International Journal of Intelligent Technologies and Applied Statistics
Vol.5, No.1 (2012) pp.35–40

Bounded Tolerance Representations for Maximal Outerplanar Graphs

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ABSTRACT

A graph $G = (V, E)$ is a *tolerance graph* if there is a set $I = \{I_v | v \in V\}$ of closed real interval and a set $\tau = \{\tau_v | v \in V\}$ of positive real numbers such that $(x, y) \in E \Leftrightarrow |I_x \cap I_y| \geq \min\{\tau_x, \tau_y\}$. We show that if G is a 2-connected maximal outerplanar graph with more than two vertices of degree 2, then G has S_3 as an induced subgraph. We provide a characterization of the class of 2-connected maximal outerplanar graphs that are bounded tolerance graphs.

Keywords: Tolerance graph; Maximal outerpanar graphs; Interval graphs; 3-sun

1. Introduction

The intervals on the real line can be very useful tools. They could represent fragments of DNA on the genome, the durations of a set of events on a time line, or an estimation [4]. *Interval graphs* were introduced in graph theory. Each vertex v in an interval graph $G = (V, E)$ is associated with an interval I_v, and two vertices are adjacent in G if and only if the intersection of their associated intervals is nonempty. Tolerance graphs, introduced by Golumbic and Monma [1], are generalizations of interval graphs. An undirected graph G is a *tolerance graph* if each vertex $x \in V(G)$ can be assigned a closed interval I_x and a tolerance $t_x > 0$ such that $xy \in E(G)$ if and only if $|I_x \cap I_y| \geq \min\{t_x, t_y\}$. Such a collection $(\mathcal{I}, t)$ of intervals and tolerances is called *a tolerance representation* of G where $\mathcal{I} = \{I_x | x \in V\}$ and $t = \{t_x | x \in V\}$. If a graph G has a tolerance representation with $t_v \leq |I_v|$ for all $v \in V$, then G is called a *bounded tolerance graph* and the representation is called a *bounded tolerance representation*.

Golumbic and Trenk [2] mentioned that their original motivation of studying tolerance graphs was the need to solve scheduling problems in which resources such

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as rooms, vehicles, support personnel, etc. may be needed on an exclusive basis, but where a measure of flexibility or tolerance would allow for sharing or relinquishing the resource if a solution is not otherwise possible. One can think of this as a model of conflicts for events occurring in a block of time, in which intervals represent a set of events on a time line and their associated tolerances are flexibility of the event. Golumbic and Trenk gave characterizations of trees and bipartite graphs, respectively, which are bounded tolerance graphs.

2. A results on maximal outerplanar graphs

A graph is *outerplanar* if it has an embedding in ther plane with every vertex on the boundary of the unbounded face. A *maximal outerplanar graph* is a simple outerplanar graph that is not a spanning subgraph of a larger simple outerplanar graph.

A *chord* is an edge joining two nonconsecutive vertices of a cycle. A chrod in a simple maximal outerpalnnar graph belongs to exactly two triangles i.e., the end vertices of any chrod have exactly two common neighbors. A triangle is called an *inner triangle* of G if all its three edges are chords.

There are two stuctures play important roles in our main result. Three vertices $v_1, v_2, v_3 \in V(G)$ form an *asteroidal triple* (AT) of graph G, if for every permutation i, j, k of 1, 2, 3, there is a path from v_i to v_j which avoids using any vertex in the closed neighborhood $N[v_k] = \{v_k\} \cup N(v_k)$. See Figure 1 for example. The *k-sun* $S_k (k \geq 3)$ is the graph with the vertex set $\{x_1, x_2, ..., x_k\} \cup \{y_1, y_2, ..., y_k\}$ and the edge set $\{x_1y_1, y_1x_2, x_2y_2, ..., x_ky_k, y_kx_1\} \cup \{y_iy_j | i \neq j\}$. See Figure 2 for S_3.

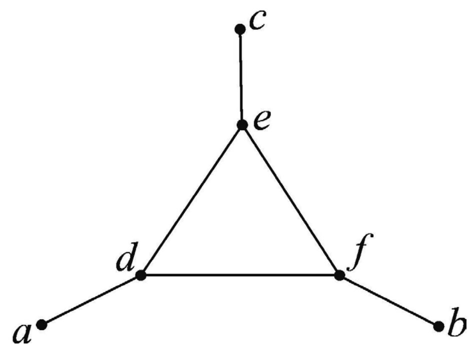

Figure 1. The set $\{a, b, c\}$ is an asteroidal triple.

Lemma 1. *Let G be a 2-connected maximal outerplanar graph. If G has more than two vertices of degree 2, then G has S_3 as an induced subgraph.*

Proof. It is sufficient to show that G has an inner triangle.

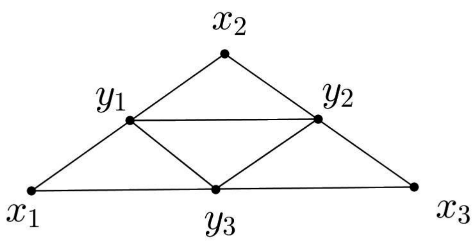

Figure 2. The graph of S_3.

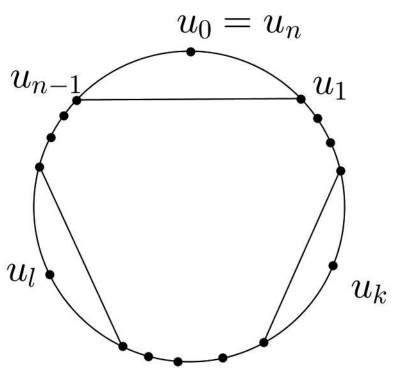

Figure 3. The graph G.

Let $u_0u_1 \ldots u_n$ be the cycle which forms the boundary of the unbounded region of G. Suppose $\deg(u_0) = \deg(u_k) = \deg(u_l) = 2$, where $2 \le k$, $k + 2 \le l \le n - 2$. See Figure 3.

Let $i' = \max\{i \mid u_iu_j \in E(G), 1 \le i \le k-1, l+1 \le j \le n-1\}$ and $j' = \min\{j \mid u_iu_j \in E(G), 1 \le i \le k-1, l+1 \le j \le n-1\}$. Then $u_{i'}u_{j'} \in E(G)$. There exists u_p $(i' < p < j')$ which is a common neighbor of $u_{i'}$ and $u_{j'}$. We see that $k + 1 \le p \le l - 1$. Then $u_{i'}u_pu_{j'}$ is an inner triangle of G.

3. Main result

Let $n(G)$ denote the number of vertices of a graph G.

Theorem 1. *Suppose that G is a 2-connected maximal outerplanar graph with $n(G) \ge 4$. The following are equivalent.*

(i) G is a bounded tolerance graph.
(ii) G is AT-free.
(iii) G has no induced subgraph S_3.
(iv) G has exactly two vertices of degree 2.
(v) G is an interval graph.

Proof. (i) ⇒ (ii): Golumbic and Trenk [2] showed that bounded tolerance graphs are cocom-parability graphs and Golumbic et al. [3] showed that all cocomparability graphs ar AT-free. Therefore, bounded tolerance graphs are AT-free.

(ii) ⇒ (iii): The reason is that S_3 contains an AT. See Figure 2.

(iii) ⇒ (iv): By Lemma 1, G has exactly two vertices of degree 2.

(iv) ⇒ (v): Suppose that G has exactly two vertices of degree 2.

Let x and y be the vertices of degree 2. We have the following three cases.

Case 1: The vertices x and y have two common neighbors.

In this case, $n(G) = 4$. It is easy to show that G is an interval graph.

Case 2: The vertices x and y have exactly one common neighbor.

Let u be the common neighbor of x and y. Let $v_1, v_2, \ldots, v_k$ be vertices between x and y. Then, there is no chords of the form v_iv_j, otherwise it will contradict to the fact that there are only two verices of degree 2. Therefore, all vertices v_i, $1 \le i \le k$ are neighbors of u. The graph G is given in Figure 4.

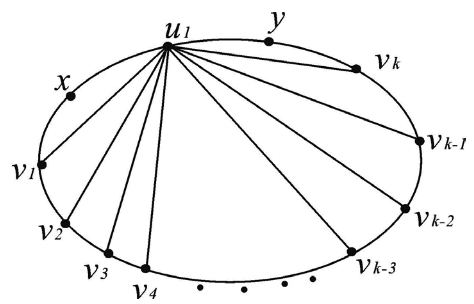

Figure 4. The graph G.

We have an interval representation of G as shown in Figure 5. Hence, G is an interval graph.

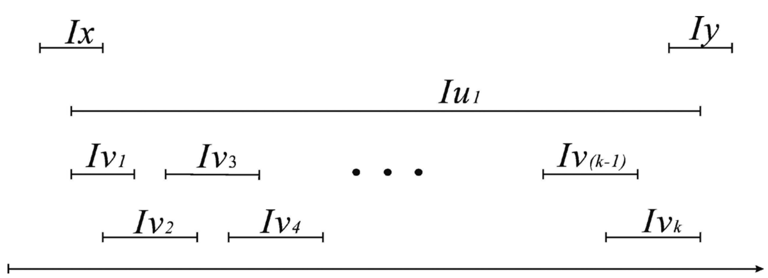

Figure 5. The interval representation of G.

Case 3: The vertices x and y have no common neighbors .

Let $u_1, u_2, \ldots, u_n$ be vertices between x and y along the boundary clockwisely, and $v_1, v_2, \ldots, v_k$ be vertices between x and y along the boundary counterclockwisely. Then, there is no chords of the form v_iv_j and u_iu_j, otherwise it will contradict to the fact that there are only two verices of degree 2.

For simplicity, we use the following examples to illustrate an interval representation of G.

Example 1: Figure 6, Figure 7.

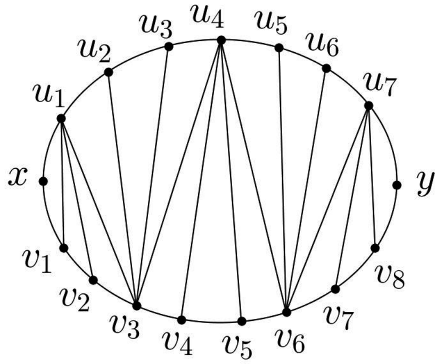

Figure 6. The graph G_1.

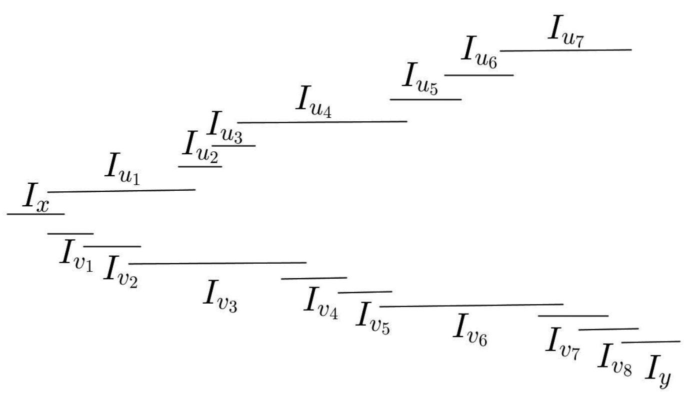

Figure 7. The interval representation of G_1.

Example 2: Figure 8, Figure 9.

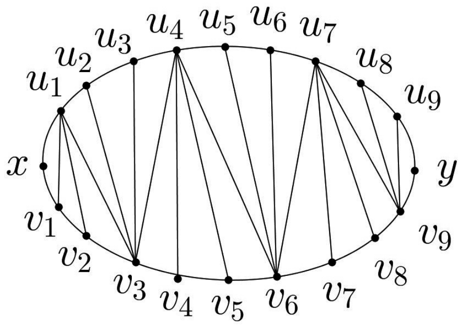

Figure 8. The graph G_2.

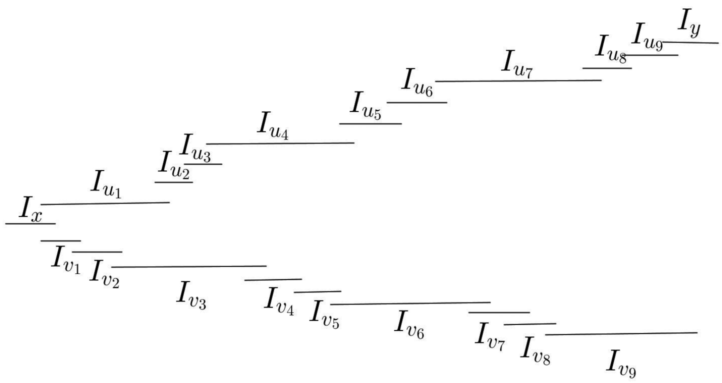

Figure 9. Interval representation of G_2.

(v) $\Rightarrow$ (i): The proof appears in [2] (see [2], Theorem 2.5).

References

[1] M. C. Golumbic and C. L. Monma (1982). A generalization of interval graphs with tolerances, *Congressus Numerantium*, 35, 321-331.

[2] M. C. Golumbic and A. N. Trenk (2004). *Tolerance Graphs*, Cambridge University Press, Cambridge, UK.

[3] M. C. Golumbic, D. Rotem and J. Urrutia (1983). Comparability graphs and intersection graphs, *Discrete Mathematics*, 43, 37-46.

[4] D. T. Shirke, S. Gulati and R. R. Kumbhar (2009). Upper tolerance intervals for exponential distribution based on grouped data, *International Journal of Intelligent Technologies and Applied Statistics*, 2, 89-107.