbf{A}+\mathbf{y}_{N-1} \mathbf{B}-\mathbf{y}_{N-1} \mathbf{A} \mathbf{B}_{1}^{-1} \mathbf{C}=\mathbf{0} \tag{4.2}
\end{align*}
$$

where $\mathbf{G}_{1}$ and $\mathbf{G}_{2}$ are defined in accord with the coefficients derived from Eqs. (3.3) and (3.4) at $n=N-1$ respectively.

$$
\mathbf{G}_{1}=\left[\begin{array}{c}
a_{2}(1) \mathbf{u}_{1} \\
a_{2}(2) \mathbf{u}_{2} \\
\vdots \\
a_{2}(t) \mathbf{u}_{t}
\end{array}\right] \otimes \boldsymbol{\beta}_{2}
$$

Rewriting (4.1) and (4.2), we have

$$
\begin{align*}
& \mathbf{y}_{1}\left[\mathbf{B}-\left(\mathbf{e}_{1}^{\prime} \boldsymbol{\beta}_{1} \otimes \mathbf{S}_{2} \mathbf{e}_{2}^{\prime} \boldsymbol{\beta}_{2}\right)\right]-\mathbf{c} \mathbf{G}_{1}=\mathbf{0},  \tag{4.3}\\
& \mathbf{c}\left(\mathbf{G}_{1}+\mathbf{G}_{2} \mathbf{B}_{1}^{-1} \mathbf{C}\right)=\mathbf{0} \tag{4.4}
\end{align*}
$$

where

$$
\mathbf{G}_{2}=\boldsymbol{\beta}_{1} \otimes\left[\begin{array}{c}
a_{1}(1) \mathbf{v}_{1} \\
a_{1}(2) \mathbf{v}_{2} \\
\vdots \\
a_{1}(t) \mathbf{v}_{t}
\end{array}\right] .
$$

Rewriting (3.6), we give

$$
\begin{equation*}
\mathbf{y}_{1}\left[\mathbf{e}^{\prime}+\left(\mathbf{S}_{1}^{-1} \mathbf{e}^{\prime} \otimes \mathbf{S}_{2} \mathbf{e}^{\prime}\right)\right]+\mathbf{c} \mathbf{G}_{3}+\mathbf{c} \mathbf{W}^{N-1} \mathbf{G}_{2} \mathbf{B}_{1}^{-1} \mathbf{e}^{\prime}=1 \tag{4.5}
\end{equation*}
$$

where

$$
\mathbf{G}_{3}=\left[\begin{array}{c}
w_{1} \frac{\left(1-w_{1}^{N-2}\right)}{1-w_{1}} \\
w_{2} \frac{\left(1-w_{2}^{N-2}\right)}{1-w_{2}} \\
\vdots \\
w_{t} \frac{\left(1-w_{-}^{N-2)}\right.}{1-w_{t}}
\end{array}\right],
$$

and $\mathbf{W}=\operatorname{diag}\left[w_{1}, w_{2}, \ldots, w_{t}\right]$. Combining with (4.5), (4.3) and (4.4), we build up a nonhomogeneous system of linear equations.

In addition to $\mathbf{c}$, there are $t+k m$ unknowns in the equations. Total number of equations are $2 \mathrm{~km}+1$ which is greater than number of unknowns. Instead of solving it by Gauss Elimination, the solution of this problem may be obtained by using some other numerical methods. For example, we introduce the least square algorithm method to find the solution.

### 4.2. The system of linear equations

Here, we are interested in only finitely many of equations and showing an efficient solution procedure in this section as a summary of discussion previously. To our best knowledge, most of methods discussed in the queueing theory focus on solving state-balanced equations with infinite number of variables, such as matrix-geometric methods in [4] and matrix analytic methods in [1]. For solving a system of finite number of equations and variables, one may use a popular computing software MATLAB for this purpose since there is a large class of numerical methods which covers a system of simultaneous equations of this kind. Although a numerical issue is beyond the scope of this paper, an illustrative example is given in next section to compare numerical results obtained by our method and MATLAB.

Let $g=t+k m, h=2 k m+1$, and denote by $\mathbf{z} \in \mathbb{C}^{1 \times g}$ a vector consisting of unknowns. Let $\mathbf{z}=\left(\mathbf{y}_{1}, \mathbf{c}\right)$. Denote by $\mathbf{T} \in \mathbb{C}^{g \times h}$ the coefficient matrix associated with $h$ equations. Denote by $\mathbf{b} \in \mathbb{R}^{1 \times h}$ the constant vector which is composed of 2 km zeros except the last element, i.e., $\mathbf{b}=(0,0, \ldots, 0,1)$. Then we can rewrite these $h$ equations as the form:

$$
\begin{equation*}
\mathbf{z T}=\mathbf{b} . \tag{4.6}
\end{equation*}
$$

We have a system of linear nonhomogeneous equation (4.6) where the number of equations is much greater than that of unknowns. Obviously, $g<h$ if $m \geqslant 2$ and $k \geqslant 2$. Regarding the solution, the uniqueness property is guaranteed only when the steady state probability of this system exists although the system of (4.6) overdetermined. For the present model, there maybe involves a more complicated procedure needed to select the appropriate solution basis from the class of available solutions. Here, we shall make this system of equations possible to be solved by a popular numerical method, e.g., the least square algorithm introduced in [7] which minimizes $\|\mathbf{z T}-\mathbf{b}\|_{2}$. Since $\mathbf{T}$ has full row rank of $g$, there is a unique least square solution $\mathbf{z}^{*}$ and it solves the symmetric positive definite linear system:

$$
\mathbf{z T T}^{\prime}=\mathbf{b T}^{\prime},
$$

where $\mathbf{T}^{\prime}$ is the transpose of $\mathbf{T}$. Because the most widely used method for solving (4.6) is a method of normal equations, we make use of a general procedure described in [7] as follows:

1. Compute the upper triangular portion of $\mathbf{U}=\mathbf{T T}^{\prime}$.
2. Set $\mathbf{d}=\mathbf{b T}^{\prime}$.
3. Compute the Cholesky factorization $\mathbf{U}=\mathbf{H}^{\prime} \mathbf{H}$.
4. Solve $\mathbf{q H}=\mathbf{d}$ and $\mathbf{z H}^{\prime}=\mathbf{q}$.

This algorithm requires $\mathrm{O}\left(k^{3} m^{3}\right)$ flops. The compression of the $g$-by- $h$ data matrix $\mathbf{T}$ into the much smaller $g$-by- $g$ cross-product matrix $\mathbf{U}$ when $k m \gg t$ is attractive. Normally, QR factorization procedure can be used to attain more efficiency by taking the advantage of structure of $\mathbf{T}$ as well as $\mathbf{B}_{1}^{-1}$. But it is beyond our focus in this paper and it is not discussed here.

We solve boundary stationary probabilities $\mathbf{y}_{0}, \mathbf{y}_{N}$ and unboundary stationary probabilities $\mathbf{y}_{n}$, $1 \leqslant n \leqslant N-1$ by this approach. Denote by $\pi(n)$ the general-time probabilities of $n$ customers in system. $\pi(n)$ is the marginal probability of $y_{n, i, r}, r=1, \ldots, k$ and $i=1, \ldots, m$. It is natural to see that

$$
\begin{aligned}
& \pi(n)=\sum_{r, i} y_{n, r, i}=\mathbf{y}_{n} \mathbf{e}^{\prime}, \\
& \pi(0)=\mathbf{y}_{0} \mathbf{e}^{\prime} .
\end{aligned}
$$

It is easy to obtain a probability of idle time $\pi(0)$, and the blocking probability $\pi(N)$ in the system. Based on them, other performance measures, e.g. the LST of waiting time and idle time distributions are derived accordingly. However, we omit their discussion here for focusing on a vector-form solution technique. One may refer to [4] for detail.

### 4.3. A summary of the algorithm

We describe the algorithm for solving stationary probabilities of a $C_{k} / C_{m} / 1 / N$ system in the following steps:

Step 1: Solve equation $f_{1}^{*}(x) f_{2}^{*}(-x)=1$, let $x_{s}$ be a solution, $s=1, \ldots, t$.
Step 2: Compute $w_{s}, \mathbf{u}_{s}, \mathbf{v}_{s}$.

1. Compute $w_{s}$ defined in $w_{s}=f_{1}^{*}\left(x_{s}\right)$.
2. Compute $\mathbf{u}_{s}$ defined in (3.11), $\mathbf{u}_{s}=a_{1}(s) \boldsymbol{\beta}_{1}\left(\mathbf{S}_{1}-x_{s} \mathbf{I}_{1}\right)^{-1}$.
3. Compute $\mathbf{v}_{s}$ defined in (3.12), $\mathbf{v}_{s}=a_{2}(s) \boldsymbol{\beta}_{2}\left(\mathbf{S}_{2}+x_{s} \mathbf{I}_{2}\right)^{-1}$.

Step 3: Compute $\mathbf{w}_{s, n}$ defined in (3.13).
Step 4: Let $\mathbf{y}_{n}$ be a linear combination of $\mathbf{w}_{s, n}$ in (3.14).
Step 5: Set a linear nonhomogeneous system consisting of Eqs. (4.3)-(4.5).
Step 6: Use the least square algorithm to solve the linear nonhomogeneous system and obtain coefficients $c_{s}$, $s=1, \ldots, t$, and $\mathbf{y}_{1}$.

Step 7: Substituting coefficients $c_{s}, i=1, \ldots, t$ to (3.14) and obtain unboundary stationary probabilities $\mathbf{y}_{n}$, $2 \leqslant n \leqslant N-1$, and boundary stationary probabilities $\mathbf{y}_{0}$ and $\mathbf{y}_{N}$.
Step 8: Compute the system-size probability $\pi(n), n=0,1, \ldots, N$.
It is important to note that no matter how large $N$ is, we only need to solve coefficients $c_{s}, s=1, \ldots, t$. Hence the computational complexity is greatly reduced.

## 5. An example

We solve the $E_{2} / E_{2} / 1 / 4$ for illustration of our approach. The system has the following features:

$$
N=4, \quad \boldsymbol{\beta}_{1}=\boldsymbol{\beta}_{2}=(1,0), \quad \lambda_{1}=\lambda_{2}=4, \quad \mu_{1}=\mu_{2}=5 .
$$

Step 1: Solve equation $f_{1}^{*}(x) f_{2}^{*}(-x)=1$, let $x_{s}$ be a solution , $s=1,2$. We have

$$
f_{1}^{*}(x)=\left(\frac{4}{x+4}\right)^{2}, \quad f_{2}^{*}(x)=\left(\frac{5}{x+5}\right)^{2}
$$

and the solutions of $f_{1}^{*}(x) f_{2}^{*}(-x)=1$ are

$$
x_{1}=1.0000, \quad x_{2}=6.8443 .
$$

Step 2: For solutions $w_{s}, \mathbf{u}_{s}, \mathbf{v}_{s}$.

1. Compute $w_{s}$ defined in $w_{s}=f_{1}^{*}\left(x_{s}\right)$.

$$
w_{1}=f_{1}^{*}\left(x_{1}\right)=0.64 \quad \text { and } w_{2}=f_{1}^{*}\left(x_{2}\right)=0.1361 .
$$

2. Compute $\mathbf{u}_{s}$ defined in (3.11), $\mathbf{u}_{s}=a_{1}(s) \boldsymbol{\beta}_{1}\left(\mathbf{S}_{1}-x_{s} \mathbf{I}_{1}\right)^{-1}$.

$$
\mathbf{u}_{1}=(0.5555,0.44445) \quad \text { and } \quad \mathbf{u}_{2}=(0.7305,0.2695) .
$$

3. Compute $\mathbf{v}_{s}$ defined in (3.12), $\mathbf{v}_{s}=a_{2}(s) \boldsymbol{\beta}_{2}\left(\mathbf{S}_{2}+x_{s} \mathbf{I}_{2}\right)^{-1}$.

$$
\mathbf{v}_{1}=(0.4445,0.5555) \quad \text { and } \quad \mathbf{v}_{2}=(-0.5844,1.5844) .
$$

Step 3: Compute $\mathbf{w}_{s, n}$ defined in (3.13), $\mathbf{w}_{s, n}=w_{s}^{n}\left(\mathbf{u}_{s} \otimes \mathbf{v}_{s}\right), n=2,3$.
$\mathbf{w}_{1,2}=(0.1011,0.1264,0.0809,0.1011)$,
$\mathbf{w}_{1,3}=(0.0647,0.0809,0.0518,0.0647)$,
$\mathbf{w}_{2,2}=(-0.0079,0.0214,-0.0029,0.0079)$,
$\mathbf{w}_{2,3}=(-0.0011,0.0029,-0.0004,0.0011)$.
Step 4: Let $\mathbf{y}_{n}$ be a linear combination of $\mathbf{w}_{s, n}$ that is $\mathbf{y}_{n}=\sum_{s=1}^{2} c_{s} \mathbf{w}_{s, n}, c_{s} \in \mathbb{C}$.
Step 5: Set a linear nonhomogeneous system consisting of Eqs. (4.3)-(4.5).
Step 6: Use the least square algorithm to solve the linear nonhomogeneous system and obtain coefficients $c_{s}$, $s=1,2$, and $\mathbf{y}_{1}$ :
$c_{1}=0.56472, \quad c_{2}=-0.32645$,
$\mathbf{y}_{1}=(0.1073,0.0596,0.0778,0.0697)$.
Step 7: Substituting coefficients $c_{s}, s=1,2$ to (3.14) and obtain stationary probabilities $\mathbf{y}_{n}, n=0,2,3,4$.

$$
\begin{aligned}
& \mathbf{y}_{0}=(0.0747,0.1618), \\
& \mathbf{y}_{2}=(0.0592,0.0639,0.0463,0.0541), \\
& \mathbf{y}_{3}=(0.0366,0.0444,0.0291,0.0359), \\
& \mathbf{y}_{4}=(0.0128,0.0228,0.0139,0.0277) .
\end{aligned}
$$

Step 8: Compute the system-size probability $\pi(n), n=1, \ldots, N$.

$$
\begin{array}{ll}
\pi(0)=0.2365, & \pi(1)=0.3144, \\
\pi(2)=0.2235, & \pi(3)=0.1460, \\
\pi(4)=0.0772 . &
\end{array}
$$

The numerical results are reconfirmed with an existing model in a textbook [8], of which numerical solutions are obtained by MATLAB and simulation as given in the following table. The maximal absolute error between each probability value compared with that in our method is within $0.5 \%$.

|  | $\pi(0)$ | $\pi(1)$ | $\pi(2)$ | $\pi(3)$ | $\pi(4)$ |
| :--- | :--- | :--- | :--- | :--- | :--- |
| MATLAB | 0.2346 | 0.3124 | 0.2225 | 0.1479 | 0.0826 |
| Simulation | 0.2396 | 0.3149 | 0.2233 | 0.1422 | 0.0799 |

## 6. Conclusions and remarks

In this paper, we have analyzed the $C_{k} / C_{m} / 1 / N$ open queueing system containing finite number of customers. We have found properties which show that a product of the LST of interarrival and service times distributions satisfies an equation derived from unboundary states of the state balance equations. According to the state balance equations, we present that the stationary probabilities on the unboundary states can be written as a linear combination of product-forms. Each component of these products can be expressed in terms of roots of the system of equations. We introduce the least square algorithm to solve the numerical problem, which is independent of $N$, hence greatly reducing the computational complexity. It is observed in practice that the greater $N$ the better numerical precision of the solution is, although the computational complexity of the algorithm is independent of the capacity $N$. We suggest that the methods presented here may be extended to a multi-server open system containing finite number of customers with phase type interarrival and service times distributions.

## References

[1] J.Y. Le Boudec, Steady-state probabilities of the $P H / P H / 1$ queue, Queueing Syst. 3 (1988) 73-88.
[2] H. Luh, Matrix product-form solutions of stationary probabilities in tandem queues, J. Oper. Res. 42-4 (1999) 436-656.
[3] D. Bertsimas, An analytic approach to a general class of $G / G / s$ queueing systems, Oper. Res. 38 (1990) 139-155.
[4] M.F. Neuts, Matrix-Geometric Solutions in Stochastic Models, The John Hopkins University Press, 1981.
[5] A. Van De Liefvoort, The waiting-time distribution and its moments of the $P H / P H / 1$ queue, Oper. Res. Lett. 9 (1990) 261-269.
[6] R. Bellman, Introduction to Matrix Analysis, McGraw-Hill, London, 1960.
[7] G.H. Golub, G.H. Van Loan, Matrix-Computations, The John Hopkins University Press, 1989.
[8] D. Gross, C.M. Harris, Fundamentals of Queueing Theory, John Wiley \& Sons, NY, 1998.


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