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Dimer-monomer model on the Sierpinski gasket

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Abstract

We present the numbers of dimer–monomers $M_d(n)$ on the Sierpinski gasket $SG_d(n)$ at stage n with dimension d equal to two, three and four. The upper and lower bounds for the asymptotic growth constant, defined as $z_{SG_d} = \lim_{v\to\infty} \ln M_d(n)/v$ where v is the number of vertices on $SG_d(n)$, are derived in terms of the results at a certain stage. As the difference between these bounds converges quickly to zero as the calculated stage increases, the numerical value of z_{SG_d} can be evaluated with more than a hundred significant figures accurate. From the results for $d = 2, 3, 4$, we conjecture the upper and lower bounds of z_{SG_d} for general dimension. The corresponding results on the generalized Sierpinski gasket $SG_{d,b}(n)$ with $d = 2$ and $b = 3, 4$ are also obtained.

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Keywords: Dimer–monomer model; Sierpinski gasket; Recursion relations; Asymptotic growth constant

1. Introduction

The enumeration of the number of dimer–monomers $N_{DM}(G)$ on a graph G is a classical model [1–3]. In the model, each diatomic molecule is regarded as a dimer which occupies two adjacent sites of the graph. The sites that are not covered by any dimers are considered as occupied by monomers. Although the close-packed dimer problem on planar lattices has been expressed in closed form almost half a century ago [4–6], the general dimer–monomer problem was shown to be computationally intractable [7]. Some recent studies on the enumeration of close-packed dimer, single-monomer and general dimer–monomer problems on regular lattices were carried out in Refs. [8–18]. It is of interest to consider dimer–monomers on self-similar fractal lattices which have scaling invariance rather than translational invariance. Fractals are geometrical structures of non-integer Hausdorff dimension realized by repeated construction of an elementary shape on progressively smaller length scales [19,20]. A well-known example of fractal is the Sierpinski gasket which has been extensively studied in several contexts [21–31]. We shall derive the recursion relations for the numbers of dimer–monomers on the Sierpinski gasket with dimension equal to two, three and four, and determine the asymptotic growth constants. We shall also consider the number of dimer–monomers on the generalized Sierpinski gasket with dimension equal to two.

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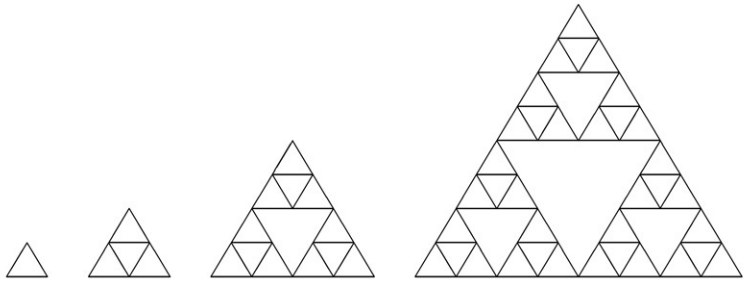

Fig. 1. The first four stages $n = 0, 1, 2, 3$ of the two-dimensional Sierpinski gasket $SG_2(n)$.

2. Preliminaries

We first recall some relevant definitions in this section. A connected graph (without loops) $G = (V, E)$ is defined by its vertex (site) and edge (bond) sets V and E [32,33]. Let $v(G) = |V|$ be the number of vertices and $e(G) = |E|$ the number of edges in G. The degree or coordination number k_i of a vertex $v_i \in V$ is the number of edges attached to it. A k-regular graph is a graph with the property that each of its vertices has the same degree k. In general, one can associate monomer and dimer weights to each monomer and dimer connecting adjacent vertices (see, for example [16]). For simplicity, all such weights are set to one throughout this paper.

When the number of dimer–monomers $N_{DM}(G)$ grows exponentially with $v(G)$ as $v(G) \to \infty$, there exists a constant z_G describing this exponential growth

$$z_G = \lim_{v(G)\to\infty} \frac{\ln N_{DM}(G)}{v(G)}, \tag{2.1}$$

where G, when used as a subscript in this manner, implicitly refers to the thermodynamic limit.

The construction of the two-dimensional Sierpinski gasket $SG_2(n)$ at stage n is shown in Fig. 1. At stage $n = 0$, it is an equilateral triangle; while stage $n + 1$ is obtained by the juxtaposition of three n-stage structures. In general, the Sierpinski gaskets SG_d can be built in any Euclidean dimension d with fractal dimensionality $D = \ln(d+1)/\ln 2$ [22]. For the Sierpinski gasket $SG_d(n)$, the numbers of edges and vertices are given by

$$e(SG_d(n)) = \binom{d+1}{2}(d+1)^n = \frac{d}{2}(d+1)^{n+1}, \tag{2.2}$$

$$v(SG_d(n)) = \frac{d+1}{2}[(d+1)^n + 1]. \tag{2.3}$$

Except the $(d + 1)$ outmost vertices which have degree d, all other vertices of $SG_d(n)$ have degree $2d$. In the large n limit, SG_d is $2d$-regular.

The Sierpinski gasket can be generalized, denoted as $SG_{d,b}(n)$, by introducing the side length b which is an integer larger or equal to two [34]. The generalized Sierpinski gasket at stage $n + 1$ is constructed with b layers of stage n hypertetrahedrons. The two-dimensional $SG_{2,b}(n)$ with $b = 3$ at stage $n = 1, 2$ and $b = 4$ at stage $n = 1$ are illustrated in Fig. 2. The ordinary Sierpinski gasket $SG_d(n)$ corresponds to the $b = 2$ case, where the index b is neglected for simplicity. The Hausdorff dimension for $SG_{d,b}$ is given by $D = \ln\binom{b+d-1}{d}/\ln b$ [34]. Notice that $SG_{d,b}$ is not k-regular even in the thermodynamic limit.

3. The number of dimer–monomers on $SG_2(n)$

In this section we derive the asymptotic growth constant for the number of dimer–monomers on the two-dimensional Sierpinski gasket $SG_2(n)$ in detail. Let us start with the definitions of the quantities to be used.

Definition 3.1. Consider the generalized two-dimensional Sierpinski gasket $SG_{2,b}(n)$ at stage n. (i) Define $M_{2,b}(n) \equiv N_{DM}(SG_{2,b}(n))$ as the number of dimer–monomers. (ii) Define $f_{2,b}(n)$ as the number of dimer–monomers such

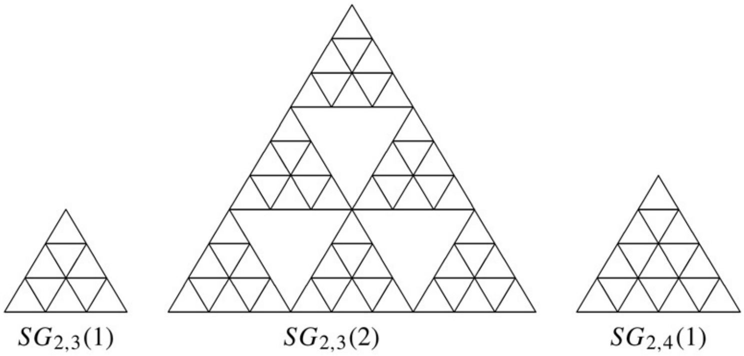

Fig. 2. The generalized two-dimensional Sierpinski gasket $SG_{2,b}(n)$ with $b = 3$ at stage $n = 1, 2$ and $b = 4$ at stage $n = 1$.

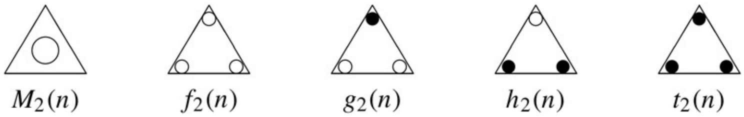

Fig. 3. Illustration for the configurations $M_2(n)$, $f_2(n)$, $g_2(n)$, $h_2(n)$, and $t_2(n)$. Only the three outmost vertices are shown explicitly for $f_2(n)$, $g_2(n)$, $h_2(n)$ and $t_2(n)$, where each open circle is occupied by a monomer and each solid circle is occupied by a dimer.

that the three outmost vertices are occupied by monomers. (iii) Define $g_{2,b}(n)$ as the numbers of dimer–monomers such that one certain outmost vertex, say the topmost vertex as illustrated in Fig. 3 for ordinary Sierpinski gasket, is occupied by a dimer while the other two outmost vertices are occupied by monomers. (iv) Define $h_{2,b}(n)$ as the numbers of dimer–monomers such that one certain outmost vertex, say the topmost vertex as illustrated in Fig. 3 for ordinary Sierpinski gasket, is occupied by a monomer while the other two outmost vertices are occupied by dimers. (v) Define $t_{2,b}(n)$ as the number of dimer–monomers such that all three outmost vertices are occupied by dimers.

Since we only consider ordinary Sierpinski gasket in this section, we use the notations $M_2(n)$, $f_2(n)$, $g_2(n)$, $h_2(n)$, and $t_2(n)$ for simplicity. They are illustrated in Fig. 3, where only the outmost vertices are shown. Because of rotational symmetry, there are three possible $g_2(n)$ and three possible $h_2(n)$ such that

$$M_2(n) = f_2(n) + 3g_2(n) + 3h_2(n) + t_2(n) \tag{3.1}$$

for nonnegative integer n. The initial values at stage zero are $f_2(0) = 1$, $g_2(0) = 0$, $h_2(0) = 1$, $t_2(0) = 0$ and $M_2(0) = 4$. The values at stage one are $f_2(1) = 4$, $g_2(1) = 4$, $h_2(1) = 3$, $t_2(1) = 2$ and $M_2(1) = 27$. The purpose of this section is to obtain the asymptotic behaviour of $M_2(n)$ as follows. The five quantities $M_2(n)$, $f_2(n)$, $g_2(n)$, $h_2(n)$ and $t_2(n)$ satisfy recursion relations.

Lemma 3.1. *For any nonnegative integer n,*

$$\begin{aligned} M_2(n+1) = {} & M_2^3(n) - 3M_2(n)[g_2(n) + 2h_2(n) + t_2(n)]^2 \\ & + 3[h_2(n) + t_2(n)][g_2(n) + 2h_2(n) + t_2(n)]^2 - [h_2(n) + t_2(n)]^3, \end{aligned} \tag{3.2}$$

$$f_2(n+1) = f_2^3(n) + 6f_2^2(n)g_2(n) + 3f_2^2(n)h_2(n) + 9f_2(n)g_2^2(n) + 2g_2^3(n) + 6f_2(n)g_2(n)h_2(n), \tag{3.3}$$

$$\begin{aligned} g_2(n+1) = {} & f_2^2(n)g_2(n) + 2f_2^2(n)h_2(n) + 4f_2(n)g_2^2(n) + f_2^2(n)t_2(n) + 8f_2(n)g_2(n)h_2(n) \\ & + 3g_2^3(n) + 2f_2(n)g_2(n)t_2(n) + 2f_2(n)h_2^2(n) + 4g_2^2(n)h_2(n), \end{aligned} \tag{3.4}$$

$$\begin{aligned} h_2(n+1) = {} & f_2(n)g_2^2(n) + 4f_2(n)g_2(n)h_2(n) + 2g_2^3(n) + 2f_2(n)g_2(n)t_2(n) + 7g_2^2(n)h_2(n) \\ & + 3f_2(n)h_2^2(n) + 2f_2(n)h_2(n)t_2(n) + 2g_2^2(n)t_2(n) + 4g_2(n)h_2^2(n), \end{aligned} \tag{3.5}$$

$$t_2(n+1) = g_2^3(n) + 6g_2^2(n)h_2(n) + 3g_2^2(n)t_2(n) + 9g_2(n)h_2^2(n) + 2h_2^3(n) + 6g_2(n)h_2(n)t_2(n). \tag{3.6}$$

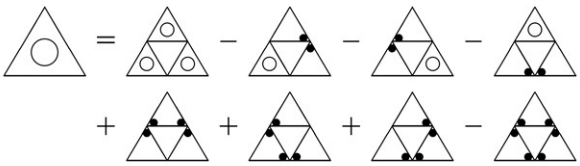

Fig. 4. Illustration for the expression of $M_2(n+1)$. Certain outmost vertices of $SG_2(n)$ are not shown means they can be occupied by either dimers or monomers.

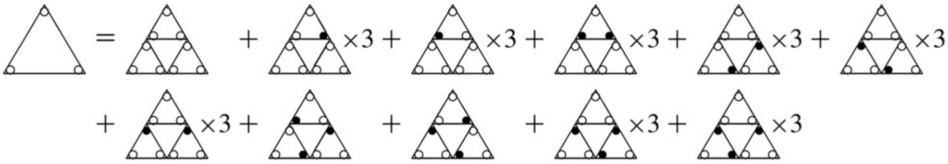

Fig. 5. Illustration for the expression of $f_2(n+1)$. The multiplication of three on the right-hand side corresponds to the three possible orientations of $SG_2(n+1)$.

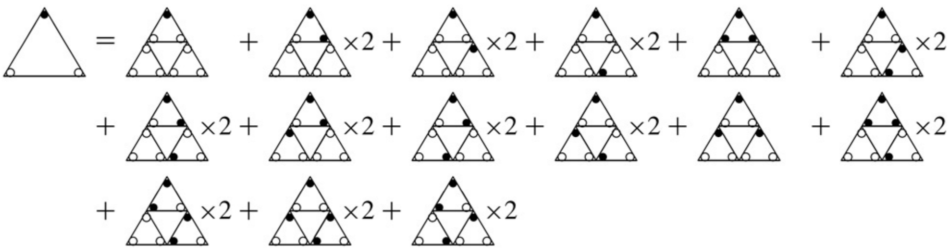

Fig. 6. Illustration for the expression of $g_2(n+1)$. The multiplication of two on the right-hand side corresponds to the reflection symmetry with respect to the central vertical axis.

Proof. The Sierpinski gaskets $SG_2(n+1)$ is composed of three $SG_2(n)$ with three pairs of vertices identified. For the number $M_2(n+1)$, the unallowable configurations are those with at least a pair of identified vertices occupied by dimers in both the original $SG_2(n)$ configurations. Therefore, the three types of configuration with a pair of identified vertices occupied by dimers should be subtracted from all possible configurations $M_2^3(n)$. The configurations with two pairs of identified vertices occupied by dimers are subtracted out twice, so that should be added back as illustrated in Fig. 4. Finally, the configuration with three pairs of identified vertices occupied by dimers should be subtracted, and Eq. (3.2) is verified.

As illustrated in Fig. 5, the number $f_2(n+1)$ consists of (i) one configuration where all three of the $SG_2(n)$ are in the $f_2(n)$ status, (ii) six configurations where two of the $SG_2(n)$ are in the $f_2(n)$ status and the other one is in the $g_2(n)$ status, (iii) three configurations where two of the $SG_2(n)$ are in the $f_2(n)$ status and the other one is in the $h_2(n)$ status, (iv) nine configurations where one of the $SG_2(n)$ is in the $f_2(n)$ status and the other two are in the $g_2(n)$ status, (v) two configuration where all three of the $SG_2(n)$ are in the $g_2(n)$ status, (vi) six configurations where one of the $SG_2(n)$ is in the $f_2(n)$ status, another one is in the $g_2(n)$ status and the other one is in the $h_2(n)$ status. Eq. (3.3) is verified by adding all possible configurations.

Similarly, $g_2(n+1)$, $h_2(n+1)$ and $t_2(n+1)$ for $SG_2(n+1)$ can be obtained with appropriate configurations of its three constituting $SG_2(n)$ as illustrated in Figs. 6–8 to verify Eqs. (3.4)–(3.6), respectively.

Eq. (3.2) can also be obtained by substituting Eqs. (3.3)–(3.6) into Eq. (3.1). □

There are always $27 = 3^3$ terms in Eqs. (3.3)–(3.6) because there are three possible choices for each of the three pairs of identified vertices: both of them are originally occupied by monomers, or either one of them is originally occupied by a monomer while the other one by a dimer. The values of $M_2(n)$, $f_2(n)$, $g_2(n)$, $h_2(n)$, $t_2(n)$ for small n can be evaluated recursively by Eqs. (3.2)–(3.6) as listed in Table 1. These numbers grow exponentially, and do not have simple integer factorizations. To estimate the value of the asymptotic growth constant defined in Eq. (2.1), we

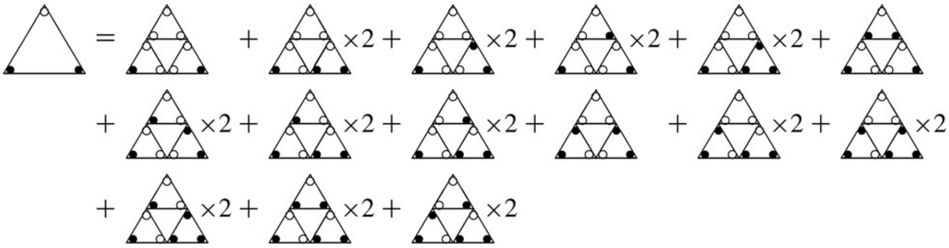

Fig. 7. Illustration for the expression of $h_2(n+1)$. The multiplication of two on the right-hand side corresponds to the reflection symmetry with respect to the central vertical axis.

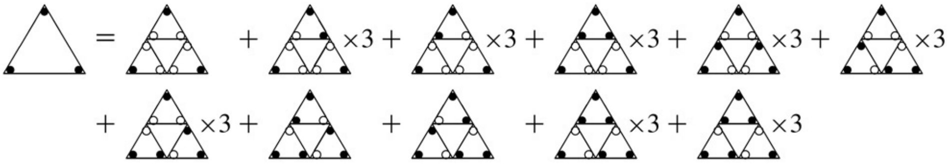

Fig. 8. Illustration for the expression of $t_2(n+1)$. The multiplication of three on the right-hand side corresponds to the three possible orientations of $SG_2(n+1)$.

Table 1
The first few values of $M_2(n)$, $f_2(n)$, $g_2(n)$, $h_2(n)$, $t_2(n)$

n	0	1	2	3	4
$M_2(n)$	4	27	10,054	499,058,851,840	60,978,122,299,433,248,924,629,725,740,007,424
$f_2(n)$	1	4	1,584	78,721,368,064	9,618,673,427,679,675,357,952,788,786,053,120
$g_2(n)$	0	4	1,352	66,974,056,448	8,183,299,472,241,085,511,976,093,040,508,928
$h_2(n)$	1	3	1,148	56,979,607,552	6,962,123,286,110,084,944,276,569,997,705,216
$t_2(n)$	0	2	970	48,476,491,776	5,923,180,596,700,062,197,918,947,839,311,872

Table 2
The first few values of $\alpha_2(n)$, $\beta_2(n)$, $\gamma_2(n)$

n	1	2	3	4
$\alpha_2(n)$	1	0.853535353535354	0.850773533223540	0.850772150002722
$\beta_2(n)$	0.75	0.849112426035503	0.850771337051088	0.850772150002159
$\gamma_2(n)$	0.666666666666667	0.844947735191638	0.850769141078411	0.850772150001597

The last digits given are rounded off.

need the following lemmas. For the generalized two-dimensional Sierpinski gasket $SG_{2,b}(n)$, define the ratios

$$\alpha_{2,b}(n) = \frac{g_{2,b}(n)}{f_{2,b}(n)}, \qquad \beta_{2,b}(n) = \frac{h_{2,b}(n)}{g_{2,b}(n)}, \qquad \gamma_{2,b}(n) = \frac{t_{2,b}(n)}{h_{2,b}(n)}. \tag{3.7}$$

For the ordinary Sierpinski gasket in this section,they are simplified to be $\alpha_2(n)$, $\beta_2(n)$, $\gamma_2(n)$ and their values for small n are listed in Table 2.

Lemma 3.2. *For any positive integer n, the magnitudes of $\alpha_2(n)$, $\beta_2(n)$, $\gamma_2(n)$ are ordered as*

$$0 \le \gamma_2(n) \le \beta_2(n) \le \alpha_2(n) \le 1\ , \tag{3.8}$$

and the magnitudes of $f_2(n)$, $g_2(n)$, $h_2(n)$, $t_2(n)$ are ordered as

$$t_2(n) \le h_2(n) \le g_2(n) \le f_2(n)\ . \tag{3.9}$$

Proof. It is clear that $0 \le \gamma_2(n)$ since all the quantities $f_2(n), g_2(n), h_2(n), t_2(n)$ are positive. $\gamma_2(n) \le \beta_2(n) \le \alpha_2(n)$ is valid for the first few positive integer n by the values given in Table 2. By Eqs. (3.3)–(3.6), we have

$$\begin{aligned}\frac{f_2(n+1)g_2(n+1)}{f_2^4(n)g_2^2(n)}[\alpha_2(n+1)-\beta_2(n+1)] &= \left[\frac{g_2(n+1)}{f_2^2(n)g_2(n)}\right]^2 - \frac{f_2(n+1)}{f_2^3(n)}\frac{h_2(n+1)}{f_2(n)g_2^2(n)}\\
&= [\alpha_2(n)-\beta_2(n)]^2[1+3\alpha_2(n)+\gamma_2(n)+4\alpha_2^2(n)[\beta_2(n)+\gamma_2(n)]+4\alpha_2^2(n)\beta_2^2(n)]\\
&\quad +[\alpha_2^2(n)-\beta_2^2(n)][[\alpha_2(n)-\gamma_2(n)][1+2\alpha_2(n)\beta_2(n)]+2\alpha_2^2(n)[\beta_2^2(n)-\gamma_2^2(n)]]\\
&\quad +[\alpha_2(n)-\beta_2(n)][4\alpha_2(n)[\alpha_2^2(n)-\beta_2(n)\gamma_2(n)]+\alpha_2(n)\beta_2(n)[\alpha_2(n)-\gamma_2(n)]\\
&\quad +\alpha_2(n)\beta_2(n)[\beta_2(n)-\gamma_2(n)]]+4\alpha_2(n)[\alpha_2^3(n)-\beta_2^3(n)][\beta_2(n)-\gamma_2(n)]\\
&\quad +[\alpha_2^2(n)-\beta_2(n)\gamma_2(n)]^2+2\alpha_2(n)[\beta_2(n)-\gamma_2(n)]^2[2\beta_2^2(n)+\alpha_2^3(n)+\alpha_2(n)\beta_2^2(n)],\end{aligned} \tag{3.10}$$

$$\begin{aligned}\frac{g_2(n+1)h_2(n+1)}{f_2^2(n)g_2^4(n)}[\beta_2(n+1)-\gamma_2(n+1)] &= \left[\frac{h_2(n+1)}{f_2(n)g_2^2(n)}\right]^2 - \frac{g_2(n+1)}{f_2^2(n)g_2(n)}\frac{t_2(n+1)}{g_2^3(n)}\\
&= [\alpha_2(n)-\beta_2(n)]^2[1+4\beta_2(n)+2\gamma_2(n)+9\beta_2^2(n)+6\beta_2^3(n)+6\beta_2^2(n)\gamma_2(n)+2\beta_2^2(n)\gamma_2^2(n)]\\
&\quad +[\alpha_2(n)-\beta_2(n)][2\alpha_2(n)[\beta_2(n)-\gamma_2(n)]+4\beta_2^2(n)[\alpha_2(n)+\beta_2(n)-2\gamma_2(n)]\\
&\quad +2\alpha_2(n)\beta_2^2(n)[\beta_2^2(n)-\gamma_2^2(n)]+2\beta_2^3(n)[\alpha_2(n)\beta_2(n)-\gamma_2^2(n)]]\\
&\quad +\beta_2(n)[\alpha_2^2(n)-\beta_2^2(n)][\beta_2(n)-\gamma_2(n)][1+8\beta_2(n)]\\
&\quad +\beta_2^2(n)[\beta_2(n)-\gamma_2(n)]^2[1+2\alpha_2(n)+2\beta_2(n)+4\alpha_2^2(n)]\end{aligned} \tag{3.11}$$

such that $\gamma_2(n) \le \beta_2(n) \le \alpha_2(n)$ is proved by mathematical induction. Next, Eq. (3.9) is valid for the first few positive integer n by the numbers given in Table 1. Furthermore, the following expression is larger or equal to zero for any positive integer n:

$$\begin{aligned}\frac{f_2(n+1)}{f_2^3(n)} - \frac{g_2(n+1)}{f_2^2(n)g_2(n)} &= [\alpha_2(n)-\beta_2(n)][2+5\alpha_2(n)+2\alpha_2(n)\beta_2(n)]\\
&\quad +[1+2\alpha_2(n)][\alpha_2^2(n)-\beta_2(n)\gamma_2(n)],\end{aligned} \tag{3.12}$$

such that $g_2(n) \le f_2(n)$ by induction, or equivalently $\alpha_2(n) \le 1$. Finally, we have $t_2(n) \le h_2(n) \le g_2(n)$ once Eq. (3.9) is established. □

Lemma 3.3. *Sequence* $\{\alpha_2(n)\}_{n=1}^\infty$ *decreases monotonically, while sequences* $\{\gamma_2(n)\}_{n=1}^\infty$ *and* $\{t_2(n)/f_2(n)\}_{n=1}^\infty$ *increase monotonically. The limits* $\alpha_2 \equiv \lim_{n\to\infty}\alpha_2(n)$, $\beta_2 \equiv \lim_{n\to\infty}\beta_2(n)$, $\gamma_2 \equiv \lim_{n\to\infty}\gamma_2(n)$ *exist.*

Proof. Eq. (3.12) implies $f_2(n+1)g_2(n) - g_2(n+1)f_2(n) \ge 0$. By induction, $\alpha_2(n)$ decreases as positive n increases using the results of Lemma 3.2. By Eqs. (3.5) and (3.6), we have

$$\begin{aligned}t_2(n+1)h_2(n) - h_2(n+1)t_2(n) &= f_2(n)h_2(n)[g_2^2(n)+4g_2(n)h_2(n)+3h_2^2(n)][\alpha_2(n)-\gamma_2(n)]\\
&\quad +2g_2^2(n)h_2(n)[g_2(n)+2h_2(n)+t_2(n)][\beta_2(n)-\gamma_2(n)]\\
&\quad +2f_2(n)h_2^2(n)[g_2(n)+h_2(n)]\left[\frac{h_2(n)}{f_2(n)} - \frac{t_2^2(n)}{h_2^2(n)}\right] \ge 0,\end{aligned} \tag{3.13}$$

such that $\gamma_2(n)$ increases as positive n increases. Finally, we have

$$\begin{aligned}t_2(n+1)f_2(n) - f_2(n+1)t_2(n) &= f_2^4(n)\left[\frac{g_2^3(n)}{f_2^3(n)} - \frac{t_2(n)}{f_2(n)}\right] + 6f_2^2(n)g_2(n)h_2(n)[\alpha_2(n)-\gamma_2(n)]\\
&\quad +3f_2^2(n)g_2(n)t_2(n)[\alpha_2(n)-\beta_2(n)]+9f_2(n)g_2^2(n)h_2(n)[\beta_2(n)-\gamma_2(n)]\\
&\quad -2f_2(n)g_2^2(n)t_2(n)\left[\frac{g_2(n)}{f_2(n)} - \frac{h_2^3(n)}{g_2^2(n)t_2(n)}\right] \ge 0,\end{aligned} \tag{3.14}$$

where the last inequality holds because of the combination of the third and the last terms:

$$2f_2(n)g_2(n)t_2(n)\left[f_2(n)\left[\frac{g_2(n)}{f_2(n)}-\frac{h_2(n)}{g_2(n)}\right]-g_2(n)\left[\frac{g_2(n)}{f_2(n)}-\frac{h_2^3(n)}{g_2^2(n)t_2(n)}\right]\right]\geq 0\,. \tag{3.15}$$

Because the sequence $\alpha_2(n)$ decreases monotonically and is bounded below, the limit α_2 exists. Similarly, sequence $\gamma_2(n)$ and $t_2(n)/f_2(n)$ increase monotonically and are bounded above so that the limits γ_2 and $\lim_{n\to\infty} t_2(n)/f_2(n)$ exist. It follows that the limit $\lim_{n\to\infty} h_2(n)/f_2(n)$ exists since

$$\lim_{n\to\infty}\frac{t_2(n)}{f_2(n)}=\lim_{n\to\infty}\frac{h_2(n)}{f_2(n)}\lim_{n\to\infty}\frac{t_2(n)}{h_2(n)}, \tag{3.16}$$

such that $\beta = \lim_{n\to\infty} h_2(n)/g_2(n)$ exists because

$$\lim_{n\to\infty}\frac{h_2(n)}{f_2(n)}=\lim_{n\to\infty}\frac{g_2(n)}{f_2(n)}\lim_{n\to\infty}\frac{h_2(n)}{g_2(n)}.\quad\square \tag{3.17}$$

With the existence of the limits α_2, β_2, γ_2, and $\gamma_2 \leq \beta_2 \leq \alpha_2$, we have

$$\begin{aligned}1 &= \lim_{n\to\infty}\frac{f_2(n+1)}{f_2(n)}\frac{g_2(n)}{g_2(n+1)}\\ &= \frac{(1+3\alpha_2)^2+2\alpha_2^3+3\alpha_2\beta_2(1+2\alpha_2)}{1+2\beta_2+4\alpha_2+\beta_2\gamma_2+8\alpha_2\beta_2+3\alpha_2^2+2\alpha_2\beta_2\gamma_2+2\alpha_2\beta_2^2+4\alpha_2^2\beta_2}\end{aligned} \tag{3.18}$$

by Eqs. (3.3) and (3.4), which leads to the following result:

Corollary 3.1. *The three limits α_2, β_2 and γ_2 are equal to one another.*

In other words, the limits α_2, β_2 and γ_2 are fix points of the recursion relations, Eqs. (3.3)–(3.6), as illustrated by Eq. (3.18). The valid solution is $\alpha_2 = \beta_2 = \gamma_2$, but the actual value cannot be obtained by solving the recursion relations. By Eq. (3.7), the numerical results give

$$\alpha_2 = \beta_2 = \gamma_2 = 0.850772150002\ldots, \tag{3.19}$$

where more than a hundred significant figures can be evaluated when stage n in Eq. (3.7) is equal to seven.

Lemma 3.4. *The asymptotic growth constant for the number of dimer–monomers on $SG_2(n)$ is bounded*

$$\frac{2}{3^{m+1}}\ln f_2(m)+\frac{\ln[1+2\gamma_2(m)]}{3^m}\leq z_{SG_2}\leq\frac{2}{3^{m+1}}\ln f_2(m)+\frac{\ln[1+2\alpha_2(m)]}{3^m}\,, \tag{3.20}$$

where m is a positive integer.

Proof. Let us define $\lambda_2(n) = f_2(n+1)/f_2^3(n)$. By Eq. (3.3), we have

$$\lambda_2(n) = [1+3\alpha_2(n)]^2+2\alpha_2^3(n)+3\alpha_2(n)\beta_2(n)[1+2\alpha_2(n)]. \tag{3.21}$$

It is clear that $1 \leq \lambda_n \leq 27$, and

$$[1+2\gamma_2(m)]^3\leq[1+2\gamma_2(n)]^3\leq[1+2\beta_2(n)]^3\leq\lambda_2(n)\leq[1+2\alpha_2(n)]^3\leq[1+2\alpha_2(m)]^3 \tag{3.22}$$

for $n \geq m$. By Eqs. (2.3) and (3.1), we have

$$\frac{\ln M_2(n)}{v(SG_2(n))}=\frac{2\ln[1+3\alpha_2(n)+3\alpha_2(n)\beta_2(n)+\alpha_2(n)\beta_2(n)\gamma_2(n)]}{3(3^n+1)}+\frac{2\ln f_2(n)}{3(3^n+1)}, \tag{3.23}$$

where

$$\begin{aligned}\ln f_2(n) &= \ln\lambda_2(n-1)+3\ln f_2(n-1)\\ &= \ln\lambda_2(n-1)+3\ln\lambda_2(n-2)+3^2\ln f_2(n-2)\end{aligned}$$

$$= \cdots$$
$$= \sum_{j=m}^{n-1} 3^{n-1-j} \ln \lambda_2(j) + 3^{n-m} \ln f_2(m) \tag{3.24}$$

for any $m < n$. By the definition of the asymptotic growth constant in Eq. (2.1)

$$\begin{aligned} z_{SG_2} &= \lim_{n\to\infty} \frac{\ln M_2(n)}{v(SG_2(n))} \\ &= \lim_{n\to\infty} \frac{2\ln[1+3\alpha_2(n)+3\alpha_2(n)\beta_2(n)+\alpha_2(n)\beta_2(n)\gamma_2(n)]}{3(3^n+1)} \\ &\quad + \lim_{n\to\infty} \frac{2\sum_{j=m}^{n-1} 3^{n-1-j}\ln\lambda_2(j) + 2[3^{n-m}\ln f_2(m)]}{3(3^n+1)} \\ &= \frac{2}{3^2}\sum_{j=m}^{\infty}\frac{\ln\lambda_2(j)}{3^j} + \frac{2}{3^{m+1}}\ln f_2(m). \end{aligned} \tag{3.25}$$

The proof is completed using the inequality (3.22). □

The difference between the upper and lower bounds for z_{SG_2} quickly converges to zero as m increases, and we have the following proposition:

Proposition 3.1. *The asymptotic growth constant for the number of dimer–monomers on the two-dimensional Sierpinski gasket $SG_2(n)$ in the large n limit is $z_{SG_2} = 0.656294236916\ldots$.*

The numerical value of z_{SG_2} can be calculated with more than a hundred significant figures accurate when m in Eq. (3.20) is equal to seven. It is too lengthy to be included here and is available from the authors on request. In passing, we notice that it is possible to tighten the lower bound in Eq. (3.20) if $\gamma_2(m)$ is replaced by $\beta_2(m)$. Numerically results show that the difference between the upper bound and such new lower bound is about half of the difference between the bounds given in Eq. (3.20). However, such improvement is insignificant compared with the convergence of the bound-difference as m increases. As it is considerably more difficult to prove that sequence $\{\beta_2(n)\}_{n=1}^{\infty}$ increases monotonically, we are satisfied by the lower bound reported here.

Define the asymptotic growth constants for the functions $f_2(n)$, $g_2(n)$, $h_2(n)$, $t_2(n)$ as in Eq. (2.1) for $M_2(n)$. Because the magnitudes of $f_2(n)$, $g_2(n)$, $h_2(n)$, $t_2(n)$ are in the same order in the large n limit by Eqs. (3.7) and (3.19), we have the following result:

Corollary 3.2. *The asymptotic growth constants for $f_2(n)$, $g_2(n)$, $h_2(n)$, $t_2(n)$ are all equal to z_{SG_2}.*

This can also be seen by Eq. (3.20) that the asymptotic growth constant for $f_2(n)$ is the same as z_{SG_2}.

4. The number of dimer–monomers on $SG_{2,b}(n)$ with $b = 3, 4$

The method given in the previous section can be applied to the number of dimer–monomers on $SG_{d,b}(n)$ with larger values of d and b. The number of configurations to be considered increases as d and b increase, and the recursion relations must be derived individually for each d and b. In this section, we consider the generalized two-dimensional Sierpinski gasket $SG_{2,b}(n)$ with the number of layers b equal to three and four. For $SG_{2,3}(n)$, the numbers of edges and vertices are given by

$$e(SG_{2,3}(n)) = 3 \times 6^n, \tag{4.1}$$
$$v(SG_{2,3}(n)) = \frac{7 \times 6^n + 8}{5}, \tag{4.2}$$

where the three outmost vertices have degree two. There are $(6^n - 1)/5$ vertices of $SG_{2,3}(n)$ with degree six and $6(6^n - 1)/5$ vertices with degree four. By Definition 3.1, the number of dimer–monomers is

Table 3
The first few values of $M_{2,3}(n)$, $f_{2,3}(n)$, $g_{2,3}(n)$, $h_{2,3}(n)$, $t_{2,3}(n)$

n	0	1	2
$M_{2,3}(n)$	4	425	755,290,432,490,932
$f_{2,3}(n)$	1	66	116,464,644,336,176
$g_{2,3}(n)$	0	56	100,722,462,529,064
$h_{2,3}(n)$	1	49	87,108,127,443,640
$t_{2,3}(n)$	0	44	75,334,018,236,644

Table 4
The first few values of $\alpha_{2,3}(n)$, $\beta_{2,3}(n)$, $\gamma_{2,3}(n)$

n	1	2	3
$\alpha_{2,3}(n)$	0.848484848484848	0.864832955126948	0.864833096846111
$\beta_{2,3}(n)$	0.875	0.864833178780796	0.864833096846111
$\gamma_{2,3}(n)$	0.897959183673469	0.864833402432925	0.864833096846111

The last digits given are rounded off.

$M_{2,3}(n) = f_{2,3}(n)+3g_{2,3}(n)+3h_{2,3}(n)+t_{2,3}(n)$. The initial values are the same as for SG_2: $f_{2,3}(0) = 1$, $g_{2,3}(0) = 0$, $h_{2,3}(0) = 1$ and $t_{2,3}(0) = 0$.

The recursion relations are lengthy and given in the Appendix. Some values of $M_{2,3}(n)$, $f_{2,3}(n)$, $g_{2,3}(n)$, $h_{2,3}(n)$, $t_{2,3}(n)$ are listed in Table 3. These numbers grow exponentially, and do not have simple integer factorizations.

The sequence of the ratio defined in Eq. (3.7) $\{\alpha_{2,3}(n)\}_{n=1}^{\infty}$ increases monotonically and $\{\gamma_{2,3}(n)\}_{n=1}^{\infty}$ decreases monotonically with $0 \le \alpha_{2,3}(n) \le \gamma_{2,3}(n) \le 1$, in contrast to the results for $SG_2(n)$. The values of $\alpha_{2,3}(n)$, $\beta_{2,3}(n)$, $\gamma_{2,3}(n)$ for small n are listed in Table 4. Define their limits as in Lemma 3.3, the numerical results give

$$\alpha_{2,3} = \beta_{2,3} = \gamma_{2,3} = 0.864833096846\ldots, \tag{4.3}$$

where more than a hundred significant figures can be evaluated when stage n in Eq. (3.7) is equal to five.

By a similar argument as Lemma 3.4, the asymptotic growth constant for the number of dimer–monomers on $SG_{2,3}(n)$ is bounded

$$\frac{5\ln f_{2,3}(m) + 6\ln[1+2\alpha_{2,3}(m)] + \ln[1+3\alpha_{2,3}(m)]}{7\times 6^m} \le z_{SG_{2,3}} \le \frac{5\ln f_{2,3}(m) + 6\ln[1+2\gamma_{2,3}(m)] + \ln[1+3\gamma_{2,3}(m)]}{7\times 6^m}, \tag{4.4}$$

with m a positive integer. We have the following proposition:

Proposition 4.1. *The asymptotic growth constant for the number of dimer–monomers on the generalized two-dimensional Sierpinski gasket $SG_{2,3}(n)$ in the large n limit is $z_{SG_{2,3}} = 0.671617161058\ldots$.*

The convergence of the upper and lower bounds remains quick. More than a hundred significant figures for $z_{SG_{2,3}}$ can be obtained when m in Eq. (4.4) is equal to five.

For $SG_{2,4}(n)$, the numbers of edges and vertices are given by

$$e(SG_{2,4}(n)) = 3\times 10^n, \tag{4.5}$$

$$v(SG_{2,4}(n)) = \frac{4\times 10^n + 5}{3}, \tag{4.6}$$

where again the three outmost vertices have degree two. There are $(10^n - 1)/3$ vertices of $SG_{2,4}(n)$ with degree six, and $(10^n - 1)$ vertices with degree four. By Definition 3.1, the number of dimer–monomers is $M_{2,4}(n) = f_{2,4}(n) + 3g_{2,4}(n) + 3h_{2,4}(n) + t_{2,4}(n)$. The initial values are the same as for SG_2: $f_{2,4}(0) = 1$, $g_{2,4}(0) = 0$, $h_{2,4}(0) = 1$ and $t_{2,4}(0) = 0$. We have written a computer program to obtain the recursion relations for $SG_{2,4}(n)$. They are too lengthy to be included here and are available from the authors on request. Some values of $M_{2,4}(n)$,

Table 5
The first few values of $M_{2,4}(n)$, $f_{2,4}(n)$, $g_{2,4}(n)$, $h_{2,4}(n)$, $t_{2,4}(n)$

n	0	1	2
$M_{2,4}(n)$	4	14,278	7,033,761,314,434,948,243,456,944,474,554,222,281,728
$f_{2,4}(n)$	1	2,220	1,095,249,688,634,151,454,219,516,689,432,826,798,080
$g_{2,4}(n)$	0	1,914	940,563,707,718,765,231,855,988,194,853,818,067,968
$h_{2,4}(n)$	1	1,640	807,724,574,091,886,425,362,687,789,454,449,995,776
$t_{2,4}(n)$	0	1,396	693,646,780,368,841,817,581,399,832,196,591,292,416

Table 6
The first few values of $\alpha_{2,4}(n)$, $\beta_{2,4}(n)$, $\gamma_{2,4}(n)$

n	1	2	3
$\alpha_{2,4}(n)$	0.862162162162162	0.858766468942539	0.858766468941692
$\beta_{2,4}(n)$	0.856844305120167	0.858766468941199	0.858766468941692
$\gamma_{2,4}(n)$	0.851219512195122	0.858766468939860	0.858766468941692

The last digits given are rounded off.

$f_{2,4}(n)$, $g_{2,4}(n)$, $h_{2,4}(n)$, $t_{2,4}(n)$ are listed in Table 5. These numbers grow exponentially, and do not have simple integer factorizations.

The sequence of the ratio defined in Eq. (3.7) $\{\alpha_{2,4}(n)\}_{n=1}^{\infty}$ decreases monotonically and $\{\gamma_{2,4}(n)\}_{n=1}^{\infty}$ increases monotonically with $0 \le \gamma_{2,4}(n) \le \alpha_{2,4}(n) \le 1$, the same as the results for $SG_2(n)$. The values of $\alpha_{2,4}(n)$, $\beta_{2,4}(n)$, $\gamma_{2,4}(n)$ for small n are listed in Table 6. Define their limits as in Lemma 3.3, the numerical results give

$$\alpha_{2,4} = \beta_{2,4} = \gamma_{2,4} = 0.858766468941\ldots \tag{4.7}$$

where more than a hundred significant figures can be evaluated when stage n in Eq. (3.7) is equal to four.

By a similar argument as Lemma 3.4, the asymptotic growth constant for the number of dimer–monomers on $SG_{2,4}(n)$ is bounded

$$\begin{aligned} &\frac{3\ln f_{2,4}(m) + 3\ln[1+2\gamma_{2,4}(m)] + \ln[1+3\gamma_{2,4}(m)]}{4\times 10^m} \le z_{SG_{2,4}} \\ &\le \frac{3\ln f_{2,4}(m) + 3\ln[1+2\alpha_{2,4}(m)] + \ln[1+3\alpha_{2,4}(m)]}{4\times 10^m}, \end{aligned} \tag{4.8}$$

with m a positive integer. We have the following proposition.

Proposition 4.2. *The asymptotic growth constant for the number of dimer–monomers on the generalized two-dimensional Sierpinski gasket $SG_{2,4}(n)$ in the large n limit is $z_{SG_{2,4}} = 0.684872262332\ldots$.*

The convergence of the upper and lower bounds is again quick. More than a hundred significant figures for $z_{SG_{2,4}}$ can be obtained when m in Eq. (4.8) is equal to four.

5. The number of dimer–monomers on $SG_d(n)$ with $d = 3, 4$

In this section, we derive the asymptotic growth constants of dimer–monomers on $SG_d(n)$ with $d = 3, 4$. For the three-dimensional Sierpinski gasket $SG_3(n)$, we use the following definitions.

Definition 5.1. Consider the three-dimensional Sierpinski gasket $SG_3(n)$ at stage n. (i) Define $M_3(n) \equiv N_{DM}(SG_3(n))$ as the number of dimer–monomers. (ii) Define $f_3(n)$ as the number of dimer–monomers such that the four outmost vertices are occupied by monomers. (iii) Define $g_3(n)$ as the number of dimer–monomers such that one certain outmost vertices is occupied by a dimer and the other three outmost vertices are occupied by monomers. (iv) Define $h_3(n)$ as the number of dimer–monomers such that two certain outmost vertices are occupied by monomers and the other two outmost vertices are occupied by dimers. (v) Define $r_3(n)$ as the number of dimer–monomers such that one certain outmost vertices is occupied by a monomer and the other three outmost vertices are occupied by

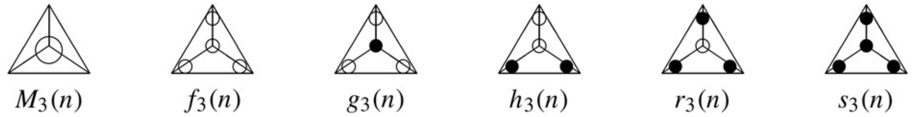

Fig. 9. Illustration for the configurations $M_3(n)$, $f_3(n)$, $g_3(n)$, $h_3(n)$, $r_3(n)$ and $s_3(n)$. Only the four outmost vertices are shown explicitly for $f_3(n)$, $g_3(n)$, $h_3(n)$, $r_3(n)$ and $s_3(n)$, where each open circle is occupied by a monomer and each solid circle is occupied by a dimer.

Table 7
The first few values of $M_3(n)$, $f_3(n)$, $g_3(n)$, $h_3(n)$, $r_3(n)$, $s_3(n)$

n	0	1	2	3
$M_3(n)$	10	945	132,820,373,046	49,123,375,811,021,432,878,640,796,802,876,545,882,185,505
$f_3(n)$	1	51	7,365,569,811	2,724,928,560,954,289,860,903,291,271,266,882,549,492,483
$g_3(n)$	0	57	7,816,070,424	2,889,924,536,764,017,260,444,663,495,693,780,813,791,233
$h_3(n)$	1	62	8,289,450,499	3,064,910,998,294,837,201,844,707,724,238,032,710,560,958
$r_3(n)$	0	60	8,786,476,992	3,250,492,861,272,219,038,243,497,885,127,347,116,333,900
$s_3(n)$	3	54	9,307,910,577	3,447,311,668,153,174,611,916,613,662,896,955,348,826,742

Table 8
The first few values of $\alpha_3(n)$, $\gamma_3(n)$ and other ratios

n	1	2	3	4
$\alpha_3(n)$	1.11764705882353	1.06116303620220	1.06055056935215	1.06055052894365
$h_3(n)/g_3(n)$	1.08771929824561	1.06056497054408	1.06055052971271	1.06055052894365
$r_3(n)/h_3(n)$	0.96774193548387	1.05995891923837	1.06055049007316	1.06055052894365
$\gamma_3(n)$	0.9	1.05934501228135	1.06055045043351	1.06055052894365

The last digits given are rounded off.

dimers. (vi) Define $s_3(n)$ as the number of dimer–monomers such that all four outmost vertices are occupied by dimers.

The quantities $M_3(n)$, $f_3(n)$, $g_3(n)$, $h_3(n)$, $r_3(n)$ and $s_3(n)$ are illustrated in Fig. 9, where only the outmost vertices are shown. There are $\binom{4}{1} = 4$ equivalent $g_3(n)$, $\binom{4}{2} = 6$ equivalent $h_3(n)$, and $\binom{4}{1} = 4$ equivalent $r_3(n)$. By definition,

$$M_3(n) = f_3(n) + 4g_3(n) + 6h_3(n) + 4r_3(n) + s_3(n). \tag{5.1}$$

The initial values at stage zero are $f_3(0) = 1$, $g_3(0) = 0$, $h_3(0) = 1$, $r_3(0) = 0$, $s_3(0) = 3$ and $M_3(0) = 10$.

The recursion relations are lengthy and given in the Appendix. Some values of $M_3(n)$, $f_3(n)$, $g_3(n)$, $h_3(n)$, $r_3(n)$, $s_3(n)$ are listed in Table 7. These numbers grow exponentially, and do not have simple integer factorizations.

Define $\alpha_3(n) = g_3(n)/f_3(n)$ and $\gamma_3(n) = s_3(n)/r_3(n)$ as in Eq. (3.7). We find $\{\alpha_3(n)\}_{n=1}^{\infty}$ decreases monotonically and $\{\gamma_3(n)\}_{n=1}^{\infty}$ increases monotonically with $1 \le \gamma_3(n) \le \alpha_3(n)$ for $n \ge 2$. The values of $\alpha_3(n)$, $\gamma_3(n)$ and other ratios for small n are listed in Table 8. Define their limits as in Lemma 3.3, the numerical results give

$$\alpha_3 = h_3/g_3 = r_3/h_3 = \gamma_3 = 1.06055052894\ldots, \tag{5.2}$$

where more than a hundred significant figures can be evaluated when stage n is equal to seven.

By a similar argument as Lemma 3.4, the asymptotic growth constant for the number of dimer–monomers on $SG_3(n)$ is bounded:

$$\frac{\ln f_3(m) + 2\ln[1+2\gamma_3(m)]}{2\times 4^m} \le z_{SG_3} \le \frac{\ln f_3(m) + 2\ln[1+2\alpha_3(m)]}{2\times 4^m}, \tag{5.3}$$

with m a positive integer. We have the following proposition.

Proposition 5.1. *The asymptotic growth constant for the number of dimer–monomers on the three-dimensional Sierpinski gasket $SG_3(n)$ in the large n limit is $z_{SG_3} = 0.781151467411\ldots$.*

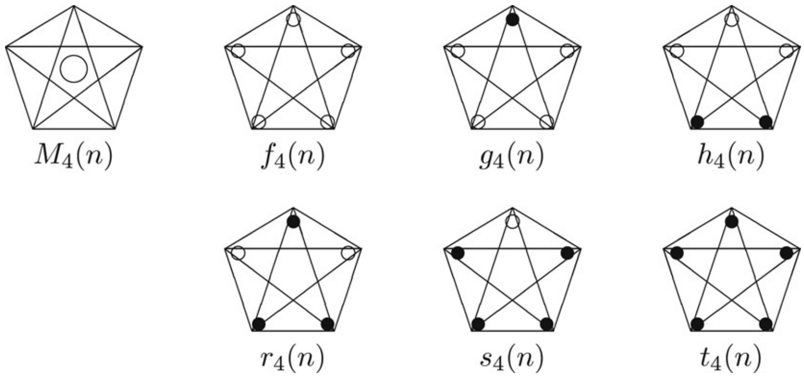

Fig. 10. Illustration for the configurations $M_4(n)$, $f_4(n)$, $g_4(n)$, $h_4(n)$, $r_4(n)$, $s_4(n)$ and $t_4(n)$. Only the five outmost vertices are shown explicitly for $f_4(n)$, $g_4(n)$, $h_4(n)$, $r_4(n)$, $s_4(n)$ and $t_4(n)$, where each open circle is occupied by a monomer and each solid circle is occupied by a dimer.

Table 9
The first few values of $M_4(n)$, $f_4(n)$, $g_4(n)$, $h_4(n)$, $r_4(n)$, $s_4(n)$, $t_4(n)$

n	0	1	2
$M_4(n)$	26	141,339	1,567,220,397,434,550,336,692,928
$f_4(n)$	1	2,460	27,951,923,701,499,685,610,752
$g_4(n)$	0	3,168	34,593,006,758,221,606,500,864
$h_4(n)$	1	3,990	42,806,033,106,111,666,338,688
$r_4(n)$	0	4,852	52,961,649,817,161,203,920,896
$s_4(n)$	3	5,683	65,517,552,720,775,495,239,744
$t_4(n)$	0	6,204	81,038,847,105,336,439,783,296

The convergence of the upper and lower bounds is as quick as for the ordinary two-dimensional case. More than a hundred significant figures for z_{SG_3} can be obtained when m in Eq. (5.3) is equal to seven.

For the four-dimensional Sierpinski gasket $SG_4(n)$, we use the following definitions.

Definition 5.2. Consider the four-dimensional Sierpinski gasket $SG_4(n)$ at stage n. (i) Define $M_4(n) \equiv N_{DM}(SG_4(n))$ as the number of dimer–monomers. (ii) Define $f_4(n)$ as the number of dimer–monomers such that the five outmost vertices are occupied by monomers. (iii) Define $g_4(n)$ as the number of dimer–monomers such that one certain outmost vertices is occupied by a dimer and the other four outmost vertices are occupied by monomers. (iv) Define $h_4(n)$ as the number of dimer–monomers such that two certain outmost vertices are occupied by dimers and the other three outmost vertices are occupied by monomers. (v) Define $r_4(n)$ as the number of dimer–monomers such that two certain outmost vertices are occupied by monomers and the other three outmost vertices are occupied by dimers. (vi) Define $s_4(n)$ as the number of dimer–monomers such that one certain outmost vertices is occupied by a monomer and the other four outmost vertices are occupied by dimers. (vii) Define $t_4(n)$ as the number of dimer–monomers such that all five outmost vertices are occupied by dimers.

The quantities $M_4(n)$, $f_4(n)$, $g_4(n)$, $h_4(n)$, $r_4(n)$, $s_4(n)$ and $t_4(n)$ are illustrated in Fig. 10, where only the outmost vertices are shown. There are $\binom{5}{1} = 5$ equivalent $g_4(n)$, $\binom{5}{2} = 10$ equivalent $h_4(n)$, $\binom{5}{3} = 10$ equivalent $r_4(n)$, $\binom{5}{1} = 5$ equivalent $s_4(n)$. By definition,

$$M_4(n) = f_4(n) + 5g_4(n) + 10h_4(n) + 10r_4(n) + 5s_4(n) + t_4(n). \tag{5.4}$$

The initial values at stage zero are $f_4(0) = 1$, $g_4(0) = 0$, $h_4(0) = 1$, $r_4(0) = 0$, $s_4(0) = 3$, $t_4(0) = 0$ and $M_4(0) = 26$.

We have written a computer program to obtain the recursion relations for $SG_4(n)$. They are too lengthy to be included here, and are available from the authors on request. Some values of $M_4(n)$, $f_4(n)$, $g_4(n)$, $h_4(n)$, $r_4(n)$, $s_4(n)$, $t_4(n)$ are listed in Table 9. These numbers grow exponentially, and do not have simple integer factorizations.

Table 10
The first few values of $\alpha_4(n)$, $\gamma_4(n)$ and other ratios

n	1	2	3	4
$\alpha_4(n)$	1.28780487804878	1.23758948141253	1.23734576161280	1.23734575732423
$h_4(n)/g_4(n)$	1.25946969696970	1.23741869000555	1.23734575860203	1.23734575732423
$r_4(n)/h_4(n)$	1.21604010025063	1.23724732179398	1.23734575559125	1.23734575732423
$s_4(n)/r_4(n)$	1.17126957955482	1.23707537334960	1.23734575258048	1.23734575732423
$\gamma_4(n)$	1.09167693119831	1.23690284114717	1.23734574956971	1.23734575732423

The last digits given are rounded off.

Define $\alpha_4(n) = g_4(n)/f_4(n)$ and $\gamma_4(n) = t_4(n)/s_4(n)$ as in Eq. (3.7). We find $\{\alpha_4(n)\}_{n=1}^{\infty}$ decreases monotonically and $\{\gamma_4(n)\}_{n=1}^{\infty}$ increases monotonically with $1 \le \gamma_4(n) \le \alpha_4(n)$ for positive integer n. The values of $\alpha_4(n)$, $\gamma_4(n)$ and other ratios for small n are listed in Table 10. Define their limits as in Lemma 3.3, the numerical results give

$$\alpha_4 = h_4/g_4 = r_4/h_4 = s_4/r_4 = \gamma_4 = 1.23734575732\ldots, \tag{5.5}$$

where more than a hundred significant figures can be evaluated when stage n is equal to seven.

By a similar argument as Lemma 3.4, the asymptotic growth constant for the number of dimer–monomers on $SG_4(n)$ is bounded:

$$\frac{2\ln f_4(m) + 5\ln[1+2\gamma_4(m)]}{5^{m+1}} \le z_{SG_4} \le \frac{2\ln f_4(m) + 5\ln[1+2\alpha_4(m)]}{5^{m+1}}, \tag{5.6}$$

with m a positive integer. We have the following proposition:

Proposition 5.2. *The asymptotic growth constant for the number of dimer–monomers on the four-dimensional Sierpinski gasket $SG_4(n)$ in the large n limit is $z_{SG_4} = 0.876779402949\ldots$.*

The convergence of the upper and lower bounds is as quick as for the ordinary two-dimensional case. More than a hundred significant figures for z_{SG_4} can be obtained when m in Eq. (5.6) is equal to seven.

6. Summary

The bounds of the asymptotic growth constants for dimer–monomers on $SG_2(n)$, $SG_3(n)$ and $SG_4(n)$ given in Sections 3 and 5 lead to the following conjecture for general $SG_d(n)$.

Conjecture 6.1. *Define $\alpha_d(n)$ as the ratio: the number of dimer–monomers on $SG_d(n)$ with all but one outmost vertices covered by monomers divided by that with all outmost vertices covered by monomers; define $\gamma_d(n)$ as the ratio: the number of dimer–monomers on $SG_d(n)$ with all outmost vertices covered by dimers divided by that with all but one outmost vertices covered by dimers. The asymptotic growth constant for the number of dimer–monomers on the d-dimensional Sierpinski gasket SG_d is bounded*

$$\frac{2\ln f_d(m) + (d+1)\ln[1+2\gamma_d(m)]}{(d+1)^{m+1}} \le z_{SG_d} \le \frac{2\ln f_d(m) + (d+1)\ln[1+2\alpha_d(m)]}{(d+1)^{m+1}}. \tag{6.1}$$

We notice that the convergence of the upper and lower bounds of the asymptotic growth constants for dimer–monomers on $SG_d(n)$ is about the same for each integer $d \ge 2$, in contrast to the results observed in Ref. [35] for spanning forests on $SG_d(n)$ where the convergence of the bounds of the asymptotic growth constants becomes slow when d increases.

We summarize the values of asymptotic growth constants $z_{SG_{d,b}}$ and the ratio $\alpha_{d,b}$ in Table 11. The value of z_{SG_d} increases as dimension d increases. Similarly for the generalized two-dimensional Sierpinski gasket, the value of $z_{SG_{2,b}}$ increases slightly as b increases. The ratio $\alpha_{d,b}$ is less than one for the generalized two-dimensional Sierpinski gasket, while it is larger than one for dimension larger or equal to three.

Compare the present results with those in Ref. [36], we find that the number of dimer–monomers on the Sierpinski gasket $SG_d(n)$ is less than that of spanning trees in general.

Table 11
Numerical values of $z_{SG_{d,b}}$ and the ratio $\alpha_{d,b}$

d	b	D	$z_{SG_{d,b}}$	$\alpha_{d,b}$
2	2	1.585	0.6562942369	0.8507721500
2	3	1.631	0.6716171611	0.8648330968
2	4	1.661	0.6848722623	0.8587664689
3	2	2	0.7811514674	1.060550529
4	2	2.322	0.8767794029	1.237345757

The last digits given are rounded off.

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Appendix A. Recursion relations for $SG_{2,3}(n)$

We give the recursion relations for the generalized two-dimensional Sierpinski gasket $SG_{2,3}(n)$ here. Since the subscript is $d = 2, b = 3$ for all the quantities throughout this section, we will use the simplified notation f_{n+1} to denote $f_{2,3}(n+1)$ and similar notations for other quantities. For any nonnegative integer n, we have

$$\begin{aligned}
f_{n+1} = {} & f_n^6 + 15 f_n^5 g_n + 12 f_n^5 h_n + 84 f_n^4 g_n^2 + 3 f_n^5 t_n + 117 f_n^4 g_n h_n + 220 f_n^3 g_n^3 + 24 f_n^4 g_n t_n + 33 f_n^4 h_n^2 \\
& + 390 f_n^3 g_n^2 h_n + 273 f_n^2 g_n^4 + 9 f_n^4 h_n t_n + 63 f_n^3 g_n^2 t_n + 180 f_n^3 g_n h_n^2 + 519 f_n^2 g_n^3 h_n + 141 f_n g_n^5 \\
& + 36 f_n^3 g_n h_n t_n + 60 f_n^2 g_n^3 t_n + 20 f_n^3 h_n^3 + 264 f_n^2 g_n^2 h_n^2 + 240 f_n g_n^4 h_n + 20 g_n^6 + 3 f_n^3 h_n^2 t_n \\
& + 30 f_n^2 g_n^2 h_n t_n + 15 f_n g_n^4 t_n + 33 f_n^2 g_n h_n^3 + 90 f_n g_n^3 h_n^2 + 21 g_n^5 h_n, \qquad (A.1)
\end{aligned}$$

$$\begin{aligned}
g_{n+1} = {} & f_n^5 g_n + 2 f_n^5 h_n + 13 f_n^4 g_n^2 + f_n^5 t_n + 35 f_n^4 g_n h_n + 60 f_n^3 g_n^3 + 14 f_n^4 g_n t_n + 18 f_n^4 h_n^2 + 188 f_n^3 g_n^2 h_n \\
& + 120 f_n^2 g_n^4 + 11 f_n^4 h_n t_n + 61 f_n^3 g_n^2 t_n + 152 f_n^3 g_n h_n^2 + 397 f_n^2 g_n^3 h_n + 99 f_n g_n^5 + f_n^4 t_n^2 \\
& + 72 f_n^3 g_n h_n t_n + 102 f_n^2 g_n^3 t_n + 30 f_n^3 h_n^3 + 372 f_n^2 g_n^2 h_n^2 + 310 f_n g_n^4 h_n + 25 g_n^6 + 4 f_n^3 g_n t_n^2 \\
& + 13 f_n^3 h_n^2 t_n + 130 f_n^2 g_n^2 h_n t_n + 57 f_n g_n^4 t_n + 107 f_n^2 g_n h_n^3 + 266 f_n g_n^3 h_n^2 + 63 g_n^5 h_n + 4 f_n^2 g_n^2 t_n^2 \\
& + 30 f_n^2 g_n h_n^2 t_n + 52 f_n g_n^3 h_n t_n + 6 g_n^5 t_n + 6 f_n^2 h_n^4 + 60 f_n g_n^2 h_n^3 + 34 g_n^4 h_n^2, \qquad (A.2)
\end{aligned}$$

$$\begin{aligned}
h_{n+1} = {} & f_n^4 g_n^2 + 4 f_n^4 g_n h_n + 11 f_n^3 g_n^3 + 2 f_n^4 g_n t_n + 4 f_n^4 h_n^2 + 50 f_n^3 g_n^2 h_n + 40 f_n^2 g_n^4 + 4 f_n^4 h_n t_n \\
& + 21 f_n^3 g_n^2 t_n + 68 f_n^3 g_n h_n^2 + 191 f_n^2 g_n^3 h_n + 56 f_n g_n^5 + f_n^4 t_n^2 + 52 f_n^3 g_n h_n t_n + 66 f_n^2 g_n^3 t_n \\
& + 25 f_n^3 h_n^3 + 289 f_n^2 g_n^2 h_n^2 + 263 f_n g_n^4 h_n + 24 g_n^6 + 9 f_n^3 g_n t_n^2 + 23 f_n^3 h_n^2 t_n + 167 f_n^2 g_n^2 h_n t_n \\
& + 72 f_n g_n^4 t_n + 150 f_n^2 g_n h_n^3 + 390 f_n g_n^3 h_n^2 + 101 g_n^5 h_n + 5 f_n^3 h_n t_n^2 + 19 f_n^2 g_n^2 t_n^2 + 94 f_n^2 g_n h_n^2 t_n \\
& + 158 f_n g_n^3 h_n t_n + 20 g_n^5 t_n + 20 f_n^2 h_n^4 + 201 f_n g_n^2 h_n^3 + 123 g_n^4 h_n^2 + 11 f_n^2 g_n h_n t_n^2 + 9 f_n g_n^3 t_n^2 \\
& + 9 f_n^2 h_n^3 t_n + 68 f_n g_n^2 h_n^2 t_n + 27 g_n^4 h_n t_n + 27 f_n g_n h_n^4 + 41 g_n^3 h_n^3, \qquad (A.3)
\end{aligned}$$

$$\begin{aligned}
t_{n+1} = {} & f_n^3 g_n^3 + 6 f_n^3 g_n^2 h_n + 9 f_n^2 g_n^4 + 3 f_n^3 g_n^2 t_n + 12 f_n^3 g_n h_n^2 + 57 f_n^2 g_n^3 h_n + 24 f_n g_n^5 + 12 f_n^3 g_n h_n t_n \\
& + 24 f_n^2 g_n^3 t_n + 8 f_n^3 h_n^3 + 126 f_n^2 g_n^2 h_n^2 + 150 f_n g_n^4 h_n + 20 g_n^6 + 3 f_n^3 g_n t_n^2 + 12 f_n^3 h_n^2 t_n \\
& + 99 f_n^2 g_n^2 h_n t_n + 51 f_n g_n^4 t_n + 111 f_n^2 g_n h_n^3 + 324 f_n g_n^3 h_n^2 + 120 g_n^5 h_n + 6 f_n^3 h_n t_n^2 + 18 f_n^2 g_n^2 t_n^2 \\
& + 117 f_n^2 g_n h_n^2 t_n + 186 f_n g_n^3 h_n t_n + 33 g_n^5 t_n + 30 f_n^2 h_n^4 + 282 f_n g_n^2 h_n^3 + 240 g_n^4 h_n^2 + f_n^3 t_n^3 \\
& + 36 f_n^2 g_n h_n t_n^2 + 21 f_n g_n^3 t_n^2 + 33 f_n^2 h_n^3 t_n + 183 f_n g_n^2 h_n^2 t_n + 99 g_n^4 h_n t_n + 87 f_n g_n h_n^4 + 180 g_n^3 h_n^3 \\
& + 3 f_n^2 g_n t_n^3 + 9 f_n^2 h_n^2 t_n^2 + 24 f_n g_n^2 h_n t_n^2 + 6 g_n^4 t_n^2 + 42 f_n g_n h_n^3 t_n + 63 g_n^3 h_n^2 t_n + 6 f_n h_n^5 \\
& + 39 g_n^2 h_n^4. \qquad (A.4)
\end{aligned}$$

There are always $2916 = 4 \times 3^6$ terms in these equations.

Appendix B. Recursion relations for $SG_3(n)$

We give the recursion relations for the three-dimensional Sierpinski gasket $SG_3(n)$ here. Since the subscript is $d = 3$ for all the quantities throughout this section, we will use the simplified notation f_{n+1} to denote $f_3(n+1)$ and similar notations for other quantities. For any nonnegative integer n, we have

$$\begin{aligned} f_{n+1} = {} & f_n^4 + 12f_n^3 g_n + 12f_n^3 h_n + 48f_n^2 g_n^2 + 4f_n^3 r_n + 84f_n^2 g_n h_n + 72f_n g_n^3 + 24f_n^2 g_n r_n + 30f_n^2 h_n^2 \\ & + 156f_n g_n^2 h_n + 30g_n^4 + 12f_n^2 h_n r_n + 36f_n g_n^2 r_n + 84f_n g_n h_n^2 + 60g_n^3 h_n + 24f_n g_n h_n r_n + 8g_n^3 r_n \\ & + 8f_n h_n^3 + 24g_n^2 h_n^2, \end{aligned} \tag{B.1}$$

$$\begin{aligned} g_{n+1} = {} & f_n^3 g_n + 3f_n^3 h_n + 9f_n^2 g_n^2 + 3f_n^3 r_n + 33f_n^2 g_n h_n + 24f_n g_n^3 + f_n^3 s_n + 24f_n^2 g_n r_n + 21f_n^2 h_n^2 \\ & + 96f_n g_n^2 h_n + 18g_n^4 + 6f_n^2 g_n s_n + 21f_n^2 h_n r_n + 51f_n g_n^2 r_n + 93f_n g_n h_n^2 + 69g_n^3 h_n + 3f_n^2 h_n s_n \\ & + 9f_n g_n^2 s_n + 3f_n^2 r_n^2 + 66f_n g_n h_n r_n + 24g_n^3 r_n + 21f_n h_n^3 + 66g_n^2 h_n^2 + 6f_n g_n h_n s_n + 2g_n^3 s_n \\ & + 6f_n g_n r_n^2 + 12f_n h_n^2 r_n + 24g_n^2 h_n r_n + 14g_n h_n^3, \end{aligned} \tag{B.2}$$

$$\begin{aligned} h_{n+1} = {} & f_n^2 g_n^2 + 6f_n^2 g_n h_n + 6f_n g_n^3 + 6f_n^2 g_n r_n + 8f_n^2 h_n^2 + 38f_n g_n^2 h_n + 8g_n^4 + 2f_n^2 g_n s_n + 14f_n^2 h_n r_n \\ & + 30f_n g_n^2 r_n + 64f_n g_n h_n^2 + 50g_n^3 h_n + 4f_n^2 h_n s_n + 8f_n g_n^2 s_n + 5f_n^2 r_n^2 + 80f_n g_n h_n r_n + 30g_n^3 r_n \\ & + 26f_n h_n^3 + 87g_n^2 h_n^2 + 2f_n^2 r_n s_n + 16f_n g_n h_n s_n + 6g_n^3 s_n + 18f_n g_n r_n^2 + 34f_n h_n^2 r_n + 72g_n^2 h_n r_n \\ & + 44g_n h_n^3 + 4f_n g_n r_n s_n + 4f_n h_n^2 s_n + 8g_n^2 h_n s_n + 8f_n h_n r_n^2 + 8g_n^2 r_n^2 + 28g_n h_n^2 r_n + 4h_n^4, \end{aligned} \tag{B.3}$$

$$\begin{aligned} r_{n+1} = {} & f_n g_n^3 + 9f_n g_n^2 h_n + 3g_n^4 + 9f_n g_n^2 r_n + 24f_n g_n h_n^2 + 27g_n^3 h_n + 3f_n g_n^2 s_n + 42f_n g_n h_n r_n \\ & + 22g_n^3 r_n + 18f_n h_n^3 + 75g_n^2 h_n^2 + 12f_n g_n h_n s_n + 6g_n^3 s_n + 15f_n g_n r_n^2 + 39f_n h_n^2 r_n + 99g_n^2 h_n r_n \\ & + 69g_n h_n^3 + 6f_n g_n r_n s_n + 9f_n h_n^2 s_n + 21g_n^2 h_n s_n + 21f_n h_n r_n^2 + 24g_n^2 r_n^2 + 96g_n h_n^2 r_n + 15h_n^4 \\ & + 6f_n h_n r_n s_n + 6g_n^2 r_n s_n + 12g_n h_n^2 s_n + 2f_n r_n^3 + 24g_n h_n r_n^2 + 14h_n^3 r_n, \end{aligned} \tag{B.4}$$

$$\begin{aligned} s_{n+1} = {} & g_n^4 + 12g_n^3 h_n + 12g_n^3 r_n + 48g_n^2 h_n^2 + 4g_n^3 s_n + 84g_n^2 h_n r_n + 72g_n h_n^3 + 24g_n^2 h_n s_n + 30g_n^2 r_n^2 \\ & + 156g_n h_n^2 r_n + 30h_n^4 + 12g_n^2 r_n s_n + 36g_n h_n^2 s_n + 84g_n h_n r_n^2 + 60h_n^3 r_n + 24g_n h_n r_n s_n + 8h_n^3 s_n \\ & + 8g_n r_n^3 + 24h_n^2 r_n^2. \end{aligned} \tag{B.5}$$

There are always $729 = 3^6$ terms in these equations.

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