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EXISTENCE AND UNIQUENESS OF SOLUTIONS OF QUASILINEAR WAVE EQUATIONS (II)

BY

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Abstract

In this work we prove a result concernig the existence and uniqueness of solutions of quasilinear wave equation and we consider also their trivial solutions.

We consider the following initial-boundary value problem for the nonlinear wave equation in the form

$$\Box u + f(u) + g(\dot{u}) = 0 \quad \text{in} \quad [0,T) \times \Omega \tag{QL}$$

with initial values $u_0 = u(0,\cdot)$, $u_1 = \dot{u}(0,\cdot)$ and boundary vale null, that is, $u(t,x) = 0$ on $[0,T] \times \partial\Omega$, where $0 < T \leq \infty$ and $\Omega \subset \mathbb{R}^n (n \in \mathbb{N})$ is a bounded domain on which the divergent theorem can be applied. $\left(L^2(\Omega), \|\cdot\|_2\right), \left(H^1(\Omega), \|\cdot\|_{1,2}\right)$ are the usual spaces of Lebesgue and Sobolev - functions. Further we use the following notation:

$$\begin{aligned}
\cdot \; &:= \frac{\partial}{\partial t}, \nabla := \left(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n}\right), \Box := \partial^2/\partial t^2 - \triangle, Du := (\dot{u}, \nabla u), \\
|Du|^2 &:= \dot{u}^2 + |\nabla u|^2, C_\Omega := \inf\left\{\|\nabla u\|_2 / \|u\|_2 : u \in H_0^1(\Omega)\right\}, \\
F(s) &:= \int_0^s f(r)\,dr, E(t) := \int_\Omega \left(|Du(t,x)|^2 + 2F(u(t,x))\right) dx.
\end{aligned}$$

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For a Banach space X and $0 < T \leq \infty$ we set

$$\begin{aligned} W^{k,p}(0,T,X) &:= \text{ Sobolev space of } W^{k,p} - \text{functions } [0,T] \to X. \\ C^k([0,T],X) &:= \text{ space of } C^k - \text{functions } [0,T] \to X. \\ H1 &:= C^0\left([0,T], H_0^1(\Omega)\right) \cap C^1\left([0,T], L^2(\Omega)\right), \\ H2 &:= C^1\left([0,T], H_0^1(\Omega)\right) \cap C^2\left([0,T], L^2(\Omega)\right). \end{aligned}$$

To the equation

$$\Box u + f(u) = 0, \tag{SL}$$

we have proved the non-existence of global solutions of the initial-boundary value problem (SL) [8] under the assumptions $E(0) < 0$, $a'(0) > 0$ and

$$\eta f(\eta) - 2(1+2\alpha)\ F(\eta) \leq 2\alpha C_\Omega^2 |\eta|^2 \quad \forall \eta \in \mathbb{R}.$$

The result of Li [8] allows the managements for the initial-boundary value problem (SL) for $f(u) = u^p$, $p \in [1, n/n-2]$. In this case Li [7] showed the uniqueness of the solutions. Further contributions to the theme "blow-up" see Racke [9].

Segal [11] applied the semi-group theory to get the global existence of solutions under the case $f(0) = 0, F(\eta) \geq -c\eta^2 \quad \forall \eta \in \mathbb{R},\ c > 0$ and $lim_{|\eta| \to \infty} |F(\eta)| / |f(\eta)| = \infty$.

In this work we consider the existence, uniqueness and triviality of the solutions of quasilinear and semilinear wave equations through the classical energy method.

1. Existence and Uniqueness of Global Solutions of Damping Wave Equations

Although there were an excellent theorem on the existence and uniqueness of solution for the problem (QL) in [3, p.91] and where Haraux A. presented an elegant proof, but the method is not elementary and it is not easy to read. That theorem says:

Suppose that $u_1 \in H_0^1(\Omega)$, $u_0 \in H_0^1(\Omega) \cap H^2(\Omega)$ *and* $h \in W^{1,1}(0,T, L^2(\Omega))$. *If* $f(u) \in L^2(\Omega), g(u) \in L^2(\Omega)$ *for each* $\forall u \in H_0^1(\Omega)$ *and*

$\|g(v)\|_2 \le c(M)$ for each $\|v\|_2 \le M$. Let g be a monotone increasing function and f be local Lipschitz-bounded; that is, there exists a function $B(\|u\|_{1,2}, \|v\|_{1,2}) \le K$ for $\|u\|_{1,2}$, $\|v\|_{1,2} \le K$ with

$$\|f(u) - f(v)\|_2 \le B\left(\|u\|_{1,2}, \|v\|_{1,2}\right) \|u - v\|_{1,2}.$$

Then there exists exactly a function $u : [0,T] \to H_0^1(\Omega)$ with $u(0,\cdot) = u_0, \dot{u}(0,\cdot) = u_1; \dot{u}(t) \in H_0^1(\Omega), u(t) \in H^2(\Omega)\ \forall t \in [0,T]$ and

$$\Box u + f(u) + g(\dot{u}) = h(t,x) \qquad a.e.\ in\ (0,T) \times \Omega.$$

Further, $\|\dot{u}(t)\|_{1,2}, \|u(t)\|_{2,2}$ are bounded in $[0,T]$.

We use the following Lemma from [[6, p.95]; [3, p.96]].

Lemma 1. *For $h \in W^{1,1}\left(0,T,L^2(\Omega)\right)$, the linear wave equation*

$$\begin{cases} \Box u = h(t,x) & in\ \times \Omega \\ u(0,\cdot) := u_0 \in H^2(\Omega) \cap H_0^1(\Omega), \\ \dot{u}(0,\cdot) := u_1 \in H_0^1(\Omega), \end{cases}$$

possesses exactly one solution $u \in H2$ with $u(t) \in H^2(\Omega)\ \forall t \in [0,T]$.

Further,

$$\frac{d}{dt}\|Du(t)\|_2^2 + 2\int_\Omega h\dot{u}(t,x)\,dx = 0 \qquad a.e. in\ [0,T]. \tag{1}$$

We have the following result

Theorem 2. *If $g \in C^1(\mathbb{R})$ is monotone increasing with $g(\dot{u}) \in W^{1,1}$ $(0,T,L^2(\Omega))$ for each $u \in H2$, and there exists consant $k \in \mathbb{R}^+$ such that $\|g(u) - g(v)\|_2 \le k\|u - v\|_2$ then the initial-boundary value problem for the damping wave equation*

$$\begin{cases} \Box u + g(\dot{u}) = 0 & in \quad \mathbb{R}^+ \times \Omega \\ u(0,\cdot) := u_0 \in H^2(\Omega) \cap H_0^1(\Omega), \\ \dot{u}(0,\cdot) := u_1 \in H_0^1(\Omega), \end{cases} \tag{DG}$$

with $u(t,x) = 0$ on $[0,T] \times \partial\Omega$ possesses exactly one global solution in $H2$, i.e. $T = \infty$.

Proof. (i) Proof the locale existence in $H1$.

1) For a $T > 0$ and $v \in H2$, we have $g(\dot{v}) \in W^{1,1}(0, T, L^2(\Omega))$. According to Lemma 1, let $w := Sv$ be the existing solution of the initial-boundary value problem for the equation

$$\begin{cases} \Box w + g(\dot{v}) = 0, \\ w(0, \cdot) := u_0 \in H^2(\Omega) \cap H_0^1(\Omega), \\ \dot{w}(0, \cdot) := u_1 \in H_0^1(\Omega), \end{cases}$$

we have $w \in H2, w(t) \in H^2(\Omega) \quad \forall t \in [0, T]$ and

$$\frac{d}{dt} \|Dw\|_2^2(t) + 2\int_\Omega \dot{w} g(\dot{v})(t, x)\, dx = 0.$$

Suppose that $v_1 := t\ u_0$, then we get $g(\dot{v}_1) = g(u_0) \in L^2(\Omega) \subset W^{1,1}\left(0, T, L^2(\Omega)\right)$ and therefore, there exists a function $v_2 \in H2$ which satisfies the initial-boundary value problem for the equation

$$\begin{cases} \Box w + g(\dot{v}_1) = 0, \\ w(0, \cdot) := u_0 \in H^2(\Omega) \cap H_0^1(\Omega), \\ \dot{w}(0, \cdot) := u_1 \in H_0^1(\Omega). \end{cases}$$

Let $v_{m+1} := Sv_m$ be the solution of the initial-boundary value problem for the linear equation

$$\begin{aligned} &\Box v_{m+1} + g(\dot{v}_m) = 0 \quad \text{in} \quad [0, T) \times \Omega, \\ &v_{m+1}(0, \cdot) = u_0 \in H^2(\Omega) \cap H_0^1(\Omega), \\ &\dot{v}_{m+1}(0, \cdot) = u_1 \in H_0^1(\Omega). \end{aligned}$$

Therefore, by Lemma 1, we have $v_{m+1}(t) \in H^2(\Omega)\ \forall t \in [0, T], v_{m+1} \in H2, m \in \mathbb{N}$ and

$$\frac{d}{dt}\int_\Omega |Dv_{m+1}(t, x)|^2\, dx + 2\int_\Omega \dot{v}_{m+1}(t, x)\, g(\dot{v}_m(t, x))\, dx = 0 \quad a.e.\ in\ [0, T].$$

Set

$$A_{m+1}(t) := \|Dv_{m+1}(t)\|_2.$$

Then we find

$$\left(A_{m+1}(t)^2\right)' \leq 2A_{m+1}(t)\,\|g(\dot{v}_m(t))\|_2 \quad a.e.\ in\ [0,T]$$

and

$$A_{m+1}(t) \leq A_{m+1}(0) + \int_0^t \|g(\dot{v}_m)(r)\|_2\,dr \tag{2}$$

for every $m \in \mathbb{N}$, almost everywhere in $[0,T]$, besides $A_{m+1}(t) = 0$.

2) Since that $g(u_0) \in L^2(\Omega)$, we get $v_2 \in H2$ and by the inequality (2), we obtain

$$A_2(t) \leq A_2(0) + \int_0^t \|g(u_0)\|_2\,dr = \|Du_0\|_2 + t\,\|g(u_0)\|_2 \leq const.$$

Set

$$\begin{aligned} M &:= \text{constant} > \|Du_0\|_2\,, k(M) := \text{local Lipschitz constant of } g, \\ \|v\|_{\infty,T} &:= \sup\left\{\|Dv(t)\|_2 : 0 \leq t \leq T\right\}, T := \frac{M - \|Du(0)\|_2}{k(M)\,M + \|g(0)\|_2}. \end{aligned}$$

Then

$$k(M)\,T = \frac{k(M)\,M - k\,\|Du_0\|_2}{k(M)\,M + \|g(0)\|_2} < 1$$

and

$$\begin{aligned} A_2(t) &= \|Dv_2\|_2(t) \leq A_2(0) + \int_0^t \|g(u_0)\|_2(r)\,dr \\ &\leq A_2(0) + \int_0^t (\|g(u_0) - g(0)\|_2 + \|g(0)\|_2)(r)\,dr \\ &\leq \|Du_0\|_2 + \int_0^t \left(k(M)\,\|u_0\|_{H_0^1} + \|g(0)\|_2\right)(r)\,dr \\ &= \|Du_0\|_2 + t\left(k(M)\,\|u_0\|_{H_0^1} + \|g(0)\|_2\right) \\ &\leq \|Du_0\|_2 + T\,(k(M)\,M + \|g(0)\|_2) \\ &= \|Du_0\|_2 + \frac{M - \|Du(0)\|_2}{k(M)\,M + \|g(0)\|_2} \cdot (k(M)\,M + \|g(0)\|_2) \\ &= M \quad \forall t \in [0,T], \end{aligned}$$

consequently

$$\|v_2\|_{\infty,T} \leq M.$$

Suppose that $\|v_m\|_{\infty,T} \leq M$, then we have

$$A_{m+1}(t) \leq \|Du_0\|_2 + (k(M)M + \|g(0)\|_2)T = M.$$

Thus we get $\|v_{m+1}\|_{\infty,T} \leq M \quad \forall m \in \mathbb{N}$.

3) We claim that v_m is a Cauchy sequence in $H1$. By inequality (2)

$$\begin{aligned} |A_{m+1} - A_m|(t) &\leq \int_0^t \|g(\dot{v}_m) - g(\dot{v}_{m-1})\|_2(r)\,dr \\ &\leq k(M)t\|v_m - v_{m-1}\|_{\infty,T} \qquad \forall t \in [0,T). \end{aligned}$$

Thus

$$\begin{aligned} &\|v_{m+1} - v_m\|_{\infty,T} \leq k(M)T\|v_m - v_{m-1}\|_{\infty,T} \\ \leq\ &(k(M)T)^{m-2}\|v_2 - v_1\|_{\infty,T} \end{aligned} \tag{3}$$

and herewith it follows

$$\|v_{m+k} - v_m\|_{\infty,T} \leq \frac{(k(M)T)^{m-1}\|v_2 - v_1\|_{\infty,T}}{1 - k(M)T} \to 0 \tag{4}$$

for m goes to ∞.

(ii) We prove the uniqueness of the solutions in $H1$.

Suppose that u is the limit of v_m, and $v \in H1$ is an another solution for (DG), then

$$\begin{aligned} &\frac{d}{dt}\int_\Omega |Dv_{m+1}(t,x) - Dv(t,x)|^2\,dx \\ &\quad\leq -2\int_\Omega (\dot{v}_{m+1}(t,x) - \dot{v}(t,x))(g(\dot{v}_m(t,x)) - g(\dot{v}(t,x)))\,dx \\ &\quad\leq 2\|D(v_{m+1} - v)(t)\|_2 \cdot \|g(\dot{v}_m(t)) - g(\dot{v}(t))\|_2. \end{aligned}$$

From (3), we obtain

$$\|v_{m+1} - v\|_{\infty,T} \leq k(M)T\|v_m - v\|_{\infty,T} \leq (k(M)T)^{m-2}\|v_2 - v\|_{\infty,T} \to 0$$

for m goes to ∞, so $u \equiv v$ in $H1$.

(iii) We show the global existence of a solution in $H1$.

Suppose that u is the limit of v_m, then by the monotone of g and Fato Lemma we get

$$\begin{aligned}\frac{d}{dt}\int_\Omega |Du(t,x)|^2\,dx &\le \lim_{m\to\infty}\frac{d}{dt}\int_\Omega |Dv_{m+1}(t,x)|^2\,dx\\ &= \lim_{m\to\infty} -2\int_\Omega \dot{v}_{m+1}(t,x)\,g(v_m(t,x))\,dx\\ &\le -2\int_\Omega \dot{u}(t,x)\,g(u(t,x))\,dx \le 0 \qquad (5)\end{aligned}$$

and

$$\int_\Omega |Du(t,x)|^2\,dx \le \int_\Omega |Du(0,x)|^2\,dx < M.$$

We set now

$$\overline{u}_0(\cdot) := u(T/2,\cdot) \in H_0^1(\Omega),\, \overline{u}_1(\cdot) := \dot{u}(T/2,\cdot) \in L^2(\Omega)$$

and construct the solutions with those initial data in an interval $[T/2,T)$.

According to the above estimates and the contraction property in 3) we can choose $T^\wedge = 3T/2$. On the interval $[0,T)$ and $[T/2,3T/2)$ the constructed solutions are equal since the uniqueness of the solutions on $[T/2,T)$. Continue these steps we obtain the existence of a solution on $[0,\infty)$.

(iv) Now we show the local existence u in $H2$ and $u(t) \in H^2(\Omega)$.

Suppose that T, M and k are the same given in (i-2).

1) For a $T>0$ and $v_m \in H2$, we have $g(\dot{v}_m) \in W^{1,1}(0,T,L^2(\Omega))$. According to Lemma 1, we have $v_{m+1}(t) \in H^2(\Omega)\ \forall t \in [0,T]$, $v_{m+1} \in H2$, $m \in \mathbb{N}$ and

$$\frac{d}{dt}\int_\Omega |Dv_{m+1}(t,x)|^2\,dx = -2\int_\Omega \dot{v}_{m+1}(t,x)\,g(\dot{v}_m(t,x))\,dx \quad a.e.\ in\ [0,T]$$

also

$$\begin{aligned}&\frac{d}{dt}\left\|Dv_{m+1}\right\|_2^2(t)+\frac{d^2}{dt^2}\int_\Omega v_{m+1}(t,x)^2\,dx\\ \le\ & 2\int_\Omega\Big(\dot v_{m+1}^2-\ (v_{m+1}+\dot v_{m+1})\,g(\dot v_m)\Big)(t,x)\,dx\\ \le\ & 2\int_\Omega \dot v_{m+1}^2(t,x)\,dx+2\left(\left\|v_{m+1}(t)\right\|_2+\left\|\dot v_{m+1}(t)\right\|_2\right)\left\|g(\dot v_m(t))\right\|_2.\end{aligned}$$

Then we find

$$\begin{aligned}&\frac{d}{dt}\left\|Dv_{m+1}\right\|_2^2(t)+\frac{d^2}{dt^2}\int_n v_{m+1}(t,x)^2dx\\ \le\ & 2A_{m+1}(t)(A_{m+1}(t)+\|g(\dot v_m(t))\|_2).\end{aligned}\tag{6}$$

for every $m\in\mathbb{N}$, almost everywhere in $[0,T]$.

2) We claim that v_m is a Cauchy sequence in $H2$. By inequalities (4) and (6)

$$\begin{aligned}&\frac{d}{dt}\left\|D(v_{m+k}-v_m)\right\|_2^2(t)+\frac{d^2}{dt^2}\int_\Omega (v_{m+k}-v_m)(t,x)^2dx\\ \le\ & 2\int_\Omega(\dot v_{m+k}-\dot v_m)^2(t,x)dx+2(\|(v_{m+k}-v_m)(t)\|_2\\ &+\|(\dot v_{m+k}-\dot v_m)(t)\|_2)\left\|g(\dot v_{m+k-1}(t))-g(\dot v_{m-1}(t))\right\|_2\\ \le\ & 2\left\|v_{m+k}-v_m\right\|_{\infty,T}\Big(\left\|v_{m+k}-v_m\right\|_{\infty,T}+k(M)\left\|v_{m+k-1}-v_{m-1}\right\|_{\infty,T}\Big)\\ \to\ & 0 \text{ for } m\to\infty.\end{aligned}\tag{7}$$

(v) We prove the uniqueness of the solutions in $H2$

Suppose that u is the limit of v_m, and $v\in H2$ is an another solution for (DG), then

$$\begin{aligned}\frac{d}{dt}\int_\Omega\left|Dv_{m+1}(t,x)-Dv(t,x)\right|^2dx\ &\le\ 2\left\|\dot v_{m+1}-\dot v\right\|_2(t)\left\|g(\dot v_m(t))-g(\dot v(t))\right\|_2\\ &\le\ 2k\left\|D(v_{m+1}-v)\right\|_2(t)\left\|\dot v_m-\dot v\right\|_2(t),\end{aligned}$$

$$\frac{d}{dt}\left\|D(v_{m+1}-v)\right\|_2\le k\left\|\dot v_m-\dot v\right\|_2(t),$$

$$\|D(v_{m+1}-v)\|_2 \le k \int_0^t \|D(v_m - v)\|_2 (t) \le k^m \frac{T^m}{m!} \|v_0\|_2 \to 0,\ m \to \infty;$$

therefore,

$$\frac{d}{dt} \int_\Omega |Dv_{m+1}(t,x) - Dv(t,x)|^2 \, dx \to 0,\ m \to \infty.$$

Hence, $u \equiv v$ in $H2$.

(vi) We show the global existence of a solution.

Suppose that u is the limit of v_m, then by the monotone of g and Fato Lemma we get

$$\begin{aligned} \frac{d}{dt} \int_\Omega |Du(t,x)|^2 \, dx &\le \lim_{m\to\infty} \frac{d}{dt} \int_\Omega |Dv_{m+1}(t,x)|^2 \, dx \\ &= \lim_{m\to\infty} -2 \int_\Omega \dot{v}_{m+1}(t,x)\, g(v_m(t,x))\, dx \\ &\le -2 \int_\Omega \dot{u}(t,x)\, g(u(t,x))\, dx \le 0 \end{aligned}$$

and

$$\int_\Omega |Du(t,x)|^2 \, dx \le \int_\Omega |Du(0,x)|^2 \, dx < M.$$

We set now

$$\overline{u}_0(\cdot) := u(T/2,\cdot) \in H_0^1(\Omega), \overline{u}_1(\cdot) := \dot{u}(T/2,\cdot) \in L^2(\Omega)$$

and construct the solutions with those initial data in an interval $[T/2, T)$. □

According to the above estimates and the contraction property in 3) we can choose $T^\wedge = 3T/2$. The on the interval $[0, T)$ and $[T/2, 3T/2)$ constructed the solutions are equal since the uniqueness of the solutions on $[T/2, T)$. Those further conditions supply us a solution on $[0, \infty)$.

2. Generalization for Theorem 2

For the semilinear wave equation we have the following simple form result.

Theorem 3. *Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a C^1- function and there exist* *constant* $k > 0$ and $p \in \mathbb{R}$ *with*

$$\left|f'(\eta)\right| \le k\left(1 + |\eta|^{p-1}\right) \qquad \forall \eta \in \mathbb{R},$$

then the boundary value problem for the semilinear wave equation (SL) possesses mostly one solution u *in* $H1$ *with* $T = \infty$, *if one of the following two conditions holds:*

case 1: $p > 1$ and $u \in C^0\left(\mathbb{R}^+, H^1(\Omega)\right) \cap C^1\left(\mathbb{R}^+, L^2(\Omega) \cap L^\infty(\Omega)\right)$

case 2: $1 \le p \le n/(n-2)$ *and* $u \in H1$ *with* $T = \infty$.

Proof of case 1. Suppose that w is the solution of the initial-boundary value problem for the linear wave equation

$$\Box w + h(t,x) = 0, w(0,\cdot) = 0 = \dot{w}(0,\cdot),$$

then

$$\left(\|Dw\|^2(t)\right)' \le \|Dw\|_2^2(t) + \|h\|_2^2(t).$$

Multiplication with e^{-t} it yields

$$\left(e^{-t}\|Dw\|_2^2(t)\right)' \le \|h\|_2^2(t) \quad \forall t \ge 0.$$

By the fact that $w(0,\cdot) = 0 = \dot{w}(0,\cdot)$ and $\|Dw\|_2(0) = 0$, it follows

$$\|Dw\|_2^2(t) \le e^t \int_0^t \|h\|_2^2(s)\,ds \quad \forall t \ge 0.$$

Suppose that u and v are two solutions of (SL), then $w := u - v$ satisfies the problem in 1) with $h := f(u) - f(v)$, thus

$$|f(u) - f(v)|^2 \le 3k^2 2^{2(p-2)}\left(1 + \left(|u|^{2(p-1)} + |v|^{2(p-1)}\right)\right) w^2.$$

For $A(t) := \|Dw\|_2^2(t)$ and $t \ge 0$ we get

$$A(t) \le 3k^2 2^{2(p-2)} e^t \int_0^t \int_\Omega \left(1 + |u|^{2(p-1)} + |v|^{2(p-1)}\right) w^2(s,x)\,dxds.$$

Now, we fix $T>0$, there exists a positive k_1 with

$$3\cdot 2^{2(p-2)}k^2e^T sup_{t\in[0,T]}\left\{\left\|1+|u|^{2(p-1)}+|v|^{2(p-1)}\right\|_\infty(t)\right\}\leq k_1(T)$$

and

$$A(t)\leq k_1\int_0^t A(s)\,ds, A(t)\leq A(0)\,e^{k_1(T)t}=0 \quad \forall t\in[0,T].$$

This means yet $u\equiv v$ in $[0,T]\times\Omega$. □

Proof of case 2. Proof for $1+1/n\leq p\leq n/(n-2)$.

1) We set $r=n/2$. By the Sobolev-imbedding $H_0^1(\Omega)\subset L^{2r(p-1)/(r-1)}$ (Ω), there are constants k_2 and $\delta:=2r(p-1)/(r-1)$ so that the inequality holds

$$1+\|u\|_\delta^\delta(t)+\|v\|_\delta^\delta(t)\leq k_2\left(1+\|u\|_{1,2}^\delta+\|v\|_{1,2}^\delta\right)(t) \qquad \forall t\geq 0.$$

There exists a constant $k_3>0$ with

$$\|u-v\|_{2r}^2(s)\leq k_3\|D(u-v)\|_2^2(s).$$

By the fact that u,v in $H1$ with $T=\infty$ the following supremum exists

$$k_4(T):=Sup_{t\in[0,T)}\left\{\left(1+\|u\|_{1,2}^\delta+\|v\|_{1,2}^\delta\right)^{(r-1)/r}(t)\right\}.$$

Let us set

$$B(t):=\int_0^t A(r)\,dr.$$

By Hölder inequality we get

$$\begin{aligned} B'(t) &\leq 2ke^T\int_0^T\left\|1+u^{2(p-1)}+v^{2(p-1)}\right\|_{r/(r-1)}\|u-v\|_{2r}^2(s)\,ds \\ &\leq 2kk_3e^TB(t)\int_0^T\left(2^{r/(r-1)}\int_\Omega\left(1+|u|^\delta+|v|^\delta\right)(s,x)\,dx\right)^{1-1/r}ds \\ &\leq 4kk_3(T)\,k_4(T)\,e^TB(t) \qquad t\in[0,T]. \end{aligned}$$

Using Gronwall inequality we obtain

$$B(t) \leq B(0)\, e^{k_5(T)\, t} = 0 \quad \forall t \in [0,T]$$

with $k_5 := 4kk_3k_4(T)\, e^T$.

By the definition of $B(t) := \int_0^t A(r)\, dr$ and $A(t) := \|D(u-v)\|_2^2(t)$ we find that

$$A(t) \equiv 0 \quad \forall t \in [0,T].$$

This means that however $u(t,x) = v(t,x)$ $\forall t \in [0,T]$, a.e. in Ω. □

2) Proof for the case: $1 + 1/n > p > 1$. Suppose that u, v are two solutions of (SL). From the condition for f for $w := u - v$ with

$$n = \frac{k}{p-1}, \;\; \alpha = \frac{2n}{n-2}, k \in \left[1\,, \frac{2n}{n-2}\right],$$

there exists $k_6 > 0$ such that

$$\|w\|_\alpha(t) \leq k_6 \|\nabla w\|_2(t)\,,\; k_6 > 0$$

and according to $u,\; v \in H$ with $T = \infty$ there exists the supremum

$$k_8(T) := \sup_{t \in [0,T)} \left\{ \left[|\Omega| + k_7 \left(\|\nabla u\|_2^{(p-1)n} + \|\nabla v\|_2^{(p-1)n} \right)(t) \right]^{1/n} \right\},$$

thus

$$A'(t) \leq 4k_6k_8(T)\, A(t)$$

and consequently

$$A(t) \leq A(0)\, e^{4\, k_6k_8(T)\cdot t} = 0 \;\; \forall t \in [0,T).$$

This means that $u(t,x) = v(t,x)$ for $t \in [0,T]$.

For the case $p = 1$ we have

$$\left|f'(\eta)\right| \leq 2k\left(1 + |\eta|^{q-1}\right) \;\; \forall \eta \in \mathbb{R},\; q \in \left[\frac{n+1}{n}, \frac{n}{n-2}\right].$$

In this case, the proof is similar to the proof in step 2). □

3. Uniqueness of The Solutions of Qusilinear Wave Equations

For some particular quasilinear wave equations we have also uniqueness result below:

Theorem 4. *Suppose that* $f,\ g : \mathbb{R} \to \mathbb{R}$ *are monotone increasing with* $f(0) = 0,\ g(0) = 0,\ F(u_0) \in L^1(\Omega)$. *Then the initial-boundary value problem for the wave equation (QL) has at most one solution* u *in* $H1$, *if* f *is local Lipschitz, that is,* $M(u) := f(u) : H_0^1(\Omega) \to L^2(\Omega)$ *with*

$$\|M(u) - M(v)\|_2 \leq B(K)\,\|u - v\|_{1,2}\,,$$

B *is bounded for* $\|u\|_{1,2}\,, \|v\|_{1,2} \leq K$. *Further, for local Lipschitz* g *there is a constant* b *with*

$$\|Du\|_2(t) \leq b \quad \forall t \in [0, T)\,.$$

Proof. Suppose that u and v are two solutions of the wave equation (QL), according to the monotone of g we get

$$\frac{d}{dt}\,\|D(u - v)\|_2^2(t) \leq 2\int_\Omega \left(|D(u - v)|^2 + |f(u) - f(v)|^2\right)(t, x)\,dx.$$

By the similar way in the proof of Theorem 2, one can conclude the uniqueness of the solutions of the wave equation (QL). We show the global boundedness. By the fact that $s\ g(s) \geq 0$ and $F(u_0) \in L^1(\Omega)$ we obtain

$$\frac{d}{dt}\int_\Omega \left(|Du|^2 + 2F(u)\right)(t, x)\,dx \leq -2\int_\Omega \dot{u}g(\dot{u})\,(t, x)\,dx \leq 0$$

Using $F(s) \geq 0$, we conclude that

$$\int_\Omega |Du|^2(t, x)\,dx \leq const. < \infty \ \ \forall t \in [0, T)\,.$$

By $w := u - v \in H1$ with $T = \infty$ it follows $w(t, \cdot) = 0$. □

4. Trivial Solution of Some Semilinear Wave Equations

In [8] we have an interesting result, the solutions of (SL) must be trivial under the conditions $u_0 \equiv 0 \equiv u_1 \equiv f(0)$ and $\eta\ f\ (\eta) + 2F(\eta) \geq -k|\eta|^p$ $\forall \eta \in \mathbb{R}$. In this section we want generalize this result.

Theorem 5. *The initial-boundary value problem for the wave equation* (SL) *has only trivial solution* $u \equiv 0$ *as global solution in* $H1$ *for* $u_0 \equiv 0 \equiv u_1$*, if the following holds*

$$f(\eta)^2 \leq k|\eta|^p \quad \forall \eta \in \mathbb{R}.$$

Proof. Let us set

$$t_1 := \sup\{t \geq 0 : A(t) \leq 1\}.$$

According to the conditions on f and using the Lemma 4 we get

$$A'(t) \leq A(t) + kA(t) + k \int_\Omega |u|^p(t, x)\, dx.$$

For $2 \leq p \leq 2n/(n-2)$, we have

$$\|u\|_p^p(t) \leq k_1 \|Du\|_2^p(t) = k_1 A(t)^{\frac{p}{2}}.$$

These together we obtain that for t in $[0, t_1]$

$$A'(t) \leq (1+k)A(t) + kk_1 A(t)^{\frac{p}{2}} \leq k_2 A(t),$$

where $k_2 := 1 + k + 2^{-1}pkk_1$. By Gronwall inequality, the assertion of this theorem follows. □

Corollary 6. *The Theorem* 5 *is applicable particular to the well-defined functions*

$$f(u) = u^{p/2} + u^{q/2}, u^{p/2} - u^{q/2}\ ,\ p, q \in [2, 2n/n-2]$$

or under the condition

$$f(\eta)^2 \le \sum_{i=1}^{m} k_i\,|\eta|^{p_i}\,\forall\eta\in\mathbb{R},$$

$k_i =$ *positive constants* , $p_i \in [2, 2n/n-2]$.

Theorem 7. *Suppose that u in $H1$ is a solution of the initial-boundary value problem for the semilinear wave equation (SL). Then $u\equiv 0$ and $f(0)=0$, if $u_0\equiv 0\equiv u_1$ and*

$$F(\eta)\ge -k\eta^2 \quad \forall\eta\in\mathbb{R}, k=\text{const.}$$

Proof. By Lemma 1 we have

$$A(t) = -2\int_\Omega F(u)(t,x)\,dx \le 2ka(t),$$

$$C_\Omega a'(t) = 2C_\Omega\int_\Omega u\dot{u}(t,x)dx \le C_\Omega^2 a(t) + \int_\Omega \dot{u}(t,x)^2 dx \le A(t) \le 2ka(t) \quad \forall t\ge 0.$$

Using Granwall inequality, we conclude that

$$a(t)\le a(0)\exp(2kt/C_\Omega) = 0 \quad \forall t\ge 0,$$

since that $u_0\equiv 0\equiv u_1$. And this implies that $u\equiv 0$ and $f(0)=0$. □

Corollary 8. *Theorem 7 holds specially for the monotone increasing function f with $f(0)=0$. For instance $f(u)=u^{2p-1}$, $-1+\exp u$.*

Theorem 9. *Let $\Omega := B_{r_2}(0) - \overline{B_{r_1}(0)}$, $r_2>r_1>0$ be an annulus in R^n and there exist constants $k>0$ and $p\ge 1$ with*

$$\eta f(\eta) + 2F(\eta) \ge -k|\eta|^p \quad \forall\eta\in\mathbb{R}.$$

Then the initial-boundary value problem for (SL) *has only $u\equiv 0$ as global radial solution in $H1$, if $u_0\equiv 0\equiv u_1\equiv f(0)$ valid.*

Remark. Here we put no condition on p. $f(u) = -mu + ku^q$ is a typical example.

Proof. Suppose that $u(t,|x|) = u(t,r)$, $r = |x|$ is a radial solution of (SL). Setting

$$u(t,r) = v(t,r)\, r^{(1-n)/2},\ G(v) := \int_0^v g(s)\, ds,$$
$$g(v) := 4^{-1}(n-1)(n-3)\, r^{-2} v + r^{(n-1)/2} f\left(r^{(n-1)/2} v\right),$$

then the equation (SL) is transformed into the following form

$$\ddot{v} - \partial^2 v/\partial r^2 + g(v) = 0 \quad \text{in} \quad [0,T) \times (r_1, r_2),$$
$$v(0,r) = u_0 r^{(n-1)/2} \equiv 0 \equiv \dot{v}(0,r),$$
$$v(t,r_1) \equiv 0 \equiv v(t,r_2).$$

Choosing $\eta = r^{(1-n)/2} v$ in the condition in Theorem 6 on f and F, then we have

$$r^{(n-1)/2} v f\left(r^{-(n-1)/2} v\right) + 2F\left(r^{-(n-1)/2} v\right) \geq -k r^{-(n-1)p/2} |v|^p.$$

From this, we conclude

$$\begin{aligned} vg(v) + 2G(v) &= 2^{-1}(n-1)(n-3)\, r^{-2} v^2 + r^{(n-1)/2} v f\left(r^{-(n-1)/2} v\right) \\ &\quad + 2r^{(n-1)/2} F\left(r^{-(n-1)/2} v\right) \\ &\geq 2^{-1}(n-1)(n-3)\, r^{-2} v^2 - k r^{(n-1)p/2} |v|^p \\ &\geq -k\left(v^2 + |v^p|\right) \end{aligned}$$

with $k := Max_{r\in[r_1,r_2]} \left\{-2^{-1}(n-1)(n-3)\, r^{-2} + k r^{(1-n)p/2}\right\}$. □

From Satz 1.2.1 in [7] for two variables t and r the assertion of this theorem follows.

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