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PARABOLA METHOD IN ORDINARY DIFFERENTIAL EQUATION

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Abstract. In this paper we used the method of parabola approximation to study some nonlinear differential equations. We derive exact, explicit solutions to the parabolic equations and use this analytical results in the numerical computations for the general equations. We then draw the comparison of between the solutions of original and approximated equations. Moreover, we apply such method to the population growth problem. The error of the difference between the solutions of the differential equations and the numerical results caused by the discrete approximations is reasonable.

1. Introduction

Consider the general differential equations

$$\frac{du}{dt} = f(t,u), \ u(0) = u_0.$$

The parabola approximation method is to approximate the function $f(t,u)$ through the second-order Taylor expansion.

By the Taylor's theory

$$f(t,u) \sim \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{\partial}{\partial t} t + \frac{\partial}{\partial u} u \right)^n f \big|_{t=t_0, u=u_0} (t,u)$$

where $\left(\frac{\partial}{\partial t} t + \frac{\partial}{\partial u} u\right)^n f \big|_{t=t_0,u=u_0} (t,u)$ denotes the binomial expansion at $t = t_0$, $u = u_0$,

$$\sum_{k=0}^{n} \frac{\partial^n f(t,u)}{\partial t^k \partial u^{n-k}} \big|_{t=t_0,u=u_0} (t-t_0)^k (u-u_0)^{n-k}.$$

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We study the second-order approximation of the problem $\frac{du}{dt} = f(t,u)$ as the following approximation problem

$$\frac{dv(t)}{dt} = A(v(t)-u_0)^2 + B(t)(v(t)-u_0) + C(t), v(t_0) = u(t_0),$$

where

$$A = \tfrac{1}{2} f_{uu}(t_0,u_0), \quad B(t) = f_{t,u}(t_0,u_0)(t-t_0) + f_u(t_0,u_0),$$

$$C(t) = \tfrac{1}{2} f_{tt}(t_0,u_0)(t-t_0)^2 + f_t(t_0,u_0)(t-t_0) + f(t_0,u_0).$$

To illustrate this, we consider in the following examples the cases of $f(t,u)$ is $f(u)$; and $f(x) = \sin x,\ \tan x,\ \sec x.$

Example 1. We consider the problem $\frac{dv(t)}{dt} = \sin v,\ v(0) = v_0$ having the solution $v(t) = \cos^{-1}\left(\frac{\cos v_0 + 1 - (1-\cos v_0)e^{2t}}{\cos v_0 + 1 + (1-\cos v_0)e^{2t}}\right)$. The associated approximate equation $\frac{d\bar{v}(t)}{dt} = \bar{v}(t) - \frac{1}{6}\bar{v}(t)^3,\ \ \bar{v}(0) = v_0$, has the solution $\bar{v}(t) = \frac{\sqrt{6}v_0 e^t}{\sqrt{6+v_0^2(e^{2t}-1)}}$. The graphs of v and $\bar{v}$ are very closed in the neighborhood of $(0.1,0) = (v_0,t_0)$. The expansion of these two functions in the neighborhood of $(0.1,0) = (v_0,t_0)$, are

$$v(t) = \frac{1}{2}\pi - \sin^{-1}(\cos v_0) + |\sin v_0|\left(t + \frac{t^2}{2}\cos v_0\right) + O\left(t^3\right),$$

$$\bar{v}(t) = v_0 + tv_0\left(1 - \frac{1}{6}v_0^2\right) + t^2 v_0\left(\frac{1}{2} - \frac{1}{3}v_0^2 + \frac{1}{24}v_0^4\right) + O\left(t^3\right);$$

it is also clear that v and $\bar{v}$ are very closed for (t,v_0) near $(0,0)$.

Example 2. We consider the problem $\frac{dv(t)}{dt} = \tan v,\ v(0) = v_0$, having the solution $2v(t) = \cos^{-1}\left(1 - (1-\cos 2v_0)e^{2t}\right)$. We treat the equation $\frac{d\bar{v}(t)}{dt} = \bar{v}(t) + \frac{1}{3}\bar{v}(t)^3, \bar{v}(t) = v_0$, having the positive solution $2\ln\bar{v}(t) - \ln\left(\bar{v}(t)^2 + 3\right) = 2\ln v_0 - \ln\left(v_0^2+3\right) + 2t$. We have seen the graphs of v and $\bar{v}$ are very closed in the neighborhood of $(0.1,0) = (v_0,t_0)$, and can see that the expansion of these two functions in the neighborhood of $(0.1,0) = (v_0,t_0)$,

$$t = -\tfrac{1}{2}\ln(1-\cos 2v_0) - 1.9577 + 9.9666(v(t)-0.1)$$
$$- 50.167(v(t)-0.1)^2 + O\left((v(t)-0.1)^3\right),$$

$$t = -\ln v_0 + \tfrac{1}{2}\ln\left(v_0^2+3\right) - 2.8536 + 9.9668(\bar{v}(t)-0.1)$$
$$- 50.165(\bar{v}(t)-0.1)^2 + 333.34 O(\bar{v}(t)-0.1)^3.$$

Example 3. We consider the problem $\frac{dv(t)}{dt} = \sec v,\ v(0) = v_0$,having the solution $\sin v(t) = \sin v_0 + t,\ v(t) = \sin^{-1}(\sin v_0 + t)$. We treat the equation $\frac{d\bar{v}(t)}{dt} = 1 + \frac{1}{2}\bar{v}(t)^2, \bar{v}(0) = v_0$, having the solution

$$\bar{v}(t) = \sqrt{2}\tan\left(\tan^{-1}\left(\tfrac{v_0}{\sqrt{2}}\right) + \tfrac{t}{\sqrt{2}}\right).$$

We have seen the graphs of v and $\bar{v}$ are very closed in the neighborhood of $(0,0) = (v_0, t_0)$, and can see that the expansion of these two functions in the neighborhood of $(0,0) = (v_0, t_0)$,

$$v(t) = v_0 + \tfrac{t}{\cos v_0} + \tfrac{1}{2}\tfrac{\sin v_0}{\cos^3 v_0}t^2 + O\left(t^3\right),$$

$$\bar{v}(t) = v_0 + \left(1 + \tfrac{1}{2}v_0^2\right)t + \tfrac{v_0}{4}\left(2 + v_0^2\right)t^2 + O\left(t^3\right).$$

In real applications, from the experimental data, the system of $t,\ u(t)$ and $f(t, u)$ are usually very dynamic and nonlinear, which make it difficult to understand the properties of a targetted object. In this article, we try to propose a computational procedure to estimate the solutions of the population problem.

Our computational procedure depends on the exact solution formula for the parabolic equations. For this, we will set-up some fundamental lemmas in Section 2. In Section 3, we study a special model equation for population. Concluding remarks are given in Section 4.

2. Fundamental Lemmas

The following lemmas consider the parabolic differential equation with given three points values y_i at time t_i, $i = 0,\ 1,\ 2$,

$$\begin{cases} \dfrac{dy(t)}{dt} = Ay(t)^2 + By(t) + C, \\ y(t_0) = y_0, y(t_1) = y_1, y(t_2) = y_2, t_0 < t_1 < t_2, \end{cases} \tag{2}$$

Lemma 2.1 *The differential equation (2)with* $y_0 \leq y_1 \leq y_2, \delta = B^2 - 4AC$ *can be solved as the following:*

$(I - i)$ *for* $\delta > 0$,

$$y(t) = x^1 + \frac{\sqrt{\delta}}{A}\frac{1}{1 - \frac{y_0 - x^2}{y_0 - x^1}e^{(x^2 - x^1)A(t - t_0)}};$$

$(I - ii)$ *for* $\delta = 0$,

$$y(t) = x^1 - \frac{1}{A}\frac{1}{t - t_0 - A^{-1}(y_0 - x^1)^{-1}};$$

$(I-iii)$ *for* $\delta<0, k=\frac{\sqrt{-\delta}}{2A}$,

$$y(t)=-\frac{B}{2A}+k\tan\left(Ak(t-t_0)+\tan^{-1}\frac{y(t_0)+\frac{B}{2A}}{k}\right).$$

Proof of lemma 2.1.

(I-i) For $\delta>0$, $\frac{dy}{dt}=A\left(y-x^1\right)\left(y-x^2\right)$, we obtain that

$$\ln\left|\frac{y(t)-x^2}{y(t)-x^1}\right|=\ln\left|\frac{y_0-x^2}{y_0-x^1}\right|+\sqrt{\delta}\,(t-t_0)\,;$$

therefore

$$y(t)=x^1+\left(x^2-x^1\right)\frac{1}{1-\frac{y_0-x^2}{y_0-x^1}\exp\left(\sqrt{\delta}\,(t-t_0)\right)}.$$

(I-ii) For $\delta=0$, $\frac{dy}{dt}=A\left(y-x^1\right)^2$, we have $\frac{1}{y(t)-x^1}=\frac{1}{y_0-x^1}-A(t-t_0)\,;$ therefore

$$y(t)=x^1+\frac{1}{\frac{1}{y_0-x^1}-A(t-t_0)}.$$

And this solution can be obtained by the limiting processing

$$\lim_{x^2\to x^1}y_{x^2}(t)=x^1+\lim_{x^2\to x^1}\frac{1}{\left(\frac{1}{y_0-x^1}-A\frac{y_0-x^2}{y_0-x^1}(t-t_0)\right)\exp\left(A\left(x^2-x^1\right)(t-t_0)\right)}$$

$$=x^1+\frac{1}{\frac{1}{y_0-x^1}-A(t-t_0)}.$$

(I-iii) For $\delta<0$, $\frac{dy}{dt}=A\left(y-x^1\right)\left(y-x^2\right)$, we conclude that

$$Ak(t-t_0)=\tan^{-1}\frac{y(t)+\frac{B}{2A}}{k}-\tan^{-1}\frac{y_0+\frac{B}{2A}}{k},$$

$$y(t)=-\frac{B}{2A}+k\tan\left(Ak(t-t_0)+\tan^{-1}\frac{y_0+\frac{B}{2A}}{k}\right)$$

$$=-\frac{B}{2A}+\frac{\sqrt{-\delta}}{2A}\tan\left(\frac{\sqrt{-\delta}}{2}(t-t_0)+\tan^{-1}\frac{2Ay_0+B}{\sqrt{-\delta}}\right). \qquad \blacksquare$$

Remark 2.1. This lemma will be used in Section 3 for the computations of every three population data obtained from the Ministry of Interior Taiwan between the following periods

(i) For female: $1953-55, 1957-57, 1967-69, 1968-70, 1972-74, 1973-75, 1974-76, 1986-88, 1989-91, 1994-96, 1998-00$;

(ii) For male: $1953-55, 1954-56, 1957-59, 1958-60, 1967-69, 1968-70, 1972-74, 1973-75, 1974-76, 1975-77, 1978-80, 1986-88, 1987-89, 1989-91, 1992-94, 1994-96, 1995-97, 1998-00, 1999-01, 02-04$

Lemma 2.2. *The differential equation* (2) *with* $y_0 \leq y_2 \leq y_1$ *can be solved as the following*

$(II-i)$ *for* $\delta > 0$,

$$y(t) = x^1 + \frac{\sqrt{\delta}}{A}\frac{1}{1-\frac{y_0-x^2}{y_0-x^1}e^{(x^2-x^1)A(t-t_0)}} for \quad t \in [t_0, t_1],$$

$$y(t) = x^1 + \frac{\sqrt{\delta}}{A}\frac{1}{1-\frac{y_1-x^2}{y_1-x^1}e^{-(x^2-x^1)A(t-t_1)}} for \quad t \in [t_1, t_2];$$

$(II-ii)$ *for* $\delta = 0$,

$$y(t) = x^1 - \frac{1}{A}\frac{1}{t-t_0-A^{-1}(y_0-x^1)^{-1}} \quad for \quad t \in [t_0, t_1],$$

$$y(t) = x^1 + \frac{1}{A}\frac{1}{(t-t_1)+A^{-1}(y_1-x^1)^{-1}} \quad for \quad t \in [t_1, t_2];$$

$(II-iii)$ *for* $B^2 - 4AC < 0, k = \frac{\sqrt{-\delta}}{2A}$,

$$y(t) = -\frac{B}{2A} + k\tan\left(Ak(t-t_0) + \tan^{-1}\frac{y(t_0)+\frac{B}{2A}}{k}\right) \quad for \quad t \in [t_0, t_1],$$

$$y(t) = -\frac{B}{2A} + k\tan\left(-Ak(t-t_1) + \tan^{-1}\frac{y_1+\frac{B}{2A}}{k}\right) \quad for \quad t \in [t_1, t_2].$$

Proof of lemma 2.2. (II-i) For $\delta > 0$, $\frac{dy}{dt} = A\left(y-x^1\right)\left(y-x^2\right)$, then we have for $t \in [t_0, t_1]$,

$$y(t) = x^1 + \left(x^2 - x^1\right)\frac{1}{1-\frac{y_0-x^2}{y_0-x^1}\exp\left(\sqrt{\delta}(t-t_0)\right)}.$$

Also we obtain

$$y_1 = x^1 + \left(x^2 - x^1\right)\frac{1}{1-\frac{y_0-x^2}{y_0-x^1}\exp\left(\sqrt{\delta}(t_1-t_0)\right)},$$

$$t_1 = t_0 + \frac{1}{\sqrt{\delta}}\ln\left(\frac{y_1-x^2}{y_1-x^1}\frac{y_0-x^1}{y_0-x^2}\right).$$

For $t \in [t_1, t_2]$, we obtain that

$$\frac{y(t) - x^2}{y(t) - x^1} = \frac{y_1 - x^2}{y_1 - x^1} e^{-\sqrt{\delta}(t - t_1)};$$

therefore

$$y(t) = x^1 + \left(x^2 - x^1\right) \frac{1}{1 - \frac{y_1 - x^2}{y_1 - x^1} \exp\left(-\sqrt{\delta}(t - t_1)\right)}.$$

Also,

$$\frac{y_2 - x^1}{x^2 - x^1} = \frac{1}{1 - \frac{y_1 - x^2}{y_1 - x^1} \exp\left(-\sqrt{\delta}(t_2 - t_1)\right)},$$

$$t_2 = t_1 - \frac{1}{\sqrt{\delta}} \ln\left(\frac{y_2 - x^2}{y_2 - x^1} \frac{y_1 - x^1}{y_1 - x^2}\right).$$

(II-ii) For $\delta = 0$, $t \in [t_0, t_1]$, $\frac{dy}{dt} = A\left(y - x^1\right)^2$, then we get that

$$y(t) = x^1 + \frac{y_0 - x^1}{1 - A(y_0 - x^1)(t - t_0)}.$$

Also we have

$$\frac{1}{y_1 - x^1} = \frac{1}{y_0 - x^1} - A(t_1 - t_0), \ t_1 = t_0 + \frac{y_1 - y_0}{A(y_1 - x^1)(y_0 - x^1)}.$$

For $t \in [t_1, t_2]$,

$$\frac{1}{y(t) - x^1} = \frac{1}{y_1 - x^1} + A(t - t_1), y(t) = x^1 + \frac{1}{\frac{1}{y_1 - x^1} + A(t - t_1)};$$

therefore

$$y_2 = x^1 + \frac{1}{\frac{1}{y_1 - x^1} + A(t_2 - t_1)}, t_2 = t_1 + \frac{y_1 - y_2}{A(y_2 - x^1)(y_1 - x^1)}.$$

(II-iii) For $\delta < 0$, $t \in [t_0, t_1]$, then we conclude that

$$Ak(t - t_0) = \tan^{-1} \frac{y(t) + \frac{B}{2A}}{k} - \tan^{-1} \frac{y(t_0) + \frac{B}{2A}}{k},$$

$$y(t) = -\frac{B}{2A} + \frac{\sqrt{-\delta}}{2A} \tan\left(\frac{\sqrt{-\delta}}{2}(t - t_0) + \tan^{-1} \frac{2Ay(t_0) + B}{\sqrt{-\delta}}\right).$$

And

$$t_1 = t_0 + \frac{2}{\sqrt{-\delta}} \left(\tan^{-1} \frac{2Ay_1 + B}{\sqrt{-\delta}} - \tan^{-1} \frac{2Ay_0 + B}{\sqrt{-\delta}} \right),$$

$$y_1 = -\frac{B}{2A} + \frac{\sqrt{-\delta}}{2A} \tan \left(\frac{\sqrt{-\delta}}{2} (t_1 - t_0) + \tan^{-1} \frac{2Ay_0 + B}{\sqrt{-\delta}} \right).$$

For $t \in [t_1, t_2]$, then

$$\int_{y(t_1)}^{y(t)} \frac{1}{\left(r + \frac{B}{2A}\right)^2 + k^2} dr = -A (t - t_1),$$

$$-Ak (t - t_1) = \tan^{-1} \frac{y(t) + \frac{B}{2A}}{k} - \tan^{-1} \frac{y_1 + \frac{B}{2A}}{k};$$

therefore

$$y(t) = -\frac{B}{2A} + \frac{\sqrt{-\delta}}{2A} \tan \left(-\frac{\sqrt{-\delta}}{2} (t - t_1) + \tan^{-1} \frac{2Ay_1 + B}{\sqrt{-\delta}} \right).$$

Also

$$t_2 = t_1 - \frac{2}{\sqrt{-\delta}} \left(\tan^{-1} \frac{2Ay_2 + B}{\sqrt{-\delta}} - \tan^{-1} \frac{2Ay_1 + B}{\sqrt{-\delta}} \right),$$

$$y_2 = -\frac{B}{2A} + \frac{\sqrt{-\delta}}{2A} \tan \left(-\frac{\sqrt{-\delta}}{2} (t_2 - t_1) + \tan^{-1} \frac{2Ay_1 + B}{\sqrt{-\delta}} \right). \quad \blacksquare$$

Remark 2.2. This lemma will be used for the computation of every three population data obtained from the Ministry of Interior Taiwan between the following periods

(i) For female: $1958 - 60, 1961 - 63, 1975 - 77, 1978 - 80, 1987 - 89, 1992 - 94, 1995 - 97, 1999 - 01, 02 - 04$.

(ii) For male: $1954 - 56, 1958 - 60, 1968 - 70, 1975 - 77, 1978 - 80, 1987 - 89, 1992 - 94, 1995 - 97, 1999 - 01, 02 - 04$.

Similar to the above proof of Lemma 2.2 we can obtain the following Lemmas; we omit the similar arguments for their proofs.

Lemma 2.3. *The differential equation* (2) *with* $y_1 \leq y_2 \leq y_0$ *can be solved as the following*

$(III-i)$ *for* $\delta>0,$

$$y\left(t\right)=x^{1}+\frac{\sqrt{\delta}}{A}\frac{1}{1-\frac{y_{0}-x^{2}}{y_{0}-x^{1}}e^{-\left(x^{2}-x^{1}\right)A\left(t-t_{0}\right)}}\quad for\quad t\in\left[t_{0},t_{1}\right],$$

$$y\left(t\right)=x^{1}+\frac{\sqrt{\delta}}{A}\frac{1}{1-\frac{y_{1}-x^{2}}{y_{1}-x^{1}}e^{\left(x^{2}-x^{1}\right)A\left(t-t_{1}\right)}}\quad for\quad t\in\left[t_{1},t_{2}\right];$$

$(III-ii)$ *for* $\delta=0,$

$$y\left(t\right)=x^{1}+\frac{1}{A}\frac{1}{t-t_{0}+A^{-1}\left(y_{0}-x^{1}\right)^{-1}}\quad for\quad t\in\left[t_{0},t_{1}\right],$$

$$y\left(t\right)=x^{1}-\frac{1}{A}\frac{1}{t-t_{0}-A^{-1}\left(y_{1}-x^{1}\right)^{-1}}\quad for\quad t\in\left[t_{1},t_{2}\right];$$

$(III-iii)$ *for* $\delta<0,k=\frac{\sqrt{-\delta}}{2A},$

$$y\left(t\right)=-\frac{B}{2A}+k\tan\left(-Ak\left(t-t_{0}\right)+\tan^{-1}\frac{y\left(t_{0}\right)+\frac{B}{2A}}{k}\right)\quad for\quad t\in\left[t_{0},t_{1}\right],$$

$$y\left(t\right)=-\frac{B}{2A}+k\tan\left(Ak\left(t-t_{1}\right)+\tan^{-1}\frac{y\left(t_{1}\right)+\frac{B}{2A}}{k}\right)\quad for\quad t\in\left[t_{1},t_{2}\right].$$

Remark 2.3. This lemma will be used to compute every three population data obtained from the Ministry of Interior Taiwan between the following periods

(i) For female: $1952-54, 1956-58, 1966-68, 1971-73, 1977-79, 1988-90, 1991-93, 1993-95, 1997-99, 01-03.$

(ii) For male: $1952-54, 1956-58, 1966-68, 1971-73, 1977-79, 1981-83, 1988-90, 1991-93, 1997-99, 01-03$

Lemma 2.4. *The differential equation* (2) *with* $y_1\leq y_0\leq y_2$ *can be solved as the following*

$(IV-i)$ *for* $\delta>0,$

$$y\left(t\right)=x^{1}+\frac{\sqrt{\delta}}{A}\frac{1}{1-\frac{y_{0}-x^{2}}{y_{0}-x^{1}}e^{-\left(x^{2}-x^{1}\right)A\left(t-t_{0}\right)}}\quad for\quad t\in\left[t_{0},t_{1}\right],$$

$$y\left(t\right)=x^{1}+\frac{\sqrt{\delta}}{A}\frac{1}{1-\frac{y_{1}-x^{2}}{y_{1}-x^{1}}e^{\left(x^{2}-x^{1}\right)A\left(t-t_{1}\right)}}\quad for\quad t\in\left[t_{1},t_{2}\right];$$

$(IV - ii)$ *for* $\delta = 0$,

$$y(t) = x^1 + \frac{1}{A}\frac{1}{t - t_0 + A^{-1}(y_0 - x^1)^{-1}} \quad for \quad t \in [t_0, t_1],$$

$$y(t) = x^1 - \frac{1}{A}\frac{1}{t - t_1 - A^{-1}(y_1 - x^1)^{-1}} \quad for \quad t \in [t_1, t_2];$$

$(IV - iii)$ *for* $\delta < 0, k = \frac{\sqrt{-\delta}}{2A}$,

$$y(t) = -\frac{B}{2A} + k\tan\left(-Ak(t - t_0) + \tan^{-1}\frac{y(t_0) + \frac{B}{2A}}{k}\right) \quad for \quad t \in [t_0, t_1],$$

$$y(t) = -\frac{B}{2A} + k\tan\left(Ak(t - t_1) + \tan^{-1}\frac{y(t_1) + \frac{B}{2A}}{k}\right) \quad for \quad t \in [t_1, t_2].$$

Remark 2.4. This lemma will be used for computing every three population data obtained from the Ministry of Interior Taiwan between the following periods

(i) For female: $1964 - 66, 1985 - 87, 03 - 05$.

(ii) For male: $1964 - 66, 1985 - 87, 1993 - 95, 03 - 05$.

Lemma 2.5. *The differential equation* (2) *with* $y_2 \le y_1 \le y_0$ *can be solved as the following*

$(V - i)$ *for* $\delta > 0$,

$$y(t) = x^1 + \frac{\sqrt{\delta}}{A}\frac{1}{1 - \frac{y_0 - x^2}{y_0 - x^1}e^{-(x^2 - x^1)A(t - t_0)}};$$

$(V - ii)$ *for* $\delta = 0$,

$$y(t) = x^1 + \frac{1}{A}\frac{1}{t - t_0 + A^{-1}(y_0 - x^1)^{-1}};$$

$(V - iii)$ *for* $\delta < 0, k = \frac{\sqrt{-\delta}}{2A}$,

$$y(t) = -\frac{B}{2A} + k\tan\left(-Ak(t - t_0) + \tan^{-1}\frac{y(t_0) + \frac{B}{2A}}{k}\right).$$

Remark 2.5. This lemma will be used for the computation of every three population data obtained from the Ministry of Interior Taiwan between the following periods

(i) For female: $1955-57, 1959-61, 1962-64, 1963-65, 1970-72, 1976-78, 1979-81, 1980-82, 1981-83, 1982-84, 1983-85, 84-86, 96-98, 00-02$.

(ii) For male: $1955-57, 59-61, 60-62, 61-63, 62-64, 63-65, 70-72, 76-78, 79-81, 80-82, 82-84, 83-85, 84-86, 96-98, 00-02$.

Lemma 2.6. *The differential equation* (2) *with* $y_2 \leq y_0 \leq y_1$ *can be solved as the following*

$(VI-i)$ *for* $\delta > 0$,

$$y(t) = x^1 + \frac{\sqrt{\delta}}{A}\frac{1}{1-\frac{y_0-x^2}{y_0-x^1}e^{(x^2-x^1)A(t-t_0)}} \quad for \quad t \in [t_0, t_1],$$

$$y(t) = x^1 + \frac{\sqrt{\delta}}{A}\frac{1}{1-\frac{y_1-x^2}{y_1-x^1}e^{-(x^2-x^1)A(t-t_1)}} \quad for \quad t \in [t_1, t_2];$$

$(VI-ii)$ *for* $\delta = 0$,

$$y(t) = x^1 - \frac{1}{A}\frac{1}{t-t_0-A^{-1}(y_0-x^1)^{-1}} \quad for \quad t \in [t_0, t_1],$$

$$y(t) = x^1 + \frac{1}{A}\frac{1}{t-t_0+A^{-1}(y_0-x^1)^{-1}} \quad for \quad t \in [t_1, t_2];$$

$(VI-iii)$ *for* $\delta < 0, k = \frac{\sqrt{-\delta}}{2A}$,

$$y(t) = -\frac{B}{2A} + k\tan\left(Ak(t-t_0) + \tan^{-1}\frac{y(t_0)+\frac{B}{2A}}{k}\right) \quad for \quad t \in [t_0, t_1],$$

$$y(t) = -\frac{B}{2A} + k\tan\left(-Ak(t-t_1) + \tan^{-1}\frac{y(t_1)+\frac{B}{2A}}{k}\right) \quad for \quad t \in [t_1, t_2].$$

Remark 2.6. This lemma will be used to compute every three population data obtained from the Ministry of Interior Taiwan between the following periods

(i) For female: $1965-67, 1969-71, 1990-92$.

(ii) For male: $1965-67, 1969-71, 1990-92$.

In the next section we want to discuss some models using the Parabola method.

As mentioned at the beginning, we approximate the differential equation $\frac{du}{dt} = f(t, u)$ by the following equation

$$\frac{dv(t)}{dt} = A(v(t) - u_0)^2 + B(t)(v(t) - u_0) + C(t), v(t_0) = u(t_0),$$
$$A = \tfrac{1}{2} f_{uu}(t_0, u_0), \quad B(t) = f_{t,u}(t_0, u_0)(t - t_0) + f_u(t_0, u_0),$$
$$C(t) = \tfrac{1}{2} f_{tt}(t_0, u_0)(t - t_0)^2 + f_t(t_0, u_0)(t - t_0) + f(t_0, u_0).$$

3. Special Population Model

We denote by:

$$b(t) = t - th \text{ year birth population},$$
$$\frac{db(t)}{dt}/b(t) := \text{birth rate} = bir(t),$$
$$dbir(t)/dt := \text{birth speed-up}.$$

For convenience, we denote $dbir(t)/dt$ by $dbir(t)$ in graph. Using the parabolic approximation curve partition scoring we will study the population growth problem from 1952 to 2005 in Taiwan and obtain some properties on the birth rate, population and a model between Birth rate and the Population.

From the population data obtained from the Ministry of Interior Taiwan and through the following substitution

$$\frac{db(t)}{dt} := b(t+1) - b(t),$$
$$dbir(t) := \frac{dbir(t)}{dt} = bir(t+1) - bir(t)$$
$$= \frac{db(t+1)}{dt}/b(t+1) - \frac{db(t)}{dt}/b(t)$$
$$= \frac{b(t+2) - b(t+1)}{b(t+1)} - \frac{b(t+1) - b(t)}{b(t)},$$

we can make the following graphs through plotting by using Maple shown as Fig. 1.

Consider the relation between $\frac{dbir(t)}{dt}$ and $bir(t)$ for $bir(t)$ lies on $[-0.12, -0.05]$ the above graph can be shown as Fig. 2.

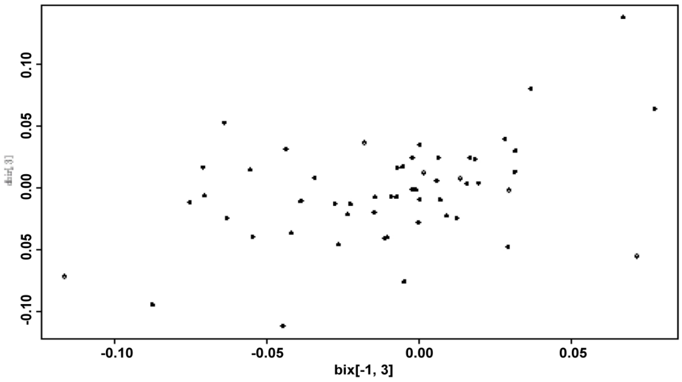

Fig. 1. Graph of dbir(t)-bir(t)-1.

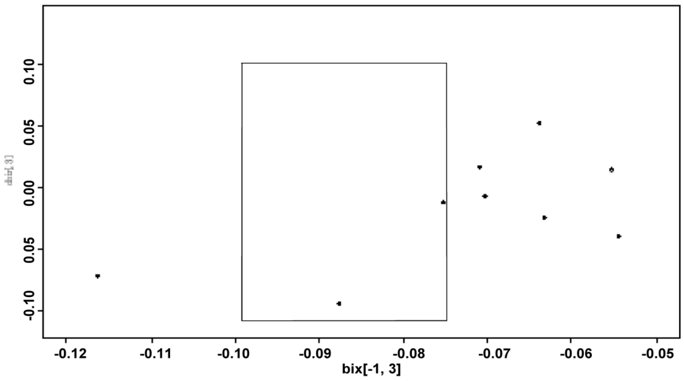

Fig. 2. Graph of dbir(t)-bir(t)-2.

We think that there should exist some reasonable reglues in such a social sciences and we introduce the method proposed in Section 3 a quadratic model to consist with the Graph of dbir(t)-bir(t)-2 given above and every three-pointwisely divide the graph dbir(t)-bir(t)-1 into several subgraphs as follows for $v(t) = bir(t)$ and $a(t) = \frac{dv(t)}{dt}$,

$$a(t) = \frac{dv(t)}{dt} = A_j v(t)^2 + B_j v(t) + C_j \quad for \quad v(t) \in I_j ,$$

where $A_j, B_j, C_j,\ j = 0, 1, \cdots, 15$ are constants and $I_0 = [-0.12, -0.07]$,

$I_1 = [-0.072, -0.06]$, $I_2 = [-0.066, -0.052]$, $I_3 = [-0.048, -0.042]$,

$I_4 = [-0.04, -0.025]$, $I_5 = [-0.0225, -0.02]$, $I_6 = [-0.02, -0.01]$,

$I_7 = [-0.013, -0.015]$, $I_8 = [-0.015, 0]$, $I_9 = [0, 0.0016]$, $I_{10} = [0.0016, 0.005]$,

$I_{11}=[0.005, 0.0133]$, $I_{12}=[0.0133, 0.0166]$, $I_{13}=[0.0166, 0.02]$, $I_{14}=[0.02, 0.03]$, $I_{15}=[0.03, 0.04]$. That is,

$$(3.1)\qquad a(t)=\begin{cases} A_0 v(t)^2+B_0 v(t)+C_0 \quad for \quad v(t)\in[-0.12,-0.07], \\ A_1 v(t)^2+B_1 v(t)+C_1 \quad for\ v(t)\in[-0.072,-0.06], \\ A_2 v(t)^2+B_2 v(t)+C_2 \quad for \quad v(t)\in[-0.066,-0.052], \end{cases}$$

$$(3.2)\qquad a(t)=\begin{cases} A_3 v(t)^2+B_3 v(t)+C_3 \quad for \quad v(t)\in[-0.048,-0.042], \\ A_4 v(t)^2+B_4 v(t)+C_4 \quad for\ v(t)\in[-0.04,-0.025], \\ A_5 v(t)^2+B_5 v(t)+C_5 \quad for \quad v(t)\in[-0.0225,-0.02], \\ A_6 v(t)^2+B_6 v(t)+C_6 \quad for\ v(t)\in[-0.02,-0.01], \\ A_7 v(t)^2+B_7 v(t)+C_7 \quad for\ v(t)\in[-0.013,-0.015], \\ A_8 v(t)^2+B_8 v(t)+C_8 \quad for\ v(t)\in[-0.015,0], \end{cases}$$

$$(4.3)\qquad a(t)=\begin{cases} A_9 v(t)^2+B_9 v(t)+C_9 \quad for\ v(t)\in[0,0.0016], \\ A_{10} v(t)^2+B_{10} v(t)+C_{10} \quad for\ v(t)\in[0.0016,0.005], \\ A_{11} v(t)^2+B_{11} v(t)+C_{11} \quad for \quad v(t)\in[0.005,0.0133], \\ A_{12} v(t)^2+B_{12} v(t)+C_{12} \quad for\ v(t)\in[0.0133,0.0166], \\ A_{13} v(t)^2+B_{13} v(t)+C_{13} \quad for\ v(t)\in[0.0166,0.02], \\ A_{14} v(t)^2+B_{14} v(t)+C_{14} \quad for\ v(t)\in[0.02,0.03], \\ A_{15} v(t)^2+B_{15} v(t)+C_{15} \quad for\ v(t)\in[0.03,0.04]. \end{cases}$$

Where $A_j, B_j, C_j,\ j=0,1,\cdots,15$ are constants. From the above equations we propose a rough approximate model as the following simple continuous type

$$(3.4)\qquad \begin{aligned} &\frac{dv(t)}{dt}=a(t)=A(t)\,v(t)^2+B(t)\,v(t)+C(t),\ v(t_0)=v_0, \\ &v(t)=\tfrac{db(t)}{dt}/b(t),\, b(t_0)=b_0,\, b(t_1)=b_1,\, v(t_1)=v_1,\, b(t_2)=b_2,\ v(t_2)=v_2, \end{aligned}$$

where $b(t)=t-th$ year birth population, $v(t):=bir(t)$ birth increasing rate, $dbir(t)/dt:=$ birth speed-up. The existence of solution of (3.4) can be got by the standard arguments.

To study the property of birth population we use the lemmas $2.1\sim 2.6$ in Section 2 to solve the function $v(t)=bir(t)=\frac{db(t)}{dt}/b(t)$ in those small time intervals and obtain the population function $b(t)$ (named "Estimated number" for forward difference method and "theoretical computational results"for backward difference method) by taking integration on $v(t)$ with respect to t, then take the square mean every three points except the first, second, last two and last (2003) years and than we obtain the results through using the forward difference method, according to the official Annals $\frac{db(t)}{dt}$ is instated by $b(t+1) - b(t)$, we obtain the result as shown below

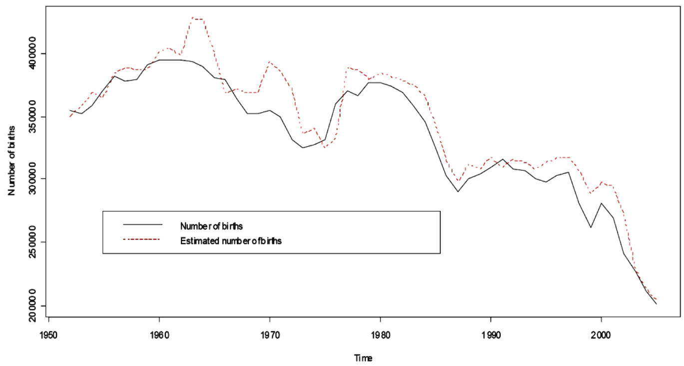

Fig. 3.

with errors

$$\frac{1}{54}\sum_{i=1}^{54}\left|\frac{B_i(t) - b_i(t)}{b_i(t)}\right| \sim 0.04266164 \sim 4.3\%,$$

$$\frac{1}{54}\sqrt{\sum\left(\frac{B_i(t) - b_i(t)}{b_i(t)}\right)^2} \sim 0.007299726 \sim 0.73\%.$$

Through the backward difference method, according to the official Annals $\frac{db(t)}{dt}$ is instated by $b(t) - b(t-1)$, and as the same above computation method we obtain the graph as below

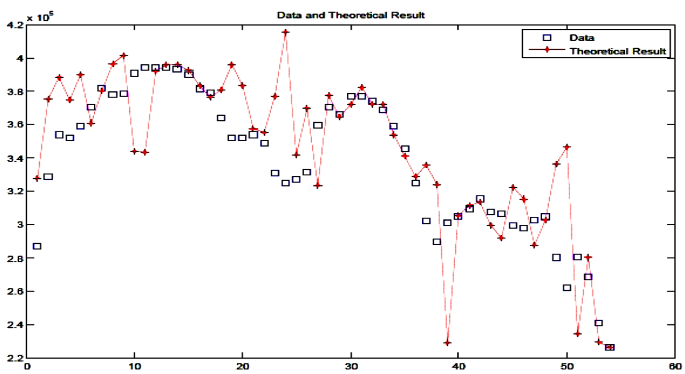

Fig. 4.

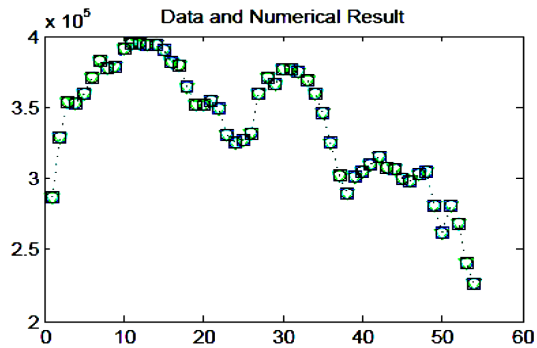

Fig. 5.

where the number 0 in x- axis represents the year 1952,with errors of case 1

$$\frac{1}{54}\sum_{i=1}^{54}\left|\frac{\hat{b}_i(t)-b_i(t)}{b_i(t)}\right| \sim 6.5837381846382\%,$$

$$\frac{1}{54}\sqrt{\sum\left(\frac{\hat{b}_i(t)-b_i(t)}{b_i(t)}\right)^2} \sim 0.0422277256212\%\,.$$

4. Conclusions

We compare these two methods-forward and backward differences-together and it show the results that If we could delete the problematic four data caused by some unregulated statistical methods on population, then through the forward method we can obtain better estimate with errors 4.27% and 0.73% in the sense of mean and square mean respectively; and 6.58% and 0.042% in the same situation through the backward difference method.

There were historical survey on the related topics, for example, Lee-Carter model for the rate of Mortality, APC model for ... , etc.

These errors result from

(i) the computational method and

(ii) the large disparity between the difference equation and differential equation when the dynamics and nonlinearity are strong.

We plan to establish new methodology to deal such nonlinear problem in the future.

The problem (3.4) for population can not be solved easily, and from the experimental point of view (at least from the data at Ministry of Interior Taiwan)

$A(t)\sim a_{1,i}t^2+b_{1,i}t+c_{1,i}$, $B(t)\sim a_{2,i}t^2+b_{2,i}t+c_{2,i}, C(t)\sim a_{3,i}t^2+b_{3,i}t+c_{3,i},$

for $t \in J_i$, J_i are some time-intervals and $a_{j,i} b_{j,i}, c_{j,i}$ are constants we will compute these constants later. We have tried to use our methods applied in [1-19] to solve this equation (3.4), but till now do not yet have definite results.

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