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ScienceDirect
Discrete Mathematics 309 (2009) 1749–1759
DISCRETE MATHEMATICS
www.elsevier.com/locate/disc

Note

A note on randomized Shepp's urn scheme

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Received 15 September 2006; received in revised form 5 December 2007; accepted 15 February 2008
Available online 25 March 2008

Abstract

Shepp's urn model is a useful tool for analyzing the stopping-rule problems in economics and finance. In [R.W. Chen, A. Zame, C.T. Lin, H. Wu, A random version of Shepp's urn scheme, SIAM J. Discrete Math. 19 (1) (2005) 149–164], Chen et al. considered a random version of Shepp's urn scheme and showed that a simple drawing policy (called "the k in the hole policy") can asymptotically maximize the expected value of the game. By extending the work done by Chen et al., this note considers a more general urn scheme that is better suited to real-life price models in which the short-term value might not fluctuate. Further, "the k in the hole policy" is shown to be asymptotically optimal for this new urn scheme.

Keywords: Shepp's urn scheme; Stopping time; Optimal drawing policy; The k in the hole policy

1. Introduction

The primary Shepp's urn scheme is described as follows. Suppose an urn contains n distinguishable balls, i of which have value $+1$ and $n-i$ of which have value -1. A player knows n and i and draws balls randomly from the urn without replacement. The goal is to find a stopping rule that maximizes the expected value of the sum of the balls drawn at the stopping time. Denote the expected value of the game under the optimal play by $V(n,i)$, in [6] Shepp showed that for a given n, there exist an integer $\gamma(n)$ and a drawing policy such that $V(n,i) > 0$ if and only if $i \geq \gamma(n)$.

It is noted that Shepp's urn model has been shown useful for analyzing the stopping-rule problems in economics and finance. We highlight some extensive works below. Motivated by a bond-selling problem, in [1] Boyce introduced a random version of Shepp's urn scheme which allows an arbitrary probability distribution on the number of plus or minus balls (i or $n-i$). Assume the player knows n and the distribution of i, Boyce also proposed an algorithm to compute the player's expected value at the stopping time when he uses an optimal drawing policy. Recently, Chen et al. [3] studied the random version of Shepp's urn scheme by assuming the distribution of i to be uniform over the set $\{0,1,2,\ldots,n\}$. Due to the computational complexity of constructing the optimal drawing policy, they proposed an easy-to-implement drawing strategy called "the k in the hole policy". The idea is to continue drawing the balls until the number of -1 balls drawn is k more than the number of $+1$ balls drawn. In the study, they showed that by choosing

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0012-365X/$ - see front matter
doi:10.1016/j.disc.2008.02.017

the optimal value of k, this drawing policy asymptotically maximizes the expected value of the game, say, $\frac{n}{4}$. Further, they provided tight bounds for finding the optimal value of k so that the required computation time is largely reduced. Another notable work was done recently by Fuh et al. in [4], where a trading strategy for the volume-weighted average price (VWAP) problems (see [5] for various definitions of VWAP) was developed by utilizing the idea of "the k in the hole policy". The simulation results indicated that their proposed strategy performs well particularly for stock prices with negative drift.

In this note, we extend the urn scheme considered by Chen et al. to a new one: Suppose an urn contains n balls (n fixed), I of which have value $+1$, Z of which have value 0, $n-I-Z$ of which have value -1, and a player wishes to maximize the expected value of the sum of the balls drawn. Suppose the player does not have any information about I and Z. Therefore, it is natural to assume that I and Z are random variables and the composition (I, Z) is uniformly distributed over the set $\{(i, z) : 0 \le i + z \le n, 0 \le i, z \le n\}$ (i.e. each composition is equally likely to happen). It is noted that the urn scheme considered by Chen et al. is a special case of this new urn scheme in which $Z = 0$ with probability one. In particular, by taking into account the "0" balls, Shepp's urn model is extended so that it can better describe real-life price models in which the short-term value might not fluctuate. In Section 2, we introduce an exact optimal drawing policy and perform a recursive formula to compute the expected game value at the stopping time for the new urn scheme. However, this optimal policy reveals to be impractical due to its computational complexity (especially when n gets large). In Section 3, we show that by choosing the optimal value of k, "the k in the hole policy" is asymptotically optimal for this new urn scheme. In addition, a simple upper bound is given for finding the optimal value of k. We also provide numerical evidence to show that this policy also performs very well even when n is small. In Section 4, we show that this new urn problem is equivalent to a random 3-sided die tossing problem. Some concluding remarks are drawn in Section 5. Note that except for the main theorems, all proofs are given in the Appendix.

2. The maximal expected game value and an optimal drawing policy

In this section, we introduce an optimal drawing policy as well as how to calculate the expected value of the game for the proposed urn scheme. Suppose k balls have been drawn from the urn ($k \le n$) and i of which have value $+1$, j of which have value -1. Denote the remaining expected value of the game by $G(n, k, i, j)$ at this stage, it is clear that the optimal drawing policy can be stated as: At the beginning of the game, the player will draw a ball if and only if $G(n, 0, 0, 0) > 0$. Suppose the player has drawn k balls, i of which have value $+1$ and j of which have value -1, he will continue to draw if $G(n, k, i, j) > 0$ and stop drawing otherwise. Lemma 1 gives a recursive formula for calculating all $G(n, k, i, j)$.

Lemma 1. *If* $0 \le i + j \le k \le n - 1$, *then*

$$G(n,k,i,j) = \max\left\{0, \frac{i+1}{k+3}G(n,k+1,i+1,j) + \frac{j+1}{k+3}G(n,k+1,i,j+1) + \frac{k-i-j+1}{k+3}G(n,k+1,i,j) + \frac{i+1}{k+3} - \frac{j+1}{k+3}\right\}.$$

It is clear that $G(n, n, i, j) = 0$ for all i, j such that $0 \le i + j \le n$, since there are no balls left in the urn. Therefore, we can calculate $G(n, 0, 0, 0)$ using the recursive formula in Lemma 1. However, in order to obtain $G(n, 0, 0, 0)$, the optimal policy needs to compute $G(n, k, i, j)$ for all possible values of k, i and j This procedure requires the computation time $O(n^3)$, which is extremely inefficient when n is large. Besides, for each new n all possible values of $G(n, k, i, j)$ need to be re-computed in order to obtain $G(n, 0, 0, 0)$. This makes the optimal drawing policy very hard to implement. In the next section we will introduce a simple drawing strategy called "the k in the hole policy", and show that it can asymptotically maximize the expected value of the game. Before we proceed, the following lemma and theorem are shown useful for the rest of this study.

Lemma 2. *Let $V(n, i)$ and $V(n, z, i)$ denote the expected value of the game under the optimal drawing policy in Shepp's urn model (with only "+1" and "−1" balls) and in the new model (with "+1", "0", and "−1" balls), respectively, where n is the total number of balls, i is the number of "+1" balls, and z is the number of "0" balls. If n, i, and z are constants, then $V(n, z, i) = V(n - z, i)$.*

Theorem 1. $G(n, 0, 0, 0) \leq \frac{n}{6} + o(n)$.

Proof. Remember that the joint distribution of I and Z is $P(I = i, Z = z) = 2/(n+1)(n+2)$, $0 \leq i + z \leq n$. This implies that

$$G(n, 0, 0, 0) \leq \frac{2}{(n+1)(n+2)} \sum_{z=0}^{n} \sum_{i=0}^{n-z} V(n, z, i) = \frac{2}{(n+1)(n+2)} \sum_{z=0}^{n} \sum_{i=0}^{n-z} V(n-z, i), \tag{1}$$

where the last equality follows by Lemma 2. Divide the summation in (1) into two parts, one with $n - z$ even and the other with $n - z$ odd. Then we have that

$$G(n, 0, 0, 0) \leq \frac{2}{(n+1)(n+2)} \left\{ \sum_{z:(n-z)\text{even}} \sum_{i=0}^{n-z} V(n-z, i) + \sum_{z:(n-z)\text{odd}} \sum_{i=0}^{n-z} V(n-z, i) \right\}.$$

We first calculate $\sum_{z:(n-z)\text{even}} \sum_{i=0}^{n-z} V(n-z, i)$ in the above bracket. By the proof of Theorem 2 in [3], we know that $V(2m, i) \leq V(2m, m)$ for all $i = 0, 1, 2, \ldots, m$ and $V(2m, i) \leq 2i - 2m + V(2m, m)$ for all $i = m+1, m+2, \ldots, 2m$. This implies that

$$\begin{aligned}
\sum_{z:(n-z)\text{even}} \sum_{i=0}^{n-z} V(n-z, i) &= \sum_{z:(n-z)\text{even}} \left\{ \sum_{i=\frac{n-z}{2}+1}^{n-z} V(n-z, i) + \sum_{i=0}^{\frac{n-z}{2}} V(n-z, i) \right\} \\
&\leq \sum_{z:(n-z)\text{even}} \left\{ \sum_{i=\frac{n-z}{2}+1}^{n-z} \left[2i - (n-z) + V\left(n-z, \frac{n-z}{2}\right) \right] + \sum_{i=0}^{\frac{n-z}{2}} V\left(n-z, \frac{n-z}{2}\right) \right\}.
\end{aligned}$$

By the proof of Theorem 2 in [3], we also know that $V(2m, m) \leq \sqrt{m} + o(\sqrt{m})$. The above inequality then becomes

$$\begin{aligned}
\sum_{z:(n-z)\text{even}} \sum_{i=0}^{n-z} V(n-z, i) &\leq \sum_{z:(n-z)\text{even}} \left\{ \sum_{i=\frac{n-z}{2}+1}^{n-z} [2i - (n-z)] + (n-z+1)\sqrt{\frac{n-z}{2}} + o(n^{3/2}) \right\} \\
&= \sum_{z:(n-z)\text{even}} \left\{ \frac{(n-z)^2}{4} + (n-z+1)\sqrt{\frac{n-z}{2}} + o(n^{3/2}) \right\}.
\end{aligned}$$

Since the calculation for the other term $\sum_{z:(n-z)\text{ is odd}} \sum_{i=0}^{n-z} V(n-z, i)$ is quite similar, the primary inequality can be summarized as

$$\begin{aligned}
G(n, 0, 0, 0) &\leq \frac{2}{(n+1)(n+2)} \sum_{z=0}^{n} \left\{ \frac{(n-z)^2}{4} + O(n^{3/2}) + o(n^{3/2}) \right\} \\
&\leq \frac{2}{(n+1)(n+2)} \left\{ \frac{1}{4} \cdot \frac{n(n+1)(2n+1)}{6} + O(n^{5/2}) + o(n^{5/2}) \right\} \\
&\leq \frac{n}{6} + o(n). \quad \square
\end{aligned}$$

3. An asymptotically optimal solution: The *k* in the hole policy

Let S_m^+ denote the number of "+1" balls and S_m^- denote the number of "−1" balls for the first m draws. "The k in the hole policy", which was proposed by Chen et al. in [3], states that the player will continue drawing the ball until $S_m^- - S_m^+ > k$. The intuition why this drawing policy works for our new urn scheme is described as follows. Note that by the assumption of uniformity on (I, Z), it can be easily shown that $E[Z] = \frac{n}{3}$. Heuristically, our new urn scheme can be viewed as the urn scheme considered by Chen et al. with total number of balls $\frac{2}{3}n$ (in average sense). Thus, the result in [3] shows that "the k in the hole policy" can asymptotically achieve the expected game value $(\frac{2}{3}n)/4 = \frac{n}{6}$, which agrees with the maximal expected game value in Theorem 1. Another intuition of this policy is that, when the observed outcome is a certain amount "in the hole", probabilistically the remaining "−1" balls should be more than the remaining "+1" balls. This means that the remaining expected value of the game is more likely to be negative, which suggests the player to stop drawing the balls.

Denote $H(n,k)$ to be the expected value of the game by using "the k in the hole policy", we next show how to calculate $H(n,k)$.

Theorem 2. *For any given integers n and k such that $1 \le k \le n$,*

$$H(n,k) = \frac{1}{\binom{n+2}{2}} \cdot \left\{ \sum_{z:(n-z-k)\text{even}} \left[\frac{(n-z-k+2)(n-z-k)}{4} - \sum_{j=\frac{(n-z-k)}{2}}^{n-z-k} (2j+k-n+z) \frac{\binom{n-z}{k+j}}{\binom{n-z}{j}} \right] \right.$$
$$\left. + \sum_{z:(n-z-k)\text{odd}} \left[\frac{(n-z-k+1)^2}{4} - \sum_{j=\frac{(n-z-k+1)}{2}}^{n-z-k} (2j+k-n+z) \frac{\binom{n-z}{k+j}}{\binom{n-z}{j}} \right] \right\}.$$

Proof. Since we assume that (I,Z) is uniformly distributed over the set $\{(i,z) : 0 \le i+z \le n,\ 0 \le i,\ z \le n\}$, this directly implies $P(I=i, Z=z) = \frac{2}{(n+1)(n+2)}$ and $P(I=i|Z=z) = \frac{1}{n-z+1}$ for all possible i and z. Thus, the marginal distribution of Z is

$$P(Z=z) = \frac{P(I=i,Z=z)}{P(I=i|Z=z)} = \frac{2}{(n+1)(n+2)}(n-z+1), \quad 0 \le z \le n.$$

Let $W(n,k)$ denote the expected value of the game by using "the k in the hole policy" in the urn scheme considered by Chen et al. [3] (i.e. $Z=0$ with probability one and I is uniformly distributed over the set $\{0,1,\ldots,n\}$). It is clear that $H(n,k|Z=z) = W(n-z,k)$, where from Theorem 3 in [3] we know that for each integer $1 \le k \le n-z$, if $n-z-k$ is even,

$$W(n-z,k) = \frac{1}{n-z+1}\left\{ \frac{(n-z-k+2)(n-z-k)}{4} - \sum_{j=(n-z-k)/2}^{n-z-k} (2j+k-n+z)\frac{\binom{n-z}{k+j}}{\binom{n-z}{j}} \right\}, \tag{2}$$

and if $n-z-k$ is odd,

$$W(n-z,k) = \frac{1}{n-z+1}\left\{ \frac{(n-z-k+1)^2}{4} - \sum_{j=(n-z-k+1)/2}^{n-z-k} (2j+k-n+z)\frac{\binom{n-z}{k+j}}{\binom{n-z}{j}} \right\}. \tag{3}$$

Note that

$$\begin{aligned} H(n,k) &= \sum_{z=0}^{n} P(Z=z)H(n,k|Z=z) = \sum_{z=0}^{n} \frac{2(n-z+1)}{(n+1)(n+2)} W(n-z,k) \\ &= \sum_{z=0}^{n-k} \frac{2(n-z+1)}{(n+1)(n+2)} W(n-z,k) + \sum_{z=n-k+1}^{n} \frac{2(n-z+1)}{(n+1)(n+2)} W(n-z,k), \end{aligned}$$

where the second summation in the last equality is clearly zero. The result is then established by dividing the first summation into two parts ($n-z-k$ is even and $n-z-k$ is odd) and plugging in (2) and (3). □

Note that for any given n, the performance of "the k in the hole policy" clearly depends on the choice of k. Let $k_n^* = \arg\max_{1 \le k \le n} H(n,k)$, we then have the following theorem.

Theorem 3. $H(n,k_n^*) \sim \frac{n}{6}$ *as $n \to \infty$.*

Proof. We first show that $H(n,k_n^*) \ge \frac{n}{6} + o(n)$. Without loss of generality, we assume that $n = m^2 \ge 9$ for some positive integer m (i.e. $\sqrt{n}$ is an integer). For each $1 \le k \le n$, we then have that

$$H(n,k_n^*) \ge H(n,k) = \sum_{z=0}^{n-\sqrt{n}} \frac{2(n-z+1)}{(n+1)(n+2)} W(n-z,k) + \sum_{z=n-\sqrt{n}+1}^{n-k} \frac{2(n-z+1)}{(n+1)(n+2)} W(n-z,k).$$

Let us examine the above equation term-by-term. Note that from Theorem 5 in [3], $\frac{4W(n,k)}{n} \sim (1-\frac{k}{n})^2 - \frac{b_k}{(k+1)(k+2)}$ as $n \to \infty$, where $\frac{1}{2} < b_k < 4$. If k is chosen appropriately, say, let $k = [n^{1/3}]$ (the largest integer less than or equal to $n^{1/3}$), then we have that $W(n-z,k) = \frac{n-z}{4} + o(n)$ in the first summation as $n \to \infty$. Therefore, we have that

$$\begin{aligned}
&\sum_{z=0}^{n-\sqrt{n}} \frac{2(n-z+1)}{(n+1)(n+2)} W(n-z,[n^{1/3}]) = \sum_{z=0}^{n-\sqrt{n}} \frac{2(n-z+1)}{(n+1)(n+2)} \left[\frac{n-z}{4} + o(n)\right] \\
&= \frac{1}{2(n+1)(n+2)} \sum_{z=0}^{n-\sqrt{n}} [(n-z)(n-z+1) + 4(n-z+1)o(n)] \\
&= \frac{1}{2(n+1)(n+2)} \sum_{i=\sqrt{n}}^{n} [i(i+1) + 4(i+1)o(n)] \\
&= \frac{1}{2(n+1)(n+2)} \left\{\frac{n(n+1)(n+2)}{3} - \frac{(\sqrt{n}-1)\sqrt{n}(\sqrt{n}+1)}{3} + \frac{4o(n)(n+\sqrt{n}+2)(n-\sqrt{n}+1)}{2}\right\} \\
&= \frac{n}{6} + o(n).
\end{aligned}$$

On the other hand, it is clear that

$$\sum_{z=n-\sqrt{n}+1}^{n-k} \frac{2(n-z+1)}{(n+1)(n+2)} W(n-z,k) \geq 0.$$

So we conclude that $H(n,k_n^*) \geq H(n,[n^{1/3}]) \geq \frac{n}{6} + o(n)$. From Theorem 1 we also know that $H(n,k_n^*) \leq \frac{n}{6} + o(n)$. These then imply $H(n,k_n^*) \sim \frac{n}{6}$ as $n \to \infty$. □

Theorem 3 shows that by choosing $k = k_n^*$, "the k in the hole policy" has the game value approximately $\frac{n}{6}$ (as $n \to \infty$), which agrees with the maximal game value shown in Theorem 1. So we conclude that "the k_n^* in the hole policy" is asymptotically optimal for the proposed urn scheme. The next question then becomes how to identify k_n^* for any given n. A naive approach is to compute $H(n,k)$ for all $1 \leq k \leq n$ and then find the best k that maximizes $H(n,k)$. However, such a procedure reveals to be quite inefficient — especially when n is large. The following theorem provides a simple upper bound for finding k_n^*, which allows us to reduce a large amount of computation.

Theorem 4. *For any given n, $k_n^* \leq n^{1/3}$.*

In order to prove Theorem 4, the following two facts are needed. We show them without proof, since the result can be obtained directly from Theorem 7 in [3].

Fact 1. *Recall that $W(n,k)$ is the expected value of the game by using "the k in the hole policy" in the urn scheme considered by Chen et al. For all k such that $1 \leq k \leq n$ and $n-k$ is even, there exists a k_n' such that $W(n,k)$ is increasing with k for $k \leq k_n'$ and $W(n,k)$ is decreasing with k for $k > k_n'$.*

Fact 2. *For all k such that $1 \leq k \leq n$ and $n-k$ is odd, there exists a k_n' such that $W(n,k)$ is increasing with k for $k < k_n'$ and $W(n,k)$ is decreasing with k for $k > k_n'$.*

Now we are ready to prove Theorem 4.

Proof of Theorem 4. For any given n, let us denote k_n to be the optimal choice of k which maximizes $W(n,k)$. The result of Theorem 8 in [3] shows that k_n has an upper bound $\bar{k}_n = \min\{k : 1 \leq k \leq n, (n+1)(n+k)^2 \leq 2(n-k)^2k^3\}$. To solve $\bar{k}_n$, note that

$$\begin{aligned}
&(n+1)(n+k)^2 \leq 2(n-k)^2k^3 \\
&\Rightarrow (k^2+2nk+n^2)(n+1) \leq 2(n^2-2nk+k^2)k^3 \\
&\Rightarrow 2k^5 - 4nk^4 + 2n^2k^3 - (n+1)k^2 - 2n(n+1)k - n^2(n+1) \geq 0.
\end{aligned}$$

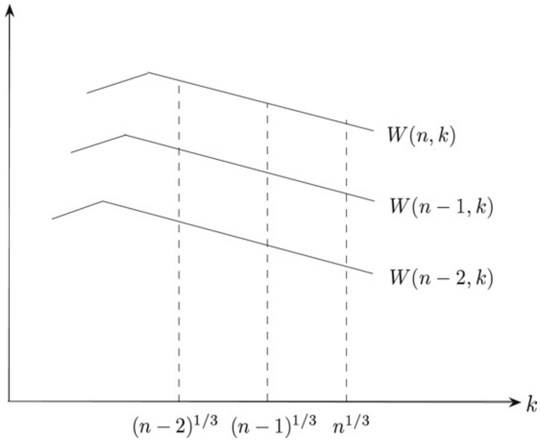

Fig. 1. A geometric display of all possible $W(n-z, k)$.

By choosing $k = n^{1/3}$, it is clear that the above inequality holds when n is large. Specifically, a simple numerical examination shows that this is true for $n \geq 17$. This implies that $k_n \leq \bar{k}_n \leq n^{1/3}$ for $n \geq 17$. In addition, a direct computation shows that $k_n \leq n^{1/3}$ for $n \leq 16$. So we conclude that $k_n \leq n^{1/3}$ for any given n. Now we show this is also true for k_n^*. Recall that (in the proof of Theorem 2) for any $1 \leq k \leq n$,

$$H(n, k) = \frac{2}{(n+1)(n+2)} \sum_{z=0}^{n} (n-z+1) W(n-z, k),$$

which appears to be a linear combination of all possible $W(n-z, k)$. In addition, $W(n-z, k)$ has the following important features. First, from Fact 1 we know that when $n-z-k$ is even, $W(n-z, k) \geq W(n-z, k+2)$ for all $k \geq (n-z)^{1/3}$. Second, from Fact 2 we know that when $n-z-k$ is odd, $W(n-z, k+1) \geq W(n-z, k+3)$ for all $k \geq (n-z)^{1/3}$. Finally, from Theorem 4 in [3] we know that $W(n, k)$ is increasing with n for any fixed k. Based on these features, a geometric display of all $W(n-z, k)$ is shown in Fig. 1. Since $n-z+1$ is independent of k and $H(n, k)$ is a linear combination of all possible $W(n-z, k)$, from Fig. 1 it is clear that the optimal k which maximizes $H(n, k)$ (i.e. k_n^*) is bounded above by $n^{1/3}$. $\square$

Theorem 4 shows that we can easily identify k_n^* by merely checking out $H(n, k)$ for $k \leq n^{1/3}$. To evaluate the performance of "the k_n^* in the hole policy", the numerical values of $H(n, k)$, $G(n, 0, 0, 0)$, k_n^*, and $n^{1/3}$ for different values of n and k, are provided in Table 1. From Table 1, we can see that for all possible values of n, the difference between the expected game value under the optimal drawing policy and the expected game value under "the k_n^* in the hole policy" is less than 1. This indicates that "the k_n^* in the hole policy" performs very well, even when n is small. Another useful result from Table 1 is that k_n^* is increasing with n. Suppose that we already know $k_{n_1}^*$ for a given urn with n_1 balls and we wish to find $k_{n_2}^*$ for another urn with n_2 balls, $n_2 > n_1$. "The k in the hole policy" needs only extra computation time $(n_2)^{1/3} - k_{n_1}^* + 1$ to identify $k_{n_2}^*$, since we know that $k_{n_2}^* \geq k_{n_1}^*$. On the other hand, the exact optimal drawing policy has to re-compute all possible $G(n_2, k, i, j)$, which requires extra computation time $O(n_2^3)$.

4. A random 3-sided die tossing problem

Motivated by a random coin tossing problem in [3], in this section we show that the proposed new urn scheme is equivalent to the following random die tossing problem. Suppose a player is given a 3-sided die with each side corresponding to a score +1, 0, and −1, respectively. He is allowed to toss the die at most n times but he can stop any time as he wishes. Suppose for each toss he gets +1 with probability P_1, 0 with probability P_2, and −1 with probability $1 - P_1 - P_2$. He does not know exactly (P_1, P_2) but he knows a priori that (P_1, P_2) is uniformly distributed over the set $\{(p_1, p_2) : 0 \leq p_1 + p_2 \leq 1, 0 \leq p_1 \leq 1, 0 \leq p_2 \leq 1\}$. This implies that (P_1, P_2) has the joint probability density function $\pi(p_1, p_2) = 2,\ 0 \leq p_1 + p_2 \leq 1, 0 \leq p_1 \leq 1, 0 \leq p_2 \leq 1$. The goal of the game is to maximize the expected value of the sum of the scores at the stopping time.

Table 1
The numerical results of $H(n,k)$, $G(n,0,0,0)$, k_n^*, and $[n^{1/3}]$ for different values of n

n	$H(n,1)$	$H(n,2)$	$H(n,3)$	$H(n,4)$	$H(n,5)$	$G(n,0,0,0)$	k_n^*	$[n^{1/3}]$
10	**1.02**	0.91	0.69	0.48	0.31	**1.02**	1	2
20	2.29	**2.35**	2.12	1.83	1.55	**2.38**	2	2
30	3.57	**3.84**	3.64	3.35	3.03	**3.85**	2	3
40	4.86	**5.34**	5.20	4.91	4.58	**5.35**	2	3
50	6.14	**6.85**	6.77	6.49	6.16	**6.87**	2	3
60	7.43	**8.36**	8.34	8.09	7.76	**8.41**	2	3
70	8.71	9.87	**9.92**	9.69	9.37	**9.97**	3	4
80	10.00	11.38	**11.50**	11.30	10.98	**11.54**	3	4
90	11.29	12.89	**13.08**	12.91	12.60	**13.12**	3	4
100	12.58	14.41	**14.67**	14.52	14.22	**14.71**	3	4
200	25.45	29.56	30.54	**30.68**	30.52	**30.77**	4	5
300	38.32	44.72	46.42	**46.87**	46.86	**47.04**	4	6
400	51.20	59.88	62.30	63.06	**63.20**	**63.39**	5	7
500	64.08	75.04	78.19	79.25	**79.55**	**79.80**	5	7
600	76.95	90.20	94.08	95.45	95.91	**96.25**	6	8
700	89.83	105.36	109.96	111.64	112.26	**112.72**	6	8
800	102.71	120.52	125.85	127.84	128.61	**129.22**	6	9
900	115.58	135.68	141.74	144.04	144.97	**145.73**	7	9
1000	128.46	150.84	157.63	160.24	161.32	**162.26**	7	10

n	$H(n,6)$	$H(n,7)$	$H(n,8)$	$H(n,9)$	$H(n,10)$	$G(n,0,0,0)$	k_n^*	$[n^{1/3}]$
10	0.19	0.10	0.04	0.01	0	**1.02**	1	2
20	1.30	1.07	0.86	0.68	0.53	**2.38**	2	2
30	2.72	2.42	2.14	1.88	1.64	**3.85**	2	3
40	4.24	3.91	3.59	3.28	2.99	**5.35**	2	3
50	5.81	5.45	5.11	4.77	4.45	**6.87**	2	3
60	7.40	7.03	6.67	6.32	5.97	**8.41**	2	3
70	9.00	8.63	8.26	7.89	7.53	**9.97**	3	4
80	10.62	10.24	9.86	9.48	9.11	**11.54**	3	4
90	12.24	11.86	11.47	11.09	10.70	**13.12**	3	4
100	13.87	13.49	13.10	12.70	12.31	**14.71**	3	4
200	30.23	29.87	29.49	29.08	28.66	**30.77**	4	5
300	46.65	46.34	45.97	45.58	45.17	**47.04**	4	6
400	63.08	62.82	62.49	62.11	61.71	**63.39**	5	7
500	79.52	79.31	79.01	78.66	78.27	**79.80**	5	7
600	**95.96**	95.81	95.54	95.21	94.84	**96.25**	6	8
700	**112.40**	112.30	112.08	11.77	111.41	**112.72**	6	8
800	**128.84**	128.80	128.61	28.33	127.99	**129.22**	6	9
900	145.29	**145.30**	145.15	44.89	144.57	**145.73**	7	9
1000	161.73	**161.80**	161.68	61.45	161.15	**162.26**	7	10

For a given positive integer n, suppose the die has been tossed k times ($k \le n$) and i of which have score $+1$, j of which have score -1. Let $D(n,k,i,j)$ denote the remaining expected value of the game, analogous to that shown in Section 2, an optimal tossing policy is: The player will continue to toss the die if and only if $D(n,k,i,j) > 0$. Given n, the following lemma states that $D(n,k,i,j) = G(n,k,i,j)$ for all possible k, i, and j.

Lemma 3. *If* $0 \le i + j \le k \le n - 1$, *then* $D(n,k,i,j) = G(n,k,i,j)$.

Lemma 3 shows that this random die tossing problem is equivalent to our urn problem. Let $T(n,k)$ be the expected value of this game using "the k in the hole policy", the next theorem shows that it is equivalent to $H(n,k)$ in the previous urn problem.

Theorem 5. *For all* $1 \le k \le n$, $T(n,k) = H(n,k)$.

Proof. Let z be the number of getting score 0 and i be the number of getting score $+1$ in n tosses. If the player knows z, then for each $0 \le i \le n-z$, the expected value at the stopping time when using "the k in the hole policy" is the same as that of tossing a random coin $n-z$ times (i of which have score $+1$ and $n-z-i$ of which have score -1). Let us denote this conditional expected value by $C(n-z,k,i)$. Also, given that (p_1, p_2), let us denote the probability of getting i tosses of $+1$ and z tosses of 0 by $P(i,z|p_1,p_2)$. It is clear that

$$P(i,z|p_1,p_2) = \frac{n!}{i!z!(n-i-z)!} p_1^i p_2^z (1-p_1-p_2)^{n-i-z}.$$

Therefore, we have that

$$\begin{aligned}
T(n,k) &= \int\int_{0\le p_1+p_2\le 1} \sum_{z=0}^{n}\sum_{i=0}^{n-z} C(n-z,k,i)\pi(p_1,p_2)P(i,z|p_1,p_2)\mathrm{d}p_1\mathrm{d}p_2 \\
&= 2\sum_{z=0}^{n}\sum_{i=0}^{n-z} \frac{n!C(n-z,k,i)}{i!z!(n-i-z)!} \int\int_{0\le p_1+p_2\le 1} p_1^i p_2^z (1-p_1-p_2)^{n-i-z}\mathrm{d}p_1\mathrm{d}p_2 \\
&= 2\sum_{z=0}^{n}\sum_{i=0}^{n-z} \frac{n!C(n-z,k,i)}{i!z!(n-i-z)!} \frac{\Gamma(i+1)\Gamma(z+1)\Gamma(n-i-z+1)}{\Gamma(n+3)} \\
&= \frac{1}{\binom{n+2}{2}} \sum_{z=0}^{n}\sum_{i=0}^{n-z} C(n-z,k,i). \qquad (4)
\end{aligned}$$

From Theorem 11 in [3], we know that if $n-z-k$ is even,

$$\sum_{i=0}^{n-z} C(n-z,k,i) = \frac{(n-z-k+2)(n-z-k)}{4} - \sum_{j=\frac{(n-z-k)}{2}}^{n-z-k} (2j+k-n+z)\frac{\binom{n-z}{k+j}}{\binom{n-z}{j}},$$

and if $n-z-k$ is odd,

$$\sum_{i=0}^{n-z} C(n-z,k,i) = \frac{(n-z-k+1)^2}{4} - \sum_{j=\frac{(n-z-k+1)}{2}}^{n-z-k} (2j+k-n+z)\frac{\binom{n-z}{k+j}}{\binom{n-z}{j}}.$$

By dividing (4) into two parts (one for $n-z-k$ is even and the other for $n-z-k$ is odd) and plugging in the above results, we see that $T(n,k) = H(n,k)$. $\square$

5. Concluding remarks

In this note, we consider a new version of Shepp's urn scheme by adding the "0" balls with the assumption of uniform random composition of the urn. The proposed urn scheme reveals to be more realistic than the existing approaches developed for modeling real-life prices, since it takes into account the fact that the short-term value might not fluctuate. Some important results are summarized next. First, the maximal expected value of the proposed urn scheme is approximately $\frac{n}{6}$, as $n \to \infty$. Second, by choosing the best value of k, "the k in the hole policy" is shown to be asymptotically optimal for this new urn scheme. Further, a simple upper bound for finding the optimal value of k is provided. Although the bound is not as tight as those provided by Chen et al. in [3] (due to the complexity of the new urn scheme), it allows us to reduce a large amount of computation so that "the k in the hole policy" can be easily implemented. Third, in comparison with the optimal drawing policy, the numerical results show that by choosing the optimal k, "the k in the hole policy" performs very well — even when n is small. Finally, we show that the proposed urn scheme is equivalent to a random 3-sided die tossing problem. We believe that our new urn scheme and the results obtained here will be useful for analyzing the volume-weighted average price (VWAP) problems. There are other issues for future research: (i) how to construct the (asymptotically) optimal drawing policy for the proposed urn scheme when the counts of the balls do not follow the uniform distribution; and (ii) how to analyze the urn problem by considering different scores of balls.

Acknowledgment

This research was supported by NSC Grant 94-2118-M-008-004.

Appendix

Proof of Lemma 1. If the player decides to stop the game after the first k draws, the remaining expected value of the game is then 0. If he decides to draw another ball and we denote the probability that the ball is $+1$ by a, the probability that the ball is -1 by b, the probability that the ball is 0 by $c = 1 - a - b$, the remaining expected value of the game is then

$$aG(n, k+1, i+1, j) + bG(n, k+1, i, j+1) + cG(n, k+1, i, j) + a - b.$$

Therefore, it suffices to show that $a = \frac{i+1}{k+3}$ and $b = \frac{j+1}{k+3}$. Define the random variable X_i to be the value of the ith ball drawn, and let S_k^+ be the number of "+1" balls, S_k^- be the number of "−1" balls in the first k draws. Then we have that

$$a = P(X_{k+1} = 1 | S_k^+ = i, S_k^- = j) = \frac{P(S_k^+ = i, S_k^- = j, X_{k+1} = 1)}{P(S_k^+ = i, S_k^- = j)}.$$

We first calculate the denominator of the above equality. Remember we denote the number of "+1" balls by I, the number of "0" balls by Z, and thus the number of "−1" balls by $n - I - Z$. Therefore, we have that

$$\begin{aligned} P(S_k^+ = i, S_k^- = j) &= \sum_{s=0}^{n} \sum_{t=0}^{n-s} P(S_k^+ = i, S_k^- = j, I = s, n - I - Z = t) \\ &= \sum_{s=0}^{n} \sum_{t=0}^{n-s} P(S_k^+ = i, S_k^- = j | I = s, Z = n - s - t) P(I = s, Z = n - s - t). \end{aligned}$$

Since $P(I = s, Z = n - s - t) = \frac{2}{(n+1)(n+2)}$, we then have that

$$\begin{aligned} P(S_k^+ = i, S_k^- = j) &= \frac{2}{(n+1)(n+2)} \sum_{s=0}^{n} \sum_{t=0}^{n-s} \frac{\binom{s}{i}\binom{t}{j}\binom{n-s-t}{k-i-j}}{\binom{n}{k}} \\ &= \frac{2}{(n+1)(n+2)\binom{n}{k}} \sum_{s=i}^{n-(k-i)} \binom{s}{i} \sum_{t=j}^{(n-s)-(k-i)+j} \binom{t}{j} \binom{(n-s)-t}{(k-i)-j} \\ &= \frac{2}{(n+1)(n+2)\binom{n}{k}} \sum_{s=i}^{n-(k-i)} \binom{s}{i} \binom{n-s+1}{k-i+1} \\ &= \frac{2}{(n+1)(n+2)\binom{n}{k}} \sum_{s=i}^{(n+1)-(k+1)+i} \binom{s}{i} \binom{(n+1)-s}{(k+1)-i} \\ &= \frac{2}{(n+1)(n+2)\binom{n}{k}} \binom{n+2}{k+2} = \frac{2}{(k+1)(k+2)}. \end{aligned}$$

Note that a similar calculation yields

$$P(S_k^+ = i, S_k^- = j, X_{k+1} = 1) = \frac{2(i+1)}{(k+1)(k+2)(k+3)}.$$

So we have that

$$a = \left\{ \frac{2(i+1)}{(k+1)(k+2)(k+3)} \right\} \Big/ \left\{ \frac{2}{(k+1)(k+2)} \right\} = \frac{i+1}{k+3}.$$

Due to the symmetric property of two events $\{X_{k+1} = 1\}$ and $\{X_{k+1} = -1\}$, the same procedure can be used to obtain that $b = \frac{j+1}{k+3}$. The proof of Lemma 1 is then complete. □

Proof of Lemma 2. We prove Lemma 2 by induction. From page 299 in [2], we know that

$$V(n-z,i)=\max\left\{0,\frac{n-z-i}{n-z}[-1+V(n-z-1,i)]+\frac{i}{n-z}[1+V(n-z-1,i-1)]\right\}. \tag{5}$$

Analogously, we can obtain that

$$V(n,z,i)=\max\left\{0,\frac{n-z-i}{n}[-1+V(n-1,z,i)]+\frac{i}{n}[1+V(n-1,z,i-1)]+\frac{z}{n}V(n-1,z-1,i)\right\}. \tag{6}$$

We start by checking the initial condition that $n=\max\{z,i\}$. If $\max\{z,i\}=z$, it is clear that $V(n,z,0)=V(z,z,0)=0=V(0,0)$. If $\max\{z,i\}=i$, it is clear that $V(n,z,i)=V(i,0,i)=i=V(i,i)$. Assume that $V(m,z,i)=V(m-z,i)$ for any given z,i, and some $m>\max\{z,i\}$, now we show that $V(m+1,z,i)=V(m+1-z,i)$. From (6), we have that

$$\begin{aligned}
&V(m+1,z,i)\\
&\quad=\max\left\{0,\frac{m+1-z-i}{m+1}[-1+V(m,z,i)]+\frac{i}{m+1}[1+V(m,z,i-1)]+\frac{z}{m+1}V(m,z-1,i)\right\}\\
&\quad=\max\left\{0,\frac{m+1-z-i}{m+1}[-1+V(m-z,i)]\right.\\
&\qquad\left.+\ \frac{i}{m+1}[1+V(m-z,i-1)]+\frac{z}{m+1}V(m+1-z,i)\right\}.
\end{aligned} \tag{7}$$

From (5), we have that

$$V(m+1-z,i)=\max\left\{0,\frac{m+1-z-i}{m+1-z}[-1+V(m-z,i)]+\frac{i}{m+1-z}[1+V(m-z,i-1)]\right\},$$

which directly implies

$$\begin{aligned}
\frac{m+1-z}{m+1}V(m+1-z,i)\ &=\ \max\left\{0,\frac{m+1-z-i}{m+1}[-1+V(m-z,i)]\right.\\
&\qquad\left.+\ \frac{i}{m+1}[1+V(m-z,i-1)]\right\}.
\end{aligned} \tag{8}$$

Adding (7) and (8) together, we have that

$$\begin{aligned}
V(m+1,z,i)\ &=\ \max\left\{0,\frac{m+1-z}{m+1}V(m+1-z,i)+\frac{z}{m+1}V(m+1-z,i)\right\}\\
&=\ V(m+1-z,i).
\end{aligned}$$

The proof is then complete. □

Proof of Lemma 3. Analogous to the proof of Lemma 1, if the player decides to stop the game after the first k tosses, the remaining expected value of the game is then 0. If he decides to toss the die once more and we denote the probability of getting $+1$ by a, the probability of getting -1 by b, the probability of getting 0 by $c=1-a-b$, the remaining expected value of the game is then

$$aD(n,k+1,i+1,j)+bD(n,k+1,i,j+1)+cD(n,k+1,i,j)+a-b.$$

Since $D(n,n,i,j)=G(n,n,i,j)=0$ for all i,j such that $0\le i+j\le n$, it suffices to show $a=\frac{i+1}{k+3}$ and $b=\frac{j+1}{k+3}$. Again, we show merely the result for a since the result for b can be obtained by symmetry. Note that

$$a=P(X_{k+1}=1|S_k^+=i,S_k^-=j)=\frac{P(S_k^+=i,S_k^-=j,X_{k+1}=1)}{P(S_k^+=i,S_k^-=j)},$$

where X_i is the score of the ith toss, S_k^+ is the number of getting score $+1$ and S_k^- is the number of getting score -1 in the first k tosses. Since

$$\begin{aligned}
P(S_k^+ = i, S_k^- = j) &= \int\int_{0\le p_1+p_2\le 1} P(S_k^+ = i, S_k^- = j, p_1, p_2)\mathrm{d}p_1\mathrm{d}p_2 \\
&= \int\int_{0\le p_1+p_2\le 1} \pi(p_1, p_2) P(S_k^+ = i, S_k^- = j | p_1, p_2)\mathrm{d}p_1\mathrm{d}p_2 \\
&= 2\int\int_{0\le p_1+p_2\le 1} \frac{k!}{i!(k-i-j)!j!} p_1^i p_2^{k-i-j}(1-p_1-p_2)^j \mathrm{d}p_1\mathrm{d}p_2 \\
&= 2\cdot \frac{k!}{i!(k-i-j)!j!}\cdot \frac{\Gamma(i+1)\Gamma(k-i-j+1)\Gamma(j+1)}{\Gamma(k+3)} \\
&= \frac{2}{(k+1)(k+2)},
\end{aligned} \tag{9}$$

and a similar calculation gives that

$$P(S_k^+ = i, S_k^- = j, X_{k+1} = 1) = \frac{2(i+1)}{(k+1)(k+2)(k+3)}. \tag{10}$$

Dividing (10) by (9), we obtain that $a = \frac{i+1}{k+3}$. □

References

[1] W.M. Boyce, Stopping rules for selling bonds, Bell J. Econ. Manage Sci. 1 (1970) 27–53.
[2] W.M. Boyce, On a simple optimal stopping problem, Discrete Math. 5 (1973) 297–312.
[3] R.W. Chen, A. Zame, C.T. Lin, H. Wu, A random version of Shepp's urn scheme, SIAM J. Discrete Math. 19 (1) (2005) 149–164.
[4] C.D. Fuh, H.W. Teng, R.H. Wang, On-line VWAP trading strategies, preprint.
[5] A. Madhavan, Transaction performance: The changing face of trading investment guides series, in: VWAP stategies, Institutional Investor Inc., 2002, pp. 32–38.
[6] L.A. Shepp, Explicit solutions to some problems of optimal stopping, Ann. Math. Statist. 40 (1969) 993–1010.