Physica A 390 (2011) 419–426

Contents lists available at ScienceDirect

Physica A

journal homepage: www.elsevier.com/locate/physa

Asymptotic behavior for a version of directed percolation on a square lattice

Lung-Chi Chen

Department of Mathematics, Fu-Jen Catholic University, Taiwan

ARTICLE INFO

Article history:
Received 28 July 2010
Received in revised form 10 September 2010
Available online 19 October 2010

Keywords:
Domany–Kinzel model
Directed percolation
Compact directed percolation
Asymptotic behavior
Critical behavior
Susceptibility
Exact solutions
Berry–Esseen theorem

ABSTRACT

We consider a version of directed bond percolation on a square lattice whose vertical edges are directed upward with probabilities p_v and horizontal edges are directed rightward with probabilities p_h and 1 in alternate rows. Let $\tau(M, N)$ be the probability that there is a connected directed path of occupied edges from $(0, 0)$ to (M, N). For each $p_h \in [0, 1]$, $p_v = (0, 1)$ and aspect ratio $\alpha = M/N$ fixed, it was established (Chen and Wu, 2006) [9] that there is an $\alpha_c = [1 - p_v^2 - p_h(1 - p_v)^2]/2p_v^2$ such that, as $N \to \infty$, $\tau(M, N)$ is 1, 0, and 1/2 for $\alpha > \alpha_c$, $\alpha < \alpha_c$, and $\alpha = \alpha_c$, respectively. In particular, for $p_h = 0$ or 1, the model reduces to the Domany–Kinzel model (Domany and Kinzel, 1981 [7]). In this article, we investigate the rate of convergence of $\tau(M, N)$ and the asymptotic behavior of $\tau(M_n^-, N)$ and $\tau(M_n^+, N)$, where $M_n^-/N \uparrow \alpha_c$ and $M_n^+/N \downarrow \alpha_c$ as $N \uparrow \infty$. Moreover, we obtain a susceptibility on the rectangular net $\{(m, n) \in \mathbb{Z}_+ \times \mathbb{Z}_+ : 0 \le m \le M \text{ and } 0 \le n \le N\}$. The proof is based on the Berry–Esseen theorem.

1. Introduction

Directed percolation (oriented percolation) can be thought of simply as a percolation process on a directed lattice in which connections are allowed only in a preferred direction. It was first studied by Broadbent and Hammersley in 1957 [1] and it has remained to this day as one of the most outstanding interesting problems in probability and statistical mechanics. Furthermore, directed percolation is closely related to the Reggeon field theory in high-energy physics and the Markov processes with branching, recombination, and absorption that occur in chemistry and biology [2,3], etc. Various properties, results, and conjectures of directed percolation can be found in [4,5] and the references therein. However, very little is known in the way of exact solutions for the directed percolation problem.

We say that the vertex (m, n) is percolating if there is a connected directed path of occupied edges from $(0, 0)$ to (m, n). Compact directed percolation [6] is a version of the universality of directed percolation class. It is defined on a square by the condition transition probabilities as follows: $P((x, y)$ is percolating: $(x - 1, y)$, $(x, y - 1)$ are not percolating$) = 0$, $P((x, y)$ is percolating:$(x - 1, y)$ is percolating and $(x, y - 1)$ is not percolating$) = p_1$, $P((x, y)$ is percolating: $(x - 1, y)$ is not percolating and $(x, y-1)$ is percolating$) = p_2$ and $P((x, y)$ is percolating: $(x-1, y)$, $(x, y-1)$ are percolating$) = 1$ for any p_1, $p_2 \in [0, 1]$ and $(x, y) \neq (0, 0)$. Hence, the system has two absorbing states, namely, the empty and the fully occupied lattice.

Domany and Kinzel [7] defined a solvable version of compact directed percolation on a square lattice in 1981, as follows. For $p \in (0, 1)$ fixed, each nearest-neighbour vertical bond is directed upward with occupation probability p (independently of the other bonds) and each nearest-neighbour horizontal bond is directed rightward with occupation probability 1. Furthermore, it is known that the boundary of the Domany–Kinzel model has the same distribution as the one-dimensional last-passage percolation model [8]. In this article, we consider a version of directed percolation on a square lattice whose

E-mail address: lcchen@math.fju.edu.tw.

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doi:10.1016/j.physa.2010.09.039

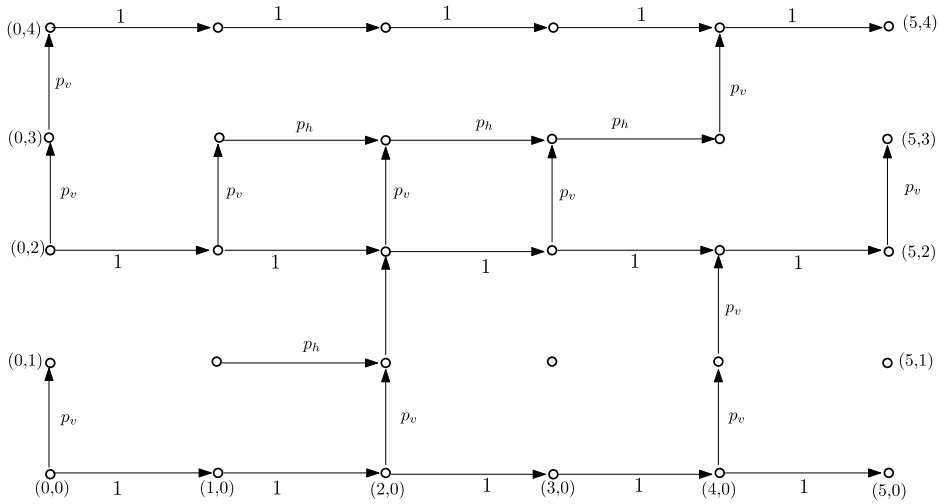

Fig. 1. A typical percolating configuration on a 5×4 lattice with hole $(3, 1)$. Open circles denote lattice sites. Oriented edges are occupied with probabilities shown. Empty edges carry probabilities $1 - p_h$ and $1 - p_v$ in the horizontal and vertical directions, respectively.

vertical edges are occupied with a probability p_v and whose horizontal edges in the n-th row are occupied with a probability 1 if n is odd and p_h if n is even. In particular, for $p_h = 0$ or 1, the model reduces to the Domany–Kinzel model, as shown below. However, the model is not a compact directed percolation class for $p_h \in (0, 1)$, since the connected cluster may has hole(s) (as shown in Fig. 1). Nevertheless, the hole(s) is (are) rectangular and its (their) width(s) is (are) 2;, it is believed that the critical behavior of the model refers to the compact directed percolation class and not to the habitual directed percolation class.

Given any $\alpha > 0$ and $p_h \in [0, 1]$, $p_v \in (0, 1)$, throughout this article, let $N_\alpha = \lfloor \alpha N \rfloor = \sup\{m \in \mathbb{Z}_+ : m \le \alpha N\}$ with $N \in \mathbb{Z}_+$ and denote a two-dimensional rectangular net $\{(m, n) \in \mathbb{Z}_+ \times \mathbb{Z}_+ : 0 \le m \le M \text{ and } 0 \le n \le N\}$ by $M \times N$. Let $\mathbb{P}$ be the probability distribution of the bond variables. Define the two-point correlation function as follows:

$$\tau(N_\alpha, N) = \mathbb{P}((N_\alpha, N) \text{ is percolating}).$$

It was shown by the method of steepest descent [9] that there is

$$\alpha_c = [1 - p_v^2 - p_h(1 - p_v)^2]/2p_v^2, \tag{1.1}$$

such that

$$\lim_{N\to\infty} \tau(N_{2\alpha}, 2N) = \begin{cases} 1 & \text{if } \alpha > \alpha_c, \\ 0 & \text{if } \alpha < \alpha_c, \\ \dfrac{1}{2} & \text{if } \alpha = \alpha_c. \end{cases} \tag{1.2}$$

For $\alpha < \alpha_c$, the critical exponent of the correlation length $\nu = 2$ is the same as that found in the Domany–Kinzel model [7,10–12].

The behavior of (1.2) is interesting, since the critical point is discontinuous. It is appropriate to define some of the standard critical exponents and to sketch the phenomenological scaling theory of $\tau(N_\alpha, N)$. For $\alpha < \alpha_c$, the scaling theory of critical behavior now asserts that the singular part of $\tau(N_\alpha, N)$ varies asymptotically as (see [13])

$$\tau(N_\alpha, N) \sim \frac{A_\alpha}{N^\eta} \exp\left(\frac{-B_\alpha N}{(\alpha_c - \alpha)^{-\nu}}\right), \tag{1.3}$$

where $f_1(N) \sim f_2(N)$ means that $\lim_{N\to\infty} f_1(N)/f_2(N) = 1$, the constants A_α and B_α depend on α, and $\eta, \nu \in (0, \infty)$ are universal constants. Furthermore, η is called the critical exponent and ν is called the critical exponent of the correlation length [14]. Note that there has been no general proof of the existence of critical exponents. To the best of our knowledge, the rate of convergence of (1.2) is unknown, and the values of A_α, B_α, and η in (1.3) for $\alpha \in (0, \alpha_c)$ are unknown too, even for Domany–Kinzel model. This allows us analyze (1.2) in detail.

Probability theory is a powerful tool to deal with this model. In fact, we can get α_c in (1.1) and the result of (1.2) by the law of large numbers rather than the method of steepest descent. Furthermore, the Berry–Esseen theorem attempts to quantify the rate at which this convergence to normality takes place. In this paper, we use the Berry–Esseen theorem to obtain sharp new results.

The rest of this paper is organized as follows. In Section 2, we state the main results (Theorems 2.1, 2.2 and 2.4) of this paper. Theorem 2.1 is proven in Section 3. In Section 4, we prove Theorem 2.2 by Theorem 2.1 and we apply Theorem 2.2 to show Theorem 2.4.

2. Main results

Let $\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \mathrm{e}^{-\frac{u^2}{2}}\, \mathrm{d}\, u$ be the standard cumulative distribution function of a *Gaussian* distribution with mean 0 and variance 1, and let $\Psi(x) = 1 - \Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{x}^{\infty} \mathrm{e}^{-\frac{u^2}{2}}\, \mathrm{d}\, u$. It is not difficult to get that

$$\Psi(x) = \frac{\mathrm{e}^{-\frac{x^2}{2}}}{\sqrt{2\pi}x}(1 + O(x^{-2})) \quad \text{for } x \text{ large}. \tag{2.1}$$

In the first result of this paper, we study the rate of convergence of $\tau(N_{2\alpha}, 2N)$ for $\alpha > 0$ fixed.

Theorem 2.1. *Let $p_v \in (0, 1)$ and $p_h \in [0, 1]$ be given. For any $\alpha > 0$, there is a critical aspect ratio α_c that is given in (1.1) such that*

$$\tau(N_{2\alpha}, 2N) = \begin{cases} \left(1 + O\left(\frac{1}{\sqrt{N}}\right)\right)\Psi(\kappa(\alpha_c - \alpha)\sqrt{N}) & \text{if } \alpha < \alpha_c, \\ \frac{1}{2} + O\left(\frac{1}{\sqrt{N}}\right) & \text{if } \alpha = \alpha_c, \\ 1 - \left(1 + O\left(\frac{1}{\sqrt{N}}\right)\right)\Psi(\kappa(\alpha - \alpha_c)\sqrt{N}) & \text{if } \alpha > \alpha_c, \end{cases}$$

where

$$\kappa = \frac{2p_v}{\sqrt{p_v^2(2\alpha_c + 1)^2 - p_h(1 - p_v)^2 - 1}} > 0. \tag{2.2}$$

For $p_h = 1$, our model reduces to a Domany–Kinzel model on $N_{2\alpha} \times 2N$ lattice, and (1.1) leads to $\alpha_c = (1 - p_v)/p_v$, in agreement with previous results [7,10], and $\kappa = \sqrt{2(1 - p_v)}/\alpha_c$. For $p_h = 0$, our model is again a Domany–Kinzel model, but on an $N_{2\alpha} \times N$ lattice with vertical edge occupation probability p_v^2. Our result gives the critical aspect ratio $2\alpha_c = (1-p_v^2)/p_v^2$, again in agreement with [7,10], and $\kappa = \sqrt{1 - p_v^2}/\alpha_c$.

By (2.1) and Theorem 2.1, our result gives that $\tau(N_{2\alpha}, 2N)$ with $\alpha < \alpha_c$ and $1 - \tau(N_{2\alpha}, 2N)$ with $\alpha > \alpha_c$ both decay exponentially to zero. Furthermore, the critical exponent of the correlation length $\nu = 2$ and $\eta = \frac{1}{2}, A_\alpha = (\sqrt{2\pi}\kappa(\alpha_c - \alpha))^{-1}$, and $B_\alpha = \kappa^2/2$ in (1.3) for $\alpha < \alpha_c$, which complements the previous asymptotic studies.

Since $A_\alpha \to \infty$ as $\alpha \to \alpha_c$, for the second result of this article, we investigate the asymptotic phenomena of $\tau(N_{2\alpha_N^-}, 2N)$ and $\tau(N_{2\alpha_N^+}, 2N)$ where $\alpha_N^+ \downarrow \alpha_c$ and $\alpha_N^- \uparrow \alpha_c$ as $N \uparrow \infty$. We say that the sequence $\{\ell_n\}_{n=1}^{\infty}$ is a regularly varying sequence if, for any $\lambda \in (0, \infty)$, $\lim_{n\to\infty} \ell_{\lfloor \lambda n \rfloor}/\ell_n = 1$. For example, $\ell_N = \log N$ or $\ell_N = c \in (0, \infty)$ for all N.

Theorem 2.2. *Given $p_v \in (0, 1)$, $p_h \in [0, 1]$, $\rho \in (0, \infty)$, and the positive regularly varying sequence $\{\ell_n\}_{n=1}^{\infty}$. Let $\alpha_N^- = \alpha_c - N^{-\rho}\kappa^{-1}\ell_N$ and $\alpha_N^+ = \alpha_c + N^{-\rho}\kappa^{-1}\ell_N$. Then both*

$$1 - \tau(N_{2\alpha_N^+}, 2N) \quad \text{and} \quad \tau(N_{2\alpha_N^-}, 2N)$$
$$= \begin{cases} \left(1 + O\left(\frac{1}{\sqrt{N}}\right)\right)\Psi\left(N^{-\rho+\frac{1}{2}}\ell_N\right) & \text{if } \rho \in \left(0, \frac{1}{2}\right) \text{ or } \rho = \frac{1}{2}, \ell_N \to \infty, \\ \Psi(\ell) + O(1)\max\left\{\frac{1}{\sqrt{N}}, |\ell - \ell_N|\right\} & \text{if } \rho = \frac{1}{2}, \ell_N \to \ell \in [0, \infty), \\ \frac{1}{2} + O\left(N^{-\rho+\frac{1}{2}}\ell_N\right) & \text{if } \rho \in \left(\frac{1}{2}, 1\right], \\ \frac{1}{2} + O\left(\frac{1}{\sqrt{N}}\right) & \text{if } \rho \in (1, \infty). \end{cases}$$

Note that $\rho = \frac{1}{2}$ is a critical value, and we have the following corollary.

Corollary 2.3. *Under the same assumptions of Theorem 2.2, we have*

$$\lim_{N\to\infty} \tau(N_{2\alpha_N^-}, 2N) = \lim_{N\to\infty}(1 - \tau(N_{2\alpha_N^+}, 2N)) = \begin{cases} 0 & \text{if} \quad \rho \in \left(0, \frac{1}{2}\right), \\ \frac{1}{2} & \text{if} \quad \rho \in \left(\frac{1}{2}, \infty\right), \end{cases}$$

and when $\rho = 1/2$, $\ell_N \to \ell \in [0, \infty]$, *we have*

$$\lim_{N\to\infty} \tau(N_{2\alpha_N^-}, 2N) = \Psi(\ell), \qquad \lim_{N\to\infty} \tau(N_{2\alpha_N^+}, 2N) = \Phi(\ell).$$

Finally, we denote the set of vertices that are connected from $(0, 0)$ on the rectangular net $N_{2\alpha} \times 2N$ by $\mathcal{C}(N_{2\alpha}, 2N)$; i.e., $\mathcal{C}(N_{2\alpha}, 2N) = \{(m, n) \in N_{2\alpha} \times 2N : (m, n) \text{ is percolating}\}$. We also denote its cardinality by $|\mathcal{C}(N_{2\alpha}, 2N)|$. Define a susceptibility on the rectangle net $N_{2\alpha} \times 2N$ as follows:

$$\chi(N_{2\alpha}, 2N) = \mathbb{E}(|\mathcal{C}(N_{2\alpha}, 2N)|),$$

where $\mathbb{E}$ is the expectation by $\mathbb{P}$. In the last result of the article, we derive the asymptotic behavior of $\chi(N_{2\alpha}, 2N)$.

Theorem 2.4. *Let* $p_v \in (0, 1)$ *and* $p_h \in [0, 1]$ *be given. For any* $\alpha > 0$, *we have*

$$\chi(N_{2\alpha}, 2N) = \begin{cases} 2(2\alpha - \alpha_c)(N+1)^2 \left(1 + O\left(\dfrac{\log N}{\sqrt{N}}\right)\right). & \text{if } \alpha > \alpha_c, \\ \dfrac{2\alpha^2}{\alpha_c}(N+1)^2 \left(1 + O\left(\dfrac{\log N}{\sqrt{N}}\right)\right) & \text{if } \alpha \le \alpha_c. \end{cases}$$

3. Proof of Theorem 2.1

The main body of this paper is the proof of Theorem 2.1. To show Theorem 2.1, we use the Berry–Esseen theorem (see [15]) whose statement is as follows.

Theorem 3.1 (*Berry–Esseen Theorem*)**.** *Given a probability space* $(\Omega, \mathcal{F}, P)$, *with its expectation denoted by* E. *Let* $Y_1, Y_2, \ldots$ *be independent and identically distributed random variables with* $E(Y_1) = 0$, $E(Y_1^2) = \sigma^2 < \infty$ *and* $E(|Y_1|^3) = \rho < \infty$. *Also let*

$$S_n = Y_1 + \cdots + Y_n \quad \text{for } n \in \mathbb{N}. \tag{3.1}$$

Then there exists a positive constant $c \in (0, \infty)$ *such that*

$$\left| P\left(\frac{S_n}{\sigma\sqrt{n}} \le x\right) - \Phi(x) \right| \le \frac{c\rho}{\sigma^3\sqrt{n}}$$

for all $x \in \mathbb{R}$ *and* $n \in \mathbb{N}$.

For any $N \in \mathbb{N}$, we say that an occupied vertical edge in a bond configuration is wet if it lies on a percolating path where $(N_{2\alpha}, 2N)$ is percolating, and is *primary* wet if it is the first wet edge (in a row of vertical edges) counting from the left. In a percolating configuration where $(N_{2\alpha}, 2N)$ is percolating, there is one *primary wet* edge in every row. Since a bond configuration is percolating whenever a vertical edge in the $2N$-th row is primary wet, which can occur at any of the m-th horizontal positions $m = 0, 1, 2, \ldots, N_{2\alpha}$, we have for $N \in \mathbb{N}$

$$\tau(N_{2\alpha}, 2N) = \sum_{m=0}^{N_{2\alpha}} P_N(m), \tag{3.2}$$

where $P_N(m)$ is the probability that the *primary wet* edge in the $2N$-th row occurs at the horizontal position m. Let $P_0(m) = \delta_{0,m}$, where δ is the Kronecker delta.

Since the vertical edges are occupied with a probability p_v and the horizontal edges in the n-th row are occupied with a probability 1 if n is odd and p_h if n is even, it is not difficult to see that P_N is the N-step transition probability for one-dimensional simple random walk starting from the origin whose 1-step distribution is given by D; i.e.,

$$P_N(m) = \sum_{k=0}^{m} P_1(k) P_{N-1}(m-k) = D^{*N}(m), \tag{3.3}$$

where D^{*N} is N-fold convolution and $D(m) = P_1(m)$ for $m = 0, 1, 2, \ldots$. Therefore we can define a random walk S_n with a probability *Prob.* as follows: Prob. $(S_0 = m) = \delta_{0,m}$ and, for $N \in \mathbb{N}$,

$$S_N = X_1 + X_2 + \cdots + X_N,$$

where *Prob.* $(X_j = m) = D(m)$ for $j = 1, 2, \ldots, N$ and *Prob.* $(S_N = m) = P_N(m)$ with $m \in \mathbb{Z}_+$. We denote the expectation using *Prob.* by *Exp.*

Let the mean of D be μ and the variance of D be σ^2. Then by Theorem 3.1 with $Y_j = X_j - \mu$, we have, for any $z \in \mathbb{R}$,

$$Prob.\left(\frac{S_N - N\mu}{\sigma\sqrt{N}} \le z\right) = \Phi(z)\left(1 + O\left(\frac{1}{\sqrt{N}}\right)\right) \quad \text{if } Exp.(|X_1|^3) < \infty.$$

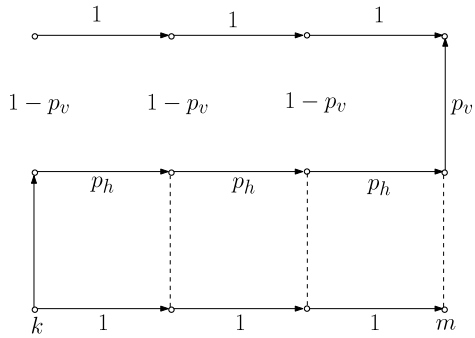

Fig. 2. Construction of (3.8). Occupied edges are shown as oriented edges; dotted edges can be either occupied or vacant.

Since $P_N(m)$ is the probability that the *primary wet* edge in the $2N$-th row occurs at the horizontal position m, from (3.2), if $2\alpha \le \mu$, we have

$$\begin{aligned}
\tau(N_{2\alpha}, 2N) &= Prob.(0 \le S_N \le N_{2\alpha}) \\
&= Prob.(-N\mu \le S_N - N\mu \le N_{2\alpha} - N\mu) \\
&= \left(1 + O\left(\frac{1}{\sqrt{N}}\right)\right) Prob.\left(-\frac{\mu\sqrt{N}}{\sigma} \le \frac{S_N - N\mu}{\sigma\sqrt{N}} \le \frac{\sqrt{N}}{\sigma}(2\alpha - \mu)\right) \\
&= \left(1 + O\left(\frac{1}{\sqrt{N}}\right)\right) \int_{-\frac{\mu\sqrt{N}}{\sigma}}^{(2\alpha-\mu)\frac{\sqrt{N}}{\sigma}} \frac{\mathrm{e}^{\frac{u^2}{2}}}{\sqrt{2\pi}}\,\mathrm{d}\,u \\
&= \left(1 + O\left(\frac{1}{\sqrt{N}}\right)\right) \Phi\left((2\alpha - \mu)\frac{\sqrt{N}}{\sigma}\right) \\
&= \left(1 + O\left(\frac{1}{\sqrt{N}}\right)\right) \Psi\left((\mu - 2\alpha)\frac{\sqrt{N}}{\sigma}\right),
\end{aligned} \tag{3.4}$$

where the last equality holds due to the symmetry of a *Gaussian* distribution ($\Phi(-x) = \Psi(x)$ for $x > 0$). Similarly, when $2\alpha > \mu$, we have

$$\begin{aligned}
\tau(N_{2\alpha}, 2N) &= 1 - Prob.(S_N > N_{2\alpha}) \\
&= 1 - \left(1 + O\left(\frac{1}{\sqrt{N}}\right)\right) Prob.\left(\frac{S_N - N\mu}{\sigma\sqrt{N}} > \frac{\sqrt{N}}{\sigma}(2\alpha - \mu)\right) \\
&= 1 - \left(1 + O\left(\frac{1}{\sqrt{N}}\right)\right) \int_{(2\alpha-\mu)\frac{\sqrt{N}}{\sigma}}^{\infty} \frac{\mathrm{e}^{-\frac{u^2}{2}}}{\sqrt{2\pi}}\,\mathrm{d}\,u \\
&= 1 - \left(1 + O\left(\frac{1}{\sqrt{N}}\right)\right) \Psi\left((2\alpha - \mu)\frac{\sqrt{N}}{\sigma}\right).
\end{aligned} \tag{3.5}$$

Then it is sufficient to show that $\mu = 2\alpha_c$ and $\sigma^2 = (2\alpha_c + 1)^2 - \frac{1+p_h(1-p_v)^2}{p_v^2}$ and $Exp.\ (|X_1|^3) < \infty$. To do this, we first define the generating function

$$\hat{D}(t) = \sum_{m=0}^{\infty} D(m)t^m. \tag{3.6}$$

As mentioned above, $D(m)$ is the probability that $(m, 2)$ is percolating with the primary wet vertical edge in the top row occurring at m. However, the primary wet vertical edge in the bottom row can be at any k in $0 \le k \le m$. Then we have

$$D(m) = \sum_{k=0}^{m} D_1(k)D_2(m-k), \tag{3.7}$$

where $D_1(k)$ and $D_2(m-k)$ are the probabilities that the primary wet edge in the bottom row is at k and the distance between the primary wet edge in the bottom row and the primary wet edge in the top row is $m-k$, respectively. Since each

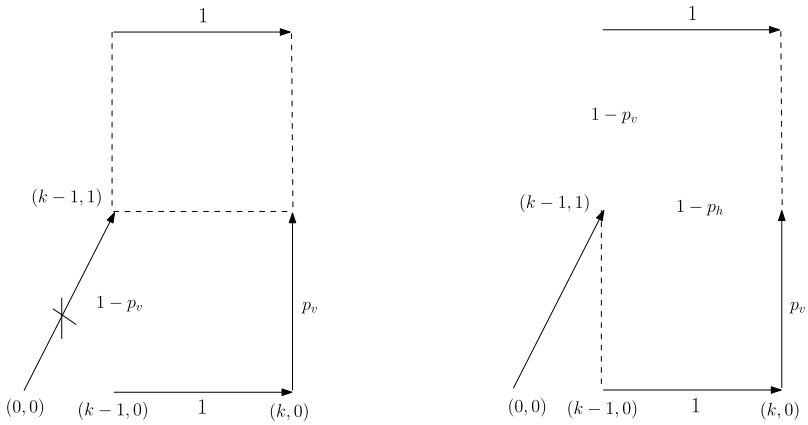

Fig. 3. Construction of (3.9). Occupied edges are shown as oriented edges; dotted edges can be either occupied or vacant.

horizontal edges connecting $(j, 2)$ and $(j+1, 2)$ with $j \in \mathbb{Z}_+$ is occupied with probability 1, it is easy to see that (as shown in Fig. 2)

$$D_2(m-k) = p_v(1-p_v)^{m-k}p_h^{m-k}. \tag{3.8}$$

To compute the factor $D_1(k)$ with $k \geq 1$ is more complicated. Note that $D_1(0) = p_v$. We decompose D_1 into two terms, as follows:

$$D_1(k) = \Delta(k) + \Sigma(k), \tag{3.9}$$

where $\Delta(k)$ is the probability that the first primary wet edge is at k such that the site $(k-1, 1)$ is disconnected with $(0, 0)$ and $\Sigma(k)$ is the probability that the first primary wet edge is at k such that the site $(k-1, 1)$ is connected with $(0, 0)$ (as shown in Fig. 3). Consider $\Delta(k)$; the vertical edge connecting $(k-1, 0)$ and $(k-1, 1)$ is empty with probability $1-p_v$. Then $\Delta(k) = p_v((1-p_v)D_1(k-1)/p_v)$. Consider $\Sigma(k)$; the factor $(1-p_h)(1-p_v)$ ensures that site $(k-1, 1)$ is not on a percolating path since the primary wet edge is at k. So $\Sigma(k) = p_v((1-p_h)(1-p_v)D(k-1)/p_v)$. Therefore,the factor $D_1(k)$ satisfies a recursive relation, which can be written as

$$D_1(k) = (1-p_v)D_1(k-1) + (1-p_h)(1-p_v)D(k-1), \quad k = 1, 2, \ldots, m. \tag{3.10}$$

Define the generating function

$$\hat{D}_1(t) = \sum_{k=0}^{\infty} D_1(k)t^k.$$

Multiplying (3.7) and (3.10) by t^m and t^{k-1}, respectively, and summing over m and $k-1$ from 0 to ∞, we obtain that

$$\hat{D}(t) = \sum_{k=0}^{\infty} D(k)t^k = \frac{p_v\hat{D}_1(t)}{1-p_h(1-p_v)t},$$

$$\frac{1}{t}(\hat{D}_1(t) - p_v) = (1-p_v)\hat{D}_1(t) + (1-p_h)(1-p_v)\hat{D}(t).$$

This gives

$$\hat{D}(t) = \frac{p_v^2}{1-at+bt^2}, \tag{3.11}$$

where

$$a = 1 - p_v^2 + p_h(1-p_v)^2, \qquad b = p_h(1-p_v)^2. \tag{3.12}$$

Then, by (3.11) and (3.12), we have the mean of D as follows:

$$\mu = \sum_{m=1}^{\infty} mD(m) = \frac{\mathrm{d}\hat{D}(t)}{\mathrm{d}t}|_{t=1} = \frac{p_v^2(a-2b)}{(1-a+b)^2} = 2\alpha_c, \tag{3.13}$$

and the variance of D as follows:

$$\begin{aligned}\sigma^2 &= \sum_{m=1}^{\infty} m^2 D(m) - \mu^2 \\ &= \frac{\mathrm{d}^2\hat{D}(t)}{\mathrm{d}t^2}|_{t=1} + \mu - \mu^2 \\ &= (2\alpha_c + 1)^2 - \frac{1 + p_h(1-p_v)^2}{p_v^2}.\end{aligned} \tag{3.14}$$

Finally, it is easy to check that $\mathbb{E}(|X_1|^3) = \sum_{m=1}^{\infty} m^3 D(m) = O(1)\frac{\mathrm{d}^3\hat{D}(t)}{\mathrm{d}t^3}|_{t=1} < \infty$. From (3.4), (3.5), (3.13) and (3.14), this completes the proof, as required. □

4. Proof of Theorems 2.2 and 2.4

We use Theorem 2.1 to show Theorem 2.2.

4.1. Proof of Theorem 2.2

Applying Theorem 2.1 with $\alpha_N^- = \alpha$, if $\rho \in \left(0, \frac{1}{2}\right)$ or $\rho = \frac{1}{2}$, $\lim_{N\to\infty} \ell_N = \infty$, we have

$$\tau(N_{2\alpha_N^-}, 2N) = \left(1 + O\left(\frac{1}{\sqrt{N}}\right)\right) \Psi\left(N^{-\rho+\frac{1}{2}}\ell_N\right). \tag{4.1}$$

If $\rho \in \left(\frac{1}{2}, \infty\right)$, since $N^{-\rho+\frac{1}{2}}\ell_N \to 0$ as $N \to \infty$, we have the following estimation:

$$\begin{aligned}\Psi\left(N^{-\rho+\frac{1}{2}}\ell_N\right) &= \frac{1}{\sqrt{2\pi}}\int_0^{\infty} \mathrm{e}^{-\frac{u^2}{2}}\,\mathrm{d}u - \frac{1}{\sqrt{2\pi}}\int_0^{\alpha_c N^{-\rho+\frac{1}{2}}\ell_N} \mathrm{e}^{-\frac{u^2}{2}}\,\mathrm{d}u \\ &= \frac{1}{2} + O\left(N^{-\rho+\frac{1}{2}}\ell_N\right).\end{aligned} \tag{4.2}$$

Similarly, if $\rho = \frac{1}{2}$ and $\lim_{N\to\infty} \ell_N = \ell \in [0, \infty)$, we obtain

$$\Psi\left(N^{-\rho+\frac{1}{2}}\ell_N\right) = \frac{1}{\sqrt{2\pi}}\int_{\ell}^{\infty} \mathrm{e}^{-\frac{u^2}{2}}\,\mathrm{d}u - \frac{1}{\sqrt{2\pi}}\int_{\ell_N}^{\ell} \mathrm{e}^{-\frac{u^2}{2}}\,\mathrm{d}u = \Psi(\ell) + O(\ell - \ell_N). \tag{4.3}$$

By (4.1)–(4.3), this completes the proof of Theorem 2.2 for $\alpha = \alpha_N^-$.

Using the same argument, it is easy to obtain the result for $\alpha = \alpha_N^+$. We omit the proof for $\alpha = \alpha_N^+$ here. □

4.2. Proof of Theorem 2.4

Let

$$\overline{\alpha}_N = \alpha_c\left(1 + \frac{\log N}{\kappa\sqrt{N}}\right), \qquad \underline{\alpha}_N = \alpha_c\left(1 - \frac{\log N}{\kappa\sqrt{N}}\right).$$

Applying Fubini's theorem, we obtain

$$\begin{aligned}\chi(N_{2\alpha}, 2N) &= \mathbb{E}\left(\sum_{(m,n)\in N_{2\alpha}\times 2N} \mathbb{I}_{\{(m,n)\in\mathcal{C}(N_{2\alpha},2N)\}}\right) \\ &= \sum_{(m,n)\in N_{2\alpha}\times 2N} \mathbb{E}(\mathbb{I}_{\{(m,n)\in\mathcal{C}(N_{2\alpha},2N)\}}) \\ &= \sum_{(m,n)\in N_{2\alpha}\times 2N} \tau(m,n),\end{aligned}$$

where $\mathbb{I}_{\{\cdot\}}$ is an indicator function. Then we decompose $\chi(N_{2\alpha}, 2N)$ into three parts, as follows:

$$\chi(N_{2\alpha}, 2N) = I_1 + I_2 + I_3, \tag{4.4}$$

where

$$I_1 = \sum_{m=0}^{N_{2\alpha}} \sum_{n=0}^{2N} \tau(m,n) \mathbb{I}_{\{m/n \geq \overline{\alpha}_N\}},$$

$$I_2 = \sum_{m=0}^{N_{2\alpha}} \sum_{n=0}^{2N} \tau(m,n) \mathbb{I}_{\{m/n \leq \underline{\alpha}_N\}},$$

$$I_3 = \sum_{m=0}^{N_{2\alpha}} \sum_{n=0}^{2N} \tau(m,n) \mathbb{I}_{\{\underline{\alpha}_N \leq m/n \leq \overline{\alpha}_N\}}.$$

If $\alpha \geq \alpha_c$, applying Theorem 2.2 with $\rho = 1/2$, $\ell_N = \log N$, we have $\tau(m,n) = 1 + O(1/\sqrt{N})$ for all $m/n \geq \overline{\alpha}_N$. It follows that

$$\begin{aligned} I_1 &= \left(1 + O\left(\frac{1}{N}\right)\right) \sum_{m=0}^{N_{2\alpha}} \sum_{n=0}^{2N} \mathbb{I}_{\{m/n \geq \overline{\alpha}_N\}} \\ &= \left(1 + O\left(\frac{1}{N}\right)\right) \left(\frac{1}{2}\left(2\overline{\alpha}_N (N+1)^2\right) + 2(2\alpha - \overline{\alpha}_N) N^2\right) \\ &= 2(2\alpha - \alpha_c)(N+1)^2 \left(1 + O\left(\frac{\log N}{\sqrt{N}}\right)\right). \end{aligned}$$

Similarly, applying Theorem 2.2 with $\rho = 1/2$, $\ell_N = \log N$, we have $\tau(m,n) = O(1/\sqrt{N})$ for all $m/n \leq \underline{\alpha}_N$ and $\tau(m,n) \leq 1$ for all $\underline{\alpha}_N \leq m/n \leq \overline{\alpha}_N$. We obtain

$$I_2,\ I_3 = O\left(\frac{N^2 \log N}{\sqrt{N}}\right).$$

Therefore, from (4.4),

$$\chi(N_{2\alpha}, 2N) = 2(2\alpha - \alpha_c)(N+1)^2 \left(1 + O\left(\frac{\log N}{\sqrt{N}}\right)\right). \tag{4.5}$$

If $\alpha < \alpha_c$, in the same way,

$$\chi(N_{2\alpha}, 2N) = \frac{2\alpha^2}{\alpha_c}(N+1)^2 \left(1 + O\left(\frac{\log N}{\sqrt{N}}\right)\right).$$

This completes the proof of Theorem 2.4, as required. □

Acknowledgements

I am grateful to F.Y. Wu and Yu Zhang for having drawn my attention to the subject of this paper. I would like to thank Akira Sakai and Shu-Chiuan Chang for useful discussions, and the anonymous referees for valuable comments and for positive suggestions. This work was partially supported by TJ&MY Foundation and an NSC grant.

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