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A closed-form approximation for valuing European basket warrants under credit risk and interest rate risk

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Over the past few years, many financial institutions have actively traded basket warrants in the over-the-counter market. Prior research has proposed an approach to valuing single-stock options subject to credit. However, this approach cannot be applied directly to the case of basket warrants. Using the martingale method, we propose a closed-form approximation for valuing European basket warrants using a continuous-time model, with credit risk and interest rate risk considered simultaneously. Finally, several numerical examples are utilized to demonstrate the characteristics of basket warrants under credit risk.

Keywords: Derivatives pricing; Derivatives securities; Stochastic interest rates; Credit risk

JEL Classification: G1, G13

1. Introduction

With the liberalization of global financial markets and the instability of the world economy, many derivatives have been developed to meet the increasing needs of investors. Among these derivative securities, basket warrants have gradually become more popular over the past decade. In essence, basket warrants are actually basket options. These options have a basket of two or more underlying assets whose prices determine basket warrants' payoffs. However, basket warrants and ordinary options are different in terms of issuing institutions. Basket warrants are normally issued by financial institutions such as investment banks. Investors thus face the credit risk of issuers. As the global financial markets are rapidly changing, investors are also concerned with interest rate risk. The purpose of this paper is to value European basket warrants, with credit risk and interest rate risk considered simultaneously.

Since securities companies which issue basket warrants may default on their obligations, investors should take into account the creditworthiness of issuing organizations when purchasing warrants. Johnson and Stulz (1987) was one of the first studies to examine the pricing of these options with default risk, also known as vulnerable options. Hull and White (1995), Jarrow and Turnbull (1995), Klein (1996), and Hung and Liu (2005) also indicate the importance of taking into account counterparty risk when pricing options traded on the over-the-counter (OTC) market due to the unavailability of a clearing mechanism.

When valuing vulnerable options, Klein (1996) does not consider the interest rate risks, while Johnson and Stulz (1987) assume a fixed interest rate. Both Gentle (1993) and Milevsky and Posner (1998) also assume fixed interest rates when valuing basket options.¶ Based on the assumption of fixed interest rates, the value of warrants would be underestimated if interest rates are highly volatile before maturity.

Hull and White (1995) and Jarrow and Turnbull (1995) assume independence between the assets of the option writer and the underlying asset of the option, especially when the option writer is a large and well diversified financial institution (Hull and White 1995). However, Klein (1996) argues that the option writer may still default due to the volatile changes in the value of the underlying asset even if the option writer has undertaken hedging. Johnson and Stulz

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¶It is noted that Gentle (1993) and Milevsky and Posner (1998) do not consider default risk when pricing basket options.

(1987) further suggest that the option writer may default, possibly due to decreases in the value of the assets of the writer and/or the growth of the value of the option. If the value of the option grows to a larger extent than that of the assets of the writer, it is likely that default may occur.

Prior studies also discuss the correlations among the assets of the option writer, the assets underlying the options, and the interest rate. Merton (1974) was one of the first to study the link between the value of the assets of the option writer and the default event, considering the relation between the interest rate and the assets underlying the options. Extending Merton's (1974) work, Longstaff and Schwartz (1995) simultaneously consider both default risk and interest rate risk. They further point out that the changes in the value of the assets of the firm and in the interest rates have a significant impact on credit spread when pricing risky bonds. However, the interest rate risk is generally not taken into account in the literature when valuing vulnerable options.

In this paper, we argue the importance of considering both risks and simultaneously model these two risks when pricing basket warrants for the following reason. In the past few years, warrant writers have been exposed to bankruptcy risk due to the Financial Crisis of 2007–2010. They are also adversely affected by interest rate volatility. Furthermore, prices of financial assets are significantly interrelated. Unlike prior studies, we therefore simultaneously take into account credit risk, interest rate risk, and correlations between assets when valuing basket warrants. Our valuation formula provides flexibility in pricing warrants with underlying assets including bonds, stocks and other types of securities. It is actually the general closed-form solution of the models of Black and Scholes (1973), Smith (1976), Gentle (1993), Hull and White (1995), Klein (1996), and Klein and Inglis (1999).

The remainder of the paper is organized as follows. The following section develops a theoretical model for valuing European basket warrants subject to interest rate risk and to financial distress on the part of the warrant writer. Next, we derive a closed-form approximation valuation formula for vulnerable basket call and put warrants. In the penultimate section, we utilize several numerical examples to show the properties of our pricing formula. Finally, the last section concludes this paper.

2. The model

This article's framework for valuing basket warrants can be considered an extension of the framework for pricing vulnerable options proposed by Klein and Inglis (1999). In their paper, Klein and Inglis (1999) value calls/puts on a single asset, while in this study we price basket warrants. The underlying of basket warrants is a basket of assets. Specifically, the basket of assets is actually a portfolio of n kinds of different tradable stocks. We assume that the warrants are traded under a continuous-time frame and that markets are perfect and frictionless, i.e. transaction costs and taxes are ignored. Suppose that the market value of each of these n stocks follows Geometric Brownian Motion (GBM). The dynamics of the market value of a particular stock are then stated as follows:

$$\frac{\mathrm{d}S_{it}}{S_{it}} = \mu_{S_i}\,\mathrm{d}t + \sigma_{S_i}\,\mathrm{d}W_{S_i}, \quad i = 1, \ldots, n, \tag{1}$$

where μ_{S_i} and σ_{S_i} are the instantaneous expected return on stock i underlying the warrant and the instantaneous standard deviation of the return (both assumed to be constants), respectively, S_{it} is the price of the ith stock at time t, W_{S_i} is a standard Wiener process, $i = 1, \ldots, n$, and $<W_{S_i}, W_{S_j}>_t = \rho_{ij}t$, for every $i, j = 1, \ldots, n$.

Credit risk, also known as default risk, is defined as the risk that warrant writers, such as investment banks or securities firms, will be unable to make the required payments when their debt obligations fall due. The bankrupt event occurs when the writer's value of assets at the expiration date (V_T) are smaller than its value of debts (D). If the warrant writer is bankrupt, its payments to investors depend on the value of its assets on the warrant's expiration date. If the value of the assets of the warrant writer V_T falls below the fixed threshold value D^* on the warrant maturity date T, then default occurs. The value of D^* is allowed to be less than the value of D, the outstanding liabilities of the writer. As in Klein and Inglis (1999), D is simplified as the value of the zero-coupon bonds issued by the writer. We assume that D is the same as D^* in the process of valuation to allow for capital forbearance of the warrant writer. Once the writer goes bankrupt, the warrant holder can only claim $(1-\alpha)V_T/D$, where $0 \le \alpha \le 1$ is the costs associated with the financial distress when the writer becomes bankrupt. Suppose that the value of the assets of the warrant writer V follows a GBM. The dynamic process of V is as follows:

$$\frac{\mathrm{d}V}{V} = \mu_V\,\mathrm{d}t + \sigma_V\,\mathrm{d}W_V, \tag{2}$$

where μ_V and σ_V are the instantaneous expected return on the assets of the writer and the instantaneous standard deviation of the return (both assumed to be constants), respectively, and W_V is a standard Wiener process.

As for the interest rate risk, let $P(t,T)$ represent the price of a zero-coupon bond at time t paying one dollar at time T, where T represents the expiration date of the warrants. Therefore,

$$\frac{\mathrm{d}P(t,T)}{P(t,T)} = \mu_P\,\mathrm{d}t + \sigma_P(T-t)\,\mathrm{d}W_P, \tag{3}$$

where μ_P and σ_P denote the instantaneous expected return on the zero-coupon bond and the instantaneous standard deviation of the return, respectively, and W_P follows a standard Wiener process.

Let $B(t)$ denote a money market account which corresponds to the future value of the wealth accumulated from an investment of \$1 at an interest rate of $r(t)$. Its dynamic process is $\mathrm{d}B(t) = r(t)B(t)\,\mathrm{d}t$, where $r(t)$ is the instantaneous interest rate at time t.

Under the risk-neutral probability measure Q, the dynamics of the zero coupon bond price are

$$\frac{\mathrm{d}P(t,T)}{P(t,T)} = r(t)\,\mathrm{d}t + \sigma_P(T-t)\,\mathrm{d}W_P^Q. \tag{4}$$

As shown in equations (3) and (4), the instantaneous expected return on the zero-coupon bond under the probability measure P is replaced by the instantaneous interest rate $r(t)$ under the risk-neutral measure Q.

In addition, the market value of the underlying stock, the value of the assets of the warrant writer, and the price of the zero-coupon bond are all assumed to be correlated with each other under the probability measure P. The instantaneous correlations between W_V and W_P, between W_S and W_P, and between W_V and W_S are ρ_{VP}, ρ_{SP}, and ρ_{VS}, respectively.

Our framework described in this section can be applied to the models proposed by Vasicek (1977), Hull and White (1990), and Heath *et al.* (1992). It is also worth noting that the valuation formula for the vulnerable European basket warrants is the general closed-form solution of the models of Black and Scholes (1973), Smith (1976), Gentle (1993), Hull and White (1995), Klein (1996), and Klein and Inglis (1999). In the 'Numerical examples' section below, we use the Cox, Ingersoll and Ross (1985) model (CIR model), which is a special case of the Hull and White (1990) model.

3. Valuation of vulnerable basket warrants

The warrant writer is considered to be bankrupt if the value of its asset V_T falls below the value D on the maturity date. If the writer defaults on its obligations at maturity, the claims of the warrant purchasers would not be completely satisfied. Thus, we use the maturity date T as a reference point in time.

Let C_T^B be the payoff of a European basket call warrant at maturity date T. C_T^B is defined as

$$C_T^B = \max\{BK_T - K, 0\},$$

where $BK_T = \sum_{i=1}^n w_i S_{iT},\ i = 1, \ldots, n$. BK_T represents the weighted average value of n kinds of different stocks at time T, K is the strike price of the basket call warrant, w_i represents the weight of the ith stock and $\sum_{i=1}^n w_i = 1$. Let $F_i^S(t,T) = S_{it}/P(t,T)$, and $F_i^S(t,T)$ is the forward price at time t of the ith asset for the settlement date T.

$$BK_T = \sum_{i=1}^n w_i S_{iT} = \sum_{i=1}^n \left[w_i F_i^S(t,T) \frac{S_{iT}}{F_i^S(t,T)} \right].$$

The payoff at maturity T of a basket call option can be rewritten as follows:

$$\begin{aligned} C_T^B &= \max\{BK_T - K, 0\} \\ &= \max\left\{ \sum_{i=1}^n \left[\frac{w_i F_i^S(t,T)}{\sum_{i=1}^n w_i F_i^S(t,T)} \frac{S_{iT}}{F_i^S(t,T)} \right.\right. \\ &\quad \left.\left. - \frac{K}{\sum_{i=1}^n w_i F_i^S(t,T)} \right] \left(\sum_{i=1}^n w_i F_i^S(t,T) \right), 0 \right\} \\ &= \sum_{i=1}^n w_i F_i^S(t,T) \cdot \left\{ \sum_{i=1}^n X_i \hat{S}_{iT} - K^*, 0 \right\}, \end{aligned} \tag{5}$$

where the modified weight is

$$X_i = \frac{w_i F_i^S(t,T)}{\sum_{i=1}^n w_i F_i^S(t,T)}, \quad \sum_{i=1}^n X_i = 1,$$

and

$$K^* = \frac{K}{\sum_{i=1}^n w_i F_i^S(t,T)}, \quad \hat{S}_{iT} = \frac{S_{iT}}{F_i^S(t,T)}.$$

If the money market account $B(t)$ is used as the numeraire, the discounted price of the asset is Q -martingale. If $P(t,T)$, the price of a zero-coupon bond with a maturity date T at time t, is used as the numeraire, the discounted price of the asset will be Q^T -martingale. Since $P(T,T) = B(T) = 1$, C_T^B, the payoff function of a vulnerable European basket call warrant with a maturity date T, can be given as follows:

$$C_T^B = \begin{cases} (BK_T - K), & \text{if } BK_T - K > 0 \text{ and } V_T \geq D, \\ (BK_T - K)\frac{(1-\alpha)V_T}{D}, & \text{if } BK_T - K > 0 \text{ and } V_T < D, \\ 0, & \text{if } BK_T - K \leq 0, \end{cases}$$

where BK_T represents the weighted average value of n kinds of different stocks at time T, K is the strike price of the basket call warrant, and V_T and D are the warrant writer's assets and liabilities, respectively. When $V_T < D$, the writer is liquidated and its residual firm value is $(1-\alpha)V_T$, where α is the percentage representing the deadweight costs associated with financial distress and $0 \leq \alpha \leq 1$. The deadweight costs include the direct and indirect costs of bankruptcy. Warrant holders can only claim back $(1-\alpha)V_T/D$, where $(1-\alpha)$ is the recovery rate. The actual payoff of a European basket call warrant with a mature date T is then $(BK_T - K)[(1-\alpha)V_T/D]$.

If $P(t,T)$ is used as the numeraire, $C_T^B/P(T,T)$ will be the martingale of the probability measure Q^T. Thus,

$$E_{Q^T}\left[\frac{C_T^B}{P(T,T)} \left(1_{\{V_T \geq D\}} + \frac{(1-\alpha)V_T}{D} 1_{\{V_T < D\}} \right) \middle| F_t \right] = \frac{C_t}{P(t,T)}.$$

The value of a vulnerable European basket call warrant at time t, C_t^B, is

$$C_t^B = P(t,T) E_{Q^T}\left[C_T^B \left(1_{(V_T \geq D)} + \frac{(1-\alpha)V_T}{D} 1_{(V_T < D)} \right) \middle| F_t \right]. \tag{6}$$

Equation (6) shows that the expected future payout on the nominal claim of amount C_T^B depends on the terminal value of the writer's assets. The nominal claim is paid out in full if the value of the assets of the option writer at maturity T, V_T, is greater than the value of the debt of the writer D. E_{Q^T} is the forward risk-neutral expectation of Q^T.

Substituting equation (5) into (6) gives

$$\begin{aligned} C_t^B = P(t,T) E_{Q^T} \Bigg[&\left(\sum_{i=1}^n X_i \hat{S}_{iT} - K^* \right)^+ \Bigg(1_{\{V_T \geq D\}} \\ &+ \frac{(1-\alpha)V_T}{D} 1_{\{V_T < D\}} \Bigg) \Bigg| F_t \Bigg] \sum_{i=1}^n w_i F_i^S(t,T). \end{aligned} \tag{7}$$

Although individual stock prices follow a log-normal distribution, the weighted average price of stocks BK_t is no longer log-normally distributed. Using the fact that an arithmetic average is always greater than a geometric average, Vorst (1992) proposes†

$$\sum_{i=1}^{n} X_i \hat{S}_{iT} \approx \prod_{i=1}^{n} \hat{S}_{iT}^{X_i} - E_{Q^T}\left[\prod_{i=1}^{n} \hat{S}_{iT}^{X_i}\right] + E_{Q^T}\left[\sum_{i=1}^{n} X_i \hat{S}_{iT}\right] \quad (8)$$

Substituting equation (8) into (7), the value of a vulnerable European call C_t^B can be rewritten as

$$\begin{aligned} C_t^B &\approx P(t,T) E_{Q^T}\left[\left(\prod_{i=1}^{n} \hat{S}_{iT}^{X_i} - K'\right)^{+}\left(1_{\{V_T \geq D\}} + \frac{(1-\alpha)V_T}{D} 1_{\{V_T < D\}}\right)\middle| F_t\right] \sum_{i=1}^{n} w_i F_i^S(t,T) \\ &= BK_t[A_1 - A_2 + A_3 - A_4], \end{aligned} \quad (9)$$

where

$$A_1 = E_{Q^T}\left[\prod_{i=1}^{n} \hat{S}_{iT}^{X_i} 1_{\{\prod \hat{S}_{iT}^{X_i} \geq K'\}} 1_{\{V_T \geq D\}} | F_t,\right]$$

$$A_2 = E_{Q^T}\left[K' 1_{\{\prod \hat{S}_{iT}^{X_i} \geq K'\}} 1_{\{V_T \geq D\}} | F_t\right],$$

$$A_3 = E_{Q^T}\left[\prod_{i=1}^{n} \hat{S}_{iT}^{X_i} \frac{(1-\alpha)V_T}{D} 1_{\{\prod \hat{S}_{iT}^{X_i} \geq K'\}} 1_{\{V_T < D\}}\middle| F_t\right],$$

$$A_4 = E_{Q^T}\left[\frac{K'(1-\alpha)V_T}{D} 1_{\{\prod \hat{S}_{iT}^{X_i} \geq K'\}} 1_{\{V_T < D\}}\middle| F_t\right]$$

$$K' = K^* + E_{Q^T}\left[\prod_{i=1}^{n} \hat{S}_{iT}^{X_i}\middle| F_t\right] - E_{Q^T}\left[\sum_{i=1}^{n} X_i \hat{S}_{iT}\middle| F_t\right].$$

Each of the above five terms can be calculated separately. A closed-form solution for valuing a vulnerable European basket call option C_t^B is as follows (see the appendix for the formal derivation):

$$\begin{aligned} C_t^B \approx BK_t\Bigg[& m_t^S \, e^{(1/2)S_S^2} N_2(a_1, a_2, \rho) - K' N_2(b_1, b_2, \rho) \\ &+ \frac{m_t^S(1-\alpha)V_t}{P(t,T)D} \, e^{(1/2)S_S^2 + S_{SV}} N_2(c_1, c_2, -\rho) \\ &- \frac{K'(1-\alpha)V_t}{P(t,T)D} N_2(d_1, d_2, -\rho)\Bigg], \end{aligned} \quad (10)$$

where

$$BK_t = \sum_{i=1}^{n} w_i S_{it}, \quad m_t^S = e^{-(1/2)\int \sum X_i \hat{\sigma}_{S_i}^2 (T-\tau)\, d\tau},$$

$$K' = \frac{KP(t,T)}{BK_t} + m_t^S \, e^{(1/2)S_S^2} - 1,$$

$$S_V^2 = \int_t^T \hat{\sigma}_V^2(T-\tau)\, d\tau, \quad S_S^2 = \int_t^T \left(\sum X_i \hat{\sigma}_{S_i}(T-\tau)\right)^2 d\tau,$$

$$S_{SV} = \sum_{i=1}^{n} X_i S_{S_i V}, \quad X_i = \frac{w_i S_{it}}{\sum w_i S_{it}}, \quad i = 1, \ldots, n,$$

$$S_{S_i V} = \int_t^T \hat{\rho}_{S_i V}(T-\tau) \hat{\sigma}_V(T-\tau) \hat{\sigma}_{S_i}(T-\tau)\, d\tau, \quad i = 1, \ldots, n,$$

$$\hat{\sigma}_V^2(T-t) = \sigma_V^2 + \sigma_P^2(T-t) - 2\sigma_V \sigma_P (T-t) \rho_{VP},$$

$$\hat{\sigma}_{S_i}^2(T-t) = \sigma_{S_i}^2 + \sigma_P^2(T-t) - 2\sigma_{S_i}\sigma_P(T-t)\rho_{S_i P}, \quad i = 1, \ldots, n,$$

$$\hat{\rho}_{S_i V}(T-t) = \frac{\sigma_{S_i}\sigma_V \rho_{S_i V} - \sigma_V \sigma_P (T-t) \rho_{VP} - \sigma_{S_i} \sigma_P (T-t) \rho_{S_i P} + \sigma_P^2 (T-t)}{\hat{\sigma}_V(T-t) \hat{\sigma}_{S_i}(T-t)}, \quad i = 1, \ldots, n,$$

$$\rho = \frac{S_{SV}}{S_S S_V},$$

$$a_1 = \frac{\ln(m_t^S/K') + S_S^2}{S_S}, \quad a_2 = \frac{\ln(V_t/P(t,T)D) - (S_V^2/2) + S_{SV}}{S_V},$$

$$b_1 = \frac{\ln(m_t^S/K')}{S_S}, \quad b_2 = \frac{\ln(V_t/P(t,T)D) - (S_V^2/2)}{S_V},$$

$$c_1 = \frac{\ln(m_t^S/K') + S_S^2 + S_{SV}}{S_S}, \quad c_2 = -\frac{\ln(V_t/P(t,T)D) + (S_V^2/2) + S_{SV}}{S_V},$$

$$d_1 = \frac{\ln(m_t^S/K') + S_{SV}}{S_S}, \quad d_2 = -\frac{\ln(V_t/P(t,T)D) + (S_V^2/2)}{S_V}.$$

†According to Vorst (1992), this approximation is very accurate. The approximation error decreases with the volatility and the length of the averaging period. In all cases the approximation error never exceeds 2%, while in many cases it is below 0.1%.

$N_2(\cdot)$ represents the bivariate normal cumulative distribution function.

The time-dependent parameters $\hat{\sigma}_V^2(T-t)$ and $\hat{\sigma}_{S_i}^2(T-t)$ represent the instantaneous variances of the return on the writer's assets and on the assets underlying the warrant in terms of the value of the zero-coupon bond, respectively. Integrating $\hat{\sigma}_V^2(T-t)$ and $\left(\sum X_i\hat{\sigma}_{S_i}(T-t)\right)^2$ over the life of the warrant from t to T gives S_V^2 and S_S^2. $\hat{\rho}_{S_iV}(T-t)$ represents the instantaneous correlation between the returns on the normalized assets of the writer and assets underlying the warrant. S_{SV} represents the covariance between the returns on these two classes of normalized assets.

Next, using the put–call parity we derive the approximate valuation formula for a vulnerable European basket put warrant. Let P_T^B be the payoff of the European basket put at maturity T:

$$P_T^B = \max\{K - BK_T, 0\}.$$

Applying the put–call parity, we obtain

$$P_t^B \approx C_t^B - BK_t + KP(t,T). \tag{11}$$

The above formula is the closed-form approximation for a vulnerable European basket put warrant.

We have already derived the formulas for vulnerable European basket calls and puts under the assumption that the underlying stock upon which a warrant is written pays no dividends during the warrant's lifetime. More specifically, we assume that the individual underlying stock continuously pays dividends at some fixed rate q_i, where $i = 1, \ldots, n$. It can also be shown that, using equations (10) and (11), we can extend our results to vulnerable basket warrants on stocks paying a continuous dividend yield, by substituting $\sum_{i=1}^n w_iS_{it}\,\mathrm{e}^{-q_i(T-t)}$ for $BK_t = \sum_{i=1}^n w_iS_{it}$.

This paper extends Klein and Inglis (1999) on pricing European options subject to financial distress and interest rate risk. Unlike Klein and Inglis (1999), we use the CIR interest rate model (Cox *et al.* 1985) to avoid negative interest rates. Moreover, Klein and Inglis (1999) evaluate vulnerable single-stock warrants, while our paper prices basket warrants. The pricing formula that we present in this paper is actually the general form of the formulae presented by Black and Scholes (1973), Klein (1996), Smith (1976), Klein and Inglis (1999), Hull and White (1995) and Gentle (1993), respectively. To be more specific, our pricing formula is the general form of the formulae for the following warrants.

(1) The single-stock warrants with fixed interest rate, not considering credit risk (that is, $n = 1$, $r(t) = r$, $V = 0$).
(2) The single-stock warrants with fixed interest rate, considering credit risk (that is, $n = 1$, $r(t) = r$).
(3) The single-stock call warrants with stochastic interest rates, not considering credit risk (that is, $n = 1$, $V = 0$).
(4) The single-stock call warrant with stochastic interest rates, considering credit risk (that is, $n = 1$).
(5) The single-stock call warrants with fixed interest rate, considering credit risk, and the unrelated assets (that is, $n = 1$, $r(t) = r$, $\rho = 0$).
(6) The basket call warrants with fixed interest rate, not considering credit risk (that is, $r(t) = r$, $V = 0$).

It is worth noting that when $n = 1$ the geometric average is equal to the arithmetic average according to equation (8). Substituting $n = 1$ into equation (10), it gives the pricing formula of Klein and Inglis (1999).

In addition, our pricing formula is also the general formula for valuing basket call warrants with stochastic interest rates, not considering credit risk; with fixed interest rate, considering credit risk; with fixed interest rate, considering credit risk, and the unrelated assets.

4. Numerical examples

Unlike Klein and Inglis (1999), who employ Vasicek's (1977) interest rate model, we use the CIR model with our numerical examples. Both models are one-factor equilibrium models and have the characteristic of mean-reversion. However, Vasicek's (1977) model can produce negative interest rates when the initial rate starts from a low value, while the CIR model does not. Due to the assumption that the future instantaneous interest rates are normally distributed, we therefore use the CIR model. The dynamics of the short-term interest rates of the CIR model under the risk-neutral measure Q are as follows:

$$\mathrm{d}r_t = a(b - r_t)\,\mathrm{d}t + \sigma\sqrt{r_t}\,\mathrm{d}W_t,$$

where b is the long-run mean level of interest rate r, a is the parameter that governs the speed of mean-reversion, and W_t is the usual Wiener process. We assume $2ab > \sigma^2$ to ensure that zero interest rates will not occur.

Based on the CIR model, the price at time t of a zero-coupon bond paying one dollar at maturity T is

$$P(t,T) = G(t,T)\,\mathrm{e}^{-H(t,T)r},$$

where

$$G(t,T) = \left(\frac{\varphi_1\,\mathrm{e}^{\varphi_2(T-t)}}{\varphi_2(\mathrm{e}^{\varphi_1(T-t)} - 1) + \varphi_1}\right)^{\varphi_3},$$

$$H(t,T) = \frac{\mathrm{e}^{\varphi_1(T-t)} - 1}{\varphi_2(\mathrm{e}^{\varphi_1(T-t)} - 1) + \varphi_1},$$

$$\varphi_1 = \sqrt{a^2 + 2\sigma^2}, \quad \varphi_2 = \frac{a + \varphi_1}{2}, \quad \varphi_3 = \frac{2ab}{\sigma^2}.$$

The parameters a, b, and σ are all positive. The volatility of the bond return is then

$$\sigma_P(T-t) = H(t,T)\sigma\sqrt{r}.$$

The formula for the value of a vulnerable basket European call option depends on a number of factors:

$$\text{basket option} = f\left(\frac{BK_t}{K}, \frac{D}{V}, P, T, \alpha, \sigma_V, \sigma_{S_i}, \sigma_r, \rho, \rho_{Vr}, \rho_{S_ir}, \rho_{ij}, a, b\right), \quad i,j = 1, \ldots, n.$$

Under several scenarios, we use a range of parameters for the valuation formula to discuss its characteristics. The choice of parameter is determined by common market situations where the basket warrants are generally issued at-the-money. That is, the value of a basket stock price is equal to the strike price. The interest rate volatility is 3%, the market interest rate is 5%, the volatility of the value of the assets of the warrant writer is 20%, and the volatility of each underlying stock price ranges from 10 to 20%. We also assume that the value of the assets of the warrant writer is larger than that of its debt, i.e. there is no default on the warrant writer at the beginning of the period. The writer's debt equals its capital forbearance. In the short-term interest rate simulation, we choose values to make $2ab>\sigma^2$ in order to avoid negative interest rates. The values of the parameters used in the numerical examples are shown in table 1.

Table 2 presents the value of vulnerable and non-vulnerable n -stock basket European call warrants under seven cases with different combinations of correlations between the value of the assets of the warrant writer, the value of the assets underlying the warrant, and the short-term interest rate, and with different levels of stock price volatility. Like Klein and Inglis (1999) and Hull and White (1995), we find that the value of a basket call warrant with credit risk is lower than that without. When a negative correlation exists between the value of the assets of the firm writing the warrant and the underlying stock prices, there is the largest reduction in the value of vulnerable basket European call warrants, as shown in case 1 (17.2% and 14%) and case 6 (16.6% and 13.5%), implying that the probability that the warrant writer will default increases when the value of the assets underlying the warrant increases while the value of the writer's asset decreases. Conversely, when the correlation is positive, there is the smallest reduction in case 1 (2.3% and 3.5%) and case 6 (2.4% and 3.6%). Moreover, we find that greater volatility in the prices of the underlying asset increases the price of the call warrant with or without considering credit risk. The reason for this result is that the payoff from the call warrant is asymmetric. Increases in the prices of the underlying asset above the exercise price lead to a higher payoff from the call warrant, but decreases in the prices below the exercise price will not result in additional losses.†

Table 3 shows the value of vulnerable and non-vulnerable two-stock basket European call warrants under seven cases with different combinations of correlations. Assume that these two stocks have the same price of 30 ($S_1 = S_2 = 30$), and that their volatilities are $\sigma_{S_1} = 0.2$ and $\sigma_{S_2} = 0.1$. Since the volatility of the stock price S_1 is greater than S_2, the volatility of the basket of underlying assets with a higher proportion of S_1 than S_2 ($B1 : w_1 = 0.7, w_2 = 0.3$) is higher than that of the basket of underlying assets with a higher proportion of S_2 than S_1 ($B2 : w_1 = 0.3, w_2 = 0.7$). We find that the value of the warrant with the $B1$ basket of underlying assets is higher than that of the warrant with the $B2$ basket of underlying assets with and without considering credit risk, again implying that higher volatility in the underlying asset prices increases the price of the call warrant. As in table 2, we also find in table 3 that when the value of the assets of the warrant writer and the value of the assets underlying the warrant are negatively related to each other, case 1 (16.4% and 15.1%) and case 6 (15.9% and 14.6%) have the largest reduction in the value of vulnerable basket European call warrants, even if the underlying stocks have different weights.

Assume that there are two basket European call warrants whose underlying assets are two stocks with the same price ($S_1 = S_2 = 30$) and equal weights $w_1 = w_2 = 0.5$. Further assume that $\rho_{S_ir} = \rho_{Vr} = 0.5$ and $\rho_{VS_i} = 0$, $i = 1, 2$. The parameters from table 1 are also utilized here. Figures 1–3 demonstrate the effect of varying the volatility of the prices of the underlying stocks ($\sigma_{s_1} = 0.1, \sigma_{s_2} = 0.1$, $\sigma_{s_1} = 0.2$, $\sigma_{s_2} = 0.1$, and $\sigma_{s_1} = 0.2, \sigma_{s_2} = 0.2$). Figures 1 and 2 plot the value of the vulnerable basket call warrant and $\mathrm{d}C/\mathrm{d}D$ as a function of the debt ratio (D/V) of the warrant writer, respectively. Figure 1 shows that the higher the debt ratio the lower the value of the call warrant. It also shows that the higher the volatility of the prices of the underlying stocks, the higher the value of the call warrant. Figure 2 shows that when the debt ratio is low (between 0 and 0.3), $\mathrm{d}C/\mathrm{d}D$ is close to zero, i.e. there is no impact on the value of the vulnerable basket call warrant. When the debt ratio is above 0.3, $\mathrm{d}C/\mathrm{d}D$ begins to decrease, implying that the value of the vulnerable basket call warrant is inversely related to the debt ratio. We also find that the reduction in the vulnerable call warrant's value increases with the volatility of the prices of the underlying stocks.

Figure 3 illustrates the relation between the ratio of underlying asset value to strike price and the value of the vulnerable basket call warrant. It appears that the volatility of prices of underlying stocks has no distinct impact on the vulnerable basket call warrant's value when the warrant is deep in-the-money (BK_t/Kranges between 0.2 and 0.4) or deep out-of-the-money (BK_t/Kranges between 1.8 and 2.0). Conversely, when the warrant is near at-the-money ($(BK_t/K) = 1$), the warrant's value increases with the volatility of the prices of the underlying stocks.

Figure 4 presents the relation between the percentage reduction in the basket call warrant due to credit risk and the debt ratio of the warrant writer under different values of σ_S ($\sigma_{s_1} = 0.1, \sigma_{s_2} = 0.1$, $\sigma_{s_1} = 0.2, \sigma_{s_2} = 0.1$ and $\sigma_{s_1} = 0.2$, $\sigma_{s_2} = 0.2$) when the value of the assets underlying the

†Our approximation aims to deal with the problem arising from the fact that the sum of log-normal variables is not log-normal. In our analysis, we assume only one issuing firm with credit risk. Therefore, the vulnerability of warrants does not affect the accuracy of our approximation.

Table 1. Definitions and values of parameters used in numerical examples.

Parameter	Definition	Value
V	Value of the assets of the warrant writer	100
S_i	Value of the assets underlying the warrant, i = 1,....,n	30
BK	Weighted average value of n kinds of stocks	30
K	Strike price	30
T	Maturity of the warrant	5
D	Total value of debt of the warrant writer	80
D^*	Fixed threshold value	80
P	Zero-coupon bond price	0.7790
w_i	Weight basket of stocks, $i = 1,....,n$	1/n
r	Short-term interest rate	0.05
α	Deadweight costs	0.25
a	Speed of mean-reversion	0.5
b	Long-term level of interest rate	0.45
σ_V	Standard deviation of the value of the assets of the warrant writer	0.2
σ_r	Standard deviation of interest rate	0.03
σ_{S_i}	Standard deviation of the value of the assets underlying the warrant, $i = 1, \ldots ,n$	
ρ_{V_r}	Correlation between W_v and W_P	
ρ_{VS_r}	Correlation between W_S and W_V, $i = 1, \ldots ,n$	
ρ_{S_ir}	Correlation between W_S and W_P, $i = 1, \ldots ,n$	

Table 2. Value of vulnerable and non-vulnerable basket European call warrants.

Case	Vol	VBECW		NVBECW		% reduction	
1. $\rho_{VS_i} = 0.5\,(\rho_{VS_i} = -0.5)$	A1:	8.542	(7.236)	8.743	(8.743)	2.3	(17.2)
$\rho_{S_ir} = \rho_{Vr} = 0, i = 1, ..., n$	A2:	6.783	(6.044)	7.028	(7.028)	3.5	(14.0)
2. $\rho_{Vr} = 0.5\,(\rho_{Vr} = -0.5)$	A1:	8.047	(8.023)	8.743	(8.743)	8.0	(8.2)
$\rho_{S_ir} = \rho_{VS_i} = 0, i = 1, ..., n$	A2:	6.452	(6.46)	7.028	(7.028)	8.2	(8.1)
3. $\rho_{S_ir} = 0.5\,(\rho_{S_ir} = -0.5)$	A1:	7.919	(8.139)	8.654	(8.830)	8.5	(7.8)
$\rho_{VS} = \rho_{Vr} = 0, i = 1, ..., n$	A2:	6.387	(6.524)	6.971	(7.084)	8.4	(7.9)
4. $\rho_{S_ir} = \rho_{Vr} = 0.5$	A1:	7.925	(8.131)	8.654	(8.830)	8.4	(7.9)
$(\rho_{S_ir} = \rho_{Vr} = -0.5)$	A2:	6.382	(6.528)	6.971	(7.084)	8.4	(7.8)
$\rho_{VS} = 0, i = 1, ..., n$							
5. $\rho_{S_ir} = 0.5, \rho_{Vr} = -0.5$	A1:	7.928	(8.148)	8.654	(8.830)	8.4	(7.7)
$(\rho_{S_ir} = -0.5, \rho_{Vr} = 0.5)$	A2:	6.392	(6.521)	6.971	(7.084)	8.3	(7.9)
$\rho_{VS_i} = 0,\ i = 1, \ldots, n$							
6. $\rho_{S_ir} = \rho_{VS_i} = \rho_{Vr} = 0.5$	A1:	8.449	(7.360)	8.654	(8.830)	2.4	(16.6)
$(\rho_{S_ir} = \rho_{VS_i} = \rho_{Vr} = -0.5)$	A2:	6.719	(6.126)	6.971	(7.084)	3.6	(13.5)
$i = 1, \ldots, n$							
7. $\rho_{VS_i} = 0,\ i = 1, \ldots, n$	A1:	8.031		8.743		8.1	
	A2:	6.457		7.028		8.1	

Vol, volatility of stock prices; A1, $\sigma_{S_i} = 0.2$; A2, $\sigma_{S_i} = 0.1$, $i = 1, ..., n$; VBECW, vulnerable basket European call warrant; NVBECW, non-vulnerable basket European call warrant;% reduction=(NVBECW–VBECW)/NVBECW*100.

warrants and the value of the warrant writer's assets are highly positively/negatively correlated ($\rho_{VS} = 0.9$, $\rho_{VS} = -0.9$). The higher the volatility of the underlying stock prices, the smaller the percentage reduction. When $\rho_{VS} = 0.9$, the percentage reduction is very limited, implying that there is no distinct variation in the basket call warrant's value as the value of the assets underlying the warrant is positively related to the value of the warrant writer's assets. In this case, the default risk on the part of the writer is relatively low. Conversely, when the relation is highly negative ($\rho_{VS} = -0.9$), the percentage reduction significantly varies from 30 to 37%. In the case of a highly negative correlation, the more volatile the underlying stock prices, the larger the percentage reduction.

Figures 5–7 plot the percentage reduction in the basket call warrant value due to credit risk as a function of the debt ratio of the warrant writer under different scenarios of volatility of the underlying stocks ($\sigma_{s_1} = 0.1, \sigma_{s_2} = 0.1$, $\sigma_{s_1} = 0.2, \sigma_{s_2} = 0.1$ and $\sigma_{s_1} = 0.2, \sigma_{s_2} = 0.2$) and correlations between the value of the asset of the warrant writer and the interest rate ($\rho_{Vr} = 0.9, \rho_{Vr} = -0.9$). These show that the percentage reduction increases with the debt ratio. In the case of low leverage, it makes no significant difference to take credit risk into account, because the probability of default risk on the part of the writer is low. Conversely, in the case of high leverage, the percentage reduction due to credit risk is high.

When the leverage is high, the value of the call warrant is higher when the correlation between the value of the asset of the warrant writer and the interest rate is highly negative than when it is highly positive. However, the percentage reduction in the call warrant value due to credit risk is lower when the correlation is highly negative than when it is highly positive. Furthermore, we find that the two lines

Table 3. Value of vulnerable and non-vulnerable basket European call warrants with underlying assets of two stocks ($\sigma_{S_1} = 0.2, \sigma_{S_2} = 0.1$).

Case	Weight	VBECW		NVBECW		% reduction	
1. $\rho_{VS_i} = 0.5$ ($\rho_{VS_i} = -0.5$)	B1:	7.940	(6.810)	8.147	(8.147)	2.5	(16.4)
$\rho_{S_ir} = \rho_{Vr} = 0,\ i = 1,2$	B2:	7.229	(6.322)	7.452	(7.452)	3.0	(15.1)
2. $\rho_{Vr} = 0.5$ ($\rho_{Vr} = -0.5$)	B1:	7.488	(7.481)	8.148	(8.148)	8.1	(8.2)
$\rho_{S_ir} = \rho_{VS_i} = 0,\ i = 1,2$	B2:	6.845	(6.846)	7.452	(7.452)	8.1	(8.1)
3. $\rho_{S_ir} = 0.5$ ($\rho_{S_ir} = -0.5$)	B1:	7.381	(7.585)	8.064	(8.230)	8.5	(7.8)
$\rho_{VS_i} = \rho_{Vr} = 0,\ i = 1,2$	B2:	6.758	(6.930)	7.380	(7.523)	8.4	(7.9)
4. $\rho_{S_ir} = \rho_{Vr} = 0.5$	B1:	7.388	(7.580)	8.064	(8.230)	8.4	(7.9)
($\rho_{S_ir} = \rho_{Vr} = -0.5$)	B2:	6.757	(6.930)	7.380	(7.523)	8.4	(7.9)
$\rho_{VS_i} = 0,\ i = 1,2$							
5. $\rho_{S_ir} = 0.5, \rho_{Vr} = -0.5$	B1:	7.378	(7.589)	8.064	(8.230)	8.5	(7.8)
($\rho_{S_ir} = -0.5, \rho_{Vr} = 0.5$)	B2:	6.760	(6.932)	7.380	(7.523)	8.4	(7.9)
$\rho_{VS_i} = 0,\ i = 1,2$							
6. $\rho_{S_ir} = \rho_{VS_i} = \rho_{Vr} = 0.5$	B1:	7.853	(6.925)	8.064	(8.230)	2.6	(15.9)
($\rho_{S_ir} = \rho_{VS_i} = \rho_{Vr} = -0.5$),	B2:	7.152	(6.422)	7.380	(7.523)	3.1	(14.6)
$i = 1,2$							
7. $\rho_{S_ir} = \rho_{VS_i} = \rho_{Vr} = 0$,	B1:	7.483		8.148		8.2	
i = 1,2	B2:	6.845		7.452		8.1	

Weight, weight of stocks; B1, $w_1 = 0.7, w_2 = 0.3$; B2, $w_1 = 0.3, w_2 = 0.7$; VBECW, vulnerable basket European call warrant; NVBECW, non-vulnerable basket European call warrant; % reduction= (NVBECW–VBECW)/NVBECW*100.

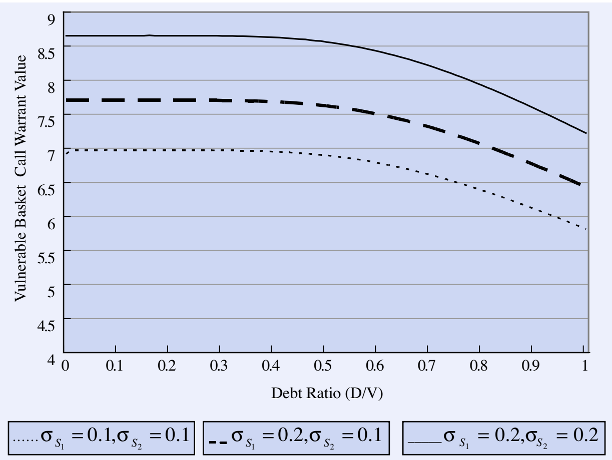

Figure 1. Relation between the value of a vulnerable basket call warrant and its debt ratio for different values of σ_s.

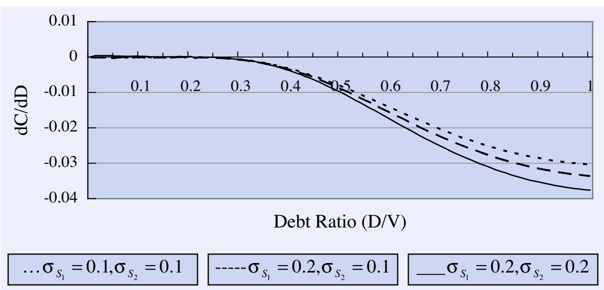

Figure 2. dC/dD of a vulnerable basket call warrant.

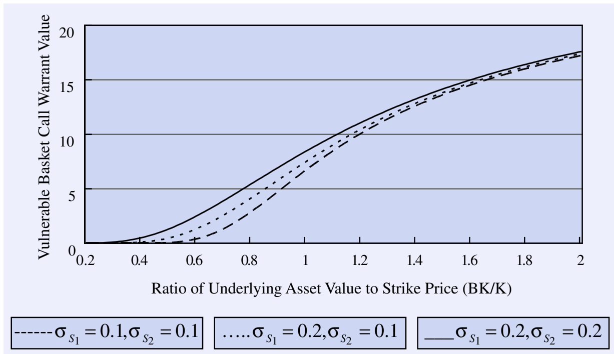

Figure 3. Relation of the value of a vulnerable basket call warrant and the ratio of the underlying asset value to the strike price for different values of σ_s.

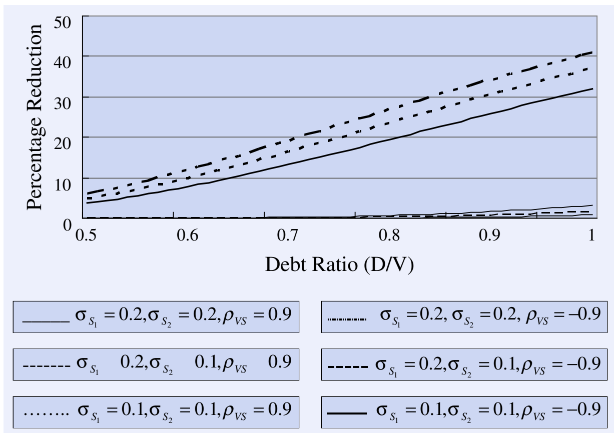

Figure 4. Percentage reduction in the value of a vulnerable basket call for different values of ρ_{vs} and σ_s. Note: Percentage reduction= (NVBECW–VBECW)/NVBECW*100, where VBECW and NVBECW represent vulnerable and non-vulnerable basket European call warrants, respectively.

in figures 5–7 intersect. The intersection of the lines can be explained using figure 1, which shows that the basket call warrant value decreases to a large (small) extent when the debt ratio is high (low). We also find that the higher the volatility of the underlying stock prices, the higher the debt ratio at which the intersections occur. Specifically, the

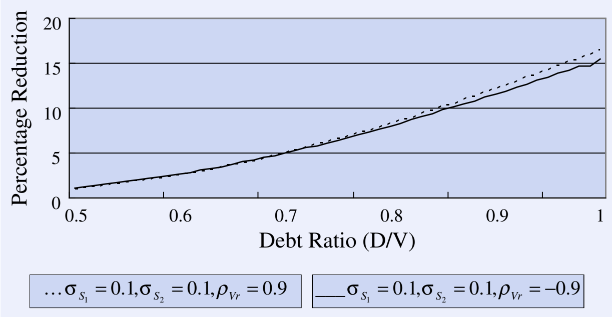

Figure 5. Percentage reduction in the value of a vulnerable basket call for different values of ρ_{vr}.

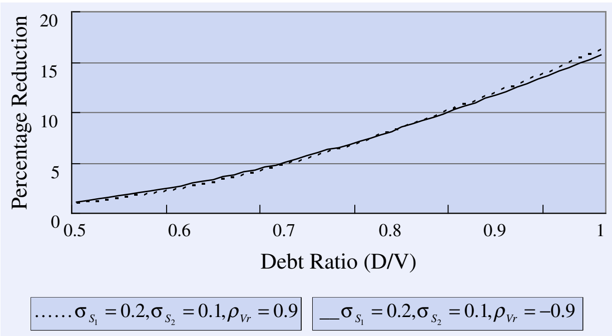

Figure 6. Percentage reduction in the value of a vulnerable basket call for different values of ρ_{vr}.

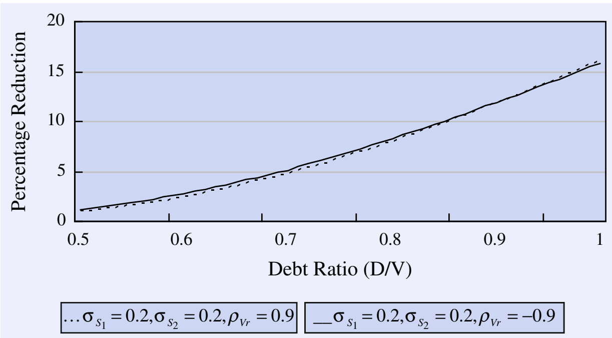

Figure 7. Percentage reduction in the value of a vulnerable basket call for different values of ρ_{vr}.

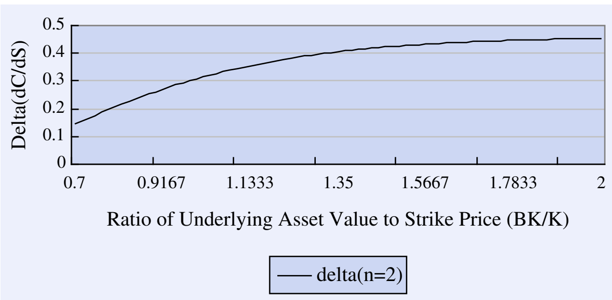

Figure 8. Delta of a vulnerable basket call warrant ($n = 2$).

intersections in figures 5–7 occur when the debt ratios are 0.75, 0.85, and 0.9, respectively. This can be explained

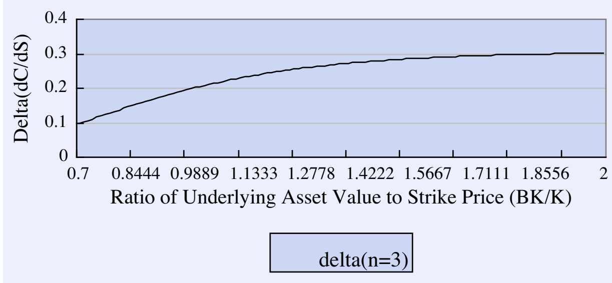

Figure 9. Delta of a vulnerable basket call warrant ($n = 3$).

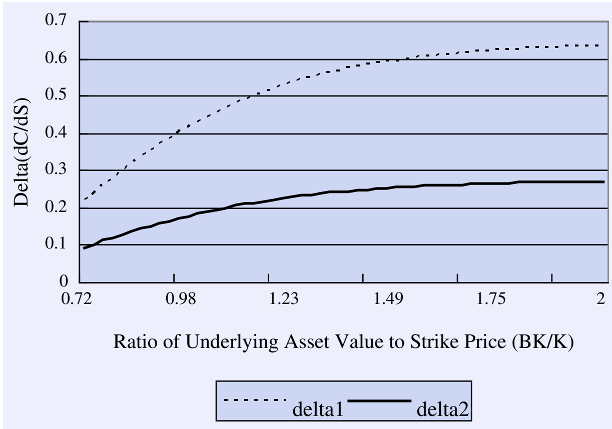

Figure 10. Delta of a vulnerable basket call warrant with unequal weights of underlying stocks.

using figure 2, which shows that the higher the volatility of the underlying stock prices and the debt ratio, the greater extent to which the basket call warrant value decreases.

Again, the parameters in table 1 are utilized. Assume that $\sigma_{s_1} = 0.2, \sigma_{s_2} = 0.2$, $\rho_{S_i r} = \rho_{Vr} = 0.5$ and $\rho_{VS_i} = 0$, $i = 1, 2$, and that there are two basket European call warrants whose underlying assets are two ($n = 2$) and three ($n = 3$) stocks with the same price of 30 and equal weights. Figures 8 and 9 show the sensitivity of the call warrant to changes in the value of the assets underlying the warrant when $n = 2$ and $n = 3$, respectively. As shown in these figures, the values of delta are 0.45 and 0.3 when $n = 2$ and $n = 3$, respectively. This finding indicates that the greater the number of stocks underlying the warrant, the less the impact of individual stocks on the price of the warrant.

Assume that there are two stocks with the same price ($S_1 = S_2 = 30$) but with unequal weights ($w_1 = 0.7$, $w_2 = 0.3$) underlying the vulnerable basket call warrant. Figure 10 shows the relation between the delta of the warrant and the ratio of the underlying asset value to the strike price. This figure illustrates that $w_1 = 0.7$, $\Delta_1 = \mathrm{d}C^B/\mathrm{d}S_1 = 0.63$, $w_2 = 0.3$, and $\Delta_2 = \mathrm{d}C^B/\mathrm{d}S_2 = 0.27$, implying that the warrant value is more sensitive to the value of the underlying stock with greater weight.

For convenience, one may simply assume that the sum of underlying asset prices obey a log-normal distribution. However, because the sum of log-normal random variables is not log-normal, the evaluation of warrants will also be an approximate solution. Table 4 compares the percentage

Table 4. Comparison of vulnerable basket European call warrants under Vorst's (1992) and the log-normal distribution.

Case	VBECW	VBECWL	Percentage reduction
1. $\sigma_{S_1} = 0.1, \sigma_{S_2} = 0.1$	4.5280	4.5280	0.00
2. $\sigma_{S_1} = 0.1, \sigma_{S_2} = 0.15$	4.8794	4.8813	0.04
3. $\sigma_{S_1} = 0.1, \sigma_{S_2} = 0.2$	5.2603	5.2698	0.18
4. $\sigma_{S_1} = 0.1, \sigma_{S_2} = 0.25$	5.6540	5.6799	0.46
5. $\sigma_{S_1} = 0.1, \sigma_{S_2} = 0.3$	6.0494	6.1036	0.89
6. $\sigma_{S_1} = 0.1, \sigma_{S_2} = 0.35$	6.4389	6.5356	1.48
7. $\sigma_{S_1} = 0.1, \sigma_{S_2} = 0.4$	6.8176	6.9737	2.24

$S_1 = S_2 = 30$, otherwise the correlation with each other asset is 0.5. VBECW, vulnerable basket European call warrant using Vorst's (1992) approximation; VBECWL, vulnerable basket European call warrant under the log-normal assumption; percentage reduction= (VBECWL–VBECW)/VBECWL*100.

Table 5. Comparison of vulnerable basket European call warrants under the CIR and Vasicek interest rate models.

ρ_{Vr}	Maturity	VBECWC	VBECWV	Percentage reduction
–0.9	1	3.2022	3.1513	–1.6
	2	5.0929	4.9849	–2.2
	3	6.8030	6.6444	–2.4
–0.4	1	3.2024	3.1505	–1.6
	2	5.0939	4.9802	–2.3
	3	6.8052	6.6347	–2.6
0	1	3.2026	3.1498	–1.7
	2	5.0954	4.9775	–2.4
	3	6.8070	6.6287	–2.7
0.4	1	3.2028	3.1492	–1.7
	2	5.0958	4.9754	–2.4
	3	6.8091	6.6237	–2.8
0.9	1	3.2031	3.1485	–1.7
	2	5.0969	4.9733	–2.5
	3	6.8123	6.6197	–2.9

$S_1 = S_2 = 30$, otherwise the correlation with each other asset is 0.5. VBECWC, vulnerable basket European call warrant under the CIR interest rate model; VBECWV, vulnerable basket European call warrant under the Vasicek interest rate model; percentage reduction= (VBECWV– BECWC)/VBECWV*100.

reduction of the vulnerable basket European call warrants value under Vorst's (1992) approximation and the log-normal distribution. We find that the percentage reduction is zero, indicating that the valuation of prices is the same for these two approximations only when the standard deviations of the two underlying asset prices are the same. The percentage reduction increases with the volatility of stocks.

Table 5 compares the results under the CIR and Vasicek interest rate models. We assume that the vulnerable basket call warrant has a basket of two underlying stocks. We find that the results of the CIR model generally resemble those of the Vasicek model, especially when time to expiration is short. Warrants normally have short maturities. Thus, the use of different interest rate models has no significant effects on our results.

Figure 11 shows the effects of interest rate risk on our results. We again assume that the vulnerable basket call warrant has a basket of two underlying stocks. Suppose that the correlation coefficient between the first underlying stock price and the interest rate is the same as that between the second and the interest rate. The correlations are 0.9, 0, and –0.9. The fixed interest rate is 0.1. The percentage reduction increases as the debt ratio increases. Unreported results show that the effects would become smaller when the interest rate is fixed at a lower level.

To use the formula proposed in this paper, one needs to come up with correct values for the parameters of the formula. We next discuss how the reader may proceed in order to estimate the parameters or calibrate the model. Because volatility is unobservable, it needs to be estimated. Data on the assets and liabilities of issuing firms can be collected from their balance sheets one season before issuing warrants. The volatility of firm assets is represented by the standard

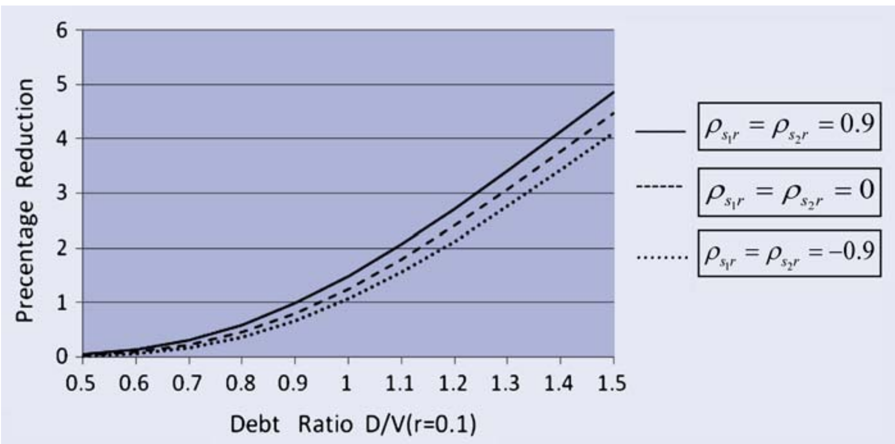

Figure 11. The effects of interest rate risk on warrant values.

deviation of the market value of the issuing firm's assets. Because the market value is difficult to estimate, we may use the stock price of the firm to compute the annual standard deviation of stock prices. In the same fashion, we can compute the variance of stock prices. The correlation coefficient is estimated using the daily stock returns of the issuing firm and the underlying firm. The time to maturity is the duration of the warrants. The risk-free rate is the T-bill rate at the issuing day. As to the exercise price, the weights of individual stocks, and their duration, one may consult the prospectus of issuing firms. We may use the recovery rates of debts published by credit rating agencies, such as Moody's Investors Service or Standard and Poor's. The interest rate data are used to calibrate the interest rate model using the generalized method of moments or quasi-maximum likelihood.

5. Conclusions

This study has proposed a model with a continuous-time framework, with credit risk and interest rate risk considered simultaneously. We extended Klein and Inglis' (1999) valuation of a vulnerable single stock option by developing a formula for pricing a vulnerable basket warrant. Under the risk-neutral measure, we have derived an approximate valuation formula for the vulnerable European basket warrant using the martingale approach. This formula was proved to be the general closed-form solution of the models proposed by Black and Scholes (1973), Smith (1976), Gentle (1993), Hull and White (1995), Klein (1996), and Klein and Inglis (1999). Finally, using the interest rate model proposed by Cox *et al.* (1985) we utilized several numerical examples to explain the valuation of a vulnerable European basket call warrant. Several findings are obtained from the simulation exercise. First, the value of a vulnerable call warrant is always less than that of a non-vulnerable one. Second, the value of the vulnerable basket call warrant increases as the volatility of the value of stocks underlying the warrant rises, or when the underlying stock with higher volatility takes up greater weight in the basket of assets. Third, the value of the call warrant is positively related to the volatility of the stocks underlying the warrant, but negatively related to the debt ratio of the warrant writer. Fourth, the volatility of the value of underlying stocks has a greater impact on the value of the call warrant when it is at-the-money than deep-in-the-money or deep-out-of-the-money. Fifth, the reduction in the call warrant value due to credit risk is relatively large when the correlation between the value of the assets of the warrant writer and the value of underlying stocks is highly negative. Also, the higher the volatility of the prices of underlying stocks, the larger the reduction in the vulnerable call warrant value.

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Appendix A

We first use the money market account $B(t)$ as a numéraire and change measure from the probability measure P to the risk-neutral probability measure Q. The instantaneous expected returns in equations (1) to (3) are then replaced by the instantaneous riskless interest rate $r(t)$. Let

$$\tilde{V}(t) = \frac{V(t)}{P(t,T)}, \quad \tilde{S}(t) = \frac{S(t)}{P(t,T)}.$$

Following Geman *et al.* (1995), we take the zero-coupon bond price $P(t,T)$ as a numéraire and change the risk-neutral probability measure Q to the forward neutral measure Q^T. Thus, $V(t)/P(t,T)$ and $S(t)/P(t,T)$ are Q^T-martingale. Under the measure Q^T, the dynamics of the forward asset prices $\tilde{V}(t)$ and $\tilde{S}(t)$ are given by

$$\frac{\mathrm{d}\tilde{V}(t)}{\tilde{V}(t)} = \hat{\sigma}_V \,\mathrm{d}W_V^{Q^T}, \tag{A1}$$

$$\frac{\mathrm{d}\tilde{S}(t)}{\tilde{S}(t)} = \hat{\sigma}_S \,\mathrm{d}W_S^{Q^T}. \tag{A2}$$

$\hat{\sigma}_V^2(T-t)$ and $\hat{\sigma}_S^2(T-t)$ are defined as

$$\hat{\sigma}_V^2(T-t) = \sigma_V^2 + \sigma_P^2(T-t) - 2\sigma_V\sigma_P(T-t)\rho_{VP},$$

$$\hat{\sigma}_S^2(T-t) = \sigma_S^2 + \sigma_P^2(T-t) - 2\sigma_S\sigma_P(T-t)\rho_{SP}.$$

The covariance and correlation between $\tilde{V}(t)$ and $\tilde{S}(t)$ can presented as follows:

$$\mathrm{Cov}\left[\frac{\mathrm{d}\tilde{V}(t)}{\tilde{V}(t)},\frac{\mathrm{d}\tilde{S}(t)}{\tilde{S}(t)}\right]=(\sigma_S\sigma_V\rho_{SV}-\sigma_V\sigma_P(T-t)\rho_{VP}-\sigma_S\sigma_P(T-t)\rho_{SP}+\sigma_P^2(T-t))\,\mathrm{d}t,$$

$$\hat{\rho}_{SV}(T-t)=\frac{\sigma_S\sigma_V\rho_{SV}-\sigma_V\sigma_P(T-t)\rho_{VP}-\sigma_S\sigma_P(T-t)\rho_{SP}+\sigma_P^2(T-t)}{\hat{\sigma}_V(T-t)\hat{\sigma}_S(T-t)}.$$

Define $f(\tilde{S})=\ln\tilde{S}^X$ and $\mathrm{d}\tilde{S}=\tilde{S}\hat{\sigma}_S\,\mathrm{d}W_S^{Q^T}$. An application of Ito's lemma for a portfolio consisting of n kinds of individual stocks shows that $\tilde{S}_{iT}^{X_i}=\tilde{S}_{it}^{X_i}\,\mathrm{e}^{-(1/2)\int X_i\hat{\sigma}_{S_i}^2+\int X_i\hat{\sigma}_{S_i}\,\mathrm{d}W_{S_i}^{Q^T}}$, $i=1,\ldots,n$.

We also have $\tilde{S}_{iT}=S_{iT}/P(T,T)=S_{iT}$, $\tilde{S}_{it}=S_{it}\,/P(t,T)=F_i^S(t,T)$, and $(\tilde{S}_{iT}/\tilde{S}_{it})^{X_i}=\hat{S}_{iT}^{X_i}$.

Under the measure Q^T, $\tilde{S}_{iT}^X$ and $\tilde{V}_T$ are given by

$$\prod_{i=1}^{n}(\hat{S}_{iT})^{X_i}=\mathrm{e}^{-(1/2)\int\sum X_i\hat{\sigma}_{S_i}^2(T-\tau)\,\mathrm{d}\tau+\int\sum X_i\hat{\sigma}_{S_i}(T-\tau)\,\mathrm{d}W_{S_i}^{Q^T}},\tag{A3}$$

$$\tilde{V}_T=\tilde{V}_t\,\mathrm{e}^{-(1/2)\int\hat{\sigma}_V^2(T-\tau)\,\mathrm{d}\tau+\int\hat{\sigma}_V(T-\tau)\,\mathrm{d}W_V^{Q^T}}.\tag{A4}$$

Evaluation of the term A_1 in equation (9). From equations (A3) and (A4), A_1 can be written as

$$A_1=m_t^S\,e^{(1/2)S_S^2}E_{Q^T}\left[\frac{\mathrm{d}\tilde{Q}}{\mathrm{d}Q^T}1_{\{\prod\hat{S}_{iT}^{X_i}\geq K'\}}1_{\{V_T\geq D\}}\middle|F_t\right].\tag{A5}$$

Define a probability measure $\tilde{Q}$ as equivalent to Q^T by means of the Radon–Nikodym derivative $\mathrm{d}\tilde{Q}/\mathrm{d}Q^T=\mathrm{e}^{-(1/2)\int(\sum X_i\hat{\sigma}_{S_i}(T-\tau))^2\,\mathrm{d}\tau+\int\sum X_i\hat{\sigma}_{S_i}(T-\tau)\,\mathrm{d}W_{S_i}^{Q^T}}$.

According to the bivariate Girsanov's Theorem, Karatzas and Shreve (1991) obtained

$$\mathrm{d}W_{S_i}^{\tilde{Q}}=\mathrm{d}W_{S_i}^{Q^T}-\sum X_i\hat{\sigma}_{S_i}\,\mathrm{d}t,\quad i=1,\ldots,n,$$

$$\mathrm{d}W_V^{\tilde{Q}}=\mathrm{d}W_V^{Q^T}-\sum X_i\hat{\sigma}_{S_i}\hat{\rho}_{S_iV}\,\mathrm{d}t.\tag{A6}$$

And

$$\mathrm{corr}\left(\frac{\mathrm{d}\tilde{S}_i}{\tilde{S}_i},\frac{\mathrm{d}\tilde{V}}{\tilde{V}}\right)=\hat{\rho}_{S_iV}\,\mathrm{d}t,\quad i=1,\ldots,n.$$

Using equation (A6) and changing measure from Q^T to $\tilde{Q}$, we obtain $\hat{S}_{iT}^{X_i}$ and $\tilde{V}_T$ as follows:

$$\prod_{i=1}^{n}\hat{S}_{iT}^{X_i}=\mathrm{e}^{\int(\sum X_i\hat{\sigma}_{S_i}(T-\tau))^2\,\mathrm{d}\tau-(1/2)\int\sum X_i\hat{\sigma}_{S_i}^2(T-\tau)\,\mathrm{d}\tau+\int\sum X_i\hat{\sigma}_{S_i}\,\mathrm{d}W_{S_i}^{\tilde{Q}}},\tag{A7}$$

$$\tilde{V}_T=\tilde{V}_t\,\mathrm{e}^{-(1/2)\int\hat{\sigma}_V^2(T-\tau)\,\mathrm{d}\tau+\int\sum X_i\hat{\rho}_{S_iV}\hat{\sigma}_{S_i}\hat{\sigma}_V(T-\tau)\,\mathrm{d}\tau+\int\hat{\sigma}_V(T-\tau)\,\mathrm{d}W_V^{\tilde{Q}}}.\tag{A8}$$

Substituting equations (A7) and (A8) into (A5), we obtain

$$\begin{aligned}A_1&=m_t^S\,\mathrm{e}^{(1/2)S_S^2}P_{\tilde{Q}}\left(\frac{-\int\sum X_i\hat{\sigma}_{S_i}\,\mathrm{d}W_{S_i}^{\tilde{Q}}}{S_S}\leq\frac{\ln(m_t^S/K')+S_S^2}{S_S},\right.\\&\qquad\left.\times\frac{-\int\hat{\sigma}_V\,\mathrm{d}W_V^{\tilde{Q}}}{S_V}\leq\frac{\ln(V/PD)-\frac{1}{2}S_V^2+S_{SV}}{S_V}\right)\\&=m_t^S\,\mathrm{e}^{(1/2)S_S^2}N_2(a_1,a_2,\rho),\end{aligned}$$

where

$$a_1=\frac{\ln(m_t^S/K')+S_S^2}{S_S},\quad a_2=\frac{\ln(V/PD)-(1/2)S_V^2+S_{SV}}{S_V},$$

$$\rho=\mathrm{corr}\left(\frac{-\int\sum X_i\hat{\sigma}_{S_i}\,\mathrm{d}W_{S_i}^{\tilde{Q}}}{S_S},\frac{-\int\hat{\sigma}_V\,\mathrm{d}W_V^{\tilde{Q}}}{S_V}\right)=\frac{S_{SV}}{S_SS_V}.$$

$N_2(\cdot)$ represents the bivariate normal cumulative distribution function.

Evaluation of the term A_2 in equation (9). Similarly, the term A_2 can be evaluated without the change of probability measure. We immediately find that $A_2=K'N_2(b_1,b_2,\rho)$, where

$$b_1=\frac{\ln(m_t^S/K')}{S_S},\quad b_2=\frac{\ln(V_t/P(t,T)D)-(S_V^2/2)}{S_V}.$$

Evaluation of the term A_3 in Equation (9). Substituting equations (A3) and (A4) into A_3, we obtain

$$A_3=\frac{(1-\alpha)m_t^S}{D}\mathrm{e}^{(1/2)S_S^2}E_{Q^T}\left[\frac{\mathrm{d}\tilde{Q}}{\mathrm{d}Q^T}V_T1_{\{\prod\hat{S}_{iT}^{X_i}\geqslant K'\}}1_{\{V_T<D\}}\middle|F_t\right].\tag{A9}$$

Substituting equations (A6) and (A7) into (A9), we have

$$\begin{aligned}A_3&=\frac{(1-\alpha)m_t^S}{D}\mathrm{e}^{(1/2)S_S^2}E_{\tilde{Q}}\left[V_T1_{\{\prod\hat{S}_{iT}^{X_i}\geq K'\}_{\{V_T<D\}}}\middle|F_t\right]\\&=\frac{(1-\alpha)m_t^s}{P(t,T)D}\;\mathrm{e}^{\frac{1}{2}S_s^2+S_{SV}}E_{\tilde{Q}}\left[\frac{\mathrm{d}Q_1}{\mathrm{d}\tilde{Q}}V_T1_{\{\prod\hat{S}_{iT}^{X_i}\geq K'\}}\{V_T<D\}\middle|F_t\right]\end{aligned}$$

Similarly, we introduce a new measure Q_1 defined by

$$\frac{\mathrm{d}Q_1}{\mathrm{d}\tilde{Q}}=\mathrm{e}^{-(1/2)\int\hat{\sigma}_V^2(T-\tau)\,\mathrm{d}\tau+\int\hat{\sigma}_V(T-\tau)\,\mathrm{d}W_V^{\tilde{Q}}}.$$

According to the bivariate Girsanov's Theorem, we obtain

$$\begin{aligned}\mathrm{d}W_V^{Q_1}&=\mathrm{d}W_V^{\tilde{Q}}-\hat{\sigma}_V\,\mathrm{d}t,\\\mathrm{d}W_{S_i}^{Q_1}&=\mathrm{d}W_{S_i}^{\tilde{Q}}-\textstyle\sum\hat{\sigma}_V\hat{\rho}_{S_iV}\,\mathrm{d}t,\quad i=1,\ldots,n.\end{aligned}\tag{A11}$$

Changing measure from $\tilde{Q}$ to Q_1 yields

$$\prod_{i=1}^{n}\hat{S}_{iT}^{X_i}=\mathrm{e}^{\int(\sum X_i\hat{\sigma}_{S_i}(T-\tau))^2\mathrm{d}\tau-(1/2)\int\sum X_i\hat{\sigma}_{S_i}^2(T-\tau)\mathrm{d}\tau+\int\sum X_i\hat{\rho}_{S_iV}(T-\tau)\hat{\sigma}_{S_i}(T-\tau)\hat{\sigma}_V(T-\tau)\mathrm{d}\tau+\int\sum X_i\hat{\sigma}_S(T-\tau)\mathrm{d}W_{S_i}^{Q_1}},\tag{A12}$$

$$\tilde{V}_T = \tilde{V}_t\,\mathrm{e}^{(1/2)\int \hat{\sigma}_V^2(T-\tau)\,\mathrm{d}\tau} + \int \sum X_i \hat{\rho}_{S_iV}(T-\tau)\hat{\sigma}_{S_i}(T-\tau)\hat{\sigma}_V(T-\tau)\,\mathrm{d}\tau + \int \hat{\sigma}_V(T-\tau)\,\mathrm{d}W_V^{Q_1} \tag{A13}$$

Substituting equations (A12) and (A13) into (A10), we obtain

$$A_3 = \frac{m_t^S(1-\alpha)V}{P(t,T)D}\,\mathrm{e}^{(1/2)S_S^2+S_{SV}}.$$

$$P_{Q_1}\left(\frac{-\int\sum X_i\hat{\sigma}_{S_i}\,\mathrm{d}W_{S_i}^{Q_1}}{S_S} \leqslant \frac{\ln(m_t^S/K')+S_S^2+S_{SV}}{S_S}, \frac{\int \hat{\sigma}_V\,\mathrm{d}W_V^{Q_1}}{S_V} \leqslant \frac{\ln(PD/V)-(1/2)S_V^2-S_{SV}}{S_V}\right) = \frac{m_t^S(1-\alpha)V}{P(t,T)D}\,\mathrm{e}^{(1/2)S_S^2+S_{SV}} N_2(c_1,c_2,-\rho),$$

where

$$c_1 = \frac{\ln(m_t^S/K')+S_S^2+S_{SV}}{S_S}, \quad c_2 = -\frac{\ln(V/PD)+\frac{1}{2}S_V^2+S_{SV}}{S_V}.$$

Evaluation of the term A_4 in equation (9). From equation (A4), A_4 can be written as

$$A_4 = \frac{K'(1-\alpha)V}{P(t,T)D} E_{Q^T}\left[\frac{\mathrm{d}Q_2}{\mathrm{d}Q^T} 1_{\{\prod \hat{S}_{iT}^{X_i}\geq K'\}} 1_{\{V_T<D\}}\middle| F_t\right]. \tag{A14}$$

Similarly, we introduce a new probability measure Q_2 defined as

$$\frac{\mathrm{d}Q_2}{\mathrm{d}Q^T} = \mathrm{e}^{-(1/2)\int \hat{\sigma}_V^2(T-\tau)\,\mathrm{d}\tau + \int \hat{\sigma}_V(T-\tau)\,\mathrm{d}W_V^{Q^T}}.$$

According to the bivariate Girsanov's Theorem, we obtain

$$\mathrm{d}W_V^{Q_2} = \mathrm{d}W_V^{Q^T} - \hat{\sigma}_V\,\mathrm{d}t,$$

$$\mathrm{d}W_{S_i}^{Q_2} = \mathrm{d}W_{S_i}^{Q^T} - \hat{\sigma}_V\hat{\rho}_{S_iV}\,\mathrm{d}t, \quad i=1,\ldots,n. \tag{A15}$$

Changing measure from Q^T to Q_2 yields

$$\prod_{i=1}^n \hat{S}_{iT}^{X_i} = \mathrm{e}^{-(1/2)\int\sum X_i\hat{\sigma}_{S_i}^2(T-\tau)\,\mathrm{d}\tau + \int\sum X_i\hat{\rho}_{S_iV}(T-\tau)\hat{\sigma}_{S_i}(T-\tau)\hat{\sigma}_V(T-\tau)\,\mathrm{d}\tau + \int\sum X_i\hat{\sigma}_{S_i}(T-\tau)\,\mathrm{d}W_{S_i}^{Q_2}}, \tag{A16}$$

$$= \frac{K'(1-\alpha)V}{B_{(t,T)}D} N_2(d_1,d_2,-\rho)\,\mathrm{e}^{(1/2)\int\hat{\sigma}_V^2(T-\tau)\,\mathrm{d}\tau + \int\hat{\sigma}_V(T-\tau)\,\mathrm{d}W_V^{Q_2}}. \tag{A17}$$

Substituting equations (A16) and (A17) into (A14), we obtain

$$A_4 = \frac{K'(1-\alpha)V}{P(t,T)D} \times P_{Q_2}\left(\frac{-\int\sum X_i\hat{\sigma}_{s_i} dW_{S_i}^{Q_i}}{S_S} \leq \frac{\ln\frac{m_t^s}{K} + S_SV}{S_S}, \frac{\int\hat{\sigma}_V dW_V^{Q^2}}{S_V}\ \frac{\hat{\sigma}_V \mathrm{d}W_V^{Q_2}}{S_V} \leq \frac{\ln(Pd/V)-\frac{1}{2}S_V^2}{S_V}\right)$$

$$= \frac{K'(1-\alpha)V}{P(t,T)D} N_2(d_1,d_2,-\rho),$$

where

$$d_1 = \frac{\ln(m_t^S/K')+S_{SV}}{S_S}, \quad d_2 = -\frac{\ln(V/PD)+\frac{1}{2}S_V^2}{S_V}.$$

Evaluation of the term K' in equation (9).

$$E_{Q^T}\left[\sum_{i=1}^n X_i\hat{S}_{iT}\middle| F_t\right] = \sum_{i=1}^n X_i\frac{P(t,T)}{S_{it}}E_{Q^T}[S_{iT}|F_t] = \sum_{i=1}^n X_i\frac{P(t,T)}{S_{it}}\frac{S_{it}}{P(t,T)} = \sum_{i=1}^n X_i = 1.$$

And

$$E_{Q^T}\left[\prod_{i=1}^n \hat{S}_{iT}^{X_i}\middle| F_t\right] = E_{Q^T}\left[\mathrm{e}^{-(1/2)\int\sum X_i\hat{\sigma}_{S_i}^2(T-\tau)\,\mathrm{d}\tau} + \int\sum X_i\hat{\sigma}_{S_i}(T-\tau)\,\mathrm{d}W_{S_i}^{Q^T}\middle| F_t\right]$$
$$= m_t^S\mathrm{e}^{(1/2)S_S^2}E_{Q^T} \times \left[\mathrm{e}^{-(1/2)}\int\left(\sum X_i\hat{\sigma}_{S_i}(T-\tau)\right)^2\mathrm{d}\tau\,\mathrm{e}^{+\int\sum X_i\hat{\sigma}_{S_i}(T-\tau)\mathrm{d}W_{S_i}^{Q^T}}\middle| F_t\right]$$
$$= m_t^S\mathrm{e}^{(1/2)S_S^2}E_{Q^T}\left[\frac{\mathrm{d}\tilde{Q}}{\mathrm{d}Q^T}\middle| F_t\right] = m_t^S\mathrm{e}^{(1/2)S_S^2}.$$

Therefore,

$$K' = \frac{KP(t,T)}{BK_t} + m_t^S\,\mathrm{e}^{(1/2)S_S^2} - 1.$$

This completes the derivation of equation (10).