An asymptotic solution for the free boundary parabolic equations

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Abstract—In this paper, we investigate the solution of a two dimensional parabolic free boundary problem. The free boundary of this problem is modelled as a nonlinear integral equation (IE). For this integral equation, we propose an asymptotic solution as time is near to maturity and develop an integral iterative method. The computational results reveal that our asymptotic solution is very close to the numerical solution as time is near to maturity.

Keywords—integral equation, asymptotic solution, free boundary problem, American exchange option

I. INTRODUCTION

IN this paper, we shall study the solutions $(P(S_1,S_2,t),X(t))$ for the free boundary problem (FBP) of parabolic equations as follows:

$$P_t+\mathcal{L}P=0,\quad 0<\frac{S_1}{S_2}<X(t), \tag{1}$$

$$P(S_1,S_2,t)=S_1-S_2,\quad X(t)\le\frac{S_1}{S_2}, \tag{2}$$

$$P(S_1,S_2,T)=(S_1-S_2)^+,\quad t=T \tag{3}$$

$$P(0,S_2,t)=0,\quad 0<S_2<\infty,\ 0<t<T, \tag{4}$$

$$P(S_1,0,t)=S_1,\quad 0<S_1<\infty,\ 0<t<T, \tag{5}$$

$$P_{S_1}(S_1,S_2,t)=1,\quad \frac{S_1}{S_2}=X(t), \tag{6}$$

$$P_{S_2}(S_1,S_2,t)=-1,\quad \frac{S_1}{S_2}=X(t). \tag{7}$$

More pricisely, we transform the FBP into an integral equation (IE) of X and provide an asymptotic solution and a numerical method for the IE.

To derive the IE, we combine (1) and (2) and get

$$P_t+\mathcal{L}P=\begin{cases}0, & 0<S_1<X(t)S_2,\\ q_1S_1-q_2S_2, & S_1\ge X(t)S_2.\end{cases}$$

The solution of this inhomogeneous linear parabolic equation with the final condition (3) is given as

$$\begin{aligned}P(S_1,S_2,\tau)= \quad & p(S_1,S_2,\tau)+S_1e^{-q_1\tau}\int_0^\tau q_1e^{q_1s}N(\tfrac{a_3}{\sigma})ds\\ & -S_2e^{-q_2\tau}\int_0^\tau q_2e^{q_2s}N(\tfrac{a_4}{\sigma})ds,\end{aligned} \tag{8}$$

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where $\tau=T-t$ and

$$\begin{aligned}a_1 &= \tfrac{1}{\sqrt{\tau}}(\log\tfrac{S_1}{S_2}+\tfrac12(\sigma^2-2q_1+2q_2)\tau),\\ a_2 &= \tfrac{1}{\sqrt{\tau}}(\log\tfrac{S_1}{S_2}-\tfrac12(\sigma^2+2q_1-2q_2)\tau),\\ a_3 &= \tfrac{1}{\sqrt{\tau-s}}(\log\tfrac{S_1}{S_2X(T-s)}+\tfrac12(\sigma^2-2q_1+2q_2)(\tau-s)),\\ a_4 &= \tfrac{1}{\sqrt{\tau-s}}(\log\tfrac{S_1}{S_2X(T-s)}-\tfrac12(\sigma^2+2q_1-2q_2)(\tau-s)).\end{aligned}$$

Let $\frac{S_1}{S_2}=X(T-\tau)$. By imposing the boundary condition (2) into (8), an implicit representation of the free boundary is obtained as follows:

$$\begin{aligned}X(T-\tau)-1= \quad & X(T-\tau)e^{-q_1\tau}N(\tfrac{\hat{a}_1}{\sigma})-e^{-q_2\tau}N(\tfrac{\hat{a}_2}{\sigma})\\ & +X(T-\tau)e^{-q_1\tau}\int_0^\tau q_1e^{q_1s}N(\tfrac{\hat{a}_3}{\sigma})ds\\ & -e^{-q_2\tau}\int_0^\tau q_2e^{q_2s}N(\tfrac{\hat{a}_4}{\sigma})ds,\end{aligned} \tag{9}$$

where

$$\begin{aligned}\hat{a}_1 &= \tfrac{1}{\sqrt{\tau}}(\log X(T-\tau)+\tfrac12(\sigma^2-2q_1+2q_2)\tau),\\ \hat{a}_2 &= \tfrac{1}{\sqrt{\tau}}(\log X(T-\tau)-\tfrac12(\sigma^2+2q_1-2q_2)\tau),\\ \hat{a}_3 &= \tfrac{1}{\sqrt{\tau-s}}(\log\tfrac{X(T-\tau)}{X(T-s)}+\tfrac12(\sigma^2-2q_1+2q_2)(\tau-s)),\\ \hat{a}_4 &= \tfrac{1}{\sqrt{\tau-s}}(\log\tfrac{X(T-\tau)}{X(T-s)}-\tfrac12(\sigma^2+2q_1-2q_2)(\tau-s)).\end{aligned}$$

In this paper we provide an asymptotic solution for the free boundary of (1)-(7). The asymptotic solution is compared to the numerical solution. The computational results reveal that our approximation is close to the numerical results as time near to maturity.

When $q_1=q_1=0$, Margrabe [10] showed that the FBP can be ragraded as a parabolic boundary value problem. When the expiration date tends to infinity, the closed form solution is obtained by [7]. For the numerical methods, Carr [1] generalized the Gesker-Johnson approach [3] to find the solution of P. Longstaff and Schwartz [9] and Rogers [12] use least squares Monte Carlo to find the solution of the FBP numerically.

This paper is organized as follows. S ection 2 provides an FBP for the AEO valuation model. In section 3, an IE is provided for the free boundary of this FBP. We propose an asymptotic solution of this IE as the remaining time near to maturity in section 4. In order to compare with our asymptotic solution, a numerical method is provided in section 5. Finally, the conclusions and comments are in section 6.

II. THE FORMULATION OF AEO

Let S_i, σ_i^2 be the underlying asset price of the i-th asset and the variance of the rate of return on the i-th asset, $i=1,2$. Let T and P be the maturity time and the pricing function of the American exchange option on the assets S_1 and S_2, respectively. Following Black and Scholes' assumption, we assume a perfect market, a constant volatilities σ_i, and the continuous dividend rate q_i of the asset i, respectively. We set current time to be zero.

Under the risk neutral probability measure, the stochastic processes for the asset price changes are assumed to be

$$\frac{dS_i}{S_i} = (r - q_i)dt + \sigma_i dw_i,\ i = 1, 2$$

where r is the constant risk-free interest rate and dw_1 and dw_2 are the differential of a Wiener processes with correlation $\mathrm{corr}(dw_1, dw_2) = \rho dt$.

The terminal payoff of the European exchange option (EEO) is given by

$$V(S_1, S_2, T) = \max(S_1 - S_2, 0), \qquad (10)$$

and $V(S_1, S_2, t)$ denotes the value of EEO at the time t. As the suggestion of Margrabe [10], the value of EEO satisfies the linear homogeneous property in S_1 and S_2, that is

$$V(\lambda S_1, \lambda S_2, t) = \lambda V(S_1, S_2, t).$$

We apply Euler Theorem to the function $V(S_1, S_2, t)$ and obtain the following equation:

$$V - S_1\frac{\partial V}{\partial S_1} - S_2\frac{\partial V}{\partial S_2} = 0. \qquad (11)$$

This means that the portfolio of holding $\frac{\partial V}{\partial S_i}$ units of asset $i,\ i = 1, 2$ becomes a replication of EEO.

By applying Itô Lemma to (11) and considering the instantaneously return with the dividends rate of both the assets, we obtain the following pricing equation

$$V_t + L_0 V = 0,\ t < T, \qquad (12)$$

where the operator L_0 is defined as

$$L_0 V = \tfrac{1}{2}\sigma_1^2 S_1^2 V_{S_1 S_1} + \rho\sigma_1\sigma_2 S_1 S_2 V_{S_1 S_2} + \tfrac{1}{2}\sigma_2^2 S_2^2 V_{S_2 S_2} - q_1 S_1 V_{S_1} - q_2 S_2 V_{S_2}.$$

If assets S_1 and S_2 both do not pay dividends, the value for American options are the same as the corresponding European option, which was discussed by Margrabe [10].

Now, we consider the case of that one of the assets pays the continuous dividends. Since the American option can be exercised at any time $t < T$, the value $P(S_1, S_2, t)$ of AEO must satisfy the following condition:

$$P(S_1, S_2, t) \geq \max(S_1 - S_2, 0)$$

for all $t \geq 0$. This is because that if $P < \max(S_1 - S_2, 0)$, we can purchase an AEO and one unit of asset two in the market at the same time; then we exercise this AEO immediately. This portfolio produces an arbitrage possibility and make a riskless profit $S_1 - S_2 - P$ which is positive. Thus, at any given time t, we separate the (S_1, S_2)-plane into two distinct regions, one is optimal to exercise prematurity $\mathbf{S}(t)$ and the other is $\mathbf{C}(t)$, where

$$\mathbf{S}(t) = \{(S_1, S_2, t) \in \mathcal{R}^+ \times \mathcal{R}^+ \times [0, T] | P(S_1, S_2, t) \leq S_1 - S_2\},$$

$$\mathbf{C}(t) = \{(S_1, S_2, t) \in \mathcal{R}^+ \times \mathcal{R}^+ \times [0, T] | P(S_1, S_2, t) > S_1 - S_2\}.$$

Here $\mathcal{R}^+$ denotes the set of all nonnegative real numbers.

Let $X(t)$ be defined as follows:

$$X(t) = \inf\{\frac{S_1}{S_2} | (S_1, S_2, t) \in \mathbf{S}(t)\}.$$

Then we have

$$P(S_1, S_2, t) = S_1 - S_2,\ \text{for}\ \frac{S_1}{S_2} > X(t) \qquad (13)$$

and

$$P(S_1, S_2, t) > \max(S_1 - S_2, 0),\ \text{for}\ \frac{S_1}{S_2} \leq X(t).$$

The function $X(t)$ is called the early exercise ratio of the assets S_1 and S_2 at time t.

The portfolio consists of longing one unit of AEO, shorting $\frac{\partial P}{\partial S_1}$ units of asset one and shorting $\frac{\partial P}{\partial S_2}$ units of asset two, that is

$$P - \frac{\partial P}{\partial S_1}S_1 - \frac{\partial P}{\partial S_2}S_2$$

and the value of AEO is equal to $S_1 - S_2$ when the AEO is exercised. According to the argument of no arbitrage, we need the following two conditions

$$\frac{\partial P}{\partial S_1} = 1 \text{ and } \frac{\partial P}{\partial S_2} = -1. \qquad (14)$$

Condition (14) is commonly called the high contact conditions, so named because conditions (13) and (14), respectively, indicate that $P(S_1, S_2, t)$, $\frac{\partial P}{\partial S_1}(S_1, S_2, t)$ and $\frac{\partial P}{\partial S_2}(S_1, S_2, t)$ are continuous across the optimal exercise boundary.

Thus, the value $P(S_1, S_2, t)$ of an AEO together with the early exercise ratio $x(t)$ are the solution of the following free boundary problem (1)-(7). The value of the alive AEO is an increasing function of S_1, a decreasing function of S_2 and of t.

III. The Integral Equation

It is convenience to define new independent variables y_1, y_2 and τ as follows:

$$y_i = \frac{-1}{\sigma_i}(q_i + \frac{1}{2}\sigma_i^2)\tau + \frac{1}{\sigma_i}\log(S_i),\ i = 1, 2,$$
$$\tau = T - t.$$

The original pricing formula (1)-(2) can be rewritten in the following dimensionless form:

$$\frac{\partial p}{\partial \tau} = L_1 p,\quad y_1 - \frac{\sigma_2}{\sigma_1}y_2 \leq x(\tau),\ 0 < \tau < T,$$
$$p(y_1, y_2, 0) = (e^{\sigma_1 y_1} - e^{\sigma_2 y_2})^+,\ \tau = 0,$$
$$p(y_1, y_2, \tau) = e^{(q_1 + \frac{1}{2}\sigma_1^2)\tau + \sigma_1 y_1} - e^{(q_2 + \frac{1}{2}\sigma_2^2)\tau + \sigma_2 y_2},$$
$$y_1 - \frac{\sigma_2}{\sigma_1}y_2 = x(\tau),\ 0 < \tau < T,$$

where $(x - y)^+ = \max(x - y, 0)$ and

$$L_1 p = \frac{1}{2}(\frac{\partial^2}{\partial y_1^2} + 2\rho\frac{\partial^2}{\partial y_1 \partial y_2} + \frac{\partial^2}{\partial y_2^2})p.$$

Under this transformation, the relation between $X(t)$ and $x(\tau)$ is defined by

$$x(\tau) = \frac{1}{\sigma_1}(\log(X(T - \tau)) + (q_2 + \frac{1}{2}\sigma_2^2 - q_1 - \frac{1}{2}\sigma_1^2)\tau). \qquad (15)$$

To solve this problem we convert equations (III)-(III) to a non-homogeneous equation by imposing (III) into (III), then we have

$$p_\tau - L_1 p = \begin{cases} 0, \text{ if } y_1 - \frac{\sigma_2}{\sigma_1} y_2 \le x(\tau) \\ q_1 e^{(q_1+\frac{1}{2}\sigma_1^2)\tau+\sigma_1 y_1} - q_2 e^{(q_2+\frac{1}{2}\sigma_2^2)\tau+\sigma_2 y_2}, \\ \text{if } y_1 - \frac{\sigma_2}{\sigma_1} y_2 \ge x(\tau). \end{cases} \tag{16}$$

By introducing the Green's function $\phi(x, y, \tau)$ for (16) yields

$$\begin{aligned} &\phi(y_1, y_2, \tau; \xi_1, \xi_2, s) \\ &= \frac{1}{2\pi(\tau-s)} \frac{1}{\sqrt{1-\rho^2}} \exp\left(-\frac{(y_1-\xi_1)^2 - 2\rho(y_1-\xi_1)(y_2-\xi_2) + (y_2-\xi_2)^2}{2(1-\rho^2)(\tau-s)}\right). \end{aligned}$$

Applying Green's function to $p(x, y, \tau)$ as well as the fact that ϕ is in a domain bounded by the optimal exercise boundary and the line $\tau = 0$, we obtain

$$\begin{aligned} &p(y_1, y_2, \tau) \\ &= \int_{-\infty}^{\infty} \int_{\frac{\sigma_2}{\sigma_1}\xi_1}^{\infty} e^{\sigma_1 \xi_1} \phi(y_1, y_2, \tau; \xi_1, \xi_2, 0) d\xi_1 d\xi_2 \\ &\quad - \int_{-\infty}^{\infty} \int_{\frac{\sigma_2}{\sigma_1}\xi_1}^{\infty} e^{\sigma_2 \xi_2} \phi(y_1, y_2, \tau; \xi_1, \xi_2, 0) d\xi_1 d\xi_2 \\ &\quad + \int_0^\tau \int_{-\infty}^{\infty} \int_{\frac{\sigma_2}{\sigma_1}\xi_1 + x(s)}^{\infty} q_1 e^{\sigma_1 \xi_1} \phi(y_1, y_2, \tau; \xi_1, \xi_2, s) d\xi_1 d\xi_2 ds \\ &\quad - \int_0^\tau \int_{-\infty}^{\infty} \int_{\frac{\sigma_2}{\sigma_1}\xi_1 + x(s)}^{\infty} q_2 e^{\sigma_2 \xi_2} \phi(y_1, y_2, \tau; \xi_1, \xi_2, s) d\xi_1 d\xi_2 ds \\ &= I^{(1)} - I^{(2)} + I^{(3)} - I^{(4)}. \end{aligned} \tag{17}$$

If the values of q_1 and q_2 are both equal to zero; that is, there is no dividends on the underlying assets, then the integral $I^{(3)}$ and $I^{(4)}$ contribute nothing. So the early exercise premium of the AEO equal to zero. This means that the value of the AEO which is written on the no dividend paying assets is the same as the European counterpart.

By letting $\sqrt{\tau}u_1 - \sigma_1\tau = y_1 - \xi_1$ and $\sqrt{\tau}u_2 - \rho\sigma_1\tau = y_2 - \xi_2$ in $I^{(1)}$, $\sqrt{\tau}u_1 - \rho\sigma_2\tau = y_1 - \xi_1$ and $\sqrt{\tau}u_2 - \sigma_2\tau = y_2 - \xi_2$ in $I^{(2)}$, $\sqrt{\tau - s}u_1 - \sigma_1(\tau - s) = y_1 - \xi_1$ and $\sqrt{\tau - s}u_2 - \rho\sigma_1(\tau - s) = y_2 - \xi_2$ in $I^{(3)}$ and $\sqrt{\tau - s}u_1 - \rho\sigma_2(\tau - s) = y_1 - \xi_1$ and $\sqrt{\tau - s}u_2 - \sigma_2(\tau - s) = y_2 - \xi_2$ in $I^{(4)}$, the integrals $I^{(1)}$-$I^{(4)}$ can be written as the following equations

$$\begin{aligned} I^{(1)} &= e^{(\sigma_1 y_1 + \frac{1}{2}\sigma_1^2\tau)} \int_{-\infty}^{\infty} \int_{-\infty}^{a_1 + bu_2} \varphi(u_1, u_2) du_1 du_2, \\ I^{(2)} &= e^{(\sigma_2 y_2 + \frac{1}{2}\sigma_2^2\tau)} \int_{-\infty}^{\infty} \int_{-\infty}^{a_2 + bu_2} \varphi(u_1, u_2) du_1 du_2, \\ I^{(3)} &= e^{(\sigma_1 y_1 + \frac{1}{2}\sigma_1^2\tau)} \int_0^\tau q_1 e^{q_1 s} \int_{-\infty}^{\infty} \int_{-\infty}^{a_3 + bu_2} \varphi(u_1, u_2) du_1 du_2 ds, \\ I^{(4)} &= e^{(\sigma_2 y_2 + \frac{1}{2}\sigma_2^2\tau)} \int_0^\tau q_2 e^{q_2 s} \int_{-\infty}^{\infty} \int_{-\infty}^{a_4 + bu_2} \varphi(u_1, u_2) du_1 du_2 ds, \end{aligned}$$

where

$$\begin{aligned} a_1 &= \frac{1}{\sigma_1\sqrt{\tau}}(\sigma_1 y_1 - \frac{\sigma_2}{\sigma_1} y_2 + (\sigma_1^2 - \rho\sigma_1\sigma_2)\tau), \\ a_2 &= \frac{1}{\sigma_1\sqrt{\tau}}(\sigma_1 y_1 - \frac{\sigma_2}{\sigma_1} y_2 + (\rho\sigma_1\sigma_2 - \sigma_2^2)\tau), \\ a_3 &= \frac{1}{\sigma_1\sqrt{\tau - s}}(\sigma_1 y_1 - \frac{\sigma_2}{\sigma_1} y_2 + (\sigma_1^2 - \rho\sigma_1\sigma_2)(\tau - s) - \sigma_1 x(s)), \\ a_4 &= \frac{1}{\sigma_1\sqrt{\tau - s}}(\sigma_1 y_1 - \frac{\sigma_2}{\sigma_1} y_2 + (\rho\sigma_1\sigma_2 - \sigma_2^2)(\tau - s) - \sigma_1 x(s)), \\ \sigma^2 &= \sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2, \end{aligned}$$

and the function $\varphi(u_1, u_2)$ is defined as

$$\varphi(u_1, u_2) = \frac{1}{2\pi} \frac{1}{\sqrt{1 - \rho^2}} \exp\left(-\frac{u_1^2 - 2\rho u_1 u_2 + u_2^2}{2(1 - \rho^2)}\right),$$

which is the probability density function of the standard bivariate normal distribution with covariant correlation ρ.

In order to reducing the double integrals in $I^{(1)}$-$I^{(4)}$ into single integrals, we first derive the following identity.

$$\begin{aligned} &\int_{-\infty}^{\infty} \int_{-\infty}^{a + \frac{\sigma_2 u_2}{\sigma_1}} \frac{1}{2\pi\sqrt{1-\rho^2}} \varphi(u_1, u_2) du_1 du_2 \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{a} \frac{1}{2\pi\sqrt{1-\rho^2}} e^{-\frac{\sigma_1^2 v_1^2}{\sigma^2}} e^{-\frac{\sigma^2}{2\sigma_1^2(1-\rho^2)}\left(v_2 + \frac{\sigma_1\sigma_2 - \sigma_1\rho}{\sigma^2} v_1\right)^2} dv_2 dv_1 \\ &= \int_{-\infty}^{a} \frac{1}{\sqrt{2\pi}} e^{-\frac{(\sigma_1 v_1)^2}{2\sigma^2}} \frac{\sigma}{\sigma_1} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{w^2}{2}} dw dv_1 \\ &= \int_{-\infty}^{\frac{a\sigma_1}{\sigma}} \frac{1}{\sqrt{2\pi}} e^{-\frac{v^2}{2}} dv = N\left(\frac{a\sigma_1}{\sigma}\right). \end{aligned}$$

Here $v_1 = u_1 - bu_2$, $v_2 = u_2$ and the function $N(x)$ is known as the distribution of the cumulative normal distribution.

Thus the double integrals in $I^{(1)}$-$I^{(4)}$ can be converted to the following single integrals:

$$\begin{aligned} I^{(1)} &= e^{(\sigma_1 y_1 + \frac{1}{2}\sigma_1^2\tau)} N\left(\frac{\sigma_1 a_1(y_1, y_2, \tau)}{\sigma}\right), \\ I^{(2)} &= e^{(\sigma_2 y_2 + \frac{1}{2}\sigma_2^2\tau)} N\left(\frac{\sigma_1 a_2(y_1, y_2, \tau)}{\sigma}\right), \\ I^{(3)} &= e^{(\sigma_1 y_1 + \frac{1}{2}\sigma_1^2\tau)} \int_0^\pi q_1 e^{q_1 s} N\left(\frac{\sigma_1 a_3(y_1, y_2, \tau, s, x(s))}{\sigma}\right) ds, \\ I^{(4)} &= e^{(\sigma_2 y_2 + \frac{1}{2}\sigma_2^2\tau)} \int_0^\pi q_2 e^{q_2 s} N\left(\frac{\sigma_1 a_4(y_1, y_2, \tau, s, x(s))}{\sigma}\right) ds. \end{aligned}$$

Following the above transformation, a simple representation of the price of AEO is obtained as follows:

$$\begin{aligned} &p(y_1, y_2, \tau) \\ &= e^{(\sigma_1 y_1 + \frac{1}{2}\sigma_1^2\tau)} N\left(\frac{\sigma_1 a_1(y_1, y_2, \tau)}{\sigma}\right) \\ &\quad - e^{(\sigma_2 y_2 + \frac{1}{2}\sigma_2^2\tau)} N\left(\frac{\sigma_1 a_2(y_1, y_2, \tau)}{\sigma}\right) \\ &\quad + e^{(\sigma_1 y_1 + \frac{1}{2}\sigma_1^2\tau)} \int_0^\pi q_1 e^{q_1 s} N\left(\frac{\sigma_1 a_3(y_1, y_2, \tau, s, x(s))}{\sigma}\right) ds \\ &\quad - e^{(\sigma_2 y_2 + \frac{1}{2}\sigma_2^2\tau)} \int_0^\pi q_2 e^{q_2 s} N\left(\frac{\sigma_1 a_4(y_1, y_2, \tau, s, x(s))}{\sigma}\right) ds. \end{aligned} \tag{18}$$

If the value of $\frac{S_1}{S_2}$ reach the early exercise ratio at the first time, that is $\frac{S_1}{S_2} = X(T - \tau)$ or $\sigma_1 y_1 - \sigma_2 y_2 = \sigma x(\tau)$, it is optimal to exercise this AEO. By imposing the boundary condition (III) into (18), we obtain an implicit representation of the early exercise ratio as follows:

$$\begin{aligned} &e^{(q_1 + \frac{1}{2}\sigma_1^2)\tau + \sigma_1 x(\tau)} - e^{(q_2 + \frac{1}{2}\sigma_2^2)\tau} \\ &= e^{\sigma_1 x(\tau) + \frac{1}{2}\sigma_1^2\tau} N\left(\frac{a_1}{\sigma}\right) - e^{\frac{1}{2}\sigma_2^2\tau} N\left(\frac{a_2}{\sigma}\right) \\ &\quad + e^{\sigma_1 x(\tau) + \frac{1}{2}\sigma_1^2\tau} \int_0^\tau q_1 e^{q_1 s} N\left(\frac{a_3}{\sigma}\right) ds \\ &\quad - e^{\frac{1}{2}\sigma_2^2\tau} \int_0^\tau q_2 e^{q_2 s} N\left(\frac{a_4}{\sigma}\right) ds, \end{aligned} \tag{19}$$

where

$$\begin{aligned} \bar{a}_1 &= \frac{1}{\sqrt{\tau}}(x(\tau) + (\sigma_1^2 - \rho\sigma_1\sigma_2)\tau), \\ \bar{a}_2 &= \frac{1}{\sqrt{\tau}}(x(\tau) + (\rho\sigma_1\sigma_2 - \sigma_2^2)\tau), \\ \bar{a}_3 &= \frac{1}{\sqrt{\tau - s}}(x(\tau) + (\sigma_1^2 - \rho\sigma_1\sigma_2)(\tau - s) - x(s)), \\ \bar{a}_4 &= \frac{1}{\sqrt{\tau - s}}(x(\tau) + (\rho\sigma_1\sigma_2 - \sigma_2^2)(\tau - s) - x(s)). \end{aligned}$$

By using the relation of (15), we replace $x(\tau)$ by $X(\tau)$ obtaining the following equation

$$\begin{aligned} X(T - \tau) - 1 = \; &X(T - \tau) e^{-q_1\tau} N\left(\frac{\hat{a}_1}{\sigma}\right) - e^{-q_2\tau} N\left(\frac{\hat{a}_2}{\sigma}\right) \\ &+ X(T - \tau) e^{-q_1\tau} \int_0^\tau q_1 e^{q_1 s} N\left(\frac{\hat{a}_3}{\sigma}\right) ds \\ &- e^{-q_2\tau} \int_0^\tau q_2 e^{q_2 s} N\left(\frac{\hat{a}_4}{\sigma}\right) ds, \end{aligned}$$

where

$$\begin{aligned}\hat{a}_1 &= \tfrac{1}{\sqrt{\tau}}(\log X(T-\tau) + \tfrac{1}{2}(\sigma^2 - 2q_1 + 2q_2)\tau),\\ \hat{a}_2 &= \tfrac{1}{\sqrt{\tau}}(\log X(T-\tau) - \tfrac{1}{2}(\sigma^2 + 2q_1 - 2q_2)\tau),\\ \hat{a}_3 &= \tfrac{1}{\sqrt{\tau-s}}(\log \tfrac{X(T-\tau)}{X(T-s)} + \tfrac{1}{2}(\sigma^2 - 2q_1 + 2q_2)(\tau - s)),\\ \hat{a}_4 &= \tfrac{1}{\sqrt{\tau-s}}(\log \tfrac{X(T-\tau)}{X(T-s)} - \tfrac{1}{2}(\sigma^2 + 2q_1 - 2q_2)(\tau - s)).\end{aligned}$$

IV. An Asymptotic Solution of Finite-Lived AEO

Unfortunately, the explicit solution of (9) is not easy to obtain when the maturity date T is finite. In this section we will apply the properties of the complementary error function to provide an asymptotic solution for (9).

Let $\mathrm{erfc}(x)$ denotes the complementary error function, i.e.

$$\mathrm{erfc}(x) = \frac{2}{\sqrt{\pi}}\int_x^\infty e^{-t^2}dt.$$

The relation between error function and normal distribution function is

$$N(x) = 1 - \frac{1}{2}\mathrm{erfc}(\frac{x}{\sqrt{2}}). \quad (20)$$

By using the Taylor expansion and integration by parts, the complementary error function is asymptotic to

$$\mathrm{erfc}(x) = \frac{e^{-x^2}}{\sqrt{\pi}x}(1 - \frac{1}{2x^2} + - \cdots) \sim \frac{e^{-x^2}}{\sqrt{\pi}x}, \text{ as } x \to \infty. \quad (21)$$

In this section, we will replace $N(x)$ in terms of $\mathrm{erfc}(\frac{x}{\sqrt{2}})$ and provide an asymptotic solution for the early exercise ratio $X(T-\tau)$ as the remaining time near to zero.

Before deriving the asymptotic expression of $X(T-\tau)$, we address the following lemma which has been provided in [2].

Lemma 1: Let $B(z,\tau)$ be a monotone decreasing function of z on $[0,1]$. Suppose that there is a $z_0 \in [0,1]$ such that $B(z_0,\tau) = 0$ for all τ and that $B^2(z,\tau) \to \infty$ for all $z \neq z_0$ as $\tau \to 0$. Then, as τ is near to 0, we have following two asymptotic formulas:

$$\int_0^1 A(z)e^{-B^2(z)}dz \sim A(z_0)\frac{\sqrt{\pi}}{|B_z(z_0)|}, \quad (22)$$

$$\frac{1}{\sqrt{\pi}}\int_0^1 B^{-1}e^{-B^2}dz \sim -\frac{B_{zz}(z_0)}{2|B_z(z_0)|^3}. \quad (23)$$

Proof: Since $B^2(z,\tau) \to \infty$ for all $z \neq z_0$ as $\tau \to 0$ and $B^2(z_0,\tau) = 0$ for all τ then $e^{-B^2(z,\tau)} \to 0$ for all $z \neq z_0$ as $\tau \to 0$ and $e^{-B^2(z_0,\tau)} = 1$ for all τ. This implies that the neighborhood of z_0 provides the main contribution to the value of the Laplace integral as τ is near to 0. Thus, we expand $B^2(z,\tau)$ at $z = z_0$ by using Taylor expansion and obtain that

$$\begin{aligned}B^2(z,\tau) = \ & B^2(z_0,\tau) + 2B(z_0,\tau)B_z(z_0,\tau)(z - z_0)\\ & + B_z^2(z_0,\tau)(z-z_0)^2 + \cdots\\ & \sim B_z^2(z_0,\tau)(z-z_0)^2\end{aligned}$$

since $B(z_0,\tau) = 0$. And then, we use this expansion formula in the exponent. As τ is near to 0, the Laplace integral will approximate to the following Gaussian integral

$$\int_0^1 A(z)e^{-B^2(z,\tau)} \sim A(z_0)\int_0^1 e^{-B_z^2(z_0,\tau)(z-z_0)^2}dz.$$

Now, we rewrite the above Gaussion integral by its asymptotic formula and obtain that

$$A(z_0)\int_0^1 e^{-B_z^2(z_0,\tau)(z-z_0)^2}dz \sim A(z_0)\frac{\sqrt{\pi}}{|B_z(z_0,\tau)|}.$$

Since $B(z_0) = 0$ then $B^{-1} \to \infty$ as $z \to z_0$. The above result can not be applied when $A(z) = B^{-1}(z,\tau)$. Now, we rewrite $B^{-1}(z,\tau)$ as follows:

$$\begin{aligned}B^{-1} &= \tfrac{B_z(z_0)(z-z_0) - B(z)}{B(z)B_z(z_0)(z-z_0)} + [B_z(z_0)(z-z_0)]^{-1}\\ &\sim -\tfrac{B_{zz}(z_0,\tau)}{2B_z^2(z_0)} + [B_z(z_0)(z-z_0)]^{-1}\end{aligned}$$

Here, the final term of above equation is obtain by applying Tayor expansion to $B(z)$ at $z = z_0$. Now, we use this asymptotic formula to substitute B^{-1} and obtain the following formula

$$\begin{aligned}&\tfrac{1}{\sqrt{\pi}}\textstyle\int_0^1 B^{-1}e^{-B^2}dz\\ &\sim \tfrac{1}{\sqrt{\pi}}\textstyle\int_0^1 \left(-\tfrac{B_{zz}(z_0,\tau)}{2B_z^2(z_0)} + [B_z(z_0)(z-z_0)]^{-1}\right)e^{-B_z^2(z_0)(z-z_0)^2}dz\\ &\sim -\tfrac{B_{zz}(z_0)}{2|B_z(z_0)|^3}.\end{aligned}$$

■

We now begin to derive the asymptotic expression of $X(T-\tau)$ as τ is near to 0.

Theorem 2: The asymptotic solution of (9), when τ closed to zero, is as follows:

1) For $q_1 > q_2$,

$$X(T-\tau) \sim (1 + \frac{\sigma^2\tau(\frac{q_1}{q_1-q_2}) + d}{1 + \sigma^2\tau(\frac{q_1}{q_1-q_2})^2})e^{(q_1-q_2)\tau}, \quad (24)$$

where $d = \sqrt{d_1 - d_2}$. $d_1 = \sigma^4\tau^2(\frac{q_1}{q_1-q_2})^2$, $d_2 = 2\sigma^2\tau\log[\frac{(q_1-q_2)\pi\sqrt{2\tau}}{\sigma^2}](1 + \sigma^2\tau(\frac{q_1}{q_1-q_2})^2)$.

2) for $q_1 = q_2$,

$$X(T-\tau) \sim e^{-\left[-2\sigma^2\tau\log\left(\sqrt{2}\pi\tau\sigma^{-2}q_1\right)\right]^{1/2}}.$$

Proof: By defining $Y(T-\tau) = X(T-\tau)e^{(q_2-q_1)\tau}$, (9) can be converted as follows:

$$\begin{aligned}Y(T-\tau)e^{q_1\tau} - e^{q_2\tau} = \ & Y(T-\tau)N(\tfrac{\tilde{a}_1(\tau)}{\sigma}) - N(\tfrac{\tilde{a}_2(\tau)}{\sigma})\\ & + Y(T-\tau)\textstyle\int_0^\tau q_1e^{q_1s}N(\tfrac{\tilde{a}_3(\tau,s)}{\sigma})ds\\ & - \textstyle\int_0^\tau q_2e^{q_2s}N(\tfrac{\tilde{a}_4(\tau,s)}{\sigma})ds,\end{aligned} \quad (25)$$

where

$$\begin{aligned}&\tilde{a}_1(\tau) = \tfrac{\log Y(T-\tau)}{\sqrt{\tau}} + \tfrac{1}{2}\sigma^2\sqrt{\tau} = \tilde{a}_2(\tau) + \sigma^2\sqrt{\tau},\\ &\tilde{a}_3(\tau,s) = \tfrac{\log\frac{Y(T-\tau)}{Y(T-s)}}{\sqrt{\tau-s}} + \tfrac{1}{2}\sigma^2\sqrt{\tau-s} = \tilde{a}_4(\tau,s) + \sigma^2\sqrt{\tau-s}.\end{aligned}$$

By applying (20), we express (25) in terms of the complementary error function as follows:

$$\begin{aligned}&Y(T-\tau)\mathrm{erfc}(\tfrac{1}{\sqrt{2}}(\tfrac{\log Y(T-\tau)}{\sigma\sqrt{\tau}}\\ &+\tfrac{1}{2}\sigma\sqrt{\tau})) - \mathrm{erfc}(\tfrac{1}{\sqrt{2}}(\tfrac{\log Y(T-\tau)}{\sigma\sqrt{\tau}} - \tfrac{1}{2}\sigma\sqrt{\tau}))\\ &= \lim_{y\to Y(T-\tau)}\{y\textstyle\int_0^\tau q_1e^{q_1s}\mathrm{erfc}(\tfrac{1}{\sqrt{2}}(\tfrac{\log\frac{y}{Y(T-s)}}{\sigma\sqrt{\tau-s}} + \tfrac{1}{2}\sigma\sqrt{\tau-s}))ds\\ &\quad - \textstyle\int_0^\tau q_2e^{q_2s}\mathrm{erfc}(\tfrac{1}{\sqrt{2}}(\tfrac{\log\frac{y}{Y(T-s)}}{\sigma\sqrt{\tau-s}} - \tfrac{1}{2}\sigma\sqrt{\tau-s}))ds\}.\end{aligned} \quad (26)$$

As τ near to zero, since $X(T-\tau) > X(T) \geq 1$ and $Y(T-\tau) = X(T-\tau)e^{(q_2-q_1)\tau}$ then $\frac{1}{\sqrt{2}}(\frac{\log Y(T-\tau)}{\sigma\sqrt{\tau}} \pm \frac{1}{2}\sigma\sqrt{\tau}) =$

$\frac{1}{\sqrt{2}}(\frac{\log X(T-\tau)}{\sigma\sqrt{\tau}} \pm \frac{1}{2}(\sigma + \frac{q_2-q_1}{\sigma})\sqrt{\tau})$ tends to infinity Thus, component of LHS of (26), $\mathrm{erfc}(\frac{1}{\sqrt{2}}(\frac{\log Y(T-\tau)}{\sigma\sqrt{\tau}} \pm \frac{1}{2}\sigma\sqrt{\tau}))$, has the following asymptotic form:

$$\begin{aligned}&\mathrm{erfc}(\tfrac{1}{\sqrt{2}}(\tfrac{\log Y(T-\tau)}{\sigma\sqrt{\tau}} \pm \tfrac{1}{2}\sigma\sqrt{\tau}))\\ &\sim \tfrac{1}{\sqrt{\pi}}\tfrac{1}{\frac{\log Y(T-\tau)}{\sigma\sqrt{\tau}}\pm\frac{1}{2}\sqrt{\sigma^2\tau}}e^{-\frac{1}{2}(\frac{\log Y(T-\tau)}{\sigma\sqrt{\tau}}\pm\frac{1}{2}\sigma\sqrt{\tau})^2},\end{aligned} \tag{27}$$

as $\tau \to 0$. By applying the integral mean value theorem, the integrand of RHS of (26) can be rewritten as

$$\begin{aligned}&\mathrm{erfc}(\tfrac{1}{\sqrt{2}}(\tfrac{\log\frac{y}{Y(T-s)}}{\sigma\sqrt{\tau-s}} \pm \tfrac{1}{2}\sigma\sqrt{\tau-s}))\\ &= \mathrm{erfc}(\tfrac{1}{\sqrt{2}}(\tfrac{\log\frac{y}{Y(T-s)}}{\sigma\sqrt{\tau-s}})) \mp \tfrac{2}{\sqrt{\pi}}\int_{\frac{1}{\sqrt{2}}(\frac{\log\frac{y}{Y(T-s)}}{\sigma\sqrt{\tau-s}})}^{\frac{1}{\sqrt{2}}(\frac{\log\frac{y}{Y(T-s)}}{\sigma\sqrt{\tau-s}}\pm\frac{1}{2}\sigma\sqrt{\tau-s})} e^{-\eta^2}d\eta\\ &= \mathrm{erfc}(\tfrac{1}{\sqrt{2}}(\tfrac{\log\frac{y}{Y(T-s)}}{\sigma\sqrt{\tau-s}})) \mp \tfrac{1}{\sqrt{\pi}}\sigma\sqrt{\tau-s}e^{-c^2},\end{aligned}$$

where c lies between $\frac{1}{\sqrt{2}}(\frac{\log\frac{y}{Y(T-s)}}{\sigma\sqrt{\tau-s}})$ and $\frac{1}{\sqrt{2}}(\frac{\log\frac{y}{Y(T-s)}}{\sigma\sqrt{\tau-s}} \pm \frac{1}{2}\sigma\sqrt{\tau-s})$. By setting $s = \tau z$, and considering τ near to zero, we have $c \sim \frac{1}{\sqrt{2}}(\frac{\log\frac{y}{Y(T-s)}}{\sigma\sqrt{\tau-s}})$ and $e^{q_i\tau} \sim 1$, $i = 1, 2$. Then the RHS of (26) has the following asymptotic form

$$\begin{aligned}&\lim_{y\to Y(T-\tau)}\{(q_1y-q_2)\tau\int_0^1 \mathrm{erfc}(\tfrac{1}{\sqrt{2}}\tfrac{\log\frac{y}{Y(T-\tau z)}}{\sqrt{\sigma^2\tau(1-z)}})dz\\ &-(q_1y-q_2)\tfrac{\sigma^2\tau^{\frac{3}{2}}}{\sqrt{\pi}}\int_0^1\sqrt{1-z}e^{\frac{-\log^2\frac{y}{Y(T-\tau z)}}{2\sigma^2\tau(1-z)}}dz\}.\end{aligned}$$

Therefore, we derive the following asymptotic equation of (26):

$$\begin{aligned}&\sqrt{\tfrac{2}{\pi}}\tfrac{\sigma^2\tau^{\frac{3}{2}}}{\log^2 Y(T-\tau)}e^{-\frac{1}{2}(\frac{\log Y(T-\tau)}{\sqrt{\sigma^2\tau}}-\frac{1}{2}\sqrt{\sigma^2\tau})^2}\\ &\sim \lim_{y\to Y(T-\tau)}\{(q_1y-q_2)\tau\int_0^1 \mathrm{erfc}(\tfrac{1}{\sqrt{2}}\tfrac{\log\frac{y}{Y(T-\tau z)}}{\sqrt{\sigma^2\tau(1-z)}})dz\\ &\quad-(q_1y-q_2)\sigma^2\tau^{\frac{3}{2}}\int_0^1\sqrt{\tfrac{1-z}{\pi}}e^{\frac{-\log^2\frac{y}{Y(T-\tau z)}}{2\sigma^2\tau(1-z)}}dz\}.\end{aligned} \tag{28}$$

Note that

$$Y(T) = \max(1, \frac{q_2}{q_1}).$$

Now we consider the case of $q_1 \geq q_2$ and let

$$\alpha(\tau) = \frac{-\log Y(T-\tau)}{\sqrt{\tau}}, \text{ for } q_1 \geq q_2,$$

and then (28) can be converted as follows:

$$\begin{aligned}&\tfrac{\sigma^2\tau^{3/2}}{\tau\alpha^2(\tau)}e^{-\frac{\tau\alpha^2(\tau)}{2\sigma^2\tau}} \sim\\ &\sqrt{\tfrac{\pi}{2}}\left(q_1e^{-\sqrt{\tau}\alpha(\tau)}-q_2\right)\tau\int_0^1 \mathrm{erfc}\left(B(z,\alpha(z\tau),y)\right)dz\\ &-\left(q_1e^{-\sqrt{\tau}\alpha(\tau)}-q_2\right)\tfrac{\sigma^2\tau^{3/2}}{\sqrt{2}}\int_0^1\sqrt{1-z}e^{-B^2(z,\alpha(z\tau),y)}dz,\end{aligned} \tag{29}$$

as $y \to Y(T-\tau)$. Here

$$B(z,\alpha(\tau z),y) = \frac{\sqrt{z}\alpha(\tau z) - \frac{\log\frac{q_2}{q_1 y}}{\sqrt{\tau}}}{\sigma\sqrt{2(1-z)}}.$$

By applying the definition of $\alpha(\tau)$ and take the limit under the integral, we have

$$B(z,\alpha(\tau z),\alpha(\tau)) = \frac{\sqrt{z}\alpha(\tau z)-\alpha(\tau)}{\sigma\sqrt{2(1-z)}}.$$

For convenient we denote $B(z,\tau,y)$ as $B(z)$. Since $Y(T-\tau z)$ is a monotone increasing function of z, then there is an unique number z_0 such that

$$Y(T-\tau z_0) = y$$

and $Y(T-\tau z_0) < y$ for $z < z_0$ and $Y(T-\tau z) > y$ for $z > z_0$. This implies that $B(z,\tau,y) \to \infty$ for all z in $[0, z_0)$ and $B(z,\tau,y) \to -\infty$ for all z in $(z_0, 1]$, as $\tau \to 0$. Thus, we replace $\mathrm{erfc}(B(z))$ by using (21) for $B(z,\tau) \to \pm\infty$ when τ is small. Then the first integral of (29) can be rewritten as the following asymptotic formula :

$$\begin{aligned}&\int_0^1 \mathrm{erfc}(B(z))dz\\ &\sim \tfrac{1}{\sqrt{\pi}}\int_0^{z_0(x)}B^{-1}(z)e^{-B^2(z)}dz + \int_{z_0(x)}^1(2+B^{-1}(z)\tfrac{e^{-B^2(z)}}{\sqrt{\pi}})dz\\ &= 2[1-z_0] + \tfrac{1}{\sqrt{\pi}}\int_0^1 B^{-1}(z)e^{-B^2(z)}dz.\end{aligned}$$

In order to find out the asymptotic solution of (29), we consider that y approaches to $Y(T-\tau)$ and sets $z_0 = 1$. The remainder is to evaluate the following two integrals

$$\frac{1}{\sqrt{\pi}}\int_0^1 B^{-1}(z)e^{-B^2(z)}dz \text{ and } \int_0^1\sqrt{1-z}e^{-B^2(z)}dz.$$

Since $B(z_0,\tau) = 0$ for all τ and, for $z \neq z_0$, $B^2(z,\tau) \to \infty$ as $\tau \to 0$ then

$$\frac{1}{\sqrt{\pi}}\int_0^1 B^{-1}(z)e^{-B^2(z)}dz \sim -\frac{B_{zz}(z_0)}{2|B_z(z_0)|^3}, \tag{30}$$

$$\int_0^1\sqrt{1-z}e^{-B^2(z)}dz \sim \sqrt{1-z_0}\frac{\sqrt{\pi}}{|B_z(z_0)|}, \tag{31}$$

by using lemma 1.

The limit of the first integral in (29) is asymptotic to the RHS of (30). We see that this asymptotic expression is

$$\int_0^1 \mathrm{erfc}B(z)dz \sim \frac{2\sqrt{\pi}}{\alpha^2(\tau)} \tag{32}$$

as $z_0 \to 1$. By applying (31), the second integral of (29) tends to zero as $z_0 \to 1$. So we obtain the following equation:

$$e^{-\frac{\alpha^2(\tau)}{2\sigma^2}} \sim \frac{\sqrt{2\tau}\pi}{\sigma^2}\left(q_1e^{-\sqrt{\tau}\alpha(\tau)}-q_2\right). \tag{33}$$

Since $Y(T-\tau) = e^{-\sqrt{\tau}\alpha(\tau)}$, (33) can be rewritten as

$$e^{-\frac{\log^2 Y(T-\tau)}{2\sigma^2\tau}} \sim \frac{\sqrt{2\tau}\pi}{\sigma^2}(q_1Y(T-\tau)-q_2). \tag{34}$$

Let $Y(T-\tau) = 1 + y(\tau)$, then (34) can be rewritten as

$$e^{-\frac{\log^2(1+y(\tau))}{2\sigma^2\tau}} \sim \frac{\sqrt{2\tau}\pi}{\sigma^2}(q_1y(\tau)+q_1-q_2).$$

And then, we have

$$-\frac{\log^2(1+y(\tau))}{2\sigma^2\tau} \sim \log[\frac{(q_1-q_2)\pi\sqrt{2\tau}}{\sigma^2}]+\log(\frac{q_1}{q_1-q_2}y(\tau)+1).$$

Multiplying above equation by $2\sigma^2\tau$ and expanding $\log^2(1+y(\tau))$ and $\log(\frac{q_1}{q_1-q_2}y(\tau)+1)$ at 1, we obtain that

$$\begin{aligned}&-y^2(\tau) \sim 2\sigma^2\tau\log[\tfrac{(q_1-q_2)\pi\sqrt{2\tau}}{\sigma^2}]\\ &\quad+2\sigma^2\tau(\tfrac{q_1}{q_1-q_2}y(\tau)+\tfrac{q_1^2}{2(q_1-q_2)^2}y^2(\tau)).\end{aligned}$$

This implies that

$$\begin{aligned}&1+\sigma^2\tau\tfrac{q_1^2}{(q_1-q_2)^2}]y^2(\tau)\\&+2\sigma^2\tau(\tfrac{q_1}{q_1-q_2})y(\tau)\\&+2\sigma^2\tau\log[\tfrac{(q_1-q_2)\pi\sqrt{2\tau}}{\sigma^2}=0,\end{aligned}$$

and that the solution of this quadratic equation is

$$y(\tau)=\frac{\sigma^2\tau(\frac{q_1}{q_1-q_2})+d}{1+\sigma^2\tau(\frac{q_1}{q_1-q_2})^2},$$

where

$$\begin{aligned}d^2=\ &\sigma^4\tau^2(\tfrac{q_1}{q_1-q_2})^2\\&-2\sigma^2\tau\log[\tfrac{(q_1-q_2)\pi\sqrt{2\tau}}{\sigma^2}](1+\sigma^2\tau(\tfrac{q_1}{q_1-q_2})^2).\end{aligned}$$

Here, we select positive term to make sure $y(\tau)\geq 0$. So, we have

$$Y(T-\tau)=1+\frac{\sigma^2\tau(\frac{q_1}{q_1-q_2})+d}{1+\sigma^2\tau(\frac{q_1}{q_1-q_2})^2},$$

and

$$X(T-\tau)=(1+\frac{\sigma^2\tau(\frac{q_1}{q_1-q_2})+d}{1+\sigma^2\tau(\frac{q_1}{q_1-q_2})^2})e^{(q_1-q_2)\tau}.$$

However, the above approximation can not be applied to the case $q_1 = q_2$. We use first order approximation to $q_1e^{-\sqrt{\tau}\alpha(\tau)}-q_2$ and obtain

$$q_1e^{-\sqrt{\tau}\alpha(\tau)}-q_1\sim -q_1\sqrt{\tau}\alpha(\tau),\ \text{as } \tau\to 0.$$

Now, (33) can be rewritten as follows:

$$e^{-\frac{\alpha^2(\tau)}{2\sigma^2}}\sim -\sqrt{2}\pi\tau\sigma^{-2}q_1\alpha(\tau).$$

Beginning the iteration scheme from the initial value $\alpha_0 = 0$, we obtain that

$$\alpha(\tau)\sim\left[-2\sigma^2\log\left(\sqrt{2}\pi\tau\sigma^{-2}q_1\right)\right]^{1/2}.$$

Then

$$Y(T-\tau)\sim e^{-\left[-2\sigma^2\tau\log\left(\sqrt{2}\pi\tau\sigma^{-2}q_1\right)\right]^{1/2}},$$

and

$$X(T-\tau)\sim e^{-\left[-2\sigma^2\tau\log\left(\sqrt{2}\pi\tau\sigma^{-2}q_1\right)\right]^{1/2}}.$$

■

V. NUMERICAL COMPARISON

The asymptotic formula is compared to the numerical solution of the IR method [6]. Figure 1 displays the graph of the case of that $\sigma_1=\sigma_2=0.5$, $q_1=0.02$, $q_2=0.01$ and $\rho=0.5$. Figure 2 displays the graph of the case of that $\sigma_1=\sigma_2=0.5$, $q_1=q_2=0.01$ and $\rho=0.5$. The solid curve is numerically computed by IR method and the dash curve is computed by our asymptotic formulas. These figures show that the results from our asymptotic formula and IR method are very close as time near to maturity.

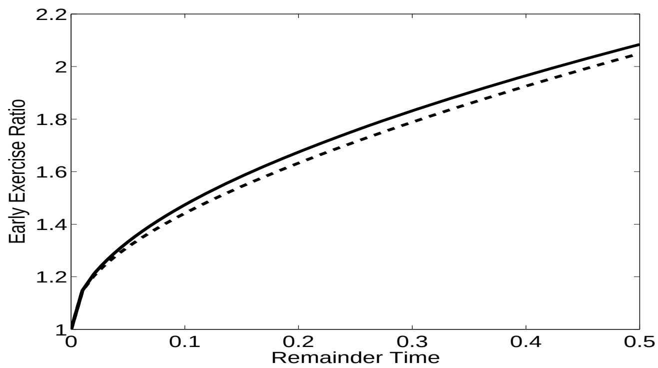

Fig. 1. The early exercise ratio $X(\tau)$ as a function of $\tau=T-t$ for $q_1=0.02$, $q_2=0.01$ with given by (24)(dash curve) and recursive integration method(solid curve)

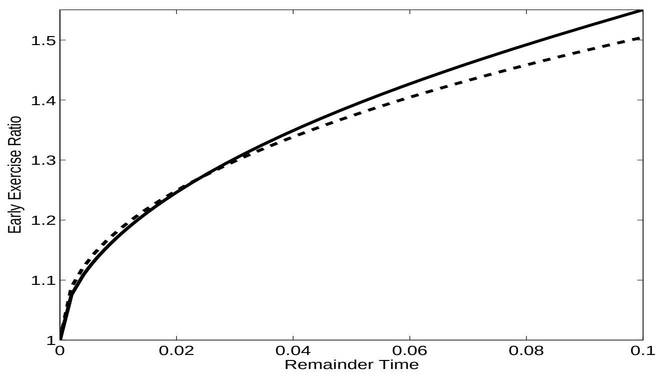

Fig. 2. The early exercise ratio $X(\tau)$ as a function of $\tau=T-t$ for $q_1=q_2=0.01$ with given by (**??**)(dash curve) and recursive integration method(solid curve)

VI. CONCLUSION

An AEO pricing model together with the early exercise ratio are modelled as a FBP and this FBP is converted into an IE. Meanwhile, the formula of this early exercise ratio is implicit in the solution of the IE. We propose an asymptotic solutions of the IE for the cases of $q_1>q_2$ and $q_1=q_2$, respectively. However, this approach can not derive an asymptotic formula for the case of $q_1<q_2$. We also extend the numerical method of one variable integral recursive method proposed by Kim [6] to the case of two variables. Compared with this numerical solution, our asymptotic solution of the IE is very close to the numerical solution as time near to maturity.

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