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Dynamic pricing and warranty policies for products with fixed lifetime

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ABSTRACT

We consider a repairable product with known market entry and departure times. A warranty policy is offered with product purchase, under which a customer can have a failed item repaired free of charge in the warranty period. It is assumed that customers are heterogeneous in their risk attitudes toward uncertain repair costs incurred after the warranty expires. The objective is to determine a joint dynamic pricing and warranty policy for the lifetime of the product, which maximizes the manufacturer's expected profit. In the first part of the analysis, we consider a linearly decreasing price function and a constant warranty length. We first study customers' purchase patterns under several different pricing strategies by the manufacturer and then discuss the optimal pricing and warranty strategy. In the second part, we assume that the warranty length can be altered once during the product lifetime in developing a joint pricing and warranty policy. Numerical studies show that a dynamic warranty policy can significantly outperform a fixed-length warranty policy.

1. Introduction

Pricing of products is a major decision for manufacturers (or sellers) and has become a challenging issue in today's marketplace, where the conditions change rapidly over time (Raman and Chatterjee, 1995). In recent years, dynamic pricing has become a common practice and attracted increased research attention. Many methods have been proposed to consider product life cycle, learning curves, diffusion and saturation effects, and uncertainties in demand and supply, in formulating optimal pricing strategies (see, for example, Lin, 2006; Polatoglu and Sahin, 2000; Zhao and Zheng, 2000; Raman and Chatterjee, 1995; Rajan et al., 1992; Kalish, 1983; Dolan and Jeuland, 1981).

In addition to product price, warranty policy is another important decision for a manufacturer. Product warranty is a contract between the manufacturer (or seller) of a product and the customers, which requires the manufacturer to provide repair or replacement services when the product fails within the warranty period (Polatoglu and Sahin, 1998; Blischke and Murthy, 1992). Besides being a protection mechanism for customers, product warranty increasingly serves as a marketing tool for durables like automobiles and high-tech goods (Menezes and Currim, 1992; DeCroix, 1999). In general, a warranty policy with a better coverage, such as a broader range and/or a longer period, can increase the manufacturer's competitive advantage; however, it will also result in higher costs for the manufacturer. Therefore, the manufacturer must deal with warranty policies using a strategic approach (Murthy and Blischke, 2000; Murthy and Djamaludin, 2002).

In this paper, we consider a product with known market entry and departure times. This situation is typical for high-tech products, where manufacturers have an incentive to introduce new versions that make old ones obsolete due to the fast development of technology (Waldman, 1993). This phenomenon is known as "planned obsolescence." As a result, the product's effective life time in the market is usually not long, which may induce more frequent sales of the new versions. Sometimes the product quits the market at a known time, signaling that the old generation is entirely eliminated from the market and replaced by a new generation. For example, the Federal television regulators announced a plan that would turn off the analog television signal on April 7, 2009, forcing all broadcasters to switch to high definition television by that time (Boliek, 2004). However, manufacturers must still produce the traditional version of televisions to meet customers' demand in the current stage. Hence, it is an important decision for them to make on how to price the product and provide a suitable warranty policy during this period so that sufficient revenue can be earned but no great cost will be incurred.

From the customers' perspective, they need an appealing warranty policy as a protection because the repair cost is expensive for high-tech products. For example, labor rates for HDTV, projection and LCD TV's can be as much as $250 per hour. Repair costs for computer problems associated with processors, hard drives,

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monitors, memory components, and DVD drives are between $150 and $900 (RepairTechinc, 2007). Therefore, warranties offer necessary protection to customers. In addition, customers are heterogeneous in their preference of the product price and warranty length. Although the market entry and departure times are the same to all customers, they may choose to purchase the product at different time because their expectations towards the product price and warranty policy are different. Therefore, the manufacturer must take into consideration the heterogeneity of customers when making decision of profit maximization.

In our study, we develop a mathematical model to consider a joint product pricing and warranty policy for a repairable high-tech product over its effective lifetime, which starts from its entry into the market and ends when it quits the market. The manufacturer offers a free warranty with product purchase, under which the customer can have the product repaired free of charge if it fails during the warranty period. If the product fails after the warranty expires, the customer has to bear the repair cost. We assume that customers are heterogeneous in their risk attitudes toward uncertain repair costs after the warranty expires. Consequently, a customer's purchasing decision, including purchase price and time of purchase, depends on the manufacturer's product pricing strategy, the product's remaining life at the time of purchase, product warranty, and the uncertainty associated with the repair cost after the warranty period.

Our objective is to determine a joint dynamic pricing and warranty policy in the product's lifetime, which will allow the manufacturer to gain the maximum profit when the target customers are heterogeneous. In the first part of our analysis, we assume a policy with a linearly decreasing price function and a fixed warranty length. We study customers' purchase patterns under several different pricing strategies by the manufacturer and then discuss the optimal pricing and warranty strategy. In the second part, we address the issue of whether a constant warranty length should be used for the entire product lifetime. In our analysis, we assume that the warranty length can be changed once at a given point in the product lifetime, and we develop a joint pricing and warranty policy. Through comparison, we find that the manufacturer will benefit from offering dynamic warranty policy, which can bring them more profit than constant warranty.

This paper is organized as follows. We give the literature in Section 2 and the assumptions and formulation of the basic model in Section 3. In Section 4, we consider a linear price function with a fixed warranty length to derive the customers' purchase patterns and the optimal pricing and warranty for the manufacturer. In Section 5, we study a dynamic warranty policy, which allows a change to the warranty length at a given point in the product lifetime. In Section 6, we discuss managerial implications of the proposed models based on the results obtained from numerical experiments. In Section 7, we summarize the main results of this paper and discuss several possible extensions of this paper. All proofs of analytical results are given in the appendix.

2. Literature review

There has been an increased research attention on the joint determination of optimal product pricing and warranty policy. Glickman and Berger (1976) proposed an early model for this important research area. They assumed that the customers were homogeneous, with their demand determined by an exponential function of price and warranty length. The optimal price and warranty length were obtained by maximizing the manufacturer's profit function. Murthy (1990) proposed a model to jointly determine the price, warranty length, and product reliability to maximize the total expected profit of a new product for a manufacturer. Menezes and Currim (1992) derived the optimal price and warranty length for particular classes of failure rate, demand, and competitor-response functions. Mitra and Patankar (1997) investigated the selection of product price, total warranty length, and initial warranty length for warranty programs involving options for extended warranty periods. DeCroix (1999) considered an oligopoly market and proposed a game-theoretic model for determining the optimal product price, reliability, and warranty length. Ladany and Shore (2007) developed a general model to make the optimal warranty decision by using the response modeling methodology, assuming that the demand follows a Cobb–Douglas-type function.

Market dynamics has also been incorporated into the pricing and warranty decisions. Mesak (1996) presented diffusion models to derive the optimal pricing policy and warranty period for a monopolist selling new products. Based on the assumption that the demand depends on the product price, warranty period, and cumulative number of adopters, Mesak (1996) obtained the optimal trajectories for both the price and warranty length over a planning horizon. Teng and Thompson (1996) developed a general framework to determine the optimal price and quality policies of new products for a monopolistic manufacturer during a planning period. In the proposed framework, they considered the learning effects on the supply (manufacturer) side and the diffusion and saturation effects on the demand (customers) side. Lin and Shue (2005), Wu et al. (2006) and Huang et al. (2007) extended the model by Teng and Thompson (1996). They considered warranty length instead of quality, assuming that the demand is determined by product price and warranty length. Optimal paths of price and warranty length of free replacement policy were derived in a pre-determined life cycle of the product.

In our proposed model, we assume that the customers are risk-averse with a random risk-aversion parameter to reflect the customers' heterogeneity in their preference on the product. Several researchers have considered risk preference in the warranty policy analysis (see, for example, Chun and Tang, 1995; Ritchken and Tapiero, 1986; Ritchken, 1985).

Note that, in Teng and Thompson's (1996) framework and those followed, the heterogeneity in customers' purchasing decisions was not considered. In other words, under given product price and warranty length, all customers have the same preference for the product. In practice, customers may have different attitudes towards the product, and, therefore, purchasing decisions will be different both across customers and over time. Furthermore, limited product lifetime, an important characteristic of high-tech products, was not considered in these studies.

When noticing the limitations of the previous research, this study adds to the literature by considering customers' heterogeneity in their risk preference and fixed product lifetime. Our aim is to examine all customers dynamically over time to help the manufacturer design the price and warranty to acquire maximal profit.

3. Basic model

Consider a product with an effective lifetime from time 0 to T. The target market of the product consists of Q customers, and the product value to a customer is proportional to the time he or she possesses the product. Let $t \geqslant 0$ be the time of purchase by a customer. Then, the product value is a function of t, determined by

$$R(t) = \begin{cases} k(T-t), & t \in [0, T], \\ 0, & t > T, \end{cases}$$

where k is a positive constant. We assume k is exogenous and the same across all customers.

The product is subject to random failure, with the time to failure following an exponential distribution with parameter λ. Note that we assume a constant failure rate to simplify our analysis. Our analysis can be modified for the Weibull distribution with an increasing failure rate. We further assume that the customers are risk-averse toward uncertain repair costs due to product failure with the following (dis-)utility function

$$u(x) = -\exp(ax), \tag{1}$$

where $a > 0$ is the risk-aversion parameter. This utility function implies that the customer is constantly risk-averse because the measure of risk aversion, $u''(x)/u'(x)$, equals a, where $u'(x)$ and $u''(x)$ are the first and second derivatives of $u(x)$, respectively (Keeney and Raiffa, 1993). A larger a implies a higher level of risk aversion. We let a follow a uniform distribution in $[a_\ell, a_u]$ to describe the variation in risk attitude among the customers.

The net value of the product to a customer is determined by $R(t)$ plus the total dis-utility associated with the total repair cost incurred in the period that is not covered by the warranty (i.e., from $t + T_w$ to T). We assume the customers do not have price expectation and will make a purchase decision right after the net product value exceeds the price.

The manufacturer is risk-neutral and uses a dynamic pricing strategy with the price as a function of the customers' purchase times. We use $P(t)$ to denote the product price at time t. The manufacturer offers a free warranty with a fixed time length, T_w, from the point of purchase, which entitles the customer to receive a minimal repair service free of charge if a product failure occurs during the warranty period. It is assumed that the repair time is negligible, and that, after the repair, the product is restored to its operational state just prior to failure. We let C_r denote the cost of a minimal repair to the manufacturer. If the product fails after the warranty expires, the customer will repair it at his or her own expense. For simplicity, we assume that the customer's unit repair cost is also C_r. After time T, the effective lifetime of the product, the manufacturer does not provide repair services.

The total profit to the manufacturer is determined by the total revenue from the product sale minus the total product manufacturing cost and the total repair cost incurred in the warranty period. We assume that the manufacturer's unit cost of manufacturing the product is a constant, C_m. The manufacturer's objective is to find the dynamic pricing and warranty policies to maximize expected total profit.

Let $C_r(t)$ denote the total repair cost to the customer who purchases the product at time t. The certainty equivalent of $u(C_r(t))$, denoted by $\overline{C}_r(t)$, is the amount such that the customer is indifferent between this fixed amount and the uncertain repair cost (Keeney and Raiffa, 1993); i.e.,

$$u(\overline{C}_r(t)) = E[u(C_r(t))]. \tag{2}$$

Since the number of product failures in the time interval $[t + T_w, T]$ is Poisson distributed with mean $m(T - (t + T_w)) = \int_{t+Tw}^{T} \lambda \, \mathrm{d}s = \lambda(T - (t + T_w))$, the expected utility in $[t + T_w, T]$ is

$$E[u(C_r(t)] = \sum_{i=0}^{\infty} (-\exp(aiC_r)) \times \frac{\lambda^i (T - T_w - t)^i \mathrm{e}^{-\lambda(T - T_w - t)}}{i!}, \tag{3}$$

which can be expressed as

$$E[u(C_r(t)] = -\mathrm{e}^{-\lambda(T - T_w - t)[1 - \exp(aCr)]}. \tag{4}$$

From (2) and (4), we obtain that

$$\overline{C}_r(t) = \frac{\exp(aCr) - 1}{a} \lambda(T - T_w - t), \tag{5}$$

where $\lambda(T - T_w - t)$ is the expected number of product failures in $[t + T_w, T]$.

We define $\widehat{C}_r = (\exp(aC_r) - 1)/a$ as the "equivalent" or perceived unit repair cost for the customer with risk-aversion parameter a. As a result, the equivalent total repair cost for the customer can be denoted as

$$\overline{C}_r(t) = \widehat{C}_r \lambda(T - T_w - t).$$

It is straightforward to verify that $\widehat{C}_r$ is an increasing function of a. Therefore, the perceived repair cost is higher for a more risk-averse customer.

We define the net product value to a customer who purchases the product at time t, denoted by $V_p(t)$, as the difference of $R(t)$ and $\overline{C}_r(t)$:

$$V_p(t) = R(t) - \overline{C}_r(t).$$

For convenience, we will use "product value" to represent the net product value hereafterward. Let a_1 be the solution to $\widehat{C}_r = k/\lambda$. It is straightforward to show that $V_p(t)$ is a decreasing function of t if $a < a_1$, a constant if $a = a_1$, and an increasing function if $a > a_1$ in $(0, T - T_w]$. Consequently, there are three cases according to the relationship between a_1, a_ℓ, and a_u:

Case 1. $a_u < a_1$: the product values to all the customers decrease in $(0, T - T_w]$.

Case 2. $a_\ell < a_1 < a_u$: the product values to the customers with $a_\ell < a < a_1$ decrease and those of the customers with $a_1 < a < a_u$ increase in $(0, T - T_w]$.

Case 3. $a_1 < a_\ell$: the product values to all the customers increase in $(0, T - T_w]$.

In this paper, we consider only Case 1 in our analysis. Results associated with Cases 2 and 3 can be found in Zhou (2006).

A customer will make a purchase when the following condition is met:

$$V_p(t) \geqslant P(t). \tag{6}$$

For a given $P(t)$, the purchase time of a customer is affected by a, since $\overline{C}_r(t)$ is a function of a. We use t_a to denote the purchase time for the customer with risk-aversion parameter a, and the profit for the manufacturer to sell an item to the customer is given by the selling price at t_a minus the unit production cost and the expected repair cost in the warranty period. Consequently, the manufacturer's total profit is

$$profit = \sum_{a \in \Omega} [P(t_a) - C_m - C_r \lambda T_w], \tag{7}$$

where Ω represents the set of the customers who make a purchase in the lifetime of the product. The optimal solution to the manufacturer is to find the product price function and warranty length to maximize total profit.

4. Linear dynamic pricing with constant warranty length

In this section, we assume that the manufacturer uses a linearly decreasing price function and a constant warranty length in the product lifetime. Let the price offered by the manufacturer be

$$P(t) = P_0 - bt, \tag{8}$$

where P_0 is the initial price at time 0, and $b > 0$ is the decreasing rate of the price over time. Assuming the customers do not have price expectation and will make a purchase immediately when (6) is satisfied, we obtain the following results.

Lemma 1. *If* $P_0 \leqslant (k-b)T_w + bT$, *then* $t_a \leqslant T - T_w$ *for all* a. *Otherwise,* $\{t_a : t_a \in (0, T - T_w]\} = \emptyset$.

The condition $P_0 \leqslant (k-b)T_w + bT$ in Lemma 1 can be met by a low initial price, a fast speed in price decrease, or both. When this condition is met, all the customers will purchase the product between times 0 and $T - T_w$. There are three possible scenarios depending on the manufacturer's pricing strategy. The first is that a group of customers purchase the product at time 0, and the remaining customers make the purchase between times 0 and $T - T_w$. In the second, all the customers purchase the product at $t = 0$. In the third, no customers purchase the product at time 0, but all of the customers make the purchase between times 0 and $T - T_w$. The following corollaries give conditions for additional purchase patterns.

Corollary 1. *If* $(k-b)T_w + bT < P_0 < bT$, *then* $t_a \in (T - T_w, T)$ *and is the same for all* a.

Corollary 2. *If* $P_0 - bT > \max(0, (k-b)T_w)$, *then* $\{t_a : t_a \in (0, T]\} = \emptyset$.

If the condition in the first corollary is met, all the customers make purchase at the same time between $T - T_w$ and T. Note that, in this corollary, $P_0 - bT < 0$ does not mean the product price will become negative at time T. It only implies that the price decreases fast enough to be 0 before T. We assume that it will remain 0 after that time. The second corollary gives the condition that no customers will purchase the product because the initial price is high, the price does not decrease fast enough, or both.

At $t = 0$, a customer will purchase the product if

$$kT - \widehat{C}_r\lambda(T - T_w) \geqslant P_0, \tag{9}$$

or,

$$\widehat{C}_r \leqslant \frac{kT - P_0}{\lambda(T - T_w)}.$$

Let a_0 be the solution to

$$\widehat{C}_r = \frac{kT - P_0}{\lambda(T - T_w)}.$$

Since $\widehat{C}_r$ is an increasing function of a, all the customers with $a \leqslant a_0$ will purchase the product at time 0. The proportion of the customers who purchase the product at time 0, denoted by δ_0, is given by

$$\delta_0 = \frac{a_0 - a_\ell}{\alpha_u - \alpha_\ell},$$

if $a_0 > a_\ell$, 0, otherwise. Let δ_1 denote the proportion of the customers who purchase the product after time 0. The value of δ_1 ranges from 0 to $1 - \delta_0$.

For the customers with risk-aversion parameter a, who makes a purchase at $t_a \in (0, T - T_w]$, we have

$$k(T - t_a) - \widehat{C}_r\lambda(T - T_w - t_a) \geqslant P_0 - bt_a, \tag{10}$$

or

$$(\lambda\widehat{C}_r - k + b)t_a \geqslant \widehat{C}_r\lambda(T - T_w) + P_0 - kT. \tag{11}$$

Since the purchase condition of these customers is not satisfied at time 0,

$$kT - \widehat{C}_r\lambda(T - T_w) < P_0. \tag{12}$$

Combine (11) and (12), and we obtain

$$\lambda\widehat{C}_r - k + b > 0.$$

This result can be written as the following lemma.

Lemma 2. *If* $t_a \in (0, T - T_w]$, *then* $\lambda\widehat{C}_r - k + b > 0$.

From (11) and Lemma 2, we obtain that

$$t_a = \frac{\widehat{C}_r\lambda(T - T_w) + P_0 - kT}{\lambda\widehat{C}_r - k + b}.$$

About this purchase time, we derive the following theorem.

Theorem 1. *If* $a_0 \in [a_\ell, a_u]$ *(or* $a_0 < a_\ell$*) and* $P_0 < (k-b)T_w + bT$, *then* t_a *is an increasing function of* $a \in (a_0, a_u]$ *(or* $a \in [a_\ell, a_u]$*).*

This theorem indicates that, when all the customers purchase the product between times 0 and $T - T_w$, more risk-averse customers make a purchase at a later time. Note that, if $P_0 = (k-b)T_w + bT$, then for $a \in (a_0, a_u]$ (or $a \in [a_\ell, a_u]$),

$$t_a = \frac{\widehat{C}_r\lambda(T - T_w) + P_0 - kT}{\lambda\widehat{C}_r - k + b} = T - T_w.$$

Therefore, in this situation, there are only two possible purchase times, 0 and $T - T_w$.

When the manufacturer can freely set both product price (b, P_0) and warranty length (T_w), we obtain the following theorem.

Theorem 2. *When* b, P_0 *and* T_w *can all be freely determined,* $profit_{\max} = Q(kT - C_m - C_r\lambda T)$ *with the optimal solution* $T_w^* = T$ *and* $P_0^* = kT$.

From this theorem, we know that, when both price (b, P_0) and warranty length (T_w) are decision variables, using a longer warranty length leads to a higher profit. This is because the customers are risk-averse, but the manufacturer is risk-neutral. The value of an increase in the warranty length to the customer is higher than the corresponding increase in the warranty cost to the manufacturer. Consequently, the manufacturer can increase the product price more than the warranty cost incurred by changing the warranty length. As a result, the manufacturer should offer the maximum warranty length as the optimal strategy.

Note that the result in Theorem 2 may not hold for Cases 2 and 3 mentioned in the last section, where $a_\ell < a_1 < a_u$ and $a_1 < a_\ell$, respectively. In other words, a combination of a finite rate of price decrease and a warranty length shorter than T can be the optimal solution. In this situation, the optimal price function and the optimal warranty length, T_w^*, can be found by a direct search procedure. Furthermore, if the manufacturer cannot change the price function, it is possible to find the optimal warranty length by a direct search procedure. When doing the search, we can obtain the following proposition.

Proposition 1. *For given* P_0, T_w^* *decreases as* b *increases, and for given* b, T_w^* *increases as* P_0 *increases.*

This result can be explained intuitively. First, when the product price decreases at a higher rate, the manufacturer does not need the same warranty length to attract customers. Furthermore, if the manufacturer charges a higher initial price, a longer warranty is necessary to compensate the additional cost to the customers.

5. Dynamic warranty policy

In this section, we will consider a warranty policy that allows the length of the warranty period to change once at a given point in the product lifetime. In Section 5.1, we assume a linear price function in deriving the optimal initial length and the optimal warranty length after the change. In Section 5.2, we assume that the price and warranty can be simultaneously changed once at a given point in the product lifetime.

5.1. Continuous price function

We assume that the linear price function (8) is given and that the manufacturer changes the warranty length once at time t_1 in the lifetime of the product. Let the warranty lengths before and after t_1 be denoted as T_{w1} and T_{w2}, respectively. The manufacturer's objective is to determine the optimal warranty lengths, denoted by T^*_{w1} and T^*_{w2}, for maximizing profit.

We report only the results associated with the case where certain customers purchase the product at time 0 and all the remaining customers purchase the product before the end of the product's lifetime. The results for other situations can be found in Zhou (2006).

Theorem 3. *For a given linear price function and* t_1, $T^*_{w1} \leqslant T^*_{w2}$ *if* $T^*_{w2} > 0$.

This theorem indicates that the manufacturer should increase the warranty length at time t_1. There are two reasons for this result: (i) all the customers have decreasing product values over time, and (ii) the customers who have not made a purchase before t_1 are more risk-averse than those who have. Consequently, the manufacturer must provide a more aggressive warranty policy to attract these customers. Closed-form solutions for T^*_{w1} and T^*_{w2} are not available. A search procedure can be used to find the solution numerically.

5.2. Discrete price function

In this section, we assume the manufacturer can change both the warranty length and product price at time t_1. We use (P_{01}, T_{w1}) and (P_{02}, T_{w2}) to denote the price-warranty combinations before and after t_1, respectively. In our analysis, we assume P_{01} is given, and T_{w1}, P_{02}, and T_{w2} are decision variables. Furthermore, we assume $P_{02} \leqslant P_{01}$ to prevent extreme results.

Since the product value to a customer decreases over time, a customer whose purchase conditions are not satisfied at the beginning will not make a purchase later if both the price and warranty length are kept the same. Therefore, customers' purchases can occur only at time 0 and t_1. Let P^*_{02}, T^*_{w1}, and T^*_{w2} be the optimal price and warranty lengths. We have the following result.

Theorem 4. *For given* P_{01}, $P^*_{02} = P_{01}$, *and* $T^*_{w1} < T^*_{w2}$ *if* $T^*_{w2} > 0$.

This result suggests that, when the manufacturer has an option to change both the product price and warranty length at a given time, the best policy is to keep the price unchanged but offer a longer warranty. As discussed previously, the value of an increase in the warranty length to the customer is higher than the corresponding increase in the warranty cost to the manufacturer. Consequently, the manufacturer should use a larger warranty length to attract remaining customers, rather than a lower price. We can find the optimal warranty lengths, T^*_{w1} and T^*_{w2}, for given P_{01} by a direct search procedure.

For comparison, we consider a special case where a fixed warranty length is used for the entire product lifetime. Let T^*_w denote the optimal warranty length. We obtain the following result.

Proposition 2. *For given* P_{01}, $T^*_{w1} \leqslant T^*_w$.

This proposition implies that, if the manufacturer can offer a dynamic warranty, the better policy is to offer a shorter warranty at the beginning to attract less risk-averse customers and then increase the warranty length to attract more risk-averse customers.

6. Numerical results

We present an example to compare the three policies discussed in the last section. For convenience in our presentation, we define policy 1 as the strategy of combining the linear price function (8) and a constant warranty length; policy 2 as a linear price function and two warranty lengths; and policy 3 as two product prices and two warranty lengths. We use the example as the basis for conducting a sensitivity analysis of the effects of several important model parameters on the optimal solution.

Example. Consider a manufacturer who sells a product with a free warranty. The product lifetime is 3 years, and the product is subject to random failure with the time to failure following an exponential distribution function with $\lambda = 2$ per year. Furthermore, the average unit product manufacturing and repair costs are \$200 and \$70, respectively. The value of the product to a customer is \$500 per unit possessing time during the product lifetime. The manufacturer is risk-neutral, and the customers are constantly risk-averse toward the repair cost, with the risk-aversion parameter uniformly distributed between 0.0001 and 0.03. Without loss of generality, we let the number of customers be 1.

For policy 1 with $P_0 = \$1000$ and $b = \$500$ per year, the optimal warranty length is 1.48 years, and the optimal profit is \$557.766. The percentages of the customers who made purchase at time 0 and afterward are 72.2% and 27.8%, respectively. For the second policy, if we allow the warranty length to change at $t_1 = 0.2$, the optimal warranty lengths before and after t_1 are $T^*_{w1} = 0.952$ and $T^*_{w2} = 1.568$, respectively. As indicated by Theorem 3, the warranty length increases after t_1 to attract more risk-averse customers who have not made a purchase. Under the optimal solution, the percentages of the customers who made a purchase at time 0 and afterward are 48.7% and 51.3%, respectively. The optimal solution of Policy 2 results in a higher average profit of \$576.50.

For policy 3, the optimal solution is $P^*_{02} = \$1000$, $T^*_{w1} = 0.807$ and $T^*_{w2} = 1.963$. The percentages of the customers who made a purchase at time 0 and t_1 are 43.0% and 57.0%, respectively. The profit is \$594.81, the highest among the three policies. Compared with policy 2, T^*_{w1} for policy 3 is shorter, but T^*_{w2} is longer. This is because the manufacturer has to use a more aggressive warranty policy after t_1 to compensate for the effect of the unchanged and relatively higher prices used in policy 3. The results of the three policies suggest that the manufacturer could benefit substantially from using a dynamic warranty policy.

Based on the example, we conduct a sensitivity analysis to further study the properties of the optimal solutions. We first consider policy 1. In order to investigate the relationship between P_0, b, and T^*_w, we obtain T^*_w under selected values of P_0 and b. The results are given in Table 1 and Fig. 1. It is evident that, for a given b value, T^*_w is positively associated with P_0, and that, for a given P_0 value, T^*_w is negatively associated with b. Actually, these results have been analytically proved in Proposition 1.

In the second analysis, we consider policy 2. In Fig. 2, the optimal warranty lengths are reported for selected values of P_0 as b is

Table 1
Effects of P_0 and b on T^*_w and profit

P_0	b	T^*_w	*profit*
900	400	1.22	491.601
900	500	1.18	493.561
900	600	1.15	494.955
900	700	1.12	495.998
1000	400	1.51	556.331
1000	500	1.48	557.969
1000	600	1.45	559.129
1000	700	1.43	559.996
1100	400	1.81	621.067
1100	500	1.78	622.373
1100	600	1.76	623.303
1100	700	1.75	623.998

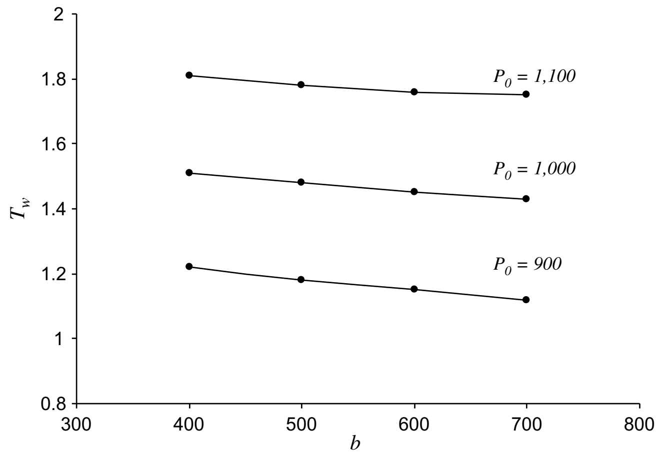

Fig. 1. Effects of P_0 and b on T_w^*.

kept constant. As P_0 increases, both T_{w1}^* and T_{w2}^* increase. This result is expected. However, the difference between T_{w1}^* and T_{w2}^* gradually becomes smaller when P_0 increases, and the benefit of allowing two warranty lengths diminishes when P_0 is very large. In other words, the dynamic warranty policy is beneficial when the initial price is relatively low.

In Table 2, we report T_{w1}^* and T_{w2}^* for selected values of t_1 ranging from 0.1 to 0.8. The results show that, as t_1 increases, the warranty length increases for period $[0, t_1)$ but decreases for $[t_1, T]$. Furthermore, the profit decreases as t_1 increases. This is because the earlier the manufacturer changes the warranty length, the shorter the warranty length needed to attract a fewer number of less risk-averse customers. However, after t_1, a long warranty length is needed because the product price remains relatively high and the remaining lifetime of the product is longer.

We also observe that, if t_1 is equal to or larger than .5, T_{w2}^* is 0, which suggests that all the customers have already made a purchase before t_1. This result suggests that it is not beneficial to change the warranty length at a late point in the product lifetime.

We now consider policy 3. From Proposition 2, we know $P_{02}^* = P_{01}$. Therefore, we need to determine only the two warranty lengths for the optimal solution. To study the effect of t_1 on the optimal solution, we obtain T_{w1}^* and T_{w2}^* for selected values of t_1 ranging from 0.1 to 1.0 under the condition $P_{01} = P_{02} = 1000$. The results are reported in Table 3.

We found that T_{w1}^* is not sensitive to changes in t_1. Since $P_{01} = P_{02}$, policy 3 is a special case of policy 2 with $b = 0$. Our computational experience shows that T_{w1}^* for policy 2 is also not sensitive to t_1 when b is small. Furthermore, we also found that T_{w2}^* increases in t_1. This is because, as t_1 increases, the product values

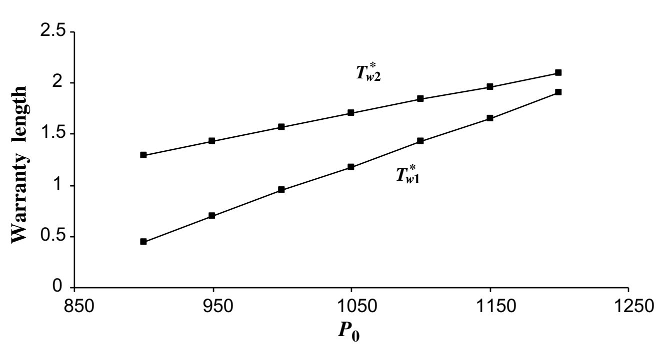

Fig. 2. T_{w1}^* and T_{w2}^* as functions of P_0.

Table 2
The effects of t_1 on T_{w1}^* and T_{w2}^* for policy 2

t_1	T_{w1}^*	T_{w2}^*	δ_0	δ_1	*Profit*
0.1	0.783	1.624	0.421	0.579	590.013
0.2	0.952	1.568	0.487	0.513	576.503
0.3	1.107	1.512	0.550	0.450	566.189
0.4	1.300	1.482	0.636	0.364	559.724
0.5	1.475	0	0.720	0.280	557.966
0.6	1.476	0	0.721	0.279	557.968
0.7	1.472	0	0.719	0.281	557.962
0.8	1.474	0	0.720	0.280	557.965

Table 3
The effects of t_1 on T_{w1}^* and T_{w2}^* for policy 3

t_1	T_{w1}^*	T_{w2}^*	δ_0	δ_1	*profit*
0.1	0.8073	1.9581	0.4301	0.5699	595.19
0.2	0.8073	1.9627	0.4301	0.5699	594.81
0.3	0.8372	1.9674	0.4418	0.5582	594.44
0.4	0.8372	1.9721	0.4418	0.5582	594.08
0.5	0.8372	1.9767	0.4418	0.5582	593.72
0.6	0.8372	1.9814	0.4418	0.5582	593.35
0.7	0.8372	1.9860	0.4418	0.5582	592.99
0.8	0.8372	1.9907	0.4418	0.5582	592.62
0.9	0.8372	1.9953	0.4418	0.5582	592.26
1.0	0.8372	2.0000	0.4418	0.5582	591.90

to those customers who did not purchase the product at $t = 0$ will be lower at time t_1. Since the price remains the same, the manufacturer has to provide a longer warranty to attract remaining customers. The results in the table also show that the profit decreases as t_1 increases. This is expected because of the general rule observed in policies 1 and 2 is to change the warranty length as soon as possible. However, the effects of t_1 on the profit are not very significant. Based on the results presented for the three policies, we conclude that a dynamic warranty policy is most effective when it is used jointly with an aggressive pricing policy.

7. Conclusions

In this paper, we consider a repairable product with known market entry and departure times. The manufacturer of the product offers a free warranty with product purchase, under which the customer can have the product repaired free of charge if it fails during the warranty period. It is assumed that the customers are heterogeneous with respect to risk aversion toward uncertain repair costs after the warranty expires. We consider both constant warranty and dynamic warranty in order to investigate the impact of dynamic warranty on the manufacturer's decision making of profit maximization.

In the first part of our analysis, we study customers' purchase patterns under several different pricing strategies by the manufacturer and then discuss the optimal pricing and warranty strategy. We obtain that if the manufacturer is a market monopolist, the best strategy for him is to charge the highest price but offer the longest warranty period which covers the entire product lifetime. However, there is often competition in the market, which imposes some restriction on the product price the manufacturer can set. In this situation, the optimal warranty length decreases with the price decreasing rate but increases with the initial product price.

In the second part, we consider the dynamic warranty by assuming that the warranty length can be changed once in the product lifetime. We find that the manufacturer should provide an even longer warranty after the change than before to acquire the maximum profit. Through the comparison between the constant warranty and dynamic warranty, it is obtained that the manufacturer will benefit from offering dynamic warranty policies. In addition, earlier changing time will bring more profit to the manufacturer.

Although there have been plausible arguments for using a dynamic warranty policy, and we have demonstrated analytically its benefits, dynamic warranty policies have not been commonly used in practice. We hope that the results of this paper could convince practitioners to recognize the heterogeneity in risk attitude among customers and consider using a dynamic warranty policy jointly with product price as a marketing tool to manage customers' purchase pattern in the product lifetime. In order to apply the proposed theoretical framework, it is essential to validate the utility function given by (1) and estimate the distribution of the risk-aversion parameter, using the methodology established by Keeney and Raiffa (1993) and in the statistics field. If the utility function does not adequately reflect customers' risk attitudes, an appropriate utility function can be identified, and the same analysis can be performed accordingly. It is strongly recommended that the manufacturer collect customer data through his or her sales process in order to evaluate and modify the price and warranty policy.

The proposed framework in this study can be extended from several possible directions. In our analysis, we find that the manufacturer will acquire more profit through earlier change time and more times of change for warranty length. Therefore, the approach of continuous dynamic programming can be involved to generalize this methodology. Through this study, we aim to develop a scheme of dynamic price and warranty policy for high-tech products when customers are heterogeneous. Hence, manufacturers can apply this scheme to guide their managerial decisions.

Second, this framework can be applied to design the personalized price and warranty policy for each customer as long as there is a mechanism to measure or obtain information on a customer's risk attitude. By doing this personalization, it is expected that the manufacturer's profit will increase.

Third, the study can be extended by considering two or multiple competing manufacturers for the same group of customers. We will analyze how the competition affects the pricing and warranty policies of individual manufacturers and what is the optimal decision policy for each manufacturer in a competitive market. This is more practical situation for most of the high-tech products. Therefore, the study results can provide useful suggestions for manufacturers in their operation and management.

Appendix A

A.1. Proof of Lemma 1

We first prove $t_a \leqslant T - T_w$ for $a \in [a_\ell, a_u]$, if $P_0 \leqslant (k-b)T_w + bT$. In the range $t \in (0, T - T_w]$, the product value to a customer with risk attitude a is $k(T-t) - \widehat{C}_r\lambda(T - T_w - t)$ or $kT - \widehat{C}_r\lambda(T - T_w) - (k - \widehat{C}_r\lambda)t$. This shows that the product value for a customer is a linear function of t in $(0, T - T_w]$. In the range $[T - T_w, T)$, the product value is $k(T-t)$ for all the customers, which is also a linear function of t. For the customers who do not make a purchase at time 0, the price is higher than the product values. The condition, $P_0 < (k-b)T_w + bT$, implies that the price is below the product value at $t = T - T_w$. Hence, there must be a point in $(0, T - T_w)$, at which the product value and price are equal for each customer. If $P_0 = (k-b)T_w + bT$, the purchase time is $T - T_w$. Therefore, $t_a \leqslant T - T_w$ for all a.

Now, we prove $\{t_a : t_a \in (0, T - T_w]\} = \emptyset$, if $P_0 > (k-b)T_w + bT$. If $t_a \in (0, T - T_w]$, then $k(T - t_a) - \widehat{C}_r\lambda(T - T_w - t_a) \geqslant P_0 - t_a$, or $(\lambda\widehat{C}_r - k + b)t_a \geqslant \widehat{C}_r\lambda(T - T_w) + P_0 - kT$. Since the purchase condition of any of these customers is not satisfied at time 0, $kT - \widehat{C}_r\lambda(T - T_w) < P_0$, i.e., $\widehat{C}_r\lambda(T - T_w) + P_0 - kT > 0$. As a result, $\lambda\widehat{C}_r - k + b > 0$. Then $t_a = \frac{\lambda\widehat{C}_r(T-T_w)+P_0kT}{\lambda\widehat{C}_r+b-k} \leqslant T - T_w$. Therefore, $P_0 \leqslant (k-b)T_w + bT$, contradicting $P_0 > (k-b)T_w + bT$. Hence, if $P_0 > (k-b)T_w + bT$, $\{t_a : t_a \in (0, T - T_w]\} = \emptyset$.

A.2. Proof of Corollary 1

In the range of $t \in [0, T - T_w]$, $(k-b)T_w + bT - (k(T-t) + bt) = (b-k)(T - T_w - t)$. Because $P_0 > (k-b)T_w + bT$ and $P_0 - bT < 0$, k is smaller than b. We obtain $(k-b)T_w + bT \geqslant k(T-t) + bt$ and $P_0 - bt > k(T-t) - \widehat{C}_r\lambda(T - T_w)$, indicating that no purchase occurs before time $T - T_w$. Furthermore, since $P_0 > (k-b)T_w + bT$, no customer will purchase the product at $T - T_w$.

The first condition, $P_0 > (k-b)T_w + bT$, implies that the price is higher than the product values for all the customers at $T - T_w$. The second condition, $P_0 - bT < 0$, implies the product value is higher than the price at time T. Since both the price and product value are linear functions in $(T - T_w, T)$, there exists a time point at which the two are equal. Since all the customers have the same product value after $T - T_w$, the interception point is the same for all customers. It has been proved above that no customer will make the purchase before $T - T_w$. Therefore, under these conditions, all the customers will make a purchase within the interval $(T - T_w, T)$ and the purchase time is the same for all customers.

A.3. Proof of Corollary 2

The conditions, $P_0 > (k-b)T_w + bT$ and $P_0 - bT > 0$, indicate that the price is higher than the product value at both $T - T_w$ and T. Since the price and product value are linear functions of t, there is no intersection of the two functions in $(T - T_w, T]$. From Lemma 1, $\{t_a : t_a \in (0, T - T_w]\} = \emptyset$. Hence, $\{t_a : t_a \in (0, T]\} = \emptyset$.

A.4. Proof of Theorem 1

It has been proved that if $P_0 < (k-b)T_w + bT, t_a < T - T_w$ for all the customers. For those customers who do not make a purchase at time 0, (i.e., those with $a \in (a_0, a_u]$ if $a_0 \in [a_\ell, a_u]$ (or $a \in [a_\ell, a_u]$ if $a_0 < a_\ell$)),

$$t_a = \frac{\widehat{C}_r\lambda(T - T_w) + P_0 - kT}{\lambda\widehat{C}_r - k + b} = (T - T_w) - \frac{(k-b)T_w + bT - P_0}{\lambda\widehat{C}_r - k + b}.$$

Since $\lambda\widehat{C}_r - k + b$ is positive (from Lemma 2) and increases in a, t_a also increases in a.

A.5. Proof of Theorem 2

It can be proved that, for given T_w, the optimal solution occurs when $P_0 \in [kT - \widehat{C}_r(a_u)\lambda(T - T_w), kT - \widehat{C}_r(a_\ell)\lambda(T - T_w)]$ and $b > \frac{P_0 - kT_w}{T - T_w}$ (see Zhou, 2006), where $kT - \widehat{C}_r\ (a_u)\lambda(T - T_w)$ and $kT - \widehat{C}_r\ (a_\ell)\lambda(T - T_w)$ are the initial prices, at which the most and least risk-averse customers would make a purchase at time 0, respectively. In this situation, some customers make a purchase at time 0, and all the others will purchase the product later before $T - T_w$. For given T_w, the manufacturer's profit is

$$profit = Q(P_0 - C_m - C_r\lambda T_w) - \frac{Qb}{a_u - a_\ell}\int_{a_0}^{a_u} t_a\,\mathrm{d}a,$$

where $t_a = \frac{\widehat{C}_r\lambda(T-T_w)+P_0-kT}{\lambda\widehat{C}_r-k+b}$. Let b^* and P_0^* be the optimal solution for given T_w, which leads to a_0^* and the maximum profit for the manufacturer. Then, we have $b^* > \frac{P_0^* - kT_w}{T - T_w}$.

Increasing T_w to $T_{w1} = T_w + \Delta T_w$, we prove that a feasible solution $(\tilde{b}, \widetilde{P}_0)$ can be found which leads to $\tilde{a}_0$, s.t. $\tilde{a}_0 = a_0^*$ and $\tilde{b} > \frac{P_0 - kT_{w1}}{T - T_{w1}}$. Let $\tilde{b} = b^*$ and $\widetilde{P}_0 = P_0^* + \Delta P_0$. Therefore, $\widehat{C}_r(\tilde{a}_0)\lambda = \frac{kT - P_0}{T - T_{w1}} = \frac{kT - (P_0^* + \Delta P_0)}{T - (T_w + \Delta T_w)}$. It is known that $\widehat{C}_r(a_0^*)\lambda = \frac{kT - P_0^*}{T - T_w}$. As a result, in order to satisfy $\tilde{a}_0 = a_0^*, \Delta P_0 = \widehat{C}_r(a_0^*)\lambda\Delta T_w$. Hence, we obtain $\tilde{b} = b^*$ and $\widetilde{P}_0 = P_0^* + \widehat{C}_r(a_0^*)\lambda\Delta T_w$, which leads to $\tilde{a}_0$ s.t. $\tilde{a}_0 = a_0^*$. Furthermore,

$$(k-\tilde{b})T_{w1} + \tilde{b}T - \widetilde{P}_0 = ((k-b^*)T_w + b^*T - P_0^*) + (k - b^* - \widehat{C}_r(a_0^*)\lambda)\Delta T_w,$$

which can be expressed as $[(k-b^*)T_w + b^*T - P_0^*](1 - \frac{\Delta T_w}{T-T_w})$. We obtain $\tilde{b} > \frac{P_0 - kT_{w1}}{T-T_{w1}}$, because $T_w + \Delta T_w \leqslant T$ and $(k-b^*)T_w + b^*T - P_0^* > 0$. Thus, a feasible solution $\tilde{b} = b^*$ and $\widetilde{P}_0 = P_0^* + \widehat{C}_r(a_0^*)\lambda\Delta T_w$ is found.

Next we prove the manufacturer will obtain a higher profit using price $(\tilde{b}, \widetilde{P}_0)$ than (b^*, P_0^*). For $a \in (a_0^*, a_u)$,

$$\int_{\tilde{a}_0}^{a_u} t_a \,da = \int_{\tilde{a}_0}^{a_u} \frac{\widehat{C}_r\lambda(T - T_{w1}) + \widetilde{P}_0 - kT}{\widehat{C}_r\lambda + \tilde{b} - k} da < \int_{a_0^*}^{a_u} \frac{\widehat{C}_r\lambda(T - T_w) + P_0^* - kT}{\widehat{C}_r\lambda + b^* - k} da = \int_{a_0^*}^{a_u} t_a \,da$$

because $\tilde{a}_0 = a_0^*$ and

$$\frac{\widehat{C}_r\lambda(T-T_{w1}) + \widetilde{P}_0 - kT}{\widehat{C}_r\lambda + \tilde{b} - k} = \frac{\widehat{C}_r\lambda(T-T_w) + P_0^* - kT - (\widehat{C}_r - \widehat{C}_r(a_0^*))\lambda\Delta T_w}{\widehat{C}_r\lambda + b^* - k} < \frac{\widehat{C}_r\lambda(T-T_w) + P_0^* - kT}{\widehat{C}_r\lambda + b^* - k}.$$

Moreover, $\widetilde{P}_0 - C_m - C_r\lambda T_{w1} = P_0^* - C_m - C_r\lambda T_w + (\widehat{C}_r(a_0^*) - C_r)\lambda\Delta T_w > P_0^* - C_m - C_r\lambda T_w$. Therefore, the manufacturer will obtain a higher profit if a longer warranty is used. As a result, when b, P_0, and T_w can all be freely set, *profit* increases in T_w, and the profit is maximized when $T_w = T$. In this situation, all the customers make a purchase at $t = 0$ if $kT \geqslant P_0$ is satisfied, and the maximum profit is $profit_{\max} = Q(kT - C_m - C_r\lambda T)$ with $T_w^* = T$ and $P_0^* = kT$.

A.6. Proof of Proposition 1

The manufacturer's profit is $profit = Q(P_0 - C_m - C_r\lambda T_w) - \frac{Qb}{a_u - a_\ell}\int_{a_0}^{a_u} t_a \,da$.

The first derivative of *profit* with respect to is

$$\frac{\partial profit}{\partial T_w} = -QC_r\lambda + \frac{Q}{a_u - a_\ell}\int_{a_0}^{a_u} \frac{b\widehat{C}_r\lambda}{\widehat{C}_r\lambda - k + b} da.$$

Due to the assumption $k > \widehat{C}_r\lambda$ for all a, $\frac{b\widehat{C}_r\lambda}{\widehat{C}_r\lambda - k + b}$ decreases in b. Therefore, to satisfy $\frac{\partial profit}{\partial T_w} = 0, a_0$ decreases. In order for a_0 to decreases, T_w has to be reduced. The second derivative of *profit* with respective to T_w is $\frac{\partial^2 profit}{\partial T_w^2} = -\frac{Qb}{a_u - a_l}\frac{\widehat{C}_r(a_0)\lambda}{\widehat{C}_r(a_0)\lambda - k + b} < 0$. Therefore, the optimal T_w decreases in b.

In $\frac{\partial profit}{\partial T_w}$, $\frac{b\widehat{C}_r\lambda}{\widehat{C}_r\lambda - k + b}$ is independent of P_0, implying that a_0 satisfying $\frac{\partial profit}{\partial T_w} = 0$ remains the same when P_0 and T_w change. Since $\widehat{C}_r(a_0) = \frac{kT - P_0}{\lambda(T - T_w)}$, T_w increases in P_0. Because $\frac{\partial^2 profit}{\partial T_w^2} < 0$, the optimal T_w increases in P_0.

A.7. Proof of Theorem 3

If $T_{w2}^* = 0$, all the customers have purchased the product before t_1, making the dynamic warranty policy unnecessary. We will focus on the case, where $T_{w2}^* > 0$.

It is evident that the policy of allowing two warranty lengths should yield at least the same profit as that of maintaining the fixed warranty length. Suppose $T_{w1}^* > T_{w2}^*$. Let a_1 be the risk attitude of the customers who make a purchase at time t_1. Then, the customers with $a \in [a_1, a_u]$ will purchase the product at an even later time than when T_{w1}^* remains the same. The profit derived from any individual customer with $a \in [a_1, a_u]$ is

$$P_0 - bt_a - C_r\lambda T_{w2} - C_m = k(T - t_a) - \widehat{C}_r\lambda(T - T_{w2} - t_a) - \widehat{C}_r\lambda T_{w2} - C_m.$$

This profit is less than that when T_{w1}^* remains the same. In addition, the profits derived from the customers with $a \in [a_\ell, a_1)$ are the same for the two cases. Hence, the profit associated with the policy that allows two warranty lengths is lower than that using one warranty length, which is possible. Therefore, we conclude $T_{w1}^* \leqslant T_{w2}^*$ for a given linear price function.

A.8. Proof of Theorem 4

Consider a price-warranty combination $(P_{01}, T_{w1}, P_{02}, T_{w2})$, in which $P_{02} < P_{01}$. Assume the customers with $a \in [a_\ell, a_{01}]$ purchase the product at time 0 and those with $a \in (a_{01}, a_{02}]$ make a purchase at time t_1, where $a_{01} \in [a_\ell, a_u]$ is the solution to $\widehat{C}_r = \frac{kT - P_{01}}{\lambda(T - T_{w1})}$ and $a_{02} \in [a_{01}, a_u]$ is the solution to $\widehat{C}_r = \frac{k(T - t_1) - P_{02}}{\lambda(T - T_{w2} - t_1)}$. Then, the manufacturer's profit is

$$\begin{aligned} profit &= \frac{a_{01} - a_\ell}{a_u - a_\ell}Q(P_{01} - C_m - C_r\lambda T_{w1}) + \frac{a_{02} - a_{01}}{a_u - a_\ell}Q(P_{02} - C_m - C_r\lambda T_{w2}) \\ &= \frac{a_{01} - a_\ell}{a_u - a_\ell}Q(P_{01} - C_m - C_r\lambda T_{w1}) + \frac{a_{02} - a_{01}}{a_u - a_\ell}Q((k - \widehat{C}_r(a_{02})\lambda) \\ &\quad \times (T - t_1) + (\widehat{C}_r(a_{02}) - C_r)\lambda T_{w2} - C_m). \end{aligned}$$

If we keep a_{02} the same and increase T_{w2} until $k(T - t_1) - \widehat{C}_r(a_{02})\lambda(T - T_{w2} - t_1) = P_{01}$, *profit* will also increase in the same range. For any price-warranty combination, this is true as long as $P_{02} < P_{01}$. Therefore, the optimal price after time t_1 is $P_{02}^* = P_{01}$. When $P_{02} = P_{01}$, no customers will make the purchase at t_1 if $T_{w1}^* \geqslant T_{w2}^*$. To obtain a larger profit, T_{w1}^* must be smaller than T_{w2}^*, if $T_{w2}^* > 0$.

A.9. Proof of Proposition 2

We consider two cases. In Case 1, we assume $a_0 < a_u$. Suppose T_{w1}^* is larger than T_w^*, and T_{w2}^* remains unchanged. The customers with $a \in [a_\ell, a_{01}]$ purchase the product at time 0, and those with $a \in (a_{01}, a_1]$ make a purchase at time t_1, where $a_{01} \in [a_\ell, a_u]$ is the solution to $\widehat{C}_r = \frac{kT - P_{01}}{\lambda(T - T_{w1})}$ and $a_1 \in (a_{01}, a_u]$ is the solution to $\widehat{C}_r = \frac{k(T - t_1) - P_{01}}{\lambda(T - T_{w2} - t_1)}$. The manufacturer's profit associated with the dynamic warranty policy is

$$profit = \frac{a_{01} - a_\ell}{a_u - a_\ell}Q(P_{01} - C_m - C_r\lambda T_{w1}^*) + \frac{a_1 - a_{01}}{a_u - a_\ell}Q(P_{01} - C_m - C_r\lambda T_{w2}^*).$$

Since $T_{w1}^* > T_w^*$, $a_{01} > a_0$. If the price does not change, the customers purchase the product only at time 0 and the time when warranty length changes. If we use T_{w1}^* in the entire product lifetime, the manufacturer's profit is lower than that when T_w^* is used instead; i.e.

$$\frac{a_0 - a_\ell}{a_u - a_\ell}Q(P_{01} - C_m - C_r\lambda T_w^*) \geqslant \frac{a_{01} - a_\ell}{a_u - a_\ell}Q(P_{01} - C_m - C_r\lambda T_{w1}^*).$$

If we decrease T_{w1}^* to T_w^*, then the customers with $a \in (a_{01}, a_1]$ will make a purchase at t_1 because $T_{w2}^* = T_{w2}^*(t_1, a_1) \geqslant T_{w2}^*(t_1, a_{01})$, where T_{w2}^* (t_1, a_1) and $T_{w2}^*(t_1, a_{01})$ are the minimum warranty lengths the customers with risk attitudes a_1 and a_{01} need at time t_1 to purchase the product, respectively. Therefore, the increase in the manufacturer's profit due to the decrease of T_{w1}^* is

$$\frac{a_0 - a_\ell}{a_u - a_\ell}Q(P_{01} - C_m - C_r\lambda T_w^*) - \frac{a_{01} - a_\ell}{a_u - a_\ell}Q(P_{01} - C_m - C_r\lambda T_{w1}^*) + \frac{a_{01} - a_0}{a_u - a_\ell}Q(P_{01} - C_m - C_r\lambda T_{w2}^*) > 0.$$

As a result, $T_{w1}^* \leqslant T_w^*$.

In Case 2, we assume $a_0 = a_u$. In this case, all the customers make a purchase at time 0. Therefore, $T_w^* = T_w(0, a_u)$, the minimum warranty length, which the customers with a_u require to purchase the product at $t = 0$. The manufacturer does not need to provide warranty longer than $T_w(0, a_u)$. Hence, $T^*_{w1} \leqslant T^*_w$.

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