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The ice model and the eight-vertex model on the two-dimensional Sierpinski gasket

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ABSTRACT

We present the numbers of ice model configurations (with Boltzmann factors equal to one) $I(n)$ on the two-dimensional Sierpinski gasket $SG(n)$ at stage n. The upper and lower bounds for the entropy per site, defined as $\lim_{v\to\infty} \ln I(n)/v$, where v is the number of vertices on $SG(n)$, are derived in terms of the results at a certain stage. As the difference between these bounds converges quickly to zero as the calculated stage increases, the numerical value of the entropy can be evaluated with more than a hundred significant figures accuracy. The corresponding result of the ice model on the generalized two-dimensional Sierpinski gasket $SG_b(n)$ with $b = 3$ is also obtained, and the general upper and lower bounds for the entropy per site for arbitrary b are conjectured. We also consider the number of eight-vertex model configurations on $SG(n)$ and the number of generalized vertex models $E_b(n)$ on $SG_b(n)$, and obtain exactly $E_b(n) = 2^{\{2(b+1)[b(b+1)/2]^n+b+4\}/(b+2)}$. It follows that the entropy per site is $\lim_{v\to\infty} \ln E_b(n)/v = \frac{2(b+1)}{b+4} \ln 2$.

1. Introduction

The ice model was introduced by Pauling to study the residual entropy of water ice [1], and was solved exactly by Lieb on the square lattice [2,3]. The eight-vertex model is a generalization of the ice-type (six-vertex) models [4,5] and was solved by Baxter for the zero-field case [6,7]. On the triangular lattice, there are 20 vertex configurations for the ice rule [8] and the generalized 32-vertex model was considered in Ref. [9]. There is a correspondence between such a model and the Ising model, while there are other related models (see, for example, [10]). It is of interest to consider the ice model, the eight-vertex model and its generalization on self-similar fractal lattices that have scaling invariance rather than translational invariance. Fractals are geometric structures of non-integer Hausdorff dimension realized by repeated construction of an elementary shape on progressively smaller length scales [11,12]. A well-known example of a fractal is the Sierpinski gasket which has been extensively studied in several contexts, e.g., phase transitions [13,14], percolation [15,16], Schrödinger equation [17,18], random walks [19], self-avoiding random walks [20], loop-erased random walks [21], Potts model [22–24], colorings [25,26], sandpiles [27], branched polymers [28] spanning trees [29,30], spanning forests [31], connected spanning subgraphs [32], dimer coverings [33], dimer–monomer model [34], Hamiltonian walks [35], acyclic orientations [36], etc. We shall derive the recursion relations for the number of ice models with Boltzmann factors equal to one on the two-dimensional Sierpinski gasket, and determine the entropies. We shall also consider the number of ice model configurations on a generalized two-dimensional Sierpinski gasket. The number of eight-vertex model configurations and its generalization on the generalized two-dimensional Sierpinski gasket will be derived exactly.

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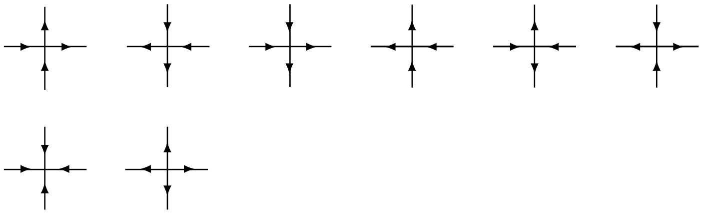

Fig. 1. (a) The six different kinds of configurations for the ice rule and (b) the other two configurations for the eight-vertex model on a degree-4 vertex.

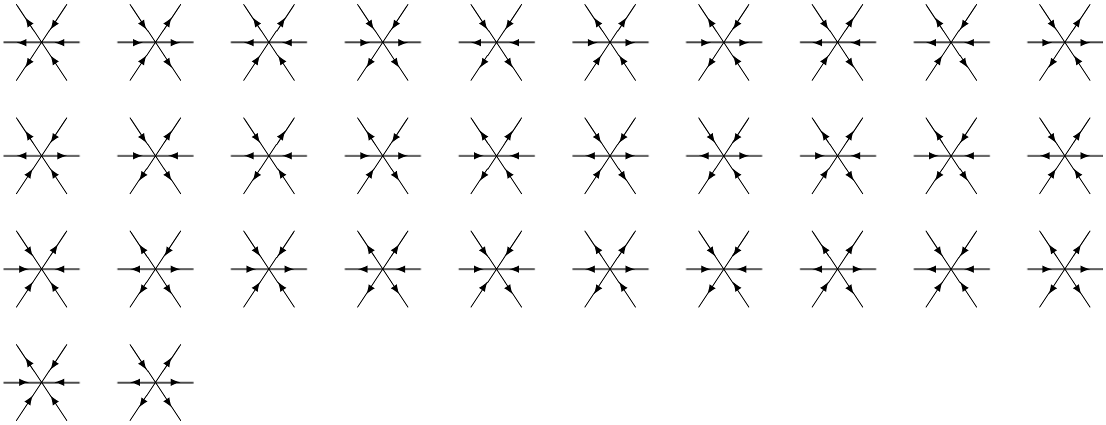

Fig. 2. (a) The 20 different kinds of configurations for the ice rule and (b) the other 12 configurations for the 32-vertex model on a degree-6 vertex.

2. Preliminaries

We first recall some relevant definitions in this section. A connected graph (without loops) $G = (V, E)$ is defined by its vertex (site) and edge (bond) sets V and E [37,38]. Let $v(G) = |V|$ be the number of vertices and $e(G) = |E|$ be the number of edges in G. The degree or coordination number k_i of a vertex $v_i \in V$ is the number of edges attached to it. A k-regular graph is a graph with the property that each of its vertices has the same degree k. Let us consider a 4-regular graph first, and assign an orientation on each edge. For the ice model, the ice rule must be satisfied at every vertex. Namely, the number of arrows pointing inward at each vertex must be two and the number of arrows pointing outward is also two. There are six possible different configurations of arrows at each vertex, as shown in Fig. 1(a). For the eight-vertex model, the number of arrows pointing inward at each vertex must be an even number. There are now eight possible different configurations of arrows at each vertex, including the six configurations of the ice model plus sink (all four arrows pointing inward) and source (all four arrows pointing outward), as shown in Fig. 1(b). For a vertex with degree six, the ice rule requires that the number of arrows pointing inward and the number of arrows pointing outward are the same and equal to three. There are 20 different arrow configurations at such vertex, as shown in Fig. 2(a). For the 32-vertex model, the number of arrows pointing inward at a degree-6 vertex must be an odd number. The 32 different arrow configurations include the previous 20 configurations plus only one arrow (six possibilities) pointing outward and only one arrow (six possibilities) pointing inward, as shown in Fig. 2(b). In general, one can associate an energy to the vertex for each configuration. All such Boltzmann weights are set to one throughout this paper.

Let us denote the total number of ice model configurations on a graph G as $I(G)$ and that of eight-vertex model configurations as $E(G)$. The entropy per site for the ice model is given by

$$S_{I,G} = \lim_{v(G)\to\infty} \frac{\ln I(G)}{v(G)}, \tag{2.1}$$

where G, when used as a subscript in this manner, implicitly refers to the thermodynamic limit. Similarly, the corresponding entropy per site for the eight-vertex model is denoted as $S_{E,G}$. We will see that the limits $S_{I,G}$ and $S_{E,G}$ exist for the Sierpinski gasket considered in this paper.

The construction of the two-dimensional Sierpinski gasket $SG(n)$ at stage n is shown in Fig. 3. At stage $n = 0$, it is an equilateral triangle, while stage $n + 1$ is obtained by the juxtaposition of three n-stage structures. The two-dimensional Sierpinski gasket has fractal dimensionality $D = \ln 3/\ln 2$ [15], and the numbers of edges and vertices are given by

$$e(SG(n)) = 3^{n+1}, \tag{2.2}$$

$$v(SG(n)) = \frac{3}{2}[3^n + 1]. \tag{2.3}$$

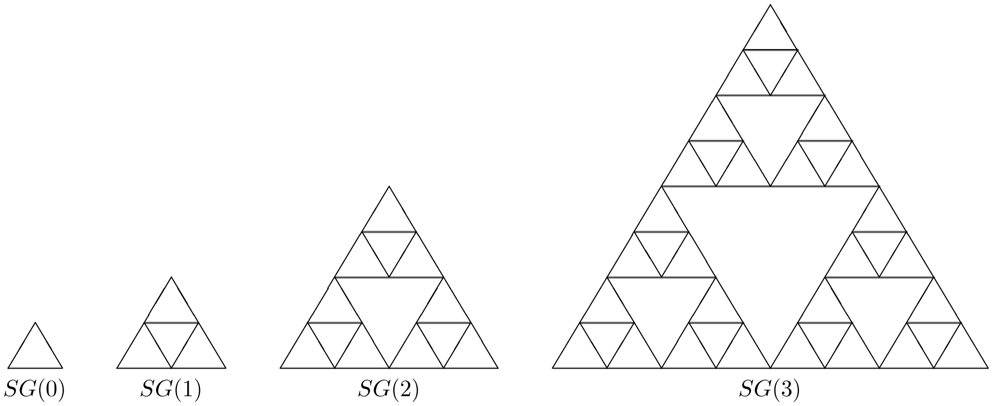

Fig. 3. The first four stages $n = 0, 1, 2, 3$ of the two-dimensional Sierpinski gasket $SG(n)$.

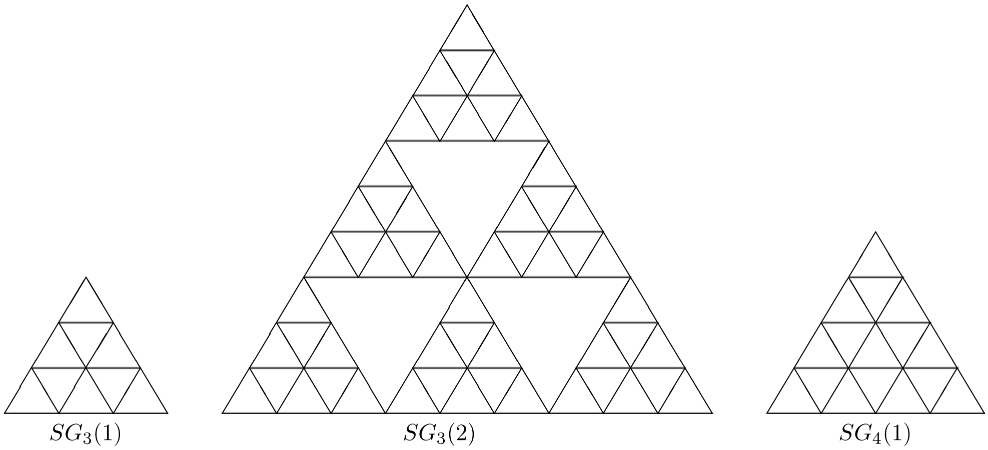

Fig. 4. The generalized two-dimensional Sierpinski gasket $SG_b(n)$ with $b = 3$ at stage $n = 1, 2$ and $b = 4$ at stage $n = 1$.

Except for the three outmost vertices which have degree 2, all other vertices of $SG(n)$ have degree 4. In the large n limit, SG is 4-regular. The three outmost vertices of the Sierpinski gasket are exempt from the ice rule in the calculation of $I(n)$ and from the eight-vertex model restriction in the calculation of $E(n)$. Namely, for the two edges connected to each of these outmost vertices, each of them can be directed either inward or outward independently.

The two-dimensional Sierpinski gasket can be generalized, denoted as $SG_b(n)$, by introducing the side length b which is an integer larger than or equal to two [39]. The generalized two-dimensional Sierpinski gasket at stage $n+1$ is constructed with b layers of stage n structures. The two-dimensional $SG_b(n)$ with $b = 3$ at stage $n = 1, 2$ and $b = 4$ at stage $n = 1$ is illustrated in Fig. 4. The ordinary two-dimensional Sierpinski gasket $SG(n)$ corresponds to the $b = 2$ case, where the index b is neglected for simplicity. The Hausdorff dimension for SG_b is given by $D = \ln\binom{b+1}{2}/\ln b$ [39]. Notice that SG_b is not k-regular even in the thermodynamic limit. Namely, there is always a fraction of vertices having degree six, while the others have degree four. We shall use the notations $I_b(n)$ and $E_b(n)$ for the numbers of ice model and generalized vertex model configurations on $SG_b(n)$.

3. The number of ice model configurations on $SG(n)$

We simply denote the number of ice model configurations on the two-dimensional Sierpinski gasket $SG(n)$ as $I(n)$. In this section we derive its entropy per site in detail. Let us define the quantities to be used.

Definition 3.1. Consider the generalized two-dimensional Sierpinski gasket $SG_b(n)$ at stage n.

- Define $g_b(n)$ as the number of ice model configurations such that one certain edge connected to an outmost vertex, say the topmost vertex in Fig. 5, is directed inward and the other five edges connected to the outmost vertices are directed outward.
- Define $pa_b(n)$ as the number of ice model configurations such that the two edges of a certain outmost vertex, say the left one in Fig. 5, are directed outward; the two edges of another certain outmost vertex, say the right one in Fig. 5, are

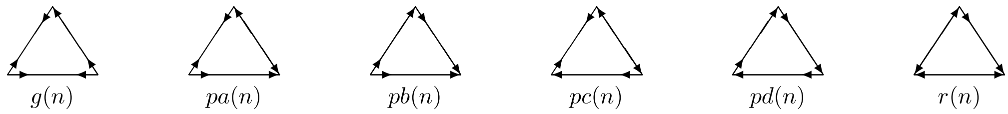

Fig. 5. Illustration of the configurations $g(n)$, $pa(n)$, $pb(n)$, $pc(n)$, $pd(n)$, and $r(n)$. Only the three outmost vertices and the directions of the edges connected to them are shown explicitly.

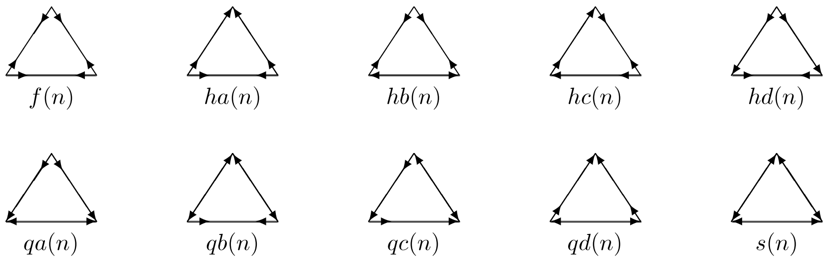

Fig. 6. Illustration of other possible edge directions which are not used here. Out of the six edges connected to the three outmost vertices, $f(n)$ has all of them directed outward; $ha(n)$, $hb(n)$, $hc(n)$, $hd(n)$ have four of them directed outward; $qa(n)$, $qb(n)$, $qc(n)$, $qd(n)$ have four of them directed inward; $s(n)$ has all of them directed inward.

directed inward; only one of the edges of the third outmost vertex is directed inward and it is on the same side of the Sierpinski gasket as one of the edges of the second outmost vertex.

- Define $pb_b(n)$ as the number of ice model configurations such that the two edges of a certain outmost vertex, say the left one in Fig. 5, are directed outward; the two edges of another certain outmost vertex, say the right one in Fig. 5, are directed inward; only one of the edges of the third outmost vertex is directed outward and it is on the same side of the Sierpinski gasket as one of the edges of the second outmost vertex.
- Define $pc_b(n)$ as the number of ice model configurations such that all three outmost vertices have one edge directed inward and one edge directed outward, while two certain directed-inward edges are on the same side of the Sierpinski gasket, say the upper-right side in Fig. 5.
- Define $pd_b(n)$ as the number of ice model configurations such that all three outmost vertices have one edge directed inward and one edge directed outward, while all three directed-inward edges are on different sides of the Sierpinski gasket with a certain direction, say clockwise in Fig. 5.
- Define $r_b(n)$ as the number of ice model configurations such that one certain edge connected to an outmost vertex, say the topmost vertex in Fig. 5, is directed outward and the other five edges connected to the outmost vertices are directed inward.

Since we only consider ordinary Sierpinski gaskets in this section, we shall use the notations $g(n)$, $pa(n)$, $pb(n)$, $pc(n)$, $pd(n)$, and $r(n)$ for simplicity. They are illustrated in Fig. 5, where only the outmost vertices and the directions of the edges connected to them are shown. In principle, the edges of the three outmost vertices have other possible directions, as illustrated in Fig. 6, but they do not appear in our consideration here, as discussed below. Because of rotational and reflection symmetries, $g(n)$, $pa(n)$, $pb(n)$, $pc(n)$ and $r(n)$ have multiplicity six, while $pd(n)$ have multiplicity two. It is clear that the initial values at stage zero are $g(0) = pa(0) = pc(0) = r(0) = 0$ and $pb(0) = pd(0) = 1$. For the purpose of obtaining the asymptotic behavior of $I(n)$ in this section, $g(n)$ and $r(n)$ are not needed, such that

$$I(n) = 6pa(n) + 6pb(n) + 6pc(n) + 2pd(n) \tag{3.1}$$

for non-negative integer n. The reason for missing $g(n)$ and $r(n)$ is due to their zero value at stage zero. For example, $g(n+1)$ may contain a term like $g(n)pb^2(n)$ in its recursion relations, but all such terms are equal to zero since $g(0) = 0$. Now the four quantities $pa(n)$, $pb(n)$, $pc(n)$ and $pd(n)$ satisfy recursion relations.

Lemma 3.1. *For any non-negative integer n,*

$$pa(n+1) = [pa(n) + pb(n)]^2[pa(n) + 3pc(n) + pd(n)], \tag{3.2}$$

$$pb(n+1) = [pa(n) + pb(n)]^2[pb(n) + 3pc(n) + pd(n)], \tag{3.3}$$

$$pc(n+1) = pa^2(n)pb(n) + pb^2(n)pa(n) + [3pc(n) + pd(n)]^3, \tag{3.4}$$

$$pd(n+1) = pa^3(n) + pb^3(n) + [3pc(n) + pd(n)]^3. \tag{3.5}$$

Proof. The Sierpinski gasket $SG(n+1)$ is composed of three $SG(n)$ with three pairs of vertices identified. At these identified vertices, the ice rule should be satisfied. As illustrated in Fig. 7, the number $pa(n+1)$ consists of two cases. For the first case, the top $SG(n)$ is always counted by $pa(n)$. The identified vertex of the lower two $SG(n)$'s has four possible configurations,

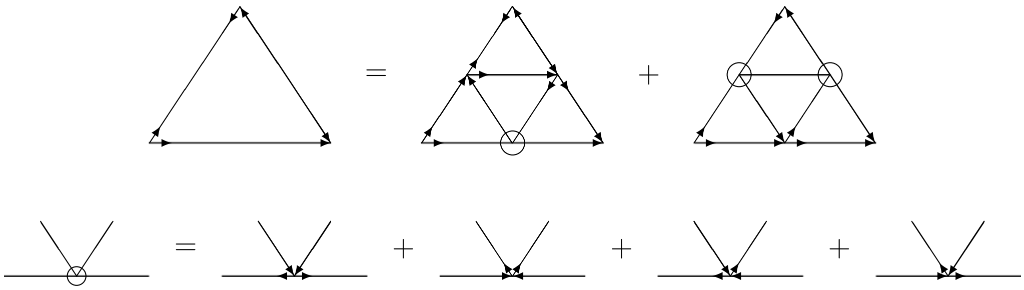

Fig. 7. Illustration of the expression for $pa(n+1)$. The representation of a big circle at an identified vertex corresponds to four possible configurations such that for each $SG(n)$ one edge is directed inward and the other edge is outward.

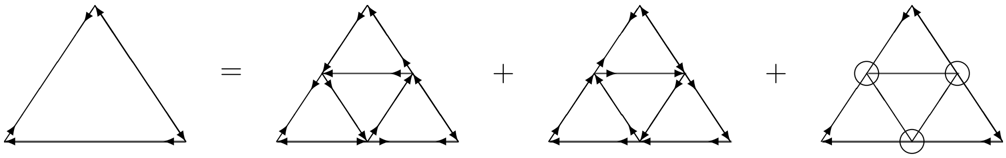

Fig. 8. Illustration of the expression for $pc(n+1)$.

Table 1
The first few values of $pa(n)$, $pb(n)$, $pc(n)$, $pd(n)$ and $I(n)$.

n	0	1	2	3	4
$pa(n)$	0	1	54	7,953,309	152,890,249,552,106,555,312,694
$pb(n)$	1	2	63	8,076,510	152,921,906,677,033,336,640,655
$pc(n)$	0	1	131	146,761,217	202,319,214,683,073,568,675,255,835
$pd(n)$	1	2	134	146,770,694	202,319,214,926,381,958,377,247,254
$I(n)$	8	28	1756	1,270,287,604	1,620,388,590,888,580,168,157,749,612

such that for each of these $SG(n)$'s one edge is directed inward and the other edge is outward, as represented by a big circle in Fig. 7. They are counted by $pa(n)[pa(n)+pb(n)]^2$. One may wonder why the other two possible configurations satisfying the ice rule are missing here. If the two edges of the left $SG(n)$ are directed outward and the two edges of the right $SG(n)$ are directed inward, then the left $SG(n)$ is counted by $ha(n)$ and the right $SG(n)$ is counted by $qa(n)$. However, such terms have no contributions because $ha(n)$, $qa(n)$ are always zero for any n for the same reason as discussed earlier. For the second case, the directions of the edges of the bottom identified vertex are fixed, as shown in Fig. 7. There are four possible configurations for the upper-left identified vertex multiplying four possible configurations for the upper-right identified vertex, and they are counted by $[pa(n)+pb(n)]^2[3pc(n)+pd(n)]$. Eq. (3.2) is verified by combining the results of these two cases.

The number $pb(n+1)$ is almost the same as $pa(n+1)$ except that the two edges of the topmost vertex change directions. It follows that the top $SG(n)$ is always counted by $pb(n)$ for the first case, while the results for the second case remain the same, so that Eq. (3.3) is verified.

As illustrated in Fig. 8, the number $pc(n+1)$ consists of three cases. For the first case, the directions of the edges of the three identified vertices are fixed and the number is counted by $pa^2(n)pb(n)$. Reversing the directions of the edges of the three identified vertices in the first case gives the second case, which is counted by $pb^2(n)pa(n)$. Finally, for the third case, each of the three identified vertex has four possible configurations such that for each related $SG(n)$ one edge is directed inward and the other is outward. This number is counted by $[3pc(n)+pd(n)]^3$, and Eq. (3.4) is verified.

The number $pd(n+1)$ is almost the same as $pc(n+1)$ except that the two edges of the topmost vertex change directions. It follows that the first case is counted by $pa^3(n)$, the second case is counted by $pb^3(n)$, while the results for the third case remain the same, so that Eq. (3.5) is verified. $\square$

The values of $pa(n)$, $pb(n)$, $pc(n)$, $pd(n)$ and $I(n)$ for small n can be evaluated recursively by Eqs. (3.2)–(3.5), as listed in Table 1. These numbers grow exponentially, and do not have simple integer factorizations. To estimate the value of the entropy defined in Eq. (2.1), we need the following lemmas.

Let us use the notations $\alpha(n) = pa(n)/pb(n)$, $\beta(n) = pd(n)/pc(n)$ and $\gamma(n) = pb(n)/pc(n)$ for $n \geq 1$.

Lemma 3.2. *$\alpha(n) \in (0,1)$ and $\beta(n) > 1$ for any positive integer n. Furthermore,*

$$0 < 1-\alpha(n+1) < \frac{\gamma(n)}{4}\big[1-\alpha(n)\big], \tag{3.6}$$

$$\frac{\alpha(n)\gamma(n)^3[1-\alpha(n)]^2}{32\beta(n)^3+\gamma(n)^3} < \beta(n+1)-1 < \frac{\gamma(n)^3[1-\alpha(n)]^2}{32}, \tag{3.7}$$

$$\frac{4\gamma(n)^2\big[4+\gamma(n)\big]}{\gamma(n)^3[1+\alpha(n)^{-1}]+64\alpha(n)^{-2}\beta(n)} < \gamma(n+1) < \frac{\gamma(n)^2\big[3+\gamma(n)+\beta(n)\big]}{16} \tag{3.8}$$

for $n \geq 1$, such that the sequence $\alpha(n)$ increases to one, $\beta(n)$ decreases to one, and $\gamma(n)$ decreases to zero as n increases.

Proof. By Eqs. (3.2) and (3.3), we have

$$pb(n+1) - pa(n+1) = [pb(n) - pa(n)][pa(n) + pb(n)]^2, \tag{3.9}$$

$pa(n) < pb(n)$ is established by mathematical induction hypothesis, $\alpha(n) \in (0, 1)$ and the initial value $\alpha(1) = 1/2$. By Eqs. (3.4) and (3.5), we have

$$pd(n+1) - pc(n+1) = [pa(n) + pb(n)][pb(n) - pa(n)]^2 > 0, \tag{3.10}$$

so that $\beta(n) > 1$ for all $n \geq 1$.

Next, using the fact that $pa(n) < pb(n)$ and $pc(n) < pd(n)$, we have

$$0 < 1 - \alpha(n+1) = 1 - \frac{pa(n) + 3pc(n) + pd(n)}{pb(n) + 3pc(n) + pd(n)} < \frac{pb(n) - pa(n)}{4pc(n)} = \frac{\gamma(n)}{4}\big[1 - \alpha(n)\big], \tag{3.11}$$

and

$$\beta(n+1) - 1 = \frac{[pa(n) + pb(n)][pb(n) - pa(n)]^2}{pa(n)pb(n)[pa(n) + pb(n)] + [3pc(n) + pd(n)]^3}, \tag{3.12}$$

such that

$$\begin{aligned} \frac{\alpha(n)\gamma(n)^3[1-\alpha(n)]^2}{32\beta(n)^3 + \gamma(n)^3} &= \frac{2pa(n)[pb(n) - pa(n)]^2}{2pb(n)^3 + [4pd(n)]^3} < \beta(n+1) - 1 \\ &< \frac{2pb(n)[pb(n) - pa(n)]^2}{[4pc(n)]^3} = \frac{\gamma(n)^3[1-\alpha(n)]^2}{32}. \end{aligned} \tag{3.13}$$

Finally, by Eqs. (3.3) and (3.4), we have

$$\gamma(n+1) = \gamma(n)^2 \times \frac{[\alpha(n) + 1]^2[\gamma(n) + 3 + \beta(n)]}{\gamma(n)^3\alpha(n)^2[1 + \alpha(n)^{-1}] + [3 + \beta(n)]^3}, \tag{3.14}$$

so that

$$\frac{4\alpha(n)^2\gamma(n)^2\big[4+\gamma(n)\big]}{\gamma(n)^3\alpha(n)^2[1+\alpha(n)^{-1}]+64\beta(n)^3} < \gamma(n+1) < \frac{\gamma(n)^2\big[3+\gamma(n)+\beta(n)\big]}{16}. \tag{3.15}$$

By Eqs. (3.11), (3.13) and (3.15), we have

$$\begin{aligned} &\frac{1-\alpha(n+1)}{1-\alpha(n)} < \frac{\gamma(n)}{4}, \qquad \frac{\gamma(n+1)}{\gamma(n)} < \frac{\gamma(n)\big[3+\gamma(n)+\beta(n)\big]}{16}, \\ &\frac{\beta(n+2)-1}{\beta(n+1)-1} < \frac{\gamma(n+1)^3[1-\alpha(n+1)]^2}{32[\beta(n+1)-1]} < \frac{\gamma(n)^5[3+\gamma(n)+\beta(n)]^3[32\beta(n)^3+\gamma(n)^3]}{2^{21}\alpha(n)}. \end{aligned} \tag{3.16}$$

With $\alpha(1) = 1/2$, $\beta(1) = 2$, $\gamma(1) = 2$, and $\alpha(2) = 6/7$, $\beta(2) = 134/131$, $\gamma(2) = 63/131$, it is easy to see that $\alpha(n)$ increases to one, $\beta(n)$ decreases to one and $\gamma(n)$ decreases to zero as n increases. $\square$

In passing, we notice that either $\gamma(n)/[1 - \alpha(n)]$ or $[\beta(n) - 1]/\gamma(n)^{5/2}$ approaches to a finite and positive value in the infinite n limit.

Lemma 3.3. *The entropy for the number of ice model configurations on SG(n) is bounded:*

$$\frac{2}{3^{m+1}} \ln pc(m) + \frac{2}{3^m} \ln 2 \leq S_{I,SG} \leq \frac{2}{3^{m+1}} \ln pd(m) + \frac{2}{3^m} \ln 2, \tag{3.17}$$

where m is a positive integer.

Proof. By Eq. (3.4) and Lemma 3.2, we have

$$pc(n) > 64pc^3(n-1) > 64[64pc^3(n-2)]^3 > \cdots > pc(m)^{3^{n-m}} \times 64^{(3^{n-m}-1)/2} \tag{3.18}$$

for any $m < n$. Similarly by Eq. (3.5) and Lemma 3.2, we obtain

$$pd(n) < 64pd^3(n-1) < 64[64pd^3(n-2)]^3 < \cdots < pd(m)^{3^{n-m}} \times 64^{(3^{n-m}-1)/2}, \tag{3.19}$$

so that

$$3^{n-m} \ln pc(m) + 3(3^{n-m}-1)\ln 2 < \ln pc(n) < \ln pd(n) < 3^{n-m} \ln pd(m) + 3(3^{n-m}-1)\ln 2. \tag{3.20}$$

By Eqs. (2.3) and (3.1), we have

$$\frac{\ln I(n)}{v(SG(n))} = \frac{2}{3(3^n+1)} \ln \left[2 + 6\frac{pc(n)}{pd(n)} + 6\frac{pb(n)}{pd(n)} + 6\frac{pa(n)}{pd(n)}\right] + \frac{2\ln pd(n)}{3(3^n+1)}. \tag{3.21}$$

By the definition of the entropy in Eq. (2.1) and Lemma 3.2,

$$\begin{aligned} S_{I,SG} &= \lim_{n\to\infty} \frac{\ln I(n)}{v(SG(n))} \\ &= \lim_{n\to\infty} \frac{2}{3(3^n+1)} \ln \left[2 + 6\frac{pc(n)}{pd(n)} + 6\frac{pb(n)}{pd(n)} + 6\frac{pa(n)}{pd(n)}\right] + \lim_{n\to\infty} \frac{2\ln pd(n)}{3(3^n+1)} \\ &= \lim_{n\to\infty} \frac{2\ln pd(n)}{3(3^n+1)}. \end{aligned} \tag{3.22}$$

The proof is completed using the inequality (3.20). □

The difference between the upper and lower bounds for $S_{I,SG}$ quickly converges to zero as m increases, and we have the following proposition.

Proposition 3.1. *The entropy per site for the number of ice model configurations on the two-dimensional Sierpinski gasket $SG(n)$ in the large n limit is $S_{I,SG} = 0.515648810655\ldots$.*

The numerical value of $S_{I,SG}$ can be calculated with more than a hundred significant figures accuracy when m in Eq. (3.17) is equal to eight. It is too lengthy to be included here and is available from the authors on request. As the square lattice (sq) also has degree 4, it is interesting to compare our result with Lieb's result on the square lattice [2,3], $S_{I,sq} = \frac{3}{2}\ln\frac{4}{3} = 0.431523108677\ldots$. We see that the entropy per site on the two-dimensional Sierpinski gasket is significantly larger.

The upper bound given in Lemma 3.3 can be improved further as follows.

Lemma 3.4. *For any integers $n > m \geq 1$,*

$$64^{(3^{n-m}-1)/2} pc(m)^{3^{n-m}} < pc(n) < 64^{(3^{n-m}-1)/2} pc(m)^{3^{n-m}} \left[\frac{\gamma(m)^3}{32} + \left(1 + \frac{\beta(m)-1}{4}\right)^3\right]^{(3^{n-m}-1)/2}. \tag{3.23}$$

Proof. By Eq. (3.4), we know

$$pc(n+1) = pc(n)^3 \left[\alpha(n)^2\gamma(n)^3 + \alpha(n)\gamma(n)^3 + \left(3+\beta(n)\right)^3\right] \tag{3.24}$$

for any $n \geq 1$. By Lemma 3.2, we have for any positive integers $n > m \geq 1$

$$64pc(n-1)^3 < pc(n) < 64pc(n-1)^3 \left[\frac{\gamma(m)^3}{32} + \left(1 + \frac{\beta(m)-1}{4}\right)^3\right]. \tag{3.25}$$

Using the formula $a_n = c^{(3^{n-m}-1)/2} a_m^{3^{n-m}}$ if $a_{j+1} = ca_j^3$ for $j = m, m+1, \ldots, n-1$, where c is a constant, then Eq. (3.23) is established. □

Lemma 3.5. *The logarithm of the number of ice model configurations divided by the number of vertices on $SG(n)$ is bounded:*

$$\begin{aligned} &\frac{2\ln 2}{3^n+1} + \frac{\left(3^{-m}-3^{-n}\right)\ln 4}{1+3^{-n}} + \frac{2}{3} \times \frac{3^{-m}}{1+3^{-n}} \ln pc(m) < \frac{\ln I(n)}{v(SG(n))} < \frac{2}{3^n+1}\left\{\ln 2 + \frac{1}{3}\ln\left[\beta(m) + \frac{3}{2}\gamma(m)\right]\right\} \\ &\quad + \frac{3^{-m}-3^{-n}}{1+3^{-n}}\left\{\ln 4 + \ln\left[1 + \frac{\beta(m)-1}{4}\right] + \frac{1}{3}\ln\left[1 + \frac{\gamma(m)^3}{32}\right]\right\} + \frac{2}{3} \times \frac{3^{-m}}{1+3^{-n}} \ln pc(m) \end{aligned} \tag{3.26}$$

for positive integers $n > m \geq 1$.

Proof. The number of ice model configurations can be rewritten as

$$I(n) = 6\big[pa(n) + pb(n) + pc(n)\big] + 2pd(n) = 2pc(n)\Big\{3\big[\alpha(n) + 1\big]\gamma(n) + 3 + \beta(n)\Big\}, \tag{3.27}$$

then by Lemma 3.2,

$$8pc(n)\left[1 + \frac{3}{2}\alpha(n)\gamma(n)\right] < I(n) < 8pc(n)\left[\beta(n) + \frac{3}{2}\gamma(n)\right]. \tag{3.28}$$

It follows that

$$\frac{2}{3(3^n+1)}\Big[3\ln 2 + \ln pc(n)\Big] < \frac{\ln I(n)}{v(SG(n))} < \frac{2}{3(3^n+1)}\left\{3\ln 2 + \ln pc(n) + \ln\left[\beta(m) + \frac{3}{2}\gamma(m)\right]\right\}. \tag{3.29}$$

By Lemma 3.4, the upper and lower bounds of $\ln pc(n)$ are given by

$$\begin{aligned}
&\frac{3^{n-m}-1}{2}(3\ln 4) + 3^{n-m}\ln pc(m) < \ln pc(n) \\
&< \frac{3^{n-m}-1}{2}\left\{3\ln 4 + \ln\left[\frac{\gamma(m)^3}{32} + \left(1 + \frac{\beta(m)-1}{4}\right)^3\right]\right\} + 3^{n-m}\ln pc(m) \\
&< \frac{3^{n-m}-1}{2}\left\{3\ln 4 + 3\ln\left[1 + \frac{\beta(m)-1}{4}\right] + \ln\left[1 + \frac{\gamma(m)^3}{32}\right]\right\} + 3^{n-m}\ln pc(m).
\end{aligned} \tag{3.30}$$

Combining Eqs. (3.29) and (3.30), the proof is completed. □

Taking the limit $n \to \infty$ in Eq. (3.26), the entropy is bounded as

$$0 < S_{I,SG} - 3^{-m}\left[\ln 4 + \frac{2}{3}\ln pc(m)\right] < 3^{-m}\left\{\ln\left[1 + \frac{\beta(m)-1}{4}\right] + \frac{1}{3}\ln\left[1 + \frac{\gamma(m)^3}{32}\right]\right\}. \tag{3.31}$$

Using the inequality $\ln(1+\epsilon) < \epsilon$ for any $\epsilon \in (0, 1)$, we obtain the improved upper bound

$$0 < S_{I,SG} - 3^{-m}\left[\ln 4 + \frac{2}{3}\ln pc(m)\right] < 3^{-m}\left[\frac{\beta(m)-1}{4} + \frac{\gamma(m)^3}{96}\right]. \tag{3.32}$$

According to Lemma 3.2, it is clear that the upper bound converges to zero quickly as m increases.

4. The number of ice model configurations on $SG_b(n)$ with $b = 3$

In this section, we consider the generalized two-dimensional Sierpinski gasket $SG_b(n)$ with the number of layers b equal to three. For $SG_3(n)$, the numbers of edges and vertices are given by

$$e(SG_3(n)) = 3 \times 6^n, \tag{4.1}$$

$$v(SG_3(n)) = \frac{7 \times 6^n + 8}{5}, \tag{4.2}$$

where the three outmost vertices have degree two. There are $(6^n - 1)/5$ vertices of $SG_3(n)$ with degree six and $6(6^n - 1)/5$ vertices with degree four. For each of the vertices with degree six, we allow three edges directed inward and the other three directed outward, just like the consideration in the 20-vertex triangular ice-rule problem [8].

By Definition 3.1, the number of ice model configurations is $I_3(n) = 6pa_3(n) + 6pb_3(n) + 6pc_3(n) + 2pd_3(n)$. The initial values are the same as for SG: $pa_3(0) = pc_3(0) = 0$ and $pb_3(0) = pd_3(0) = 1$. Again, $g_3(n)$ and $r_3(n)$ are zero for any nonnegative n. The recursion relations are lengthy and are given in Appendix A. Some values of $pa_3(n)$, $pb_3(n)$, $pc_3(n)$, $pd_3(n)$ and $I_3(n)$ are listed in Table 2. These numbers grow exponentially, and do not have simple integer factorizations.

By a similar argument to Lemma 3.3, the entropy for the number of ice model configurations on $SG_3(n)$ is bounded:

$$\frac{5}{7 \times 6^m}\ln pc_3(m) + \frac{15}{7 \times 6^m}\ln 2 \le S_{I,SG_3} \le \frac{5}{7 \times 6^m}\ln pd_3(m) + \frac{15}{7 \times 6^m}\ln 2, \tag{4.3}$$

where m is a positive integer. We have the following proposition.

Table 2
The first few values of $pa_3(n)$, $pb_3(n)$, $pc_3(n)$, $pd_3(n)$ and $I_3(n)$.

n	0	1	2	3
$pa_3(n)$	0	11	29,665,405,536	8,329,624,787,357,979,293,987,412,541,852,867,738,187,867,580,946,289,512,122,074,041,155,584
$pb_3(n)$	1	15	29,990,772,448	8,329,642,677,826,723,066,417,765,699,958,803,959,673,796,982,684,471,740,765,855,746,621,440
$pc_3(n)$	0	15	527,746,306,872	708,045,663,245,136,812,838,888,349,048,042,396,698,172,710,195,432,765,544,715,130,291,741,523,968
$pd_3(n)$	1	18	527,789,051,704	708,045,663,245,233,047,276,563,406,582,556,247,659,516,105,436,007,524,358,474,481,355,710,267,392
$I_3(n)$	8	282	4,579,993,012,544	5,664,465,261,566,078,079,800,619,338,522,817,745,538,255,642,031,993,426,552,757,072,040,596,340,736

Proposition 4.1. *The entropy per site for the number of ice model configurations on the generalized two-dimensional Sierpinski gasket $SG_3(n)$ in the large n limit is $S_{I,SG_3} = 0.576812423363\ldots$.*

The convergence of the upper and lower bounds remains quick. More than a hundred significant figures for S_{I,SG_3} can be obtained when m in Eq. (4.3) is equal to five. For the ice model on square and triangular lattices, recall that $S_{I,sq} = \frac{3}{2}\ln\frac{4}{3}$ and $S_{I,tri} = \ln\frac{3\sqrt{3}}{2}$ [8]. As the number of vertices with degree four is six times the number of vertices with degree six for $SG_3(n)$, it is interesting to compare our results with $\frac{6}{7}S_{I,sq} + \frac{1}{7}S_{I,tri} = 0.506272843501\ldots$. We see that the entropy per site on $SG_3(n)$ is again larger.

5. Discussion of ice model configurations

For the generalized two-dimensional Sierpinski gasket $SG_b(n)$, the numbers of edges and vertices are given by

$$e(SG_b(n)) = 3\left[\frac{b(b+1)}{2}\right]^n, \tag{5.1}$$

$$v(SG_b(n)) = \frac{b+4}{b+2}\left[\frac{b(b+1)}{2}\right]^n + \frac{2(b+1)}{b+2}. \tag{5.2}$$

The bounds of the entropies for the ice model on $SG(n)$ and $SG_3(n)$ given in Sections 3 and 4 lead to the following conjecture for general $SG_b(n)$.

Conjecture 5.1. *The entropy per site for the number of ice model configurations on the generalized two-dimensional Sierpinski gasket SG_b is bounded:*

$$\frac{b+2}{(b+4)\left[\frac{b(b+1)}{2}\right]^m}\left[\ln pc_b(m) + 3\ln 2\right] \le S_{I,SG_b} \le \frac{b+2}{(b+4)\left[\frac{b(b+1)}{2}\right]^m}\left[\ln pd_b(m) + 3\ln 2\right], \tag{5.3}$$

where m is a positive integer.

However, calculation of $pc_b(m)$ and $pd_b(m)$ for general b may be difficult. We notice that S_{I,SG_3} is a bit larger than $S_{I,SG}$. It is expected that the value of S_{I,SG_b} increases slightly as b increases for the generalized two-dimensional Sierpinski gasket.

6. The number of eight-vertex model or generalized vertex model configurations on $SG_b(n)$

In this section, we derive the number of generalized vertex model configurations, denoted as $E_b(n)$, on $SG_b(n)$. The eight-vertex model configurations on $SG(n)$ correspond to the $b = 2$ case. As there are vertices with degree six when $b > 2$, the eight-vertex model should be generalized. For the vertices with degree four, the number of arrows pointing inward is even just like the eight-vertex model on the square lattice, while for the vertices with degree six, the number of arrows pointing inward is odd just like the 32-vertex model on the triangular lattice. Although we could again derive recursion relations including the configurations $g(n)$ and $r(n)$ shown in Fig. 5, here we shall provide two better methods to obtain $E_b(n)$ exactly for arbitrary b.

The first method to calculate the number of generalized vertex model configurations is to use the well-known map due to F. Y. Wu [40] and Kadanoff and Wegner [41], i.e. to map the eight-vertex model to an Ising spin model on the faces [42]. Since all the weights are set to one, which corresponds to the infinite-temperature limit, the spins do not interact. One can set the adjacent spins to be parallel for the following three cases: the orientation of the edge is (i) to the right; (ii) from the lower-right to the upper-left; (iii) from the upper-right to the lower-left. On the other hand, the adjacent spins are antiparallel when the orientation of the edge is (i) to the left; (ii) from the lower-left to the upper-right; (iii) from the upper-left to the lower-right. Let us first consider the case where there is one arrow pointing inward and one arrow pointing outward at each of the three outmost vertices. We denote the number of interior faces as $F_b(n)$ for the generalized two-dimensional Sierpinski gasket $SG_b(n)$, then there are $2^{F_b(n)+1}$ spin states. As the map between the Ising spin model and the eight-vertex model is two to one (flipping all the spins simultaneously gives the same edge orientations), the number of eight-vertex models is $2^{F_b(n)}$. For the degree-6 vertices of $SG_b(n)$ with $b > 2$, the above statement is still held since a degree-6 vertices can be resolved as a triangle of three degree-4 vertices. Now, to relax the restriction at the three outmost vertices, one can flip the arrows on all edges of one exterior side of $SG_b(n)$ separately. That gives an extra factor of four, such that the number of eight-vertex models becomes $2^{F_b(n)+2}$. The problem reduces to the counting of $F_b(n)$. We denote by $u_b(n)$ the number of up-pointing triangles and by $d_b(n)$ the number of down-pointing ones surrounding an interior face. The initial values are $u_b(0) = 1$, $d_b(0) = 0$, $u_b(1) = b(b+1)/2$, $d_b(1) = b(b-1)/2$, so that

$$\begin{aligned} u_b(n) &= u_b(1)u_b(n-1) = [u_b(1)]^n, \\ \mathrm{d}_b(n) &= \mathrm{d}_b(1)u_b(n-1) + \mathrm{d}_b(n-1) = \frac{\mathrm{d}_b(1)\{[u_b(1)]^n - 1\}}{u_b(1)-1}, \\ F_b(n) + 2 &= u_b(n) + d_b(n) + 2 = \frac{2(b+1)[b(b+1)/2]^n + b + 4}{b+2}. \end{aligned} \tag{6.1}$$

With the number of vertices of $SG_b(n)$ given in Eq. (5.2), we have the following proposition.

Proposition 6.1. *The number of generalized vertex model configurations on the generalized two-dimensional Sierpinski gasket $SG_b(n)$ is $E_b(n) = 2^{\{2(b+1)[b(b+1)/2]^n+b+4\}/(b+2)}$ and the entropy per site in the large n limit is $S_{E,SG_b} = \frac{2(b+1)}{b+4}\ln 2$.*

A corollary is that in the infinite b limit, the entropy per site approaches $\lim_{b\to\infty} S_{E,SG_b} = 2\ln 2$.

For the ordinary two-dimensional Sierpinski gasket $SG(n)$, the number of eight-vertex model configurations is $E(n) = 2^{3(3^n+1)/2}$ and the entropy per site in the large n limit is $S_{E,SG} = \ln 2$. Compare again with the square lattice (sq). The special case of Baxter's result [7] with Boltzmann weights equal to one also gives $S_{E,sq} = \ln 2$. It is intriguing to observe that the entropies per site for the two-dimensional Sierpinski gasket and the square lattice are the same for the eight-vertex model, in contrast with the comparison for the ice model given in Section 3.

The second method to calculate the number of generalized vertex model configurations is to consider edge orientations directly. We denote such a number by $2^{P_b(n)}$, and let us first consider the ordinary two-dimensional Sierpinski gasket ($b = 2$). As there are no restrictions for the orientations of edges connected to the outmost vertices, it is clear that $P_2(0) = 3$. Recall that $SG(n+1)$ is constructed from three $SG(n)$ by identifying three pairs of vertices. For each identified vertex, the orientations of its four edges were arbitrary before the identification and now can only assume the eight configurations given in Fig. 1. Therefore, the number of eight-vertex model configurations should reduce by a factor two whenever a pair of outmost vertices is identified. It follows that $P_2(1) = 3\times 3 - 3 = 6$, $P_2(2) = 6\times 3 - 3 = 3^3 - 3^2 - 3 = 15$, etc. In general,

$$P_2(n) = 3^{n+1} - 3^n - 3^{n-1} - \cdots - 3 = 3\left[3^n - \frac{3^n-1}{3-1}\right] = \frac{3}{2}(3^n+1). \tag{6.2}$$

Similarly, for $b = 3$, $SG_3(n+1)$ is constructed from six $SG_3(n)$ by identifying six pairs of vertices and a set of three vertices. When three vertices are identified, the orientations of their six edges were arbitrary before the identification and can only assume the thirty-two configurations given in Fig. 2 afterwards. The number of generalized vertex model configurations again reduces by a factor two whenever three outmost vertices are identified. As the initial value $P_3(0) = 3$ is the same, we have

$$P_3(n) = 3\times 6^n - 7(6^{n-1} + \cdots + 1) = 3\times 6^n - 7\times\frac{6^n-1}{6-1} = \frac{1}{5}(8\times 6^n + 7). \tag{6.3}$$

For general b, $SG_b(n+1)$ is constructed from $b(b+1)/2SG_b(n)$ by identifying $2+3+\cdots+b+(b-1) = (b-1)(b+4)/2$ sets of vertices, such that

$$P_b(n) = 3\left[\frac{b(b+1)}{2}\right]^n - \frac{(b-1)(b+4)}{2}\times\frac{[b(b+1)/2]^n - 1}{b(b+1)/2-1} = \frac{2b+2}{b+2}\left[\frac{b(b+1)}{2}\right]^n + \frac{b+4}{b+2}, \tag{6.4}$$

which agrees with Proposition 6.1.

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Appendix A. Recursion relations for the ice model on $SG_3(n)$

We give here the recursion relations for the ice model on the generalized two-dimensional Sierpinski gasket $SG_3(n)$. Since the subscript is $b = 3$ for all the quantities throughout this section, we will use the simplified notation pa_{n+1} to denote $pa_3(n+1)$ and similar notations for other quantities. For any non-negative integer n, we have

$$\begin{aligned} pa_{n+1} = {} & 3pa_n^6 + pb_n^6 + 16pa_n^5pb_n + 6pa_n^5pc_n + 2pa_n^5pd_n + 8pb_n^5pa_n + 35pa_n^4pb_n^2 + 54pa_n^4pc_n^2 \\ & + 6pa_n^4pd_n^2 + 25pb_n^4pa_n^2 + 54pb_n^4pc_n^2 + 6pb_n^4pd_n^2 + 40pa_n^3pb_n^3 + 108pa_n^3pc_n^3 + 4pa_n^3pd_n^3 \\ & + 108pb_n^3pc_n^3 + 4pb_n^3pd_n^3 + 24pa_n^4pb_npc_n + 8pa_n^4pb_npd_n + 36pa_n^4pc_npd_n + 6pb_n^4pa_npc_n \\ & + 2pb_n^4pa_npd_n + 36pb_n^4pc_npd_n + 36pa_n^3pb_n^2pc_n + 12pa_n^3pb_n^2pd_n + 216pa_n^3pc_n^2pb_n \\ & + 108pa_n^3pc_n^2pd_n + 24pa_n^3pd_n^2pb_n + 36pa_n^3pd_n^2pc_n + 24pb_n^3pa_n^2pc_n + 8pb_n^3pa_n^2pd_n \\ & + 216pb_n^3pc_n^2pa_n + 108pb_n^3pc_n^2pd_n + 24pb_n^3pd_n^2pa_n + 36pb_n^3pd_n^2pc_n + 324pc_n^3pa_n^2pb_n \\ & + 324pc_n^3pb_n^2pa_n + 12pd_n^3pa_n^2pb_n + 12pd_n^3pb_n^2pa_n + 324pa_n^2pb_n^2pc_n^2 + 36pa_n^2pb_n^2pd_n^2 \\ & + 144pa_n^3pb_npc_npd_n + 144pb_n^3pa_npc_npd_n + 216pa_n^2pb_n^2pc_npd_n + 324pa_n^2pc_n^2pb_npd_n \\ & + 108pa_n^2pd_n^2pb_npc_n + 324pb_n^2pc_n^2pa_npd_n + 108pb_n^2pd_n^2pa_npc_n, \end{aligned} \tag{A.1}$$

$$\begin{aligned} pb_{n+1} = {} & pa_n^6 + 3pb_n^6 + 8pa_n^5pb_n + 16pb_n^5pa_n + 6pb_n^5pc_n + 2pb_n^5pd_n + 25pa_n^4pb_n^2 + 54pa_n^4pc_n^2 \\ & + 6pa_n^4pd_n^2 + 35pb_n^4pa_n^2 + 54pb_n^4pc_n^2 + 6pb_n^4pd_n^2 + 40pa_n^3pb_n^3 + 108pa_n^3pc_n^3 + 4pa_n^3pd_n^3 \\ & + 108pb_n^3pc_n^3 + 4pb_n^3pd_n^3 + 6pa_n^4pb_npc_n + 2pa_n^4pb_npd_n + 36pa_n^4pc_npd_n + 24pb_n^4pa_npc_n \\ & + 8pb_n^4pa_npd_n + 36pb_n^4pc_npd_n + 24pa_n^3pb_n^2pc_n + 8pa_n^3pb_n^2pd_n + 216pa_n^3pc_n^2pb_n \\ & + 108pa_n^3pc_n^2pd_n + 24pa_n^3pd_n^2pb_n + 36pa_n^3pd_n^2pc_n + 36pb_n^3pa_n^2pc_n + 12pb_n^3pa_n^2pd_n \\ & + 216pb_n^3pc_n^2pa_n + 108pb_n^3pc_n^2pd_n + 24pb_n^3pd_n^2pa_n + 36pb_n^3pd_n^2pc_n + 324pc_n^3pa_n^2pb_n \\ & + 324pc_n^3pb_n^2pa_n + 12pd_n^3pa_n^2pb_n + 12pd_n^3pb_n^2pa_n + 324pa_n^2pb_n^2pc_n^2 + 36pa_n^2pb_n^2pd_n^2 \\ & + 144pa_n^3pb_npc_npd_n + 144pb_n^3pa_npc_npd_n + 216pa_n^2pb_n^2pc_npd_n + 324pa_n^2pc_n^2pb_npd_n \\ & + 108pa_n^2pd_n^2pb_npc_n + 324pb_n^2pc_n^2pa_npd_n + 108pb_n^2pd_n^2pa_npc_n, \end{aligned} \tag{A.2}$$

$$\begin{aligned} pc_{n+1} = {} & 5832pc_n^6 + 8pd_n^6 + pa_n^5pb_n + 3pa_n^5pc_n + pa_n^5pd_n + pb_n^5pa_n + 3pb_n^5pc_n + pb_n^5pd_n \\ & + 11664pc_n^5pd_n + 144pd_n^5pc_n + 4pa_n^4pb_n^2 + 4pb_n^4pa_n^2 + 9720pc_n^4pd_n^2 + 1080pd_n^4pc_n^2 \\ & + 6pa_n^3pb_n^3 + 162pa_n^3pc_n^3 + 6pa_n^3pd_n^3 + 162pb_n^3pc_n^3 + 6pb_n^3pd_n^3 + 4320pc_n^3pd_n^3 \\ & + 21pa_n^4pb_npc_n + 7pa_n^4pb_npd_n + 21pb_n^4pa_npc_n + 7pb_n^4pa_npd_n + 48pa_n^3pb_n^2pc_n \\ & + 16pa_n^3pb_n^2pd_n + 162pa_n^3pc_n^2pd_n + 54pa_n^3pd_n^2pc_n + 48pb_n^3pa_n^2pc_n + 16pb_n^3pa_n^2pd_n \\ & + 162pb_n^3pc_n^2pd_n + 54pb_n^3pd_n^2pc_n + 486pc_n^3pa_n^2pb_n + 486pc_n^3pb_n^2pa_n + 18pd_n^3pa_n^2pb_n \\ & + 18pd_n^3pb_n^2pa_n + 486pa_n^2pc_n^2pb_npd_n + 162pa_n^2pd_n^2pb_npc_n + 486pb_n^2pc_n^2pa_npd_n \\ & + 162pb_n^2pd_n^2pa_npc_n, \end{aligned} \tag{A.3}$$

$$\begin{aligned} pd_{n+1} = {} & pa_n^6 + pb_n^6 + 5832pc_n^6 + 8pd_n^6 + 3pa_n^5pb_n + 9pa_n^5pc_n + 3pa_n^5pd_n + 3pb_n^5pa_n + 9pb_n^5pc_n \\ & + 3pb_n^5pd_n + 11664pc_n^5pd_n + 144pd_n^5pc_n + 3pa_n^4pb_n^2 + 3pb_n^4pa_n^2 + 9720pc_n^4pd_n^2 \\ & + 1080pd_n^4pc_n^2 + 2pa_n^3pb_n^3 + 162pa_n^3pc_n^3 + 6pa_n^3pd_n^3 + 162pb_n^3pc_n^3 + 6pb_n^3pd_n^3 \\ & + 4320pc_n^3pd_n^3 + 27pa_n^4pb_npc_n + 9pa_n^4pb_npd_n + 27pb_n^4pa_npc_n + 9pb_n^4pa_npd_n \\ & + 36pa_n^3pb_n^2pc_n + 12pa_n^3pb_n^2pd_n + 162pa_n^3pc_n^2pd_n + 54pa_n^3pd_n^2pc_n + 36pb_n^3pa_n^2pc_n \\ & + 12pb_n^3pa_n^2pd_n + 162pb_n^3pc_n^2pd_n + 54pb_n^3pd_n^2pc_n + 486pc_n^3pa_n^2pb_n + 486pc_n^3pb_n^2pa_n \\ & + 18pd_n^3pa_n^2pb_n + 18pd_n^3pb_n^2pa_n + 486pa_n^2pc_n^2pb_npd_n + 162pa_n^2pd_n^2pb_npc_n \\ & + 486pb_n^2pc_n^2pa_npd_n + 162pb_n^2pd_n^2pa_npc_n. \end{aligned} \tag{A.4}$$

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