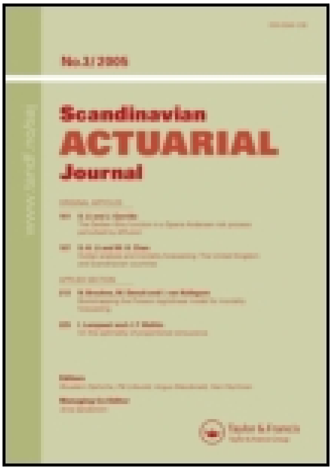

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Optimal insurance contract with stochastic background wealth

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Optimal insurance contract with stochastic background wealth

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This study presents an optimal insurance contract developed endogenously when insured individuals face two mutually dependent risks, background wealth and insurable loss. If background wealth is conditionally normally distributed given insurable loss, the optimal insurance contract may be proportional coinsurance above a straight deductible for a quadratic, negative exponential, or mean-variance utility function. Additionally, when the insured has a quadratic utility or mean-variance utility, the optimal retained schedule is a function of conditional expected value of background wealth given insurable loss. Moreover, the optimal insurance contracts for quadratic and negative exponential utility functions need not to be mean-variance efficient, even when the conditional normal distribution is assumed. Finally, when a portfolio problem is considered, the calculation about the optimal insurance contract remains almost unchanged.

Keywords: optimal insurance; background risk; mean-variance efficient

JEL classification: G22

1. Introduction

Individuals or corporations usually encounter insurable and uninsurable risks in non-life insurance. Insurable risks include building fire, automobile damage, and airplane crashes, while uninsurable risks include the volatility of share returns, variations in corporate or individual income and changes in economic conditions. For an insured party, these uninsurable risks can be viewed as background risks. Without considering background risk, several studies have yielded some important results regarding an optimal insurance contract.[1] Assume that the insurer is risk neutral and the insured is risk averse. In the absence of background risk, Arrow (1963) and Raviv (1979) show that the optimal insurance is a deductible contract. Since a deductible insurance limits the loss of the insured below a pre-specified level (deductible) whatever the magnitude of the loss is, this contract is compatible with the characteristic that the marginal utility of the insured is

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[1]Following Wang, Shyu, & Huang (2005), we define an optimal insurance contract as a policy that is Pareto-efficient. The optimal insurance form with the indemnity schedule must sufficiently satisfy the insured's objective and meet the premium request of the insurer.

decreasing in wealth or equivalently the insured is risk averse. However, if there exists background risk, a deductible insurance can only limit the loss of the insurable risk but not the background risk. Hence, the total losses could be above the pre-specified level. For instance, if the background wealth is positively related to the insurable risk, the background wealth can offset the insurable risk.[2] In this case, a deductible insurance may be dominated by a coinsurance contract. Accordingly, the results in Arrow (1963) and Raviv (1979) do not necessarily hold if background risks exist.

As pointed out by Schlesinger & Doherty (1985), uninsurable background risk might arise due to any number of the following: social risks, general market risk, informational asymmetries, transaction costs of insurance, search costs for insurance, nonmarketable assets, risk versus uncertainty. Since background risks may exist in many situations, an important extension of determining an optimal insurance contract would consider the existence of background risks from both practical and theoretical perspectives. In addition, both the insured and insurer may have some form of background risk. For instance, the uninsurable income or human capital risk can be viewed as a background risk for the insured. Besides taking the risks from the insured, the insurer has background risks such as the asset risks and other risks which are not reinsured. Hence, both the insurer and insured have to consider background risks when making their insurance decisions.[3]

Borch (1983, 1990) indicates that insurers are rational economic agents seeking to optimize their utility function and argue that if the insurer is risk neutral, there must exist a premium (above the actuarially fair value) which will induce a risk averse buyer to take the full insurance cover. Recently, numerous studies have examined insurance demand in the presence of background risks. These studies can be divided into two categories based on whether contractual forms are endogenous or exogenous.[4] Exogenous studies attempt to identify the optimal coverage level, usually given contractual forms of deductible insurance or proportional coinsurance – for example, Doherty & Schlesinger (1983a, 1983b), Mayers & Smith (1983), Aboudi & Thon (1995), Schlesinger (1997), Meyer & Meyer (1998), Guiso & Jappelli (1998), Jeleva (2000), and Luciano & Kast (2001). Doherty & Schlesinger (1983a) show that the conditions, sufficient for the optimality of full coverage or deducible insurances, depend on the correlation between insurable and uninsurable risks. They further demonstrate that the results in Arrow (1963) and Raviv (1979) may not hold for non-independence of loss and initial wealth. Mayers & Smith (1983) note that the purchase of insurance cannot be determined without considering the

[2]In this study, background wealth is defined as the insured's uninsurable wealth, i.e., the wealth which is exposed to uninsurable background risk.
[3]As discussed above, incorporating background risks is important. However, since most existing studies relating to optimal insurance do not consider background risk, their results obtained do not necessarily hold in the presence of background risk. Additionally, if background risk is present, the traditional approach deriving the optimal insurance may not be applicable.
[4]Prior studies relating to the background risk can also be classified into two groups. The first is concerned with the case where this risk is additive (e.g., Doherty & Schlesinger (1983a); Gollier (1996); Gollier & Pratt (1996); Rey (2003); Dana & Scarsini (2007), whereas the second is related to the case where the risk is multiplicative (e.g., Pratt (1988); Franke, Schlesinger, & Stapleton (2006)). This paper belongs to the first group.

investment in other assets in the portfolio when the payoffs of the insurance policy and those from other assets are correlated with each other.

The endogenous studies, however, focus on developing optimal contractual forms of insurance. Notable examples include Gollier (1996), Mahul (2000), and Vercammen (2001). Gollier (1996) shows that the optimal insurance policy displays a disappearing deductible if the convexity of marginal utility is positive and the risk associated with the uninsurable asset increases with the loss magnitude of the insurable asset. Mahul (2000) relaxes Raviv's (1979) assumption of one source of risk by considering an optimal insurance contract when the insured party faces background risks and an insurable risk. He shows that Raviv's (1979) finding still holds if these two risks are independent of each other. Moreover, using the stochastic dominance theory, he further demonstrates that the optimal insurance contract contains a disappearing deductible if both risks are dependent. Vercammen (2001) also considers the situation in which both risks are nonseparable due to the positive relation between the marginal insurable loss and the loss associated with the background risk. Contrary to Gollier's (1996) result of disappearing deductible, he shows that the optimal contract requires coinsurance above a deductible minimum when the agent is prudent.

Individual preference is usually defined using an expected utility, mean-variance, or stochastic dominance framework. The expected utility framework assumes that the individual has a von Neumann-Morgenstern utility function, which is a cardinal measure (Ingersoll 1987). In the mean-variance framework proposed by Markowitz (1952), an individual chooses a portfolio to be mean-variance efficient.[5] These two frameworks are commonly used when analyzing decision behavior under uncertainty. Literature (e.g., Meyer, 1987; Huang & Litzenberger, 1988, Chapter 3) has shown that in most cases the expected wealth utility cannot be defined by only expected value and variance, unless the utility function or the wealth probability distribution is further specified. Moreover, the expected utility and mean-variance models can coincide if the utility function is of a quadratic form or the wealth is normally distributed. Meyer (1987) further derives 7 compatible properties and some comparative statistic connecting these two models without these two restrictions but under the location and scale (LS) parameter condition.[6] Using Meyer's (1987) results, Sinn (1990) proves that the indifference curve slope in mean-standard deviation ($\mu - \sigma$) space increases with σ, given μ, when absolute risk aversion satisfies some regular conditions. Moreover, Boyle & Conniffe (2008) extend the compatibility of expected utility and mean-variance models to probability distributions that are not location-scale, but can be transformed to that family. Additionally,

[5]Preferences cannot usually be represented as a function of mean and variance only. However, for asset choice, the mean-variance framework is popular because of its analytical tractability and its rich empirical implications (Huang & Litzenberger 1988).

[6]The LS condition can be presented as a linear function, $y = \mu + \sigma x$, where y, x, μ and σ denote the uncertain wealth, random return, location and scale parameters, respectively. Accordingly, the portfolio payoff invested in one riskless asset and one risky asset satisfies the LS condition. Additionally, if the insurance contractual form is of proportional coinsurance, the insured's final wealth is a linear function of random loss and hence also satisfies this condition. It is noted, however, that policy-limit and deductible insurance does not satisfy the LS condition. Since this study aims to endogenously develop the optimal insurance contract with stochastic background wealth, the derived contract under the expected utility or mean-variance framework is not necessarily a proportional coinsurance and hence our results cannot be fully compared with those derived under the Meyer's LS condition.

a stochastic dominance framework can also identify the individual preference, corresponding to a certain family of utility functions. For instance, if Good X is second-degree dominated by Good Y, then all risk averters would prefer Y to X.[7]

Consider two stochastic payoffs A and B, which may be viewed as the payoffs from two different insurance contracts. Using the stochastic dominance approach, an individual chooses payoffs A or B by only comparing their associated probability distributions. The result obtained from this approach can be widely applied to most utility functions. However, its drawback is that the stochastic dominance between A and B may not exist. In this case, we cannot determine which one is better. The expected utility approach says an individual strictly prefers A to B if and only if $E[U(A)] > E[U(B)]$. Unlike the stochastic dominance approach, the preference relations for the expected utility approach are of completeness. However, using this approach requires information about the probability distributions of A and B as well as the parametric form of the utility function. The mean-variance approach only requires information about the means, variances and covariance for A and B. Hence, this approach is analytically tractable. However, like the stochastic dominance approach, the mean-variance efficient relation of A and B may not exist.

The expected utility framework is adopted in Doherty & Schlesinger (1983a, 1983b), Gollier (1996), Guiso & Jappelli (1998), Mahul (2000), and Vercammen (2001). The mean-variance framework is adopted in Mayers & Smith (1983). The stochastic dominance framework is adopted in Aboudi & Thon (1995), Gollier & Schlesinger (1996), Schlesinger (1997), and Meyer & Meyer (1998).

This study presents an optimal insurance contract developed endogenously when the insured agent faces an insurable risk in the presence of an uninsurable background risk. This uninsurable risk is represented by background wealth, consisting of stochastic initial wealth and a portfolio of financial assets. As in most prior studies (e.g., Raviv, 1979; Gollier, 1996; Mahul, 2000), the optimal insurance contract must generate the maximal expected utility of final wealth for the insured, given that insurance premiums meet insurer's requirements. Specifically, this study respectively adopts expected utility and mean-variance frameworks to define the preferences of insured individuals. Restated, insured individuals choose an optimal indemnity schedule for maximizing the expected utility or mean-variance efficiency corresponding to the two frameworks.

Following Gollier (1996), Mahul (2000), and Vercammen (2001), this study assumes that background risk depends on insurable loss and endogenously derives the optimal insurance contract. We extend the literature on the design of optimal insurance in the following ways. First, for comparison purposes, this study designs an optimal insurance policy simultaneously under the expected utility and mean-variance frameworks. The results are compared and discussed. Second, under an expected utility framework, this study only assumes that the utility function $U(\cdot)$ of the insured is strictly increasing and concave, i.e., $U'(\cdot) > 0 > U''(\cdot)$. Hence, the result concerning the optimal insurance design can be compared with previous studies, which developed several important results based on more specific assumptions. For instance, in addition to $U'(\cdot) > 0 > U''(\cdot)$, Gollier

[7]A lower-degree stochastic dominance implies a higher-degree one. The related argument and mathematical proof can be found in Ingersoll (1987, Chapter 5) and Huang & Litzenberger (1988, Chapter 2).

(1996) and Vercammen (2001) assume that the insured is prudent $U'''(\cdot) > 0$), whereas Mahul (2000) assume that the background risk becomes riskier, according to any degree of stochastic dominance, as the insurable loss increases. Third, under the mean-variance framework, we develop an optimal insurance policy endogenously rather than limiting the contractual form as in Mayers & Smith (1983).[8] Finally, this study derives an optimal insurance contract, assuming that the insured has mean-variance, quadratic, or negative exponential utility functions. To obtain an explicit solution, we further assume that the background wealth obeys a conditional normal distribution given the insurable loss. The main contributions of this paper are with respect to results established for the mean-variance utility function, inclusion of the portfolio problem and examination of optimal insurance with specific utility functions combined with a bivariate normal distribution assumption. In addition, unlike previous research such as Mahul (2000) we utilize Taylor's theorem to show that the Raviv's result holds when background risk is independent of insurable loss.

The main results of this paper are summarized in six propositions presented below. If background wealth is conditionally normally distributed given insurable loss, the optimal insurance contract may be proportional coinsurance above a straight deductible for quadratic, negative exponential, or mean-variance utility functions. Additionally, when the insured's utility function is quadratic or mean-variance, the optimal retained schedule is a function of conditional expected value of background wealth given insurable loss. Moreover, the optimal insurance contracts for quadratic and negative exponential utility functions need not to be mean-variance efficient, even when the conditional normal distribution is assumed.

The remainder of this paper is organized as follows. Section 2 presents the assumptions regarding insurable loss, background wealth, and insurance premiums. Sections 3 and 4 then derive the optimal insurance contract corresponding to expected utility and mean-variance frameworks, respectively. Section 5 demonstrates further analytical results for the portfolio problem and multivariate normal distribution, and determines the insurance premium and investment amount. The last section draws the conclusion.

2. Assumptions

An insured party has a stochastic initial wealth $\tilde{y}$ and faces a risk of loss $\tilde{x}$, where '$\sim$' denotes a random variable hereinafter. $\tilde{x}$ is nonnegative and perfectly insurable, and has a probability density function $f(x)$, $x \geq 0$. Variable $\tilde{y}$, which can be viewed as a random endowment, includes salary income and some uninsurable risk; hence, $\tilde{y}$ may be positive or negative.

The insured can utilize an insurance contract to compensate for loss $\tilde{x}$. The insurance contract costs a premium P and pays an indemnity schedule $I(x)$, $0 \leq I(x) \leq x$ for all x,

[8] Mayers & Smith (1983) limited the contractual form to proportional coinsurance. Accordingly, the optimal coinsurance ratio and portfolio components can be determined explicitly, and the calculation resembles the traditional mean-variance frontier proposed by Markowitz (1952).

where x is an outcome of $\tilde{x}$. According to Raviv (1979), Gollier (1996), Wang *et al.* (2005), and Huang (2006), the premium P is equivalent to the expected indemnity plus a proportional loading. Mathematically,

$$P = (1 + l)E[I(\tilde{x})] \tag{1}$$

where l denotes the percentage of loading. For convenience, let $R(x) \equiv x - I(x)$ represent the retained loss schedule, consequently, $0 \leq R(x) \leq x$ for all x.

In addition to the insurance market, the insured may invest his/her money in a financial market. Assume that $a_1, a_2, \ldots,$ and a_n, denote the respective investment amounts in assets 1, 2,..., n. The financial market consists of n assets with the following rate of returns: $\tilde{z}_1, \tilde{z}_2, \ldots, \tilde{z}_n$. Additionally, let the column vectors $\mathbf{a} = (a_1, a_2, \ldots, a_n)^T$ and $\tilde{\mathbf{z}} = (\tilde{z}_1, \tilde{z}_2, \ldots, \tilde{z}_n)^T$, where the superscript 'T' represents the transpose of a vector or matrix. Therefore, the total wealth of the insured is

$$\tilde{W} = \tilde{y} + \mathbf{a}^T\tilde{\mathbf{z}} - P - R(\tilde{x}) \tag{2}$$

where $\tilde{y} + \mathbf{a}^T\tilde{\mathbf{z}}$ is the uninsurable background wealth.

3. Expected utility framework

Assume that insured individuals have a von Neumann-Morgenstern utility function, i.e., their preferences can be represented as an expected utility (Huang & Litzenberger, 1988). Insured individuals are risk averse and not satiable, equivalently, their utility $U(\tilde{W})$ satisfies $U'(\tilde{W}) > 0 > U''(\tilde{W})$. According to Raviv (1979), Gollier (1996), and Mayers & Smith (1983), the optimality problem of insured individuals is to select a decision bundle (**a**, P, $R(x)$) for maximizing the expected wealth utility $\mathrm{E}[U(\tilde{W})]$. Formally,

$$\underset{\mathbf{a}, P \geq 0, 0 \leq R(x) \leq x}{\text{Maximize}} \; E[U(\tilde{W})] = \mathrm{E}[U\{\tilde{y} + \mathbf{a}^T\tilde{\mathbf{z}} - P - R(\tilde{x})\}] \tag{3}$$

$$\text{subject to} \quad P = (1 + l)\mathrm{E}[I(\tilde{x})] = (1 + l)\mathrm{E}[\tilde{x} - R(\tilde{x})] \tag{3a}$$

As in Raviv (1979), Gollier (1996), Wang *et al.* (2005), and Huang (2006), Equation (3) can be solved in two steps. Given **a** and P, the first step is to derive $R^*(x)$ as a function of **a** and P, $R^*(x; \mathbf{a}, P)$. Then in the second step we solve $R^*(x)$ by finding the optimal $\mathbf{a}^*$ and P^* and $R^*(x; \mathbf{a}^*, P^*)$.

In Section 3, we assume that the insured does not have opportunities to invest his/her funds in financial assets. Subsection 3.1 derives the closed form solution of $R^*(x)$. The next subsection assumes that the insured has a quadratic utility and a negative exponential utility, respectively. A more explicit solution for $R^*(x)$ is accordingly obtained.

3.1. Preliminary results

Due to the assumption of no investment opportunity available for the insured, we set $\mathbf{a} = \mathbf{0} \equiv n \times 1$ zero vector. Equation (3) is then simplified as:

$$\underset{P \geq 0, 0 \leq R(x) \leq x}{\text{Maximize}}\, E[U(\tilde{W})] = \mathrm{E}[U\{\tilde{y} - P - R(\tilde{x})\}] \tag{4}$$

$$\text{subject to} \quad P = (1 + l)\mathrm{E}[I(\tilde{x})] = (1 + l)\mathrm{E}[\tilde{x} - R(\tilde{x})] \tag{4a}$$

If the initial wealth $\tilde{y}$ is assumed deterministic rather than stochastic, $R^*(x)$ would be a deductible insurance, as shown in Arrow (1963) and Raviv (1979). We further demonstrate that $R^*(x)$ is also a deductible insurance as long as $\tilde{y}$ is independent of $\tilde{x}$. The arguments are displayed in Proposition 1 and Corollary 1 as follows:

PROPOSITION 1 *Suppose that the insured with a strictly increasing and strictly concave utility function has a stochastic initial wealth that is independent of the insurable loss, and that the insurance premium is a strictly increasing function of expected indemnity. The optimal insurance contract would be a deductible insurance.*
Proof. See Appendix 1.

The result in Proposition 1 resembles that obtained by Gollier (1996) and Mahul (2000), in which the direct utility of the insured is replaced by an indirect utility, and thus the optimal insurance problem is reduced to the model presented by Raviv (1979). Since a deductible insurance limits the loss of insured below a pre-specified level (deductible), this contract is compatible with the characteristic that the marginal utility of the insured is decreasing in wealth, or equivalently, the insured is risk averse. Note that the presence of an independent background risk causes the insured to increase his/her risk aversion. As a consequence, the optimal deductible size with an independent background risk is less than that without an independent background risk (Gollier, 1996; Mahul, 2000). However, unlike Gollier (1996) and Mahul (2000), we directly apply Taylor's Theorem to verify this proposition. We believe that the proof in Appendix 1 provides an alternative approach. The result in Proposition 1 can be directly applied to Equation (4) and the following corollary is accordingly obtained.

COROLLARY 1 *If $\tilde{y}$ and $\tilde{x}$ are independent, then the optimal retained loss schedule in Equation (4) is*

$$R^*(x) = \min\{x, d^*\} \quad \text{for all}\, x \tag{5}$$

where d^* is the solution of

$$P - (1 + l)\mathrm{E}[\max\{\tilde{x} - d^*, 0\}] = 0 \tag{5a}$$

Proof. See Appendix 2.

This study next considers a case in which $\tilde{y}$ depends on $\tilde{x}$. By using integral representations for all expected values, Equation (4) resembles a continuous optimal control problem. Following Fryer & Greenman (1987) and Raviv (1979), this equation can be then solved. The Lagrangian for Equation (4) is

$$L = \mathrm{E}[U\{\tilde{y} - P - R(\tilde{x})\}] + \lambda\{P - (1 + l)\mathrm{E}[\tilde{x} - R(\tilde{x})]\} \tag{6}$$

where λ represents the Lagrange multiplier. By using an iterated expectation rule, Equation (6) can be rewritten as

$$L = \mathrm{E}[\mathrm{E}[U\{\tilde{y} - P - R(\tilde{x})\}|x]] + \lambda\{P - (1+l)\mathrm{E}[\tilde{x} - R(\tilde{x})]\} \tag{6a}$$

where $\mathrm{E}[\cdot|x]$ denotes the conditional expectation given $\tilde{x} = x$. Since $\mathrm{E}[\cdot|x]$ is a function of x, Equation (6a) can be reformulated as

$$L = \int_0^{\infty} (\mathrm{E}[U\{\tilde{y} - P - R(\tilde{x})\}|x] + \lambda\{P - (1+l)[x - R(x)]\})f(x)dx \tag{6b}$$

The corresponding Hamiltonian in Equation (6b) is

$$H = (\mathrm{E}[U\{\tilde{y} - P - R(\tilde{x})\}|x] + \lambda\{P - (1+l)[x - R(x)]\})f(x) \tag{7}$$

Now $R^*(x)$ can be derived by maximizing Equation (7). The result is summarized in Proposition 2.

PROPOSITION 2: *The optimal retained loss schedule in Equation (4) is*

$$\begin{cases} R^*(x) = \min\{x, \max\{\hat{R}(x), 0\}\} \\ E[U'\{\tilde{y} - P - \hat{R}(\tilde{x})\}|x] = c(\text{constant}) \quad \text{for all } x \end{cases}. \tag{8}$$

Proof. See Appendix 3.

The economic implication of Proposition 2 is as follows. $\hat{R}(x)$ is the optimal retained loss schedule when $R(x)$ is not limited. Consider arbitrary loss outcomes x_1 and x_2. If the expected marginal utility for $\tilde{x} = x_1$ exceeds that for $\tilde{x} = x_2$, then simultaneously increasing indemnity in $\tilde{x} = x_1$ and decreasing indemnity in $\tilde{x} = x_2$ may enhance expected utility. Accordingly, insured individuals would select a particular $R(x)$ such that the expected marginal utility is fixed for all x. Additionally, the retained loss schedule is limited to $0 \le R(x) \le x$. Thus, the final result is presented as Equation (8).

Since the assumption for Proposition 2 is only that the utility function of the insured is increasing and concave, Equation (8) can be applied to most situations. Actually, Equation (8) contains a rich family of contractual forms and hence can widely correspond to the results obtained in the existing literature. For instance, if $\hat{R}(x) = 0$ for $x \ge \hat{x}$ and $\hat{R}(x) > 0$ for $x < \hat{x}$, the optimal insurance displays a disappearing deducible resembling the results in Gollier (1996) and Mahul (2000), where $\hat{x}$ represents some disappearing deducible. Additionally, if $\hat{R}(x) = \delta x + d$, $\delta > 0$, $d > 0$, the optimal insurance contract is a coinsurance above a straight deductible, resembling the results in Vercammen (2001), where $1 - \delta$ represents the coinsurance proportion and d represents the straight deductible.

3.2. Quadratic utility and negative exponential utility

The hyperbolic absolute risk aversion (HARA) or equivalently linear risk tolerance (LRT) class of utility functions is conventionally adopted. In this subsection, we use the quadratic utility and negative exponential utility functions as examples of HARA, since they are the most commonly seen in the literature (see, e.g., Huberman, Mayers, & Smith (1983); Bowers *et al.* (1986); Huang & Litzenberger (1988)).

A. Quadratic utility function

A quadratic utility function is defined by $U(W) = W - \frac{b}{2}W^2, \quad 0 < b < W^{-1}$. Thus, the corresponding marginal utility function is

$$U'(W) = 1 - bW, \quad 0 < b < W^{-1}. \tag{9}$$

Substituting Equation (9) into Equation (8) yields

$$\begin{cases} R^*(x) = \min\{x, \max\{\hat{R}(x), 0\}\} \\ E[1 - b\{\tilde{y} - P - \hat{R}(x)\}|x] = c \end{cases} \tag{10}$$

Equation (10) can be rewritten as

$$\begin{cases} R^*(x) = \min\{x, \max\{\hat{R}(x), 0\}\} \\ \hat{R}(x) = \mathrm{E}[\tilde{y}|x] + d_1 \\ d_1 = -P + (c - 1)/b \end{cases} \tag{11}$$

For the quadratic utility, the marginal utility is linear. Incorporating this fact into Equation (8) yields that the optimal insurance depends on the conditional expectation $\mathrm{E}[\tilde{y}|x]$. Consider the normal situation $\hat{R}(x) > 0$. In this case, $R^*(x) = \min\{x, \hat{R}(x)\}$, where the deductible $\hat{R}(x) = \mathrm{E}[\tilde{y}|x] + d_1$. If the background risk $\tilde{y}$ is independent of $\tilde{x}$, then $\mathrm{E}[\tilde{y}|x] = \mathrm{E}[\tilde{y}]$ and $R^*(x) = \min\{x, \mathrm{E}[\tilde{y}] + d_1\}$. This result, as in Proposition 1, demonstrates that the straight deductible insurance with deductible $\mathrm{E}[\tilde{y}] + d_1$ would be optimal. Additionally, for example, if $\tilde{x}$ and $\tilde{y}$ are positively correlated, then $\mathrm{E}[\tilde{y}|x] > \mathrm{E}[\tilde{y}]$ for relatively large values of x. In this case, the deductible would increase to $\hat{R}(x) = \mathrm{E}[\tilde{y}|x] + d_1$ since the conditional stochastic initial income $\mathrm{E}[\tilde{y}|x]$ can be used to offset the loss x. Conversely, the deductible would decrease when $\mathrm{E}[\tilde{y}|x] < \mathrm{E}[\tilde{y}]$ for relatively small values of x.

B. Negative exponential utility function

A negative exponential utility function is defined by $U(W) = -\frac{1}{\gamma}\exp\{-\gamma W\}, \quad \gamma > 0.$ Consequently, the corresponding marginal utility function is

$$U'(W) = \exp\{-\gamma W\}, \quad \gamma > 0 \tag{12}$$

Substituting Equation (12) into Equation (8) produces

$$\begin{cases} R^*(x) = \min\{x, \max\{\hat{R}(x), 0\}\} \\ E[\exp\{-\gamma[\tilde{y} - P - \hat{R}(x)]\}|x] = c \end{cases} \tag{13}$$

Equation (13) can be rewritten as

$$\begin{cases} R^*(x) = \min\{x, \max\{\hat{R}(x), 0\}\} \\ \hat{R}(x) = -\gamma^{-1}\log\{\mathrm{E}[\exp\{-\gamma\tilde{y}\}|x]\} + d_2 \\ d_2 = -P + \gamma^{-1}\log c \end{cases} \tag{14}$$

For the negative exponential utility, the marginal utility is also exponential, shown on Equation (12). Incorporating this fact into Equation (8) yields that the optimal insurance depends on conditional expectation $\mathrm{E}[\exp\{-\gamma\tilde{y}\}|x]$. Actually, Equation (11) is similar to Equation (14) if $\tilde{y}$ and $\hat{R}(x)$ are substituted by $\exp\{-\gamma\tilde{y}\}$ and $\exp\{-\gamma\hat{R}(x)\}$, respectively. Hence, the intuition of the design of optimal insurance for negative exponential utility is similar to that for quadratic utility.

4. Mean-variance utility framework

Suppose that the insured has a mean-variance utility function. Thus, the insured would prefer the expected wealth with minimum variance, given the same expected wealth. Conversely, given the same variance, the insured prefers the maximum expected wealth. The optimality problem of the insured is to choose a decision bundle (**a**, P, $R(x)$) to achieve a mean-variance efficiency. Mathematically,

$$\underset{\mathbf{a}, P\geq 0, 0\leq R(x)\leq x}{\text{Minimize}}\ \sigma_w^2 = \mathrm{Var}[\tilde{W}] = \mathrm{Var}[\tilde{y} + \mathbf{a}^T\tilde{\mathbf{z}} - P - R(\tilde{x})] \tag{15}$$

$$\text{subject to}\quad P = (1+l)\mathrm{E}[I(\tilde{x})] = (1+l)\mathrm{E}[\tilde{x} - R(\tilde{x})] \tag{15a}$$

$$\mathrm{E}[\tilde{W}] = \mathrm{E}[\tilde{y} + \mathbf{a}^T\tilde{\mathbf{z}} - P - R(\tilde{x})] = \mu_w \tag{15b}$$

As in the previous section, we do not consider the portfolio decision in this section. Thus, taking $\mathbf{a} = \mathbf{0}$, Equation (15) is reduced to

$$\underset{P\geq 0, 0\leq R(x)\leq x}{\text{Minimize}}\ \sigma_w^2 = \mathrm{Var}[\tilde{W}] = Var[\tilde{y} - P - R(\tilde{x})] \tag{16}$$

$$\text{subject to}\quad P = (1+l)\mathrm{E}[I(\tilde{x})] = (1+l)\mathrm{E}[\tilde{x} - R(\tilde{x})] \tag{16a}$$

$$\mathrm{E}[\tilde{W}] = \mathrm{E}[\tilde{y} - P - R(\tilde{x})] = \mu_w \tag{16b}$$

Equation (16) actually is the mean-variance frontier problem suggested by Markowitz (1952), demonstrating that the insured should choose an optimal couple $(P^*, R^*(x))$ for minimizing σ_w^2, given $\mathrm{E}[\tilde{W}] = \mu_w$. Based on Equations (16a) and (16b),

$$\begin{cases} \mathrm{E}[R(\tilde{x})] = l^{-1}[(1+l)\mu_x - \mu_y + \mu_w] = \mu_R \\ P = l^{-1}(1+l)(-\mu_x + \mu_y - \mu_w) \end{cases} \tag{17}$$

where $\mu_x = \mathrm{E}[\tilde{x}]$, $\mu_y = \mathrm{E}[\tilde{y}]$, and $\mu_R = \mathrm{E}[R(\tilde{x})]$. This equation suggests that both μ_R and P are known constants once μ_w is given. Using Equation (17), the optimality problem in Equation (16) is simplified as

$$\underset{0\leq R(x)\leq x}{\text{Minimize}}\,\sigma_w^2 = \sigma_y^2 - 2\mathrm{E}[(\tilde{y} - \mu_y)(R(\tilde{x}) - \mu_R)] + \mathrm{E}[(R(\tilde{x}) - \mu_R)^2] \tag{18}$$

where $\sigma_y^2 = \mathrm{Var}[\tilde{y}]$. Using the iterated expectation rule, Equation (18) is rewritten as

$$\underset{0\le R(x)\le x}{\text{Minimize}}\, \sigma_w^2 = \sigma_y^2 - 2\mathrm{E}[(\mathrm{E}[\tilde{y}|x] - \mu_y)(R(x) - \mu_R)] + \mathrm{E}[(R(\tilde{x}) - \mu_R)^2] \tag{19}$$

Since $\mathrm{E}[\tilde{y}|x]$ is a function of x, Equation (19) can be presented as

$$\underset{0\le R(x)\le x}{\text{Minimize}}\, \sigma_w^2 = \int_0^\infty \left(\sigma_y^2 - 2[(\mathrm{E}[\tilde{y}|x] - \mu_y)(R(x) - \mu_R)] + (R(x) - \mu_R)^2\right) f(x)dx \tag{20}$$

Equation (20) is an optimal control problem; the corresponding Hamiltonian is

$$\underset{0R(x)x}{\mathit{Minimize}}\, H = \left(\sigma_y^2 - 2[(E[\tilde{y}|x] - \mu_y)(R(x) - \mu_R)] + (R(x) - \mu_R)^2\right) f(x) \tag{21}$$

Solving Equation (21), the following proposition is obtained.

Proposition 3: *The optimal retained loss schedule in Equation (16) is*

$$\begin{cases} R^*(x) = \min\{x, \max\{\hat{R}(x), 0\}\} \\ \hat{R}(x) = \mathrm{E}[\tilde{y}|x] + \mu_R - \mu_y & \text{for all } x \\ \mu_R = l^{-1}[(1+l)\mu_x - \mu_y + \mu_w] \end{cases} \tag{22}$$

Proof. See Appendix 4.

Proposition 3 reveals that $\hat{R}(x) = \mu_R$ and $R^*(x) = \min\{x, \mu_R\}$ when $\tilde{y}$ is independent of $\tilde{x}$, implying that the optimal contract is deductible insurance with a deductible μ_R. Thus, even with existing background risk, deductible insurance is mean-variance efficient as long as the background risk does not depend on insurable loss. Without taking background risk into account, Gollier & Schlesinger (1996) show that any feasible insurance contract would be dominated by a deductible insurance policy, according to the second-order stochastic dominance criterion. Since second-degree stochastic dominance implies mean-variance efficiency, the Proposition 3 result can be utilized to supplement the argument of Gollier & Schlesinger (1996), in which only existing insurable loss is considered.

Comparing Equations (11) and (22), we find that the optimal insurance contractual form for the mean-variance utility framework resembles that for the quadratic utility function. This result is not surprising, since the expected utility for the quadratic utility function depends on only the first two moments (mean and variance); that is, $\mathrm{E}[\tilde{W} - \frac{b}{2}\tilde{W}^2] = \mu_w - \frac{b}{2}[\mu_w^2 + \sigma_w^2]$.

5. Further analytical results

This section discusses further analytical results for the optimal insurance contract. In Subsection 5.1, we take into account the portfolio decision in financial markets. In the next subsection, we attempt to obtain a more explicit solution by assuming that the random vector $[\tilde{y}\tilde{z}]^T$ given $\tilde{x} = x$ follows a multivariate normal distribution. In the final subsection, the optimal insurance premium and investment amount are determined.

5.1. Portfolio problem

Assume that the insured is able to invest his/her funds on assets in the financial markets. The derivation of $R^*(x)$ given $\mathbf{a}$ and P is similar to that in Sections 3 and 4. Propositions 2 and 3 can then be easily extended to Propositions 4 and 5 as follows.

PROPOSITION 4: *The optimal retained loss schedule in Equation (3) is*

$$\begin{cases} R^*(x) = \min\{x, \max\{\hat{R}(x), 0\}\} \\ \mathrm{E}[U'\{\tilde{y} + \mathbf{a}^T\tilde{\mathbf{z}} - P - \hat{R}(\tilde{x})\}|x] = c(\text{constant}) \end{cases} \quad \text{for all } x \tag{23}$$

PROPOSITION 5: *The optimal retained loss schedule in Equation (15)*

$$\begin{cases} R^*(x) = \min\{x, \max\{\hat{R}(x), 0\}\} \\ \hat{R}(x) = \mathrm{E}[\tilde{y} + \mathbf{a}^T\tilde{\mathbf{z}}|x] + \mu_R - \mu_y \\ \mu_R = l^{-1}[(1+l)\mu_x - \mu_y - \mathbf{a}^T\boldsymbol{\mu}_{\mathbf{z}} + \mu_w] \end{cases} \quad \text{for all } x \tag{24}$$

where $\mu_{\mathbf{z}} = \mathrm{E}[\tilde{z}]$ *represents the expectation of the vector* $\tilde{\mathbf{z}}$*.*

Since investment risk is, in general, uninsurable, it can be viewed as one source of background risks. The optimal contractual form in the portfolio problem is similar to that without considering this problem. Propositions 2 and 3, in fact, can be extended to Propositions 4 and 5. In essence, the intuition for Propositions 4 and 5 is analogous to Propositions 2 and 3.

Particularly, if the insured has a quadratic utility function, referring to Equations (11) and (23), the optimal retained loss schedule is

$$\begin{cases} R^*(x) = \min\{x, \max\{\hat{R}(x), 0\}\} \\ \hat{R}(x) = E[\tilde{y} + \mathbf{a}^T\tilde{\mathbf{z}}|x] + d_1 \\ d_1 = -P + (c-1)/b \end{cases} \tag{25}$$

Additionally, if the insured has a negative exponential utility function, referring to Equations (14) and (23), the optimal retained loss schedule is

$$\begin{cases} R^*(x) = \min\{x, \max\{\hat{R}(x), 0\}\} \\ \hat{R}(x) = -\gamma^{-1}\log\{\mathrm{E}[\exp\{-\gamma(\tilde{y} + a^T\tilde{z})\}|x]\} + d_2 \\ d_2 = -P + \gamma^{-1}\log c \end{cases} \tag{26}$$

In addition to deterministic terms, the stochastic terms (depending on $\tilde{x}$) of $\hat{R}(x)$ are the same as $\mathrm{E}[\tilde{y} + \mathbf{a}^T\tilde{\mathbf{z}}|x]$ in Equations (24) and (25). Based on Equations (11), (22), and (25), the optimal insurance contractual form with background risk for the quadratic utility is also similar to that for the mean-variance utility, no matter what the portfolio problem is considered.

5.2. Multivariate normal distribution

Assumes that random vector $[\tilde{y}\,\tilde{\mathbf{z}}]^T$ given $\tilde{x} = x$ obeys a multivariate normal distribution. Mathematically,

$$\begin{bmatrix} \tilde{y} \\ \tilde{\mathbf{z}} \end{bmatrix} | x \sim N\left(\begin{bmatrix} \mu_{y|x} \\ \boldsymbol{\mu}_{\mathbf{z}|x} \end{bmatrix}, \begin{bmatrix} \sigma^2_{y|x} & \Sigma_{yz|x} \\ \Sigma_{zy|x} & \Sigma_{zz|x} \end{bmatrix} \right) \tag{27}$$

According to Greene (1997, Chapter 3), the conditional expectation vector and conditional variance matrix are defined by

$$\mu_{y|x} = \mu_y + (\sigma_{xy}/\sigma_x^2)(x - \mu_x) = \mu_y + \rho(\sigma_y/\sigma_x)(x - \mu_x) \tag{28}$$

$$\boldsymbol{\mu}_{z|x} = \boldsymbol{\mu}_{\mathbf{z}} + (\Sigma_{zx}/\sigma_x^2)(x - \mu_x) \tag{29}$$

$$\begin{bmatrix} \sigma^2_{y|x} & \Sigma_{yz|x} \\ \Sigma_{zy|x} & \Sigma_{zz|x} \end{bmatrix} = \begin{bmatrix} \sigma^2_y & \Sigma_{yz} \\ \Sigma_{zy} & \Sigma_{zz} \end{bmatrix} - \begin{bmatrix} \sigma_{xy}/\sigma^2_x \\ \Sigma_{zx}/\sigma^2_x \end{bmatrix} \begin{bmatrix} \sigma_{xy} & \Sigma_{xz} \end{bmatrix} \tag{30}$$

where $\sigma_x^2 = \mathrm{Var}[\tilde{x}]$, $\sigma_y^2 = \mathrm{Var}[\tilde{y}]$, $\sigma_{xy} = \mathrm{Cov}[\tilde{x}, \tilde{y}] = \rho\sigma_x\sigma_y$, $\boldsymbol{\mu}_{\mathbf{z}} = \mathrm{E}[\tilde{\mathbf{z}}]$, $\Sigma_{zz} = \mathrm{Var}[\tilde{\mathbf{z}}]$, $\Sigma_{x\mathbf{z}} = \mathrm{Cov}[\tilde{x}, \tilde{\mathbf{z}}] = \Sigma^T_{\mathbf{z}x}$, and $\Sigma_{y\mathbf{z}} = \mathrm{Cov}[\tilde{y}, \tilde{\mathbf{z}}] = \Sigma^T_{\mathbf{z}y}$.

Assume that the insured has a quadratic utility. Substituting Equations (28) and (29) into expression (25), it yields

$$\begin{cases} R^*(x) = \min\{x, \max\{\hat{R}(x), 0\}\} \\ \hat{R}(x) = (\sigma_{xy}/\sigma_x^2 + \mathbf{a}^T\Sigma_{zx})x + d_1^* \\ d_1^* = -P + (c-1)/b - \mu_x(\sigma_{xy} + \mathbf{a}^T\Sigma_{\mathbf{z}x})/\sigma_x^2 + \mu_y + \mathbf{a}^T\boldsymbol{\mu}_{\mathbf{z}} \end{cases} \tag{31}$$

If the insured has a negative exponential utility, the moment generating function can be utilized to solve Equation (26). Assume that any two random vectors $\tilde{\mathbf{x}}_1$ and $\tilde{\mathbf{x}}_2$ are normally distributed, the joint moment generating function is then

$$\mathrm{E}[\exp\{\mathbf{t}_1^T\tilde{\mathbf{x}}_1 + \mathbf{t}_2^T\tilde{\mathbf{x}}_2\}] = \exp\{\mathbf{t}_1^T\boldsymbol{\mu}_1 + \mathbf{t}_2^T\boldsymbol{\mu}_2 + \tfrac{1}{2}[\mathbf{t}_1^T\Sigma_{11}\mathbf{t}_1 + 2\mathbf{t}_1^T\Sigma_{12}\mathbf{t}_2 + \mathbf{t}_2^T\Sigma_{22}\mathbf{t}_2\} \tag{32}$$

where $\boldsymbol{\mu}_1 = \mathrm{E}[\tilde{\mathbf{x}}_1]$, $\boldsymbol{\mu}_2 = \mathrm{E}[\tilde{\mathbf{x}}_2]$, $\Sigma_{11} = \mathrm{Var}(\tilde{\mathbf{x}}_1)$, $\Sigma_{22} = \mathrm{Var}(\tilde{\mathbf{x}}_2)$, $\Sigma_{12} = \mathrm{Cov}(\tilde{\mathbf{x}}_1, \tilde{\mathbf{x}}_2)$, $\mathbf{t_1}$ and $\mathbf{t_2}$ are some vectors of deterministic numbers. From Equations (27) and (32),

$$\mathrm{E}[\exp\{-\gamma(\tilde{y} + \mathbf{a}^T\tilde{\mathbf{z}})\}|x] = \exp\{-\gamma[\mu_{y|x} + \mathbf{a}^T\boldsymbol{\mu}_{z|x} + \tfrac{-\gamma}{2}(\sigma^2_{y|x} + 2\mathbf{a}^T\Sigma_{zy|x} + \mathbf{a}^T\Sigma_{\mathbf{zz}|x}\mathbf{a})]\} \tag{33}$$

Incorporating Equations (14), (28), (29), and (33) yields

$$\begin{cases} R^*(x) = \min\{x, \max\{\hat{R}(x), 0\}\} \\ \hat{R}(x) = (\sigma_{xy}/\sigma_x^2 + \mathbf{a}^T\Sigma_{\mathbf{zx}})x + d_2^* \\ d_2^* = -P + \gamma^{-1}\log c - \mu_x(\sigma_{xy} + \mathbf{a}^T\Sigma_{\mathbf{z}x})/\sigma_x^2 \\ \qquad -\tfrac{\gamma}{2}(\sigma^2_{y|x} + 2\mathbf{a}^T\Sigma_{zy|x} + \mathbf{a}^T\Sigma_{\mathbf{zz}|x}\mathbf{a}) + \mathbf{a}^T\boldsymbol{\mu}_{\mathbf{z}} + \mu_y \end{cases} \tag{34}$$

If the insured has a mean-variance utility, substituting Equations (28) and (29) into Equation (24) generates

$$\begin{cases} R^*(x) = \min\{x, \max\{\hat{R}(x), 0\}\} \\ \hat{R}(x) = (\sigma_{xy}/\sigma_x^2 + \mathbf{a}^T\Sigma_{\mathbf{z}x})x + \mu_R - \mu_y \\ \mu_R = l^{-1}[(1+l)\mu_x - \mu_y - \mathbf{a}^T\boldsymbol{\mu}_{\mathbf{z}} + \mu_w] \end{cases} \tag{35}$$

Comparing Equations (31), (34), and (35), the optimal insurance contracts for quadratic and negative exponential utility functions need not to be mean-variance efficient, even when a conditional normal distribution is assumed. Additionally, to obtain more explicit result, the portfolio problem is now excluded from our model. Accordingly, substituting **a** = **0** into Equations (31), (34), and (35) yields the following three expressions.

$$\begin{cases} R^*(x) = \min\{x, \max\{\hat{R}(x), 0\}\} \\ \hat{R}(x) = \rho(\sigma_y/\sigma_x)(x - \mu_x) + d_1^* \\ d_1^* = \mu_y + (c-1)/b - P \end{cases} \quad \text{for quadratic utility} \tag{36}$$

$$\begin{cases} R^*(x) = \min\{x, \max\{\hat{R}(x), 0\}\} \\ \hat{R}(x) = \rho(\sigma_y/\sigma_x)(x - \mu_x) + d_2^* \\ d_2^* = \mu_y + \gamma^{-1}\log c - \frac{\gamma}{2}\sigma_{y|x}^2 - P \end{cases} \quad \text{for negative exponential utility} \tag{37}$$

$$\begin{cases} R^*(x) = \min\{x, \max\{\hat{R}(x), 0\}\} \\ \hat{R}(x) = \rho(\sigma_y/\sigma_x)(x - \mu_x)] + d_3^* \\ d_3^* = l^{-1}[(1+l)\mu_x - \mu_y + \mu_w] \end{cases} \quad \text{for mean} - \text{variance utility} \tag{38}$$

Incorporating Equations (36), (37) and (38) yields the following proposition.

Proposition 6: *Suppose that the insurance premium equals expected indemnity plus a proportional loading and that the initial wealth of the insured obeys a conditional normal distribution given insurable loss. The optimal retained loss schedule would then be*

$$\begin{cases} R^*(x) = \min\{x, \max\{\hat{R}(x), 0\}\} \\ \hat{R}(x) = \rho(\sigma_y/\sigma_x)(x - \mu_x) + d^* \end{cases} \tag{39}$$

for the insured with quadratic, negative exponential, or mean-variance utility functions, where ρ is the correlation coefficient between stochastic initial wealth and insurable loss, and d^* is a constant and generally varies for different utility functions.

Proposition 6 is discussed as follows. Assume that $\hat{d} \equiv d^* - \rho\mu_x(\sigma_y/\sigma_x) > 0$, hence $\hat{R}(x) = (\rho\sigma_y/\sigma_x)x + \hat{d}$. When $\rho > 0$, i.e., the stochastic initial wealth is positively correlated with the insurable loss, the stochastic wealth may offset the losses, hence providing an implicit insurance or natural hedge against losses. A positive ρ raises the sense of *homemade insurance*, as suggested by Mayers & Smith (1983). If $0 < \rho\sigma_y/\sigma_x < 1$, then the optimal insurance contract is a proportional coinsurance above a straight deductible, similar to the result in Vercammen (2001), where the coinsurance proportion equals $1 - \rho\sigma_y/\sigma_x$ and the deductible equals $\hat{d}$. In the case of $\rho < 0$, the optimal insurance contract displays a disappearing deductible, similar to the results in Gollier (1996) and Mahul (2000), where the disappearing deductible equals $\hat{d}\sigma_x/(-\rho\sigma_y)$. In this case, the insured would purchase more insurance to cover this risk arising from diminishing wealth.

Now assume that $\hat{d} < 0$, then $\hat{R}(x)$ can be positive for sufficiently large x if $\rho > 0$, while $\hat{R}(x)$ is always negative for all x if $\rho < 0$. Accordingly, when $\hat{d} < 0$ and $\rho < 0$, the optimal insurance contract would be a full insurance. The restriction $I(x) \leq x$ is most likely to bind in the $\rho < 0$ case, rather than in the $\rho > 0$ case.[9]

Assume that the insurer is risk neutral and the insured is risk averse. In the absence of background risk, Arrow (1963) and Raviv (1979) show the optimal insurance is a deductible contract and Gollier & Schlesinger (1996) further suggest that this deductible insurance dominates all other contractual forms in the sense of second-order stochastic dominance. Accordingly, this deductible insurance would also be mean-variance efficient, since second-order stochastic dominance implies mean-variance efficient. However, in the presence of background risk, the deductible insurance need not to be of second-order stochastic dominance and mean-variance efficiency, even though the normality is assumed. The intuition can be obtained using the following simple case: Suppose $\tilde{W} \sim N(\mu, \sigma^2)$. The objective function for the mean-variance framework is $E[\tilde{W}] - \lambda \mathrm{Var}(\tilde{W}) = \mu - \lambda\sigma^2$. The objective function for quadratic utility is $E[\tilde{W} - \frac{b}{2}\tilde{W}^2] = \mu - \frac{b}{2}(\mu^2 + \sigma^2)$. The objective function for negative exponential utility is $E[-\frac{1}{\gamma}\exp\{-\gamma\tilde{W}\}] = -\frac{1}{\gamma}\exp\{-\gamma\mu + \frac{1}{2}\gamma^2\sigma^2\}$. The objective functions for quadratic and negative exponential utilities are not the same as the mean-variance framework. As a consequence, the derived optimal insurances for quadratic and negative exponential utilities need not to be mean-variance efficient under an assumption of normality.

5.3. Insurance premium and investment amount

After $R^*(x)$ is obtained, the next step is to determine $\mathbf{a}_*$ and P^*, the optimal investment amount and insurance premium. From Proposition 4, it follows that in the expected utility framework $R^*(x)$ depends on $\mathbf{a}$ and P. If $R(\mathbf{a}, P, x)$ is the solution $R^*(x)$, then Equation (3) can be rewritten as

$$\underset{\mathbf{a},P}{\mathrm{Maximize}}\, \mathrm{E}[U(\tilde{W})] = \mathrm{E}[U\{\tilde{y} + \mathbf{a}^T\tilde{\mathbf{z}} - P - R(\mathbf{a}, P, \tilde{x})\}] \tag{40}$$

In the mean-variance framework, Equations (17) and (24) implies that both μ_R and P are known constants once μ_w and $\mathbf{a}$ are given. That is, $P^* = P(a, \mu_w)$ and $R^*(x) = R(\mathbf{a}, \mu_w, x)$. Substituting $P^* = P(\mathbf{a}, \mu_w)$ and $R^*(x) = R(\mathbf{a}, \mu_w, x)$into Equation (15) yields

$$\begin{aligned}\underset{\mathbf{a}}{\mathrm{Minimize}}\, \sigma_w^2 &= \mathrm{Var}[\tilde{y} + \mathbf{a}^T\tilde{\mathbf{z}} - P(\mathbf{a}, \mu_w) - R(\mathbf{a}, \mu_w, \tilde{x})] \\ &= \mathrm{Var}[\tilde{y} + \mathbf{a}^T\tilde{\mathbf{z}} - R(\mathbf{a}, \mu_w, \tilde{x})]\end{aligned}$$

where $P(\mathrm{a}, \mu_w)$ is deterministic and, hence, does not affect the variance.

Unlike Equations (3) and (15), Equations (40) and (41) are common calculus problems rather than optimal control problems. Unless the utility function $U(\cdot)$ and the joint

[9]In fact, both the cases of positive and negative correlations are commonly seen in the real world. For example, a gas station is exposed to the risk of receiving counterfeit money from customers. However, the more revenue it has (the more cash it probably receives), the more likely it will receive counterfeit money. In this case of positive correlation, the gas station does not need to purchase full insurance against this risk. However, for the negative correlation case, the insured would buy full insurance. For instance, a department store faces fire risk. If a fire does occurs, in addition to property losses, some uninsurable losses may also be incurred. Thus, the department store will attempt to purchase over insurance. If the purchase of over insurance is not allowed by the regulations, it will buy full insurance.

probability distribution of $[\tilde{x}, \tilde{y}, \tilde{\mathbf{z}}]$ obey particular specifications, Equation (40) does not have an explicit solution for $\mathbf{a}_*$ and P^*. Similarly, unless the joint probability distribution of $[\tilde{x}, \tilde{y}, \tilde{\mathbf{z}}]$ is particularly specified, $\mathbf{a}^*$ in Equation (38) cannot be solved explicitly. Nevertheless, for Equations (40) and (41), the numerical solutions are not difficult to obtain in most cases.

6. Conclusion

This study developed endogenously an optimal insurance contract under the situation in which background wealth depends on insurable loss. It was assumed that the insured is risk averse and not satiable and that the insurance premium equals expected indemnity plus a proportional loading. With these assumptions, the following results are obtained.

First, when the background wealth of the insured obeys a conditional normal distribution given insurable loss, the optimal insurance contract may be a proportional coinsurance above a straight deductible for the insured with a quadratic, negative exponential, or mean-variance utility function. Second, the contractual form concerning the optimal insurance for an insured with a quadratic utility function resembles that for an insured with a mean-variance utility function. The optimal retained schedule is a function of conditional expected value of background wealth given insurable loss. Third, the optimal insurance contracts for quadratic and negative exponential utility functions need not to be mean-variance efficient, even when a conditional normal distribution is assumed. Moreover, when background wealth is independent of insurable loss, it is shown that a deductible insurance is optimal among other feasible insurance contracts under the expected utility framework. In this situation, the deductible insurance is mean-variance efficient. In addition, when the portfolio investment problem is considered, we find that the design of an optimal insurance policy remains almost unchanged.

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Appendix 1

Proof of Proposition 1:

Let $\tilde{x}$, $\tilde{y}$, P, $R(\tilde{x})$ be the insurable loss, stochastic initial wealth, insurance premium, and retained loss schedule. The final wealth of the insured is $\tilde{W} = \tilde{y} - P - R(\tilde{x})$. The utility function $U(\tilde{W})$ is assumed to be $U'(\tilde{W}) > 0 > U''(\tilde{W})$. Additionally, the insurance premium is $P = G\{\mathrm{E}[\tilde{x} - R(\tilde{x})]\}$, where $G\{\cdot\}$ is a strictly increasing function. Accordingly, Proposition 1 claims that the optimal retained loss schedule is

$$R^*(x) = \min\{x, d^*\} \quad \text{for all}\, x \tag{A1}$$

where d^* is the solution of

$$P = G\{E[\tilde{x} - \min\{\tilde{x}, d^*\}]\} \quad \text{(A2)}$$

That is, for any retained loss schedule $R\ (x)$ such that

$$P = G\{\mathrm{E}[\tilde{x} - R(\tilde{x})]\} = G\{\mathrm{E}[\tilde{x} - R^*(\tilde{x})]\} \quad \text{(A3)}$$

One must claim that

$$\mathrm{E}[U(\tilde{y} - P - R(\tilde{x}))] \leq \mathrm{E}[U(\tilde{y} - P - R^*(\tilde{x}))] \quad \text{for all } R(x) \quad \text{(A4)}$$

For arbitrarily real values a and b, by Taylor's Theorem, we have

$$U(b) = U(a) + (b - a)U'(a) + \tfrac{1}{2}(b - a)^2 U''(c) \quad \text{(A5a)}$$

for some value c between a and b. Incorporating the fact $U''(\cdot) < 0$ into Equation (A5a) yields

$$U(b) - U(a)(b - a)U'(a) \quad \text{(A5b)}$$

Substituting a and b with $a = y - P - R^*(x)$ and $b = y - P - R(x)$ into Equation (A5b), we obtain

$$U(y - P - R(x)) - U(y - P - R^*(x))[R^*(x) - R(x)]U'(y - P - R^*(x)) \quad \text{for all } y \text{ and } x \quad \text{(A5)}$$

Additionally, we claim

$$[R^*(x) - R(x)]U'(y - P - R^*(x)) \leq [R^*(x) - R(x)]U'(y - P - d^*) \quad \text{(A6)}$$

To confirm Equation (A6), two cases, $x \geq d^*$ and $x < d^*$, are considered.
Case 1: $x \geq d^*$

If xd^*, then $R^*(x) = d^*$ and the left-hand side equals the right-hand side for Equation (A6).

Case 2. $x < d^*$

If $x < d^*$, then $R^*(x) = x \geq R(x)$. This implies $R^*(x) - R(x) \geq 0$ and $R^*(x) < d^*$. Since $U''(\cdot) < 0$, $R^*(x) < d^*$ implies

$$U'(y - P - R^*(x)) < U'(y - P - d^*) \quad \text{(A7)}$$

Combining Equation (A7) with $R^*(x) - R(x) \geq 0$ yields the inequality of Equation (A6).

Therefore, Equation (A6) is verified for all cases of x. Combining Equations (A5) and (A6) produces

$$U(y - P - R(x)) - U(y - P - R^*(x)) \leq [R^*(x) - R(x)]U'(y - P - d^*) \quad \text{(A8)}$$

Taking expectations of Equation (A8) generates

$$\mathrm{E}[U(\tilde{y} - P - R(\tilde{x}))] - \mathrm{E}[U(\tilde{y} - P - R^*(\tilde{x}))] \leq E[(R(\tilde{x}) - R^*(\tilde{x}))U'(\tilde{y} - P - d^*)] \quad \text{(A9)}$$

Since $\tilde{y}$ and $\tilde{x}$ are independent,

$$\mathrm{E}[(R(\tilde{x}) - R^*(\tilde{x}))U'(\tilde{y} - P - d^*)] = E[(R(\tilde{x}) - R^*(\tilde{x}))]E[U'(\tilde{y} - P - d^*)] \quad \text{(A10)}$$

With the assumption that $G\{\cdot\}$ is a strictly increasing function, incorporating Equation (A3) obtains

$$\mathrm{E}[\tilde{x}] - \mathrm{E}[R(\tilde{x})] = \mathrm{E}[\tilde{x} - R(\tilde{x})] = \mathrm{E}[\tilde{x} - R^{*}(\tilde{x})] = \mathrm{E}[\tilde{x}] - \mathrm{E}[R^{*}(\tilde{x})] \tag{A11}$$

Simplifying Equation (A11) yields

$$\mathrm{E}[R(\tilde{x})] = \mathrm{E}[R^{*}(\tilde{x})] \tag{A12}$$

Combining Equations (A9), (A10) and (A12) yields the result of Equation (A4). Q.E.D.

Appendix 2

Proof of Corollary 1:

First, Equation (4a) implies that P is a monotonic and increasing function of expected indemnity. Second, $\tilde{y}$ and P in Equation (4) may be considered as the stochastic initial wealth and the insurance premium for Proposition 1. These are sufficient conditions for Equation (5), based on Proposition 1. Additionally, incorporating Equations (5) and (4a) generates Equation (5a). Q.E.D.

Appendix 3

Proof of Proposition 2:

The problem is to choose an optimal $R^{*}(x)$ to maximize Equation (7), i.e.,

$$\underset{0 \le R(x) \le x}{\text{Maximize}} \ H = (\mathrm{E}[U\{\tilde{y} - P - R(\tilde{x})\}|x] + \lambda\{P - (1+l)[x - R(x)]\})f(x) \tag{A13}$$

The first order derivative is[10]

$$\partial H/\partial R(x) = (-\mathrm{E}[U'\{\tilde{y} - P - R(\tilde{x})\}|x] + \lambda(1+l))f(x) \tag{A14}$$

The second order derivative is

$$\partial^2 H/\partial R(x)^2 = \mathrm{E}[U''\{\tilde{y} - P - R(\tilde{x})\}|x]f(x) < 0 \tag{A15}$$

Referring to Equation (A14), $\hat{R}(x)$ is assumed to be the solution of

$$-\mathrm{E}[U'\{\tilde{y} - P - R(\tilde{x})\}|x] + \lambda(1+l) = 0 \tag{A16}$$

The above equation can be rewritten as

$$\mathrm{E}[U'\{\tilde{y} - P - \hat{R}(\tilde{x})\}|x] = \lambda(1+l) = c \tag{A17}$$

where the constant $c = \lambda\,(1+l)$ is independent of x. From Equations (A14), (A15), and (A16), H is globally concave in $R(x)$ and has a maximum $H_{\max}$ at $R(x) = \hat{R}(x)$. Figure 1 presents the feature of the Hamiltonian H.

[10]The Hamiltonian problem here is to find an optimal $R(x)$ for a fixed loss x. Since the loss x is fixed, $E[\tilde{y}\,|\,x]$ is accordingly fixed. Moreover, since $R(x)$ is only a posterior claim, $\mathrm{E}[\tilde{y}\,|\,x]$ is not affected by $R(x)$, implying $\partial \mathrm{E}[\tilde{y}|x]/\partial R(x) = 0$ in this case. Therefore, Equation (A14) does not include the term $\partial \mathrm{E}[\tilde{y}\,|\,x]/\partial R(x)$.

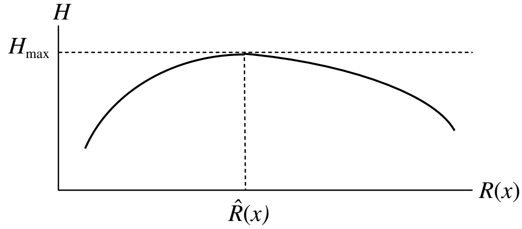

Figure 1. The feature of the Hamiltonian H in Equation (7).

Figure 1 shows that $\partial H/\partial R(x) > 0$ if $R(x) < \hat{R}(x)$, and $\partial H/\partial R(x) < 0$ if $R(x) > \hat{R}(x)$. Combining the result with the constraint of $0 \leq R(x)x$, the optimal retained schedule is

$$R^{*}(x) = \min\{x, \max\{\hat{R}(x), 0\}\} \tag{A18}$$

Incorporating Equations (A17) and (A18) yields Equation (8). Q.E.D.

Appendix 4

Proof of Proposition 3:

For convenience, Equation (21) is rewritten as follows.

$$\underset{0\leq R(x)\leq}{\text{Minimize}}\, H = \left(\sigma_y^2 - 2[(\mathrm{E}[\tilde{y}|x] - \mu_y)(R(x) - \mu_R)] + (R(x) - \mu_R)^2\right)f(x) \tag{A19}$$

The first order derivative is[11]

$$\partial H/\partial R(x) = \left(-2[(E[\tilde{y}|x] - \mu_y)] + 2(R(x) - \mu_R)\right)f(x) \tag{A20}$$

The second order derivative is

$$\partial^2 H/\partial R(x)^2 = 2f(x) > 0 \tag{A21}$$

Referring to Equation (A20), $\hat{R}(x)$ is assumed to be the solution of

$$-2[(\mathrm{E}[\tilde{y}|x] - \mu_y)] + 2(R(x) - \mu_R)] = 0 \tag{A22}$$

Equation (A22) implies that

$$\hat{R}(x) = \mathrm{E}[\tilde{y}|x] + \mu_R - \mu_y \tag{A23}$$

From Equations (A20–A23), H is globally convex in $R(x)$ and has a minimum $H_{\min}$ at $R(x) = \hat{R}(x)$. Figure 2 depicts the characteristic of the Hamiltonian H.

[11]As explained in the previous footnote, $\partial \mathrm{E}[\tilde{y}|x]/\partial R(x) = 0$ in this case.

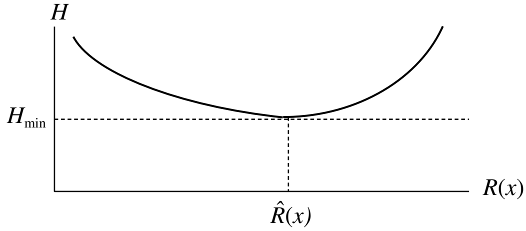

Figure 2. The feature of the Hamiltonian H in Equation (21).

The above figure shows that $\partial H/\partial R(x) < 0$ if $R(x) < \hat{R}(x)$, and $\partial H/\partial R(x) > 0$ if $R(x) > \hat{R}(x)$. Combining the result with the constraint of $0 \leq R(x) \leq x$, the optimal retained schedule is

$$R^*(x) = \min\{x, \max\{\hat{R}(x), 0\}\} \tag{A24}$$

Incorporating Equations (17), (A23), and (A24) yields Equation (22). Q.E.D.