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Valuation of the interest rate guarantee embedded in defined contribution pension plans

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Abstract

In this research, we derive the valuation formulae for a defined contribution pension plan associated with the minimum rate of return guarantees. Different from the previous studies, we work on the rate of return guarantee which is linked to the δ-year spot rate. The payoffs of interest rate guarantees can be viewed as a function of the exchange option. By employing Margrabe's [Margrabe, W., 1978. The value of an option to exchange one asset for another. Journal of Finance 33, 177–186] option pricing approach, we derive general pricing formulae under the assumptions that the interest rate dynamics follow a single-factor HJM (1992) [Heath. D. et al., 1992. Bond pricing and the term structure of interest rates: a new methodology for contingent claims valuation. Econometrica 60, 77–105] interest rate model and the asset prices follow a geometric Brownian motion. The volatility of the forward rates is assumed to be exponentially decaying. The formula is explicit for valuing maturity guarantee (type-I guarantee). For multi-period guarantee (type-II guarantee), the analytical formula only exists when the guaranteed rate is the one-year spot rate. The accuracy of the valuation formulae is illustrated with numerical analysis. We also investigate the effect of mortality and the sensitivity of key parameters on the value of the guarantee. We find that type-II guarantee is much more costly than the type-I guarantee, especially with a long duration policy. The closed form solution provides the advantage in valuing pension guarantees.

1. Introduction

In the final decades of the twentieth century, a global wave of pension reforms privatized pension obligations. The retirement pension system had traditionally been tied to defined benefit pension (DB) plans, but the critical financial burdens for DB pension plan providers caused countries to convert their retirement systems from DB to defined contribution (DC) plans. The main difference between DB and DC pension plans relates to the way they treat financial risk. With a DB plan, the retirement benefit is promised in advance, according to some predetermined formulae. Thus, employers bear the risks of poor investment performance by the pension fund. In contrast, employees with a DC plan bear the investment risk, because the retirement benefits depend on the performance of investment portfolios. The main argument in support of converting from DB to DC is that the employee must suffer investment risks. Thus, to avoid the downside risk for the employee, some guarantees have been provided with DC plans, the cost of which normally is covered by the pension plan provider. Estimating the value of the guarantee is very important for the pension plan provider to set its budget, because a poor estimation may cause it to suffer significant financial problems. Within this context, we analyze the values of guarantees in the DC plans.

In practice, there are a variety of guarantee designs in DC plans. Some are deterministic manner, whereas others employ a stochastic rate of return on a reference portfolio or interest rates. The former is called an absolute guarantee, whereas the latter is referred to as a relative guarantee in the literature (Lindset, 2004). Previous research into valuing guarantees for pension funds or life insurance products focuses on absolute guarantees, with which a fund delivers a fixed or prespecified minimum rate of return. Prior studies on absolute guarantees include Brennan and Schwartz (1976), Boyle and Schwartz (1977), Boyle and Hardy (1997), Persson and Aase (1997), Miltersen and Persson (1999), Grosen and Jørgensen (1997), Grosen and Jørgensen

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(2000), Hansen and Miltersen (2002), and Schrager and Pelsser (2004). However, setting a deterministic minimum rate of return in advance makes these ostensibly defined contribution plans more like defined benefit schemes. The problem with granting a deterministic guaranteed rate is that low guaranteed rates are not attractive to plan participants, but high guaranteed rates might cause financial burdens to the plan provider. Therefore, a stochastic guaranteed rate, such as rate of return guarantees relative to the reference portfolio, is more popular in recent development. Although relative rate of return guarantees are very common, especially in Latin America, they have not received the same research focus as absolute guarantees, with the exceptions of Ekern and Persson (1996), Pennacchi (1999) and Lindset (2004). Ekern and Persson (1996) use martingale approach to deal with the relative guarantee for a single premium unit-linked policy. Pennacchi (1999) uses a contingent claim analysis to obtain the values of both the absolute and the relative guarantee provided in the Chilean and Uruguayan DC plans. Lindset (2004) studies the various minimum guaranteed rates of return and adopts the Heath–Jarrow–Morton (Heath et al., 1992) framework to derive explicit formulae for a single deposit case. For all these three papers, the guaranteed rate of return represents the performance of the equity market. Thus, they all assume that the dynamics of the price of the reference portfolio follow a geometric Brownian motion. We extend their analysis by setting up a theoretical framework to study the rate of return guarantees relative to a return measured by the market realized δ-year spot rates and the guarantee applies to all contributions in the accumulation period of a pension plan.

Rate of return guarantees consists of two fundamentally different types. A maturity guarantee is binding only at the expiration of the contract and ensures a minimum pension payment to the participants in the contract, whereas a multi-period guarantee works on a periodic basis and secures a periodic return not less than a guaranteed minimum return. We consider both maturity and multi-period relative rate of return guarantees herein. In particular, we analyze the value of the relative rate of return guarantee on a defined contribution scheme, according to which a fixed proportion of a participant's wage is assumed to be credited to an underlying investment portfolio. The guaranteed rate is set as the market δ-year spot rates over the contract period, which is quite different from the existing literature, which links the guaranteed rate to the stochastic return of equity markets. After relating the guaranteed rate to the market realized δ-year spot rate, we attain essentially an interest rate guarantee. This method of valuing the interest rate type of relative guarantees is more complicated than that of valuing equity-type relative guarantees, because both of the rate of return process for the investment portfolio of the pension fund and the interest rate process of the guarantee must be incorporated into the model. We further assume that the value of the underlying investment portfolio follows a geometric Brownian motion and adopt Heath, Jarrow and Morton's (HJM) (Heath et al., 1992) framework with an exponentially decaying forward rate volatility to model the term structure of interest rates, from which the stochastic guaranteed δ-year spot rates can be derived. After showing that the values of both types of relative guarantees have similar payoff structures of an exchange option, we employ Margrabe (1978) exchange option pricing formula to derive the values of both guarantees explicitly. The values of both guarantees are demonstrated in the numerical results. In addition, mortality improvement has caused a lot of financial problem for the life insurer or pension fund provider. We study the effect of mortality improvement on the value of the guarantee in our numerical analysis.

The contributions of this paper relative to the previous works on relative guarantees are fourfold. First, we obtain a valuation formula for the guarantee allowed for a stochastic guaranteed rate of return linked to the interest rate, in addition to a stochastic rate of return on the investment portfolio of a pension fund. Specifically, we consider a minimum rate of return guarantee based on the market realized δ-year spot rates. Second, we consider guarantees associated with DC plans. In the literature, works on interest rate guarantees are based more often on single-premium insurance contracts. For the guarantee with a DC pension plan, its value depends on the contribution during the working period. The contributing payments make the guarantee depend on the the asset price and guaranteed δ-year spot rates at different time points. Incorporating these two main points, we provide closed-form solutions for interest rate guarantees, both maturity and multi-period guarantees. Third, we value the pension guarantee more efficiently by using a closed-form solution. In practice, the duration during which a participant joins the pension is very long, say 30 years, which might require a lot of time to compute the value of the guarantee using simulations. In this research, we illustrate the valuation using both a closed-form solution and simulation. Fourth, the effect of mortality is investigated in this research. Because pension contracts have long maturities, the effect of mortality can't be ignored.

The structure of this paper is as follows. Section 2 describes both maturity and multi-period rate of return guarantees embedded in a defined contribution pension plan and presents the underlying economic model. In Section 3, we derive the values of two types of relative guarantees using Margrabe's exchange option pricing formula. In Section 4, we provide numerical results and sensitivity analysis for the values of these guarantees. Section 5 concludes this paper.

2. Structure of guarantee and financial model settings

Under a DC pension plan associated with a relative guarantee, the accumulation of the pension fund depends on the contributions, investment performance, guaranteed rate, and working period. We analyze the relative rates of return guarantees linked to the market δ-year spot rates and consider two methods for crediting the guaranteed rate to the pension fund. If the guaranteed minimum rate is binding only at the expiration of the contract, we call it a maturity guarantee and refer it as a type-I guarantee; if the guaranteed minimum rate of return is binding for each period, we refer to it as a multi-period guarantee or type-II guarantee. Intuitively, a type-II guarantee is more valuable than a type-I guarantee.

Assume that an employee contributes a fixed proportion of his or her annual salary to the pension fund each year and the employee's salary increases each year. To express the terminal payoffs of the two types of guaranteed pension plans in a more precise way, we denote c as the contribution rate, Y_0 as the initial yearly wage, i as the yearly wage growth rate, R_t as the actual rate of return on the underlying investment portfolio at time t, and $\overline{R}(t-1, t-1+\delta)$ as the δ-year spot rate at time $t-1$, that is, the minimum guaranteed rate of return on the pension fund at time $t-1$. The δ-year spot rate, $\overline{R}(t-1, t-1+\delta)$, is chosen along with R_t, the realized return on an investment fund, to define the contract payoff. $\overline{R}(t-1, t-1+\delta)$ can be regarded as the δ-year constant maturity Treasury (CMT) rate, which focuses on an interest rate on securities with specified maturities. We choose the spot rate of constant maturity δ-year, $\overline{R}(t-1, t-1+\delta)$, as the reference rate for interest rate guarantees considered in this paper. The fixed and constant maturity feature of $\overline{R}(t-1, t-1+\delta)$ avoids the difficulties in calculating returns of a physical bond with diminishing remaining life to maturities. Moreover, rates with constant maturity features generally facilitate the trading and hedging of pure interest rate instruments for a selected maturity, without reducing time to maturity associated with physical bonds. The CMT rate could also be used to speculate on an increase or decrease in the market-driven risk premium built into LIBOR. It allows the counterparties to participate in changes in the shape of the yield curve.[1]

Under a type-I guarantee, the pension fund at retirement date after T-year's accumulation, $H_T(I)$, can be expressed as

$$\begin{aligned} H_T(I) &= \sum_{n=1}^{T} c \cdot Y_0 (1+i)^{n-1} \\ &\quad \times \max\left[\prod_{t=n}^{T} \exp(\overline{R}(t-1, t-1+\delta)), \prod_{t=n}^{T} \exp(R_t)\right] \\ &= \sum_{n=1}^{T} c \cdot Y_0 (1+i)^{n-1} \\ &\quad \times \left\{\max\left[\prod_{t=n}^{T} \exp(\overline{R}(t-1, t-1+\delta)) - \prod_{t=n}^{T} \exp(R_t), 0\right] + \prod_{t=n}^{T} \exp(R_t)\right\}. \end{aligned} \tag{1}$$

Under a type-II guarantee, the pension fund at retirement date after T-year's accumulation, $H_T(II)$, can be expressed as

$$\begin{aligned} H_T(II) &= \sum_{n=1}^{T} c \cdot Y_0 (1+i)^{n-1} \\ &\quad \times \prod_{t=n}^{T} \max\left[\exp(\overline{R}(t-1, t-1+\delta)), \exp(R_t)\right] \\ &= \sum_{n=1}^{T} c \cdot Y_0 (1+i)^{n-1} \\ &\quad \times \left\{\left[\prod_{t=n}^{T} \max\left[\exp\left(\overline{R}(t-1, t-1+\delta)\right), \exp(R_t)\right] - \prod_{t=n}^{T} \exp(R_t)\right] + \prod_{t=n}^{T} \exp(R_t)\right\}. \end{aligned} \tag{2}$$

Eqs. (1) and (2) show that the payoffs of the defined contribution pension plans with rate of return guarantees can be viewed as a function of the exchange option. We make use of Margrabe (1978) option pricing model to derive the values of the guarantees. We describe the modelling framework and derive the values of both guarantees in the next section.

In Eqs. (1) and (2), we find that the payoffs of the guaranteed defined contribution pension plans depend on both the actual rate of return of the underlying investment portfolio (R_t) and the minimum guaranteed rate of the δ-year spot rate ($\overline{R}(t-1, t-1+\delta)$) at time $t-1$. The actual rate of return is measured by the market value of the investment portfolio. To value the guarantee, we assume that the market value of the underlying investment portfolio $S_\bullet$ under the risk-neutral probability measure Q follows the stochastic process

$$\mathrm{d}S_t = r_t S_t \mathrm{d}t + \sigma_S S_t \mathrm{d}Z_t, \tag{3}$$

where r_t is the risk-free rate of return on the investment portfolio, $\sigma_S \in R^+$ is the volatility of investment return, and Z_t is a one-dimensional standard Brownian motion under the risk-neutral measure Q. The actual rate of return on the investment portfolio in period j, R_j, therefore is defined by

$$R_j = \ln\left(\frac{S_j}{S_{j-1}}\right), \quad j = 1, 2, \ldots, T.$$

When the underlying investment portfolio $S_\bullet$ follows the geometric Brownian motion, R_j is normally distributed.

In addition, the pension fund is guaranteed a δ-year spot rate. We assume that the evolution of the term structure of the interest rate is captured by a model from the single-factor Heath, Jarrow, and Morton (HJM) (Heath et al., 1992) framework. The HJM framework describes the dynamics of the forward rate curve $f(t, T)$, $0 \le t \le T$. In the HJM setting, the evolution of the forward rate curve under the risk-neutral measure Q is modeled as

$$\mathrm{d}f(t, u) = \mu_f(t, u)\mathrm{d}t + \sigma_f(t, u)\mathrm{d}W_t,$$

where W_t is a one-dimensional standard Brownian motion under measure Q, $\mu_f(t, u)$ is the drift term, and $\sigma_f(t, u)$ is the instantaneous volatility of the forward rate with maturity u. r_t, the short rate at time t, is given by

$$r(t) = f(t, t) = f(0, t) + \int_0^t \mu_f(s, t)\mathrm{d}s + \int_0^t \sigma_f(s, t)\mathrm{d}W_s$$

according to the HJM framework.

[1] Due to the liquidity of interest rate swaps relative to Treasury securities, swap rates with constant maturities, commonly known as CMS rates, are widely used in the market. The calculation of CMS rates involves convexity corrections. Pelsser (2003) proposes a theoretical framework to demonstrate that convexity adjustment can be interpreted as the side-effect of a change of probability measure.

In this framework, the time t price of a zero-coupon bond maturing at time u is computed as

$$P(t,u) = \exp\left[-\int_t^u f(t,s)\mathrm{d}s\right].$$

HJM also derive the restriction on the drift term $\mu_f(t,u)$ imposed by the absence of arbitrage as $\mu_f(t,u) = \sigma_f(t,u)\int_t^u \sigma_f(t,x)\mathrm{d}x$. Thus, arbitrage-free dynamics of the forward rate curve under the risk-neutral probability measure Q become

$$\mathrm{d}f(t,u) = \left(\sigma_f(t,u)\int_t^u \sigma_f(t,x)\mathrm{d}x\right)\mathrm{d}t + \sigma_f(t,u)\mathrm{d}W_t. \quad (4)$$

Eq. (4) indicates that the critical factor in determining the behaviour of a model under the HJM framework is the forward rate volatility $\sigma_f(t,u)$. Therefore an HJM model is essentially a specification of the forward rate volatility structure.

In this paper, we consider the specific case of exponentially decaying forward rate volatility, that is $\sigma_f(t,u) = \sigma \mathrm{e}^{-\lambda(u-t)} > 0$, where σ and λ are constants, and $\sigma_f(t,u)$ is independent of the level of the forward rate. In the literature, the constant volatility assumption underlying the HJM model has been rejected by the empirical research, for example Flesaker (1993). Moreover, the empirical research shows that long-maturity rates have a lower volatility than short-maturity rates. Therefore, we incorporate the exponentially decaying feature in the forward rate volatility of the HJM model.

By virtue of Eq. (4), the dynamics of the forward rate process under the measure Q are given by

$$\mathrm{d}f(t,u) = \left(\sigma^2 \mathrm{e}^{-\lambda(u-t)}\int_t^u \mathrm{e}^{-\lambda(x-t)}\mathrm{d}x\right)\mathrm{d}t + \sigma \mathrm{e}^{-\lambda(u-t)}\mathrm{d}W_t,$$

and the short rate r_t satisfies

$$r_t = f(0,t) + \frac{\sigma^2}{2\lambda^2}\left(1-\mathrm{e}^{-\lambda t}\right)^2 + \sigma\int_0^t \mathrm{e}^{-\lambda(t-v)}W_v \quad (5)$$

The time t price of a zero-coupon bond maturing at time u equals

$$\begin{aligned} P(t,u) = {} & \frac{P(0,u)}{P(0,t)} \times \exp\left\{-\frac{\sigma^2}{\lambda^3}\left[\left(\mathrm{e}^{\lambda t}-1\right)\left(\mathrm{e}^{-\lambda t}-\mathrm{e}^{-\lambda u}\right) - \frac{\left(\mathrm{e}^{2\lambda t}-1\right)\left(\mathrm{e}^{-2\lambda t}-\mathrm{e}^{-2\lambda u}\right)}{4}\right]\right\} \\ & \times \exp\left[-\frac{\sigma}{\lambda}\int_0^t \mathrm{e}^{\lambda v}\left(\mathrm{e}^{-\lambda t}-\mathrm{e}^{-\lambda u}\right)\mathrm{d}W_v\right], \end{aligned}$$

and the equivalent continuously compounded yield, or the time t spot rate for maturity u, is given by

$$\begin{aligned} \overline{R}(t,u) &= -\frac{\ln P(t,u)}{u-t} \\ &= \frac{\ln P(0,t) - \ln P(0,u)}{u-t} + \frac{\sigma}{\lambda(u-t)}\int_0^t \mathrm{e}^{\lambda v}\left(\mathrm{e}^{-\lambda t}-\mathrm{e}^{-\lambda u}\right)\mathrm{d}W_v \\ &\quad + \frac{\sigma^2}{\lambda^3(u-t)}\left[\left(\mathrm{e}^{\lambda t}-1\right)\left(\mathrm{e}^{-\lambda t}-\mathrm{e}^{-\lambda u}\right) - \frac{\left(\mathrm{e}^{2\lambda t}-1\right)\left(\mathrm{e}^{-2\lambda t}-\mathrm{e}^{-2\lambda u}\right)}{4}\right]. \end{aligned} \quad (6)$$

In summary, the stochastic processes for the investment portfolio S_t and the forward rate $f(t,u)$ are of the forms

$$\mathrm{d}S_t = r_t S_t \mathrm{d}t + \sigma_S S_t \mathrm{d}Z_t$$

and

$$\mathrm{d}f(t,u) = \left(\sigma^2 \mathrm{e}^{-\lambda(u-t)}\int_t^u \mathrm{e}^{-\lambda(x-t)}\mathrm{d}x\right)\mathrm{d}t + \sigma \mathrm{e}^{-\lambda(u-t)}\mathrm{d}W_t,$$

where Z_t and W_t are the two correlated Brownian motions satisfying $\mathrm{d}Z_t\mathrm{d}W_t = \rho\mathrm{d}t$ for all $\rho \in [-1,+1]$. This implies that $\mathrm{d}Z_t = \rho\mathrm{d}W_t + \sqrt{1-\rho^2}\mathrm{d}\widehat{W}_t$, where $\widehat{W}_t$ is a Q-Brownian motion independent of W_t.

Both investment portfolio return rates and interest rates are assumed to be Gaussian in this research. In this setting, analytical results for valuing interest rate guarantees embedded in the defined contribution pension plan can be derived, as we do in the next section.

3. Valuations of relative guarantees

In Section 2, we show that the payoffs of the interest rate guarantees under a defined contribution pension plan can be viewed as a function of exchange options. In this section, we apply Margrabe (1978) option pricing model to the derivation of the values of both the maturity and multi-period guarantees embedded in a defined contribution pension plan.

3.1. Maturity guarantee (type-I guarantee)

Within the setting for a maturity guarantee, we define $V_T(I)$ as the terminal account value of the type-I guarantee embedded in the defined contribution pension plan, which is

$$\begin{aligned} V_T(I) = {} & \sum_{n=1}^{T} c \cdot Y_0 (1+i)^{n-1} \\ & \times \left\{\max\left[\prod_{t=n}^{T}\exp(\overline{R}(t-1,t-1+\delta)) - \prod_{t=n}^{T}\exp(R_t), 0\right]\right\}. \end{aligned} \quad (7)$$

To value the interest rate guarantee embedded in the defined contribution pension plan, we first focus on the value of per unit of the embedded guarantee. Define $\pi_T^{(n)}(I)$ as the time T value of the guarantee for \$1 contributed in the nth year, $n \le T$; thus, the value of $\pi_T^{(n)}(I)$ is

$$\pi_T^{(n)}(I) = \max\left[\prod_{t=n}^{T}\exp(\overline{R}(t-1,t-1+\delta)) - \prod_{t=n}^{T}\exp(R_t), 0\right]. \quad (8)$$

Expressed in this way, the payoff structure of this interest rate guarantee is analogous to that of an exchange option that gives the participant the right to exchange the risky asset $\prod_{t=n}^{T}\exp(R_t)$ for another asset $\prod_{t=n}^{T}\exp(\overline{R}(t-1,t-1+\delta))$.

We denote the first asset in the exchange option as $A_{1,T}^{(n)}$, which is the time T value of the payoff from the guarantee for \$1 contributed in the nth year, that is, $A_{1,T}^{(n)} = \prod_{t=n}^{T}\exp(\overline{R}(t-1,t-1+\delta))$. We use $A_{2,T}^{(n)}$ to express the second asset in the exchange option, which is the time T payoff of investing \$1 in the nth year in the investment portfolio $S_\bullet$ with return rate R_j for the jth period, or $A_{2,T}^{(n)} = \prod_{t=n}^{T}\exp(R_t)$. Accordingly, the time T value of an embedded interest rate guarantee for \$1 contributed in the nth year can be viewed as an exchange option with payoff structure $\pi_T^{(n)}(I) = \max\left[A_{1,T}^{(n)} - A_{2,T}^{(n)}, 0\right]$ at time T. In this section, we attempt to find the initial market value of the embedded rate of return guarantee $V_0(I)$, which is equal to $V_0(I) = \sum_{n=1}^{T} c\cdot Y_0(1+i)^{n-1}\times\pi_0^{(n)}(I)$.

After some straightforward calculation, we can show that $A_{2,T}^{(n)} = \prod_{t=n}^{T}\exp(R_t) = (\frac{S_T}{S_{n-1}})$, with the defined return rate $R_t = \ln(\frac{S_t}{S_{t-1}})$. Note that $A_{2,T}^{(n)}$ is a lognormally distributed variable. If we treat the asset of $A_{2,T}^{(n)}$ as numeraire, the valuation problem can be reduced to that of a one-asset option, as

$$\frac{\pi_0^{(n)}(I)}{A_{2,0}^{(n)}} = E_0^{\widehat{P}}\left[\frac{\pi_T^{(n)}(I)}{A_{2,T}^{(n)}}\right],$$

and

$$\pi_0^{(n)}(I) = A_{2,0}^{(n)}\times E^{\widehat{P}}\left[\frac{\max\left(A_{1,T}^{(n)} - A_{2,T}^{(n)}, 0\right)}{A_{2,T}^{(n)}}\right] = A_{2,0}^{(n)}\times E^{\widehat{P}}\left[\max\left(\frac{A_{1,T}^{(n)}}{A_{2,T}^{(n)}} - 1, 0\right)\right],$$

where $E_0^{\widehat{P}}[\cdot]$ denotes the expectation function under $\widehat{P}$ measure conditional on the market information up to time 0.

In other words, given the $A_{2,T}^{(n)}$ as numeraire, the process of $\frac{\pi_\bullet^{(n)}(I)}{A_{2,\bullet}^{(n)}}$ under $\widehat{P}$ measure is a martingale. Accordingly, we define a new random variable $U_T = \frac{A_{1,T}^{(n)}}{A_{2,T}^{(n)}}$, which is again lognormal as both $A_{1,T}^{(n)}$ and $A_{2,T}^{(n)}$ are lognormally distributed under our setting. To this end, the valuation problem can be considered as the pricing of a standard European call option with a lognormally distributed underlying asset $U_\bullet$ and with unit strike price.

Proposition 1. *The initial market value of the type-I guarantee with time T payoff as* $\pi_T^{(n)}(I) = \max\left[\prod_{t=n}^{T}\exp\left(\overline{R}(t-1, t-1+\delta)\right) - \prod_{t=n}^{T}\exp(R_t), 0\right] = \max\left[A_{1,T}^{(n)} - A_{2,T}^{(n)}, 0\right]$ *is given by*

$$\pi_0^{(n)}(I) = A_{1,0}^{(n)}N\left(d_1^{(n)}\right) - A_{2,0}^{(n)}N\left(d_2^{(n)}\right),$$

where

$$\begin{aligned}A_{1,0}^{(n)} = {} & \exp[g_2(n,T)]\cdot\exp\left\{\frac{1}{2}\left(\frac{1-e^{-\lambda\delta}}{\lambda\delta}\right)^2\right.\\ & \times \left.\mathrm{Var}_0^Q\left[\sum_{t=n}^{T}\left(\sigma\int_0^{t-1}e^{-\lambda(t-1-v)}\mathrm{d}W_v\right)\right]\right\}\\ & \times\exp\left\{\frac{1}{2}\mathrm{Var}_0^Q\left[\sigma\int_0^T\int_0^u e^{-\lambda(u-v)}\mathrm{d}W_v\mathrm{d}u\right]\right\}\\ & \times\exp\left\{-\left(\frac{1-e^{-\lambda\delta}}{\lambda\delta}\right)\right.\\ & \mathrm{Cov}_0^Q\left[\sum_{t=n}^{T}\left(\sigma\int_0^{t-1}e^{-\lambda(t-1-v)}\mathrm{d}W_v\right),\right.\\ & \left.\left.\sigma\int_0^T\int_0^u e^{-\lambda(u-v)}\mathrm{d}W_v\mathrm{d}u\right]\right\}\end{aligned}$$

$$A_{2,0}^{(n)} = \exp\left(-\int_0^{n-1}f(0,u)\,\mathrm{d}u\right)$$

$$d_1^{(n)} = \frac{\ln\left(\frac{A_{1,0}^{(n)}}{A_{2,0}^{(n)}}\right) + \frac{\left(\widehat{\sigma}_I^{(n)}\right)^2 T}{2}}{\left(\widehat{\sigma}_I^{(n)}\right)\sqrt{T}},\qquad d_2^{(n)} = d_1^{(n)} - \left(\widehat{\sigma}_I^{(n)}\right)\sqrt{T}$$

$$\begin{aligned}\left(\widehat{\sigma}_I^{(n)}\right)^2 T = {} & \mathrm{Var}_0^Q\left(\ln A_{1,T}^{(n)}\right) + \mathrm{Var}_0^Q\left(\ln A_{2,T}^{(n)}\right)\\ & - 2\mathrm{Cov}_0^Q\left(\ln A_{1,T}^{(n)}, \ln A_{2,T}^{(n)}\right),\end{aligned}$$

$$\begin{aligned}g_1(t) = {} & \frac{\ln P(0,t-1) - \ln P(0,t-1+\delta)}{\delta}\\ & + \frac{\sigma^2}{\lambda^3\delta}\left[\left(e^{\lambda(t-1)} - 1\right)\left(e^{-\lambda(t-1)} - e^{-\lambda(t-1+\delta)}\right)\right.\\ & \left. - \frac{\left(e^{2\lambda(t-1)} - 1\right)\left(e^{-2\lambda(t-1)} - e^{-2\lambda(t-1+\delta)}\right)}{4}\right]\end{aligned}$$

$$\begin{aligned}g_2(n,T) = {} & \sum_{t=n}^{T}g_1(t) - \int_0^T f(0,u)\,\mathrm{d}u - \frac{\sigma^2}{2\lambda^2}\\ & \times\left[T - \frac{2}{\lambda}\left(1 - e^{-\lambda T}\right) + \frac{1}{2\lambda}\left(1 - e^{-2\lambda T}\right)\right]\end{aligned}$$

$$\begin{aligned}&\mathrm{Var}_0^Q\left[\sum_{t=n}^{T}\left(\sigma\int_0^{t-1}e^{-\lambda(t-1-v)}\mathrm{d}W_v\right)\right]\\ &\quad = \frac{\sigma^2}{2\lambda}\sum_{i=n}^{T}\sum_{j=n}^{T}\left[e^{-\lambda(i+j-2)}\left(e^{2\lambda\min(i-1,j-1)} - 1\right)\right]\end{aligned}$$

$$\begin{aligned}&\mathrm{Var}_0^Q\left[\sigma\int_0^T\int_0^u e^{-\lambda(u-v)}\mathrm{d}W_v\mathrm{d}u\right]\\ &\quad = \frac{\sigma^2}{\lambda^2}\left[T - \frac{2}{\lambda}\left(1 - e^{-\lambda T}\right) + \frac{1}{2\lambda}\left(1 - e^{-2\lambda T}\right)\right]\end{aligned}$$

$$\mathrm{Cov}_0^Q\left[\sum_{t=n}^{T}\left(\sigma\int_0^{t-1}\mathrm{e}^{-\lambda(t-1-v)}\mathrm{d}W_v\right),\right.$$
$$\left.\sigma\int_0^T\int_0^u \mathrm{e}^{-\lambda(u-v)}\mathrm{d}W_v\mathrm{d}u\right]$$
$$=\frac{\sigma^2}{\lambda^2}\left[(T-n+1)-\left(\frac{\mathrm{e}^{-\lambda T}-\mathrm{e}^{-\lambda(n-1)}}{\mathrm{e}^{-\lambda}-1}\right)\right.$$
$$\left.-\left(\frac{1-\mathrm{e}^{-\lambda(T-n+1)}}{2(\mathrm{e}^{\lambda}-1)}\right)+\left(\frac{\mathrm{e}^{-2\lambda T}-\mathrm{e}^{-\lambda(T+n-1)}}{2(\mathrm{e}^{-\lambda}-1)}\right)\right]$$

$$\mathrm{Var}_0^Q\left(\ln A_{1,T}^{(n)}\right)=\frac{\sigma^2}{2\lambda}\left(\frac{1-\mathrm{e}^{-\lambda\delta}}{\lambda\delta}\right)^2$$
$$\times\sum_{i=n}^{T}\sum_{j=n}^{T}\left[\mathrm{e}^{-\lambda(i+j-2)}\left(\mathrm{e}^{2\lambda\min(i-1,j-1)}-1\right)\right]$$

$$\mathrm{Var}_0^Q\left(\ln A_{2,T}^{(n)}\right)=\frac{\sigma^2}{2\lambda^3}\left[\mathrm{e}^{-\lambda T}-\mathrm{e}^{-\lambda(n-1)}\right]^2\left[\mathrm{e}^{2\lambda(n-1)}-1\right]$$
$$+\frac{\sigma^2}{\lambda^2}\left[(T-n+1)-\left(\frac{2(1-\mathrm{e}^{-\lambda(T-n+1)})}{\lambda}\right)\right.$$
$$\left.+\left(\frac{1-\mathrm{e}^{-2\lambda(T-n+1)}}{2\lambda}\right)\right]+\sigma_S^2(T-n+1)$$
$$+\frac{2\rho\sigma\sigma_S}{\lambda}\left[(T-n+1)-\left(\frac{1-\mathrm{e}^{-\lambda(T-n+1)}}{\lambda}\right)\right]$$

$$\mathrm{Cov}_0^Q\left(\ln A_{1,T}^{(n)},\ln A_{2,T}^{(n)}\right)=\sigma\left(\frac{1-\mathrm{e}^{-\lambda\delta}}{\lambda\delta}\right)$$
$$\times\sum_{t=n}^{T}\left[\left(\frac{1-\mathrm{e}^{-\lambda(t-n)}}{\lambda}\right)\left(\frac{\sigma}{\lambda}+\rho\sigma_S\right)\right.$$
$$+\frac{\sigma}{2\lambda^2}\left(\mathrm{e}^{-\lambda(T+t-1)}+\mathrm{e}^{-\lambda(t-n)}\right.$$
$$\left.\left.-\ \mathrm{e}^{-\lambda(t+n-2)}-\mathrm{e}^{-\lambda(T-t+1)}\right)\right].$$

$N(\cdot)$ is the cumulative probability function of a standardized normal distributed variable, and $\widehat{\sigma}_I^{(n)}\sqrt{T}$ is the volatility of the composite asset $U_T=\frac{A_{1,T}^{(n)}}{A_{2,T}^{(n)}}$.

Proof. The proof follows the same lines as that offered by Margrabe (1978). However, because neither both $A_{1,0}^{(n)}$ or $A_{2,0}^{(n)}$ is a tradable asset in the market, we derive their values in Appendices A and B. The variances of $\ln A_{1,0}^{(n)}$ and $\ln A_{2,0}^{(n)}$ are derived in Appendix C, and the covariance between $\ln A_{1,0}^{(n)}$ and $\ln A_{2,0}^{(n)}$ is calculated in Appendix D. Finally, we calculate the volatility term $\widehat{\sigma}_I^{(n)}\sqrt{T}$ of the composite asset $U_T=\frac{A_{1,T}^{(n)}}{A_{2,T}^{(n)}}$, which is the relative value of $A_{1,T}^{(n)}$ with respect to $A_{2,T}^{(n)}$, in Appendix E. □

We further consider the effect of mortality on the market value of a type-I guarantee in Corollary 1.

Corollary 1. *Assume that the financial market is independent of the employee's mortality risk. For an employee aged x at time 0, the initial market value of the type-I guarantee with payoff $\widetilde{\pi}_T^{(n)}(I)$ at time T is*

$$\widetilde{\pi}_0^{(n)}(I)={}_Tp_x\cdot\left[A_{1,0}^{(n)}N\left(d_1^{(n)}\right)-A_{2,0}^{(n)}N\left(d_2^{(n)}\right)\right],$$

where ${}_Tp_x$ denotes the survival probability that an employee aged x remains alive after T years. ${}_Tp_x$ can be expressed as $\Pr(T_x>T)$, where the random variable T_x denotes the remaining life time of an x-year-old employee.

3.2. Multi-period guarantee (type-II guarantee)

In this section, we derive the value of the multi-period guarantee embedded in the defined contribution pension plan.

We define $V_T(II)$ as the terminal account value of the multi-period guarantee embedded in the defined contribution pension plan, which is

$$V_T(II)=\sum_{n=1}^{T}c\cdot Y_0(1+i)^{n-1}\times\left\{\prod_{t=n}^{T}\max\left[\exp\left(\overline{R}(t-1,t-1+\delta)\right),\right.\right.$$
$$\left.\left.\exp\ (R_t)\right]-\prod_{t=n}^{T}\exp(R_t)\right\}. \quad (9)$$

To value the multi-period guarantee embedded in the defined contribution pension plan, we first focus on the value of per unit of the embedded guarantee. Define $\pi_T^{(n)}(II)$ as the time T value of the type-II guarantee for \$1 contributed in the nth year, $n\leq T$; then, the value of $\pi_T^{(n)}(II)$ is

$$\pi_T^{(n)}(II)=\prod_{t=n}^{T}\max\left[\exp\left(\overline{R}(t-1,t-1+\delta)\right),\exp(R_t)\right]-\prod_{t=n}^{T}\exp(R_t). \quad (10)$$

Expressed in this way, the payoff structure of a multi-period rate of return guarantee can be viewed as the difference between the accumulated value of a series of exchange options and the accumulated realized pension portfolio value. Using the no-arbitrage pricing principle, the initial market value of $\pi_0^{(n)}(II)$ can be calculated as $\frac{\pi_0^{(n)}(II)}{B_0}=E_0^Q\left[\frac{\pi_T^{(n)}(II)}{B_T}\right]$. The initial value of $V_T(II)$ is equal to $V_0(II)=\sum_{n=1}^{T}Y_0\times c(1+i)^{n-1}\times\pi_0^{(n)}(II)$.

Unfortunately, the analytical formula for $\pi_0^{(n)}(II)$ cannot be derived for arbitrary values of δ. The analytical formula is only available for the special case that the interest rate guarantee is linked to the one-year spot rate, i.e. $\delta=1$. Therefore, to value a type-II guarantee when $\delta\neq 1$, the simulation method has to be employed. In the following proposition, we calculate the initial market value of the type-II guarantee for the case of $\delta=1$.

Proposition 2. *The initial market value of the type-II guarantee with time T payoff as* $\pi_T^{(n)}(II) = \prod_{t=n}^{T} \max\left[\exp\left(\overline{R}(t-1,t)\right), \exp(R_t)\right] - \prod_{t=n}^{T}\exp(R_t)$ *is given by*

$$\pi_0^{(n)}(II) = \exp\left(-\int_0^{n-1} f(0,u)\,\mathrm{d}u\right) \times \left\{\prod_{t=n}^{T}\left[A_{3,0}N(d_3(t)) - A_{4,0}N(d_4(t)) + A_{4,0}\right] - 1\right\},$$

where

$$A_{3,0} = \exp\left\{g_3(t) + \frac{\sigma^2}{2\lambda^2}\left[1 - \frac{2}{\lambda}\left(1-\mathrm{e}^{-\lambda}\right) + \frac{1}{2\lambda}\left(1-\mathrm{e}^{-2\lambda}\right)\right]\right\}$$

$$A_{4,0} = 1$$

$$g_3(t) = g_1(t) - \int_{t-1}^{t} f(0,u)\,\mathrm{d}u - \frac{\sigma^2}{2\lambda^2} \times \left[1 + \frac{2}{\lambda}\left(\mathrm{e}^{-\lambda t} - \mathrm{e}^{-\lambda(t-1)}\right) - \frac{1}{2\lambda}\left(\mathrm{e}^{-2\lambda t} - \mathrm{e}^{-2\lambda(t-1)}\right)\right]$$

$$g_1(t) = \ln P(0,t-1) - \ln P(0,t) + \frac{\sigma^2}{\lambda^3}\left[\left(\mathrm{e}^{\lambda(t-1)} - 1\right)\left(\mathrm{e}^{-\lambda(t-1)} - \mathrm{e}^{-\lambda t}\right) - \frac{\left(\mathrm{e}^{2\lambda(t-1)} - 1\right)\left(\mathrm{e}^{-2\lambda(t-1)} - \mathrm{e}^{-2\lambda t}\right)}{4}\right]$$

$$d_3(t) = \frac{\ln\left(A_{3,0}/A_{4,0}\right) + \widehat{\sigma}_{II}^2 t/2}{\widehat{\sigma}_{II}\sqrt{t}}$$

$$d_4(t) = d_3(t) - \widehat{\sigma}_{II}\sqrt{t}$$

$$\widehat{\sigma}_{II}^2 t = \frac{\sigma^2}{\lambda^2}\left[1 - \frac{2}{\lambda}\left(1-\mathrm{e}^{-\lambda}\right) + \frac{1}{2\lambda}\left(1-\mathrm{e}^{-2\lambda}\right)\right] + \sigma_S^2 + \frac{2\rho\sigma\sigma_S}{\lambda}\left[1 - \left(\frac{1-\mathrm{e}^{-\lambda}}{\lambda}\right)\right].$$

Proof. For $\delta = 1$, the calculation of the initial market value of a type-II guarantee involves determining the expected value of time T payoff $\pi_T^{(n)}(II)$ under a risk-neutral probability measure Q. $\pi_T^{(n)}(II)$ is composed of two terms, $\prod_{t=n}^{T}\exp(R_t)$ and $\prod_{t=n}^{T}\max\left[\exp\left(\overline{R}(t-1,t)\right), \exp(R_t)\right]$. The initial market value of the $\prod_{t=n}^{T}\exp(R_t)$ term, $E_0^Q\left[\prod_{t=n}^{T}\mathrm{e}^{R_t}B_T^{-1}\right]$, which is the initial value of the accumulated realized pension portfolio value, appears in Appendix B, and is equal to $\exp\left(-\int_0^{n-1} f(0,u)\,du\right)$. We compute the initial value of $\prod_{t=n}^{T}\max\left[\mathrm{e}^{\overline{R}(t-1,t)}, \mathrm{e}^{R_t}\right]$ in Appendix F, along with the initial market value of the guarantee $\pi_0^{(n)}(II)$. □

The following corollary considers the effect of mortality on the market value of a type-II guarantee.

Corollary 2. *Assume that the financial market is independent of the employee's mortality risk. For an employee aged x at time 0, the initial market value of the type-II guarantee with payoff* $\widetilde{\pi}_T^{(n)}(II)$ *at time T for the case of* $\delta = 1$ *is given by*

$$\widetilde{\pi}_0^{(n)}(II) = {}_T p_x \cdot \exp\left(-\int_0^{n-1} f(0,u)\,\mathrm{d}u\right) \times \left\{\prod_{t=n}^{T}\left[A_{3,0}N(d_3(t)) - A_{4,0}N(d_4(t)) + A_{4,0}\right] - 1\right\},$$

where ${}_T p_x$ *is defined in Corollary 1.*

4. Numerical results and sensitivity analysis

We investigate the values for two types of rate of return guarantees in this section. In particular, we examine the guarantee values for the case that the guaranteed rate is the one-year spot rate, i.e. $\delta = 1$. In the following numerical analyses, we assume that an employee aged 30 will retire at the age of 60 for our base illustration case. The contribution rate is 6% of salary, the initial yearly wage is 100, and the employee's salary grows at 2% annually. Furthermore, we assume that the initial term structure is flat and fixed at 3%. The most important assumption affecting the value of a guarantee is the dynamics of the future asset return and the interest rate. For a robustness check, we carry out a sensitivity analysis for the key parameters underlying the financial models. Continuing the notations from the previous sections, we set the parameters for the base illustration case as follows: $\sigma_s = 0.1, \sigma_f = 0.01, \lambda = 0.1, \rho = -0.2, \delta = 1$.

Table 1 shows the value of the guarantee for the employee with different working periods. In general, the numerical results show that a type-I guarantee is cheaper than a type-II guarantee. The effect is more obvious for an employee who works for a longer period. For example, in the base case, the value of a type-I guarantee is 23.519; for a type-II guarantee, it is 153.546. However, for an employee aged 20 years, the value of a type-I guarantee is 34.565; and 310.709 for a type-II guarantee. The effect is intuitive. For a type-I guarantee, the effect of large guaranteed rates in some periods can be mitigated by smaller guaranteed rates in other periods. However, for a type-II guarantee, such mitigation does not work. This calculation implies that a type-II guarantee is more costly than a type-I guarantee.

In addition, to study the accuracy of our derived formulae for the two types of interest rate guarantees, we provide the simulation results based on 50,000 paths along with the analytical ones in Table 1. The difference between the simulated values and closed form solutions is tiny. Therefore, the formulae provide a precise and efficient way to value the embedded interest rate guarantee, especially when the employee works for a long period. Using simulations to find the guarantee values is very time consuming. Thus our derived explicit pricing formula provides a distinct advantage in terms of pricing long-duration guarantees.

We investigate the effects of mortality for both males and females and report the results in Table 2. We analyze the

Table 1
Value of guarantees with different working periods

Working period T	Type-I guarantee $V_0\,(I)$			Type-II guarantee $V_0\,(II)$		
	Formula	Simulation	Difference	Formula	Simulation	Difference
10	5.128	5.089	0.039	14.309	14.252	0.057
15	9.042	9.036	0.006	32.987	32.947	0.040
20	13.490	13.468	0.022	61.180	61.187	−0.007
25	18.345	18.222	0.123	100.649	100.605	0.044
30	23.519	23.473	0.046	153.546	153.332	0.214
35	28.943	28.807	0.136	222.500	222.438	0.062
40	34.565	34.433	0.132	310.709	310.655	0.054

Table 2
The effect of mortality on the value of guarantees

Working period T	Male			Female		
	$V_0\,(I)$	$V_0\,(II)$	${}_Tp_{30}$	$V_0\,(I)$	$V_0\,(II)$	${}_Tp_{30}$
10	5.097	14.223	0.9940	5.112	14.265	0.9970
15	8.954	32.668	0.9903	8.999	32.833	0.9953
20	13.290	60.273	0.9852	13.398	60.765	0.9932
25	17.933	98.384	0.9775	18.163	99.647	0.9901
30	22.712	148.281	0.9657	23.163	151.221	0.9849
35	27.371	210.412	0.9457	28.223	216.963	0.9751
40	31.377	282.047	0.9078	32.990	296.545	0.9544

Note: ${}_Tp_{30}$ denotes the survival probability that an employee starts working at the age of 30 and remains alive after T years.

effect of mortality using the UK standard tables for annuitant and pensioner populations for the period 1991–1994 proposed by CMI Bureau (1999). For illustration purpose, we present the methodology of constructing the mortality table by CMI Bureau (1999) in Appendix G. Comparing the results in Table 2 with those without mortality effect in Table 1, we find that the mortality effects reduce the value of the guarantee since the guarantee only applies to those who survive to retirement. The effect is less significant for females because the survival probability is higher for females than it is for males of the same age. The corresponding survival probability for males and females are listed in Table 2.

In the following tables, we depict the sensitivity analyses conducted for key parameters. In order to study the effects of key parameters, we use the case without considering the mortality effect for illustration. Table 3 shows the values of the guarantees, given different correlation estimates between the asset prices and interest rates. All values are increasing functions of the correlation parameter, because $\widehat{\sigma}_I^{(n)}$ in Proposition 1 and $\widehat{\sigma}_{II}$ in Proposition 2 both increase with the correlation estimate.

Table 4 describes how the values of both guarantees change with σ. Two effects come up when σ becomes higher. First, high guaranteed rates appear more frequently. Second, high short rates also emerge more often and actual rates of return in the risk-neutral world tend to be higher. As these effects work together, the values do not necessarily monotonically change with σ. Table 5 indicates that the values are not very sensitive to the exponentially decaying parameter (λ) under the base case. For example, as λ increases from 0.025 to 0.25, the value of the type-I guarantee only increases 0.01, and the value of the type-II guarantee only increases 0.1.

Table 3
Value of guarantees with different correlation estimates

Correlation estimates ρ	Type-I guarantee $V_0\,(I)$	Type-II guarantee $V_0\,(II)$
−1	22.588	144.700
−0.8	22.825	146.918
−0.6	23.059	149.132
−0.4	23.290	151.341
−0.2	23.519	153.546
0	23.745	155.748
0.2	23.970	157.945
0.4	24.192	160.139
0.6	24.412	162.330
0.8	24.630	164.518
1	24.845	166.703

Table 4
Value of guarantees with different forward rate volatility estimates

Forward rate volatility σ	Type-I guarantee $V_0\,(I)$	Type-II guarantee $V_0\,(II)$
0	23.709	155.396
0.005	23.605	154.383
0.01	23.519	153.546
0.015	23.450	152.885
0.02	23.400	152.400
0.025	23.368	152.091
0.03	23.354	151.958
0.035	23.359	152.001
0.04	23.382	152.221
0.045	23.423	152.618

Table 5
Value of guarantees with different estimates of exponentially decaying parameters

Decaying volatility parameters λ	Type-I guarantee $V_0\,(I)$	Type-II guarantee $V_0\,(II)$
0.025	23.515	153.511
0.05	23.517	153.523
0.075	23.518	153.535
0.1	23.519	153.546
0.125	23.520	153.558
0.15	23.521	153.569
0.175	23.522	153.581
0.2	23.524	153.592
0.225	23.525	153.603
0.25	23.526	153.614

Table 6
Value of guarantees with different asset return volatility estimates

Asset return volatility σ_S	Type-I guarantee $V_0\,(I)$	Type-II guarantee $V_0\,(II)$
0.02	4.731	21.690
0.04	9.407	46.834
0.06	14.128	76.717
0.08	18.838	111.972
0.1	23.519	153.546
0.12	28.159	202.597
0.14	32.749	260.504
0.16	37.281	328.909
0.18	41.746	409.753
0.2	46.137	505.334

The effects of the volatility of the asset return on the value of the guarantee are presented in Table 6. The values of both guarantees increase with the volatility of the asset return. This pattern is more significant for type-II than for type-I guarantees, due to the method used to calculate the guarantee. Regarding these parameters we exam in the sensitive analysis, it reflects that the asset return volatility estimate is the most sensitive parameter to the value of guarantees.

5. Conclusion

In recent years, defined contribution pension plans have emerged as a major part of the retirement income system. To transfer part of the investment risk inherent in DC plans from employees to another entity, guarantees commonly have been embedded in DC plans. Therefore, the way in which the guarantee is valued is very critical for the pension plan provider. The difficulty associated with valuing the guarantee embedded in a pension plan is that any such valuation must cope with the contributions made at different time points during the employee's work duration.

In this research, we tackle a specific type of rate of return guarantee linked to δ-year spot rates, a type of pension guarantee that the previous literature has not investigated. We assume that the asset price follows the geometric Brownian motion and the interest rate dynamic follows the HJM interest rate model with exponentially decaying volatility. The two processes can be correlated through their random terms. We employ Margrabe (1978) option pricing model to find the values of guarantees. For the guaranteed rate to be the δ-year spot rate, we derive a closed-form formula for valuing maturity interest rate guarantees. The explicit formula does not always exist for multi-period guarantees. We thus provide an analytical formula for multi-period guarantees under the special case that the guaranteed rate is the one-year spot rate. Using a closed-form solution to value pension guarantees offers the benefits of succinctness and decreased computing time. Because an employee's work duration usually is very long, a simulation framework requires far more time to value the guarantee embedded in the long-duration contract than would the closed-form solution.

In our numerical analysis, for an illustration purpose, we present the results for the special case only. We demonstrate the accuracy of our closed-form solutions with simulated results. In addition, the effect of deterministic mortality on the value of guarantee is investigated through the UK experience for annuitant and pensioner populations. We find that the mortality effect is not significant for females. However, we only consider the effect of mortality on deterministic basis. Recently, the mortality risk has been widely discussed by employing a stochastic mortality model in the literature, for example Renshaw and Haberman (2006), Cairns et al. (2006) and Cairns et al. (2007). The unanticipated change of mortality pattern may play an important role in valuing the long duration guarantees. The stochastic mortality effect is thus worth further investigating. We also provide sensitivity analyses for the key parameters that drive the values of both guarantees, and find that type-II guarantees are more costly than type-I guarantees, especially for long-term employees. The value of the guarantee is sensitive to the changes of the asset return volatility estimates and the effect is more obvious for type-II guarantees. Thus, the provider of the guarantee cannot ignore the parameter risk when valuing the guarantee and should be especially careful in valuing type-II guarantees.

In the light of the analysis in this paper, we point out some areas and issues to carry out for further research. First, we do not investigate the issues of parameter risk and mortality risk, which would be a valuable extension in further research. Second, we assume that the diffusion factor in the HJM framework is exponentially decaying. However, a more flexible volatility structure for valuing the interest rate guarantee is worth studying. Third, from the point of view of a pension plan provider, hedging the risks associated with issuing relative guarantees linked to spot rates is critical. Therefore, explorations of suitable hedging strategies could help ensure the financial safety of a pension fund.

Appendix A

In this appendix, we show the initial guaranteed accumulative value $A_{1,0}^{(n)}$, the time 0 market value of the guaranteed accumulative value for \$1 contributed in the nth year. Define a risk-free money market account $B_\bullet$ as $B_t = \exp\left(\int_0^t r_u \mathrm{d}u\right)$. Following Eq. (6), the δ-year spot rate at time $t-1$ is

$$\overline{R}(t-1,t-1+\delta) = -\frac{\ln P(t-1,t-1+\delta)}{(t-1+\delta)-(t-1)}$$
$$= \frac{\ln P(0,t-1) - \ln P(0,t-1+\delta)}{\delta} + \frac{\sigma^2}{\lambda^3\delta}\left[\left(e^{\lambda(t-1)}-1\right)\left(e^{-\lambda(t-1)} - e^{-\lambda(t-1+\delta)}\right) - \frac{\left(e^{2\lambda(t-1)}-1\right)\left(e^{-2\lambda(t-1)}-e^{-2\lambda(t-1+\delta)}\right)}{4}\right] + \left(\frac{1-e^{-\lambda\delta}}{\lambda\delta}\right)\sigma\int_0^{t-1} e^{-\lambda(t-1-v)}dW_v.$$

Let

$$g_1(t) = \frac{\ln P(0,t-1) - \ln P(0,t-1+\delta)}{\delta} + \frac{\sigma^2}{\lambda^3\delta}\left[\left(e^{\lambda(t-1)}-1\right)\left(e^{-\lambda(t-1)} - e^{-\lambda(t-1+\delta)}\right) - \frac{\left(e^{2\lambda(t-1)}-1\right)\left(e^{-2\lambda(t-1)}-e^{-2\lambda(t-1+\delta)}\right)}{4}\right]$$

which would be a constant. Therefore, $\overline{R}(t-1,t-1+\delta) = g_1(t) + \left(\frac{1-e^{-\lambda\delta}}{\lambda\delta}\right)\sigma\int_0^{t-1} e^{-\lambda(t-1-v)}dW_v$.

The time T value of the payoff from the type-I guarantee for \$1 contributed in the nth year is $A_{1,T}^{(n)} = \prod_{t=n}^{T}\exp\left(\overline{R}(t-1,t-1+\delta)\right)$. By the martingale pricing theory, we know that $\frac{A_{1,0}^{(n)}}{B_0} = E_0^Q\left(\frac{A_{1,T}^{(n)}}{B_T}\right)$, where the expectation is taken under a risk-neutral probability measure Q, conditional on the information up to time 0.

Because $B_0 = 1$,

$$A_{1,0}^{(n)} = E_0^Q\left(\frac{A_{1,T}^{(n)}}{B_T}\right) = E_0^Q\left(\prod_{t=n}^{T}\exp\left(\overline{R}(t-1,t-1+\delta)\right)/B_T\right)$$
$$= E_0^Q\left\{\exp\left[\sum_{t=n}^{T}\left(g_1(t) + \left(\frac{1-e^{-\lambda\delta}}{\lambda\delta}\right)\sigma \times \int_0^{t-1} e^{-\lambda(t-1-v)}dW_v\right) - \int_0^T\left(f(0,u) + \frac{\sigma^2}{2\lambda^2}\left(1-e^{-\lambda u}\right)^2 + \sigma\int_0^u e^{-\lambda(u-v)}dW_v\right)du\right]\right\}.$$

We further define $g_2(n,T) = \sum_{t=n}^{T} g_1(t) - \int_0^T f(0,u)\,du - \int_0^T \frac{\sigma^2}{2\lambda^2}\left(1-e^{-\lambda u}\right)^2 du = \sum_{t=n}^{T} g_1(t) - \int_0^T f(0,u)\,du - \frac{\sigma^2}{2\lambda^2}\left[T - \frac{2}{\lambda}\left(1-e^{-\lambda T}\right) + \frac{1}{2\lambda}\left(1-e^{-2\lambda T}\right)\right]$, which is a constant, and then note that

$$A_{1,0}^{(n)} = E_0^Q\left[\exp\left(g_2(n,T) + \left(\frac{1-e^{-\lambda\delta}}{\lambda\delta}\right) \times \sum_{t=n}^{T}\left(\sigma\int_0^{t-1} e^{-\lambda(t-1-v)}dW_v\right) - \sigma\int_0^T\int_0^u e^{-\lambda(u-v)}dW_v du\right)\right]$$
$$= \exp\left\{g_2(n,T) + \frac{1}{2}\left(\frac{1-e^{-\lambda\delta}}{\lambda\delta}\right)^2 \times \mathrm{Var}_0^Q\left[\sum_{t=n}^{T}\left(\sigma\int_0^{t-1} e^{-\lambda(t-1-v)}dW_v\right)\right]\right\} \times \exp\left\{\frac{1}{2}\mathrm{Var}_0^Q\left[\sigma\int_0^T\int_0^u e^{-\lambda(u-v)}dW_v du\right]\right\}$$
$$\times \exp\left\{-\left(\frac{1-e^{-\lambda\delta}}{\lambda\delta}\right) \times \mathrm{Cov}_0^Q\left[\sum_{t=n}^{T}\left(\sigma\int_0^{t-1} e^{-\lambda(t-1-v)}dW_v\right),\ \sigma\int_0^T\int_0^u e^{-\lambda(u-v)}dW_v du\right]\right\}.$$

We then calculate (i) $\mathrm{Var}_0^Q\left[\sum_{t=n}^{T}\left(\sigma\int_0^{t-1} e^{-\lambda(t-1-v)}dW_v\right)\right]$, (ii) $\mathrm{Var}_0^Q\left[\sigma\int_0^T\int_0^u e^{-\lambda(u-v)}dW_v du\right]$, and (iii) $\mathrm{Cov}_0^Q\left[\sum_{t=n}^{T}\left(\sigma\int_0^{t-1} e^{-\lambda(t-1-v)}dW_v\right), \sigma\int_0^T\int_0^u e^{-\lambda(u-v)}dW_v du\right]$.

(i)

$$\mathrm{Var}_0^Q\left[\sum_{t=n}^{T}\left(\sigma\int_0^{t-1} e^{-\lambda(t-1-v)}dW_v\right)\right] = \sigma^2\sum_{i=n}^{T}\sum_{j=n}^{T}\mathrm{Cov}_0^Q\left(\int_0^{i-1} e^{-\lambda(i-1-v)}dW_v, \int_0^{j-1} e^{-\lambda(j-1-v)}dW_v\right)$$
$$= \sigma^2\sum_{i=n}^{T}\sum_{j=n}^{T}\int_0^{\min(i-1,j-1)} e^{-\lambda(i+j-2-2v)}dv = \frac{\sigma^2}{2\lambda}\sum_{i=n}^{T}\sum_{j=n}^{T} e^{-\lambda(i+j-2)}\left(e^{2\lambda\min(i-1,j-1)}-1\right),$$

(ii)

$$\mathrm{Var}_0^Q\left[\sigma\int_0^T\int_0^u e^{-\lambda(u-v)}dW_v du\right] = \mathrm{Var}_0^Q\left[\sigma\int_0^T\int_v^T e^{-\lambda(u-v)}du\,dW_v\right] = \mathrm{Var}_0^Q\left[\frac{\sigma}{\lambda}\int_0^T\left(1-e^{-\lambda(T-v)}\right)dW_v\right]$$

$$= \frac{\sigma^2}{\lambda^2}\int_0^T \left(1-\mathrm{e}^{-\lambda(T-v)}\right)^2 \mathrm{d}v$$
$$= \frac{\sigma^2}{\lambda^2}\left[T-\frac{2}{\lambda}\left(1-\mathrm{e}^{-\lambda T}\right)+\frac{1}{2\lambda}\left(1-\mathrm{e}^{-2\lambda T}\right)\right],$$

(iii)

$$\mathrm{Cov}_0^Q\left[\sum_{t=n}^{T}\left(\sigma\int_0^{t-1}\mathrm{e}^{-\lambda(t-1-v)}\mathrm{d}W_v\right),\ \sigma\int_0^T\int_0^u \mathrm{e}^{-\lambda(u-v)}\mathrm{d}W_v\mathrm{d}u\right]$$
$$=\sum_{t=n}^{T}\mathrm{Cov}_0^Q\left[\sigma\int_0^{t-1}\mathrm{e}^{-\lambda(t-1-v)}\mathrm{d}W_v,\ \frac{\sigma}{\lambda}\times\int_0^T\left(1-\mathrm{e}^{-\lambda(T-v)}\right)\mathrm{d}W_v\right]$$
$$=\frac{\sigma^2}{\lambda}\sum_{t=n}^{T}\mathrm{Cov}_0^Q\left[\int_0^{t-1}\mathrm{e}^{-\lambda(t-1-v)}\mathrm{d}W_v\ ,\ \int_0^T\left(1-\mathrm{e}^{-\lambda(T-v)}\right)\mathrm{d}W_v\right]$$
$$=\frac{\sigma^2}{\lambda}\sum_{t=n}^{T}\left[\int_0^{t-1}\left(\mathrm{e}^{-\lambda(t-1-v)}-\mathrm{e}^{-\lambda(T+t-1-2v)}\right)\mathrm{d}v\right]$$
$$=\frac{\sigma^2}{\lambda^2}\sum_{t=n}^{T}\left[1-\mathrm{e}^{-\lambda(t-1)}-\left(\frac{\mathrm{e}^{-\lambda(T-t+1)}-\mathrm{e}^{-\lambda(T+t-1)}}{2}\right)\right]$$
$$=\frac{\sigma^2}{\lambda^2}\left[(T-n+1)-\left(\frac{\mathrm{e}^{-\lambda T}-\mathrm{e}^{-\lambda(n-1)}}{\mathrm{e}^{-\lambda}-1}\right)-\left(\frac{1-\mathrm{e}^{-\lambda(T-n+1)}}{2(\mathrm{e}^{\lambda}-1)}\right)+\left(\frac{\mathrm{e}^{-2\lambda T}-\mathrm{e}^{-\lambda(T+n-1)}}{2(\mathrm{e}^{-\lambda}-1)}\right)\right].$$

Appendix B

In this appendix we demonstrate that the initial realized accumulative value $A_{2,0}^{(n)}$, or the time 0 market value of the realized accumulative value of investing \$1 in the nth year in the pension portfolio, can be calculated analytically as $A_{2,0}^{(n)} = \exp\left(-\int_0^{n-1} f(0,u)\,\mathrm{d}u\right)$.

By the martingale pricing theory, $\frac{A_{2,0}^{(n)}}{B_0} = E_0^Q\left(\frac{A_{2,T}^{(n)}}{B_T}\right)$. Therefore, $A_{2,0}^{(n)} = E_0^Q\left(\frac{A_{2,T}^{(n)}}{B_T}\right) = E_0^Q\left(\frac{S_T}{S_{n-1}}B_T^{-1}\right) = E_0^Q\left[\frac{S_T}{S_{n-1}}\exp\left(-\int_0^T r_u\mathrm{d}u\right)\right]$. S_T, or the portfolio value at time T, is equal to $S_0\exp\left(\int_0^T r_u\mathrm{d}u-\frac{1}{2}\sigma_S^2 T+\sigma_S Z_T\right)$, the solution to the SDE of Eq. (3).

Because $A_{2,0}^{(n)} = E_0^Q\{\exp[-\int_0^{n-1} r_u\mathrm{d}u-\frac{1}{2}\sigma_S^2(T-n+1)+\sigma_S(Z_T-Z_{n-1})]\}$, after substituting Eq. (5) into r_u and defining $g_4(n,T)$ as $-\int_0^{n-1}[f(0,u)+\frac{\sigma^2}{2\lambda^2}(1-\mathrm{e}^{-\lambda u})^2]\mathrm{d}u - \frac{1}{2}\sigma_S^2(T-n+1)$, we get

$$A_{2,0}^{(n)} = E_0^Q\left[\exp\left(g_4(n,T)-\sigma\int_0^{n-1}\int_0^u \mathrm{e}^{-\lambda(u-v)}\mathrm{d}W_v\mathrm{d}u+\sigma_S\int_{n-1}^T \mathrm{d}Z_v\right)\right]$$
$$= E_0^Q\left[\exp\left(g_4(n,T)-\frac{\sigma}{\lambda}\int_0^{n-1}\left(1-\mathrm{e}^{-\lambda(n-1-v)}\right)\mathrm{d}W_v+\sigma_S\int_{n-1}^T\mathrm{d}Z_v\right)\right]$$
$$= \exp\left[g_4(n,T)+\frac{1}{2}\mathrm{Var}_0^Q\left(-\frac{\sigma}{\lambda}\int_0^{n-1}\left(1-\mathrm{e}^{-\lambda(n-1-v)}\right)\mathrm{d}W_v+\sigma_S\int_{n-1}^T\mathrm{d}Z_v\right)\right]$$
$$= \exp\left[g_4(n,T)+\frac{1}{2}\mathrm{Var}_0^Q\left(\frac{\sigma}{\lambda}\int_0^{n-1}\left(1-\mathrm{e}^{-\lambda(n-1-v)}\right)\mathrm{d}W_v\right)+\frac{1}{2}\mathrm{Var}_0^Q\left(\sigma_S\int_{n-1}^T\mathrm{d}Z_v\right)\right]$$
$$= \exp\left[g_4(n,T)+\frac{1}{2}\left(\frac{\sigma^2}{\lambda^2}\int_0^{n-1}\left(1-\mathrm{e}^{-\lambda(n-1-v)}\right)^2\mathrm{d}v\right)+\frac{1}{2}\left(\sigma_S^2\int_{n-1}^T\mathrm{d}v\right)\right]$$
$$= \exp\left[-\int_0^{n-1} f(0,u)\,\mathrm{d}u\right].$$

The last equality sign holds as

$$\int_0^{n-1}\left(1-\mathrm{e}^{-\lambda(n-1-v)}\right)^2\mathrm{d}v = \int_0^{n-1}(1-\mathrm{e}^{-\lambda u})^2\mathrm{d}u.$$

Appendix C

In this appendix, we calculate the variances of $\ln A_{1,T}^{(n)}$ and $\ln A_{2,T}^{(n)}$.

First,

$$\mathrm{Var}_0^Q\left(\ln A_{1,T}^{(n)}\right) = \mathrm{Var}_0^Q\left[\ln\prod_{t=n}^{T}\exp\left(\overline{R}(t-1,t-1+\delta)\right)\right]$$

$$
\begin{aligned}
&= \mathrm{Var}_0^Q\left[\sum_{t=n}^{T}\overline{R}\,(t-1,t-1+\delta)\right]\\
&= \mathrm{Var}_0^Q\left[\sum_{t=n}^{T}\left(\left(\frac{1-\mathrm{e}^{-\lambda\delta}}{\lambda\delta}\right)\right.\right.\\
&\qquad\left.\left.\times\ \sigma\int_0^{t-1}\mathrm{e}^{-\lambda(t-1-v)}\mathrm{d}W_v\right)\right]\\
&= \left(\frac{1-\mathrm{e}^{-\lambda\delta}}{\lambda\delta}\right)^2\\
&\qquad\times\mathrm{Var}_0^Q\left[\sum_{t=n}^{T}\left(\sigma\int_0^{t-1}\mathrm{e}^{-\lambda(t-1-v)}\mathrm{d}W_v\right)\right]\\
&= \frac{\sigma^2}{2\lambda}\left(\frac{1-\mathrm{e}^{-\lambda\delta}}{\lambda\delta}\right)^2\sum_{i=n}^{T}\sum_{j=n}^{T}\mathrm{e}^{-\lambda(i+j-2)}\\
&\qquad\times\left(\mathrm{e}^{2\lambda\min(i-1,j-1)}-1\right).
\end{aligned}
$$

The justification for the last equality sign is provided in Appendix A.

Second, we derive the variance of $\ln A_{2,T}^{(n)}$.

$$
\begin{aligned}
&\mathrm{Var}_0^Q\left(\ln A_{2,T}^{(n)}\right)=\mathrm{Var}_0^Q\left[\ln\left(\frac{S_T}{S_{n-1}}\right)\right]\\
&= \mathrm{Var}_0^Q\left[\int_{n-1}^{T}r_u\mathrm{d}u-\frac{1}{2}\sigma_S^2\,(T-n+1)\right.\\
&\qquad\left.+\ \sigma_S\,(Z_T-Z_{n-1})\right]\\
&= \mathrm{Var}_0^Q\left(\sigma\int_{n-1}^{T}\int_0^{u}\mathrm{e}^{-\lambda(u-v)}\mathrm{d}W_v\mathrm{d}u+\sigma_S\int_{n-1}^{T}\mathrm{d}Z_v\right)\\
&= \mathrm{Var}_0^Q\left(\sigma\int_0^{n-1}\int_{n-1}^{T}\mathrm{e}^{-\lambda(u-v)}\mathrm{d}u\mathrm{d}W_v\right.\\
&\qquad\left.+\sigma\int_{n-1}^{T}\int_v^{T}\mathrm{e}^{-\lambda(u-v)}\mathrm{d}u\mathrm{d}W_v+\sigma_S\int_{n-1}^{T}\mathrm{d}Z_v\right)\\
&= \mathrm{Var}_0^Q\left[-\frac{\sigma}{\lambda}\left(\mathrm{e}^{-\lambda T}-\mathrm{e}^{-\lambda(n-1)}\right)\int_0^{n-1}\mathrm{e}^{\lambda v}\mathrm{d}W_v\right.\\
&\qquad\left.-\frac{\sigma}{\lambda}\int_{n-1}^{T}\left(\mathrm{e}^{-\lambda(T-v)}-1\right)\mathrm{d}W_v+\sigma_S\int_{n-1}^{T}\mathrm{d}Z_v\right]\\
&= \mathrm{Var}_0^Q\left[-\frac{\sigma}{\lambda}\left(\mathrm{e}^{-\lambda T}-\mathrm{e}^{-\lambda(n-1)}\right)\int_0^{n-1}\mathrm{e}^{\lambda v}\mathrm{d}W_v\right]\\
&\qquad+\mathrm{Var}_0^Q\left[\sigma_S\int_{n-1}^{T}\mathrm{d}Z_v\right]\\
&\qquad+\mathrm{Var}_0^Q\left[-\frac{\sigma}{\lambda}\int_{n-1}^{T}\left(\mathrm{e}^{-\lambda(T-v)}-1\right)\mathrm{d}W_v\right]\\
&\qquad+2\mathrm{Cov}_0^Q\left[-\frac{\sigma}{\lambda}\int_{n-1}^{T}\left(\mathrm{e}^{-\lambda(T-v)}-1\right)\mathrm{d}W_v\ ,\right.\\
&\qquad\left.\sigma_S\int_{n-1}^{T}\mathrm{d}Z_v\right]\\
&= \frac{\sigma^2}{2\lambda^3}\left[\mathrm{e}^{-\lambda T}-\mathrm{e}^{-\lambda(n-1)}\right]^2\left[\mathrm{e}^{2\lambda(n-1)}-1\right]\\
&\qquad+\frac{\sigma^2}{\lambda^2}\left[(T-n+1)-\left(\frac{2(1-\mathrm{e}^{-\lambda(T-n+1)})}{\lambda}\right)\right.\\
&\qquad\left.+\left(\frac{1-\mathrm{e}^{-2\lambda(T-n+1)}}{2\lambda}\right)\right]\\
&\qquad+\sigma_S^2(T-n+1)+\frac{2\rho\sigma\sigma_S}{\lambda}\left[(T-n+1)\right.\\
&\qquad\left.-\left(\frac{1-\mathrm{e}^{-\lambda(T-n+1)}}{\lambda}\right)\right].
\end{aligned}
$$

Appendix D

In this section, we show how to compute $\mathrm{Cov}_0^Q[\ln A_{1,T}^{(n)},\ln A_{2,T}^{(n)}]$:

$$
\begin{aligned}
&\mathrm{Cov}_0^Q\left(\ln A_{1,T}^{(n)},\ln A_{2,T}^{(n)}\right)\\
&= \mathrm{Cov}_0^Q\left(\ln\prod_{t=n}^{T}\exp\left(\overline{R}\,(t-1,t-1+\delta)\right),\ln\left(\frac{S_T}{S_{n-1}}\right)\right)\\
&= \mathrm{Cov}_0^Q\left[\sum_{t=n}^{T}\overline{R}\,(t-1,t-1+\delta)\,,\int_{n-1}^{T}r_u\mathrm{d}u\right.\\
&\qquad\left.+\ \sigma_S\,(Z_T-Z_{n-1})\right]\\
&= \mathrm{Cov}_0^Q\left[\sum_{t=n}^{T}\left(\left(\frac{1-\mathrm{e}^{-\lambda\delta}}{\lambda\delta}\right)\sigma\int_0^{t-1}\mathrm{e}^{-\lambda(t-1-v)}\mathrm{d}W_v\right),\right.\\
&\qquad-\frac{\sigma}{\lambda}\left(\mathrm{e}^{-\lambda T}-\mathrm{e}^{-\lambda(n-1)}\right)\int_0^{n-1}\mathrm{e}^{\lambda v}\mathrm{d}W_v\\
&\qquad\left.-\frac{\sigma}{\lambda}\int_{n-1}^{T}\left(\mathrm{e}^{-\lambda(T-v)}-1\right)\mathrm{d}W_v+\sigma_S\int_{n-1}^{T}\mathrm{d}Z_v\right]\\
&= \left(\frac{1-\mathrm{e}^{-\lambda\delta}}{\lambda\delta}\right)\cdot\frac{\sigma^2}{\lambda}\cdot\left(-\mathrm{e}^{-\lambda T}+\mathrm{e}^{-\lambda(n-1)}\right)\\
&\qquad\times\sum_{t=n}^{T}\mathrm{Cov}_0^Q\left[\int_0^{t-1}e^{-\lambda(t-1-v)}\mathrm{d}W_v,\int_0^{n-1}\mathrm{e}^{\lambda v}\mathrm{d}W_v\right]\\
&\qquad+\left(\frac{1-\mathrm{e}^{-\lambda\delta}}{\lambda\delta}\right)\cdot\frac{\sigma^2}{\lambda}\\
&\qquad\times\sum_{t=n}^{T}\mathrm{Cov}_0^Q\left[\int_0^{t-1}\mathrm{e}^{-\lambda(t-1-v)}\mathrm{d}W_v,\right.\\
&\qquad\left.-\int_{n-1}^{T}\left(\mathrm{e}^{-\lambda(T-v)}-1\right)\mathrm{d}W_v\right]\\
&\qquad+\left(\frac{1-\mathrm{e}^{-\lambda\delta}}{\lambda\delta}\right)\cdot\sigma\sigma_S\cdot\sum_{t=n}^{T}\mathrm{Cov}_0^Q\\
&\qquad\times\left[\int_0^{t-1}\mathrm{e}^{-\lambda(t-1-v)}\mathrm{d}W_v,\int_{n-1}^{T}\mathrm{d}Z_v\right]
\end{aligned}
$$

$$= \left(\frac{1-e^{-\lambda\delta}}{\lambda\delta}\right)\cdot\frac{\sigma^2}{\lambda}\cdot\left(-e^{-\lambda T}+e^{-\lambda(n-1)}\right)$$
$$\times\sum_{t=n}^{T}\int_0^{n-1}e^{-\lambda(t-1-2v)}dv+\left(\frac{1-e^{-\lambda\delta}}{\lambda\delta}\right)$$
$$\times\frac{\sigma^2}{\lambda}\sum_{t=n}^{T}\int_{n-1}^{t-1}e^{-\lambda(t-1-v)}\left(1-e^{-\lambda(T-v)}\right)dv$$
$$+\left(\frac{1-e^{-\lambda\delta}}{\lambda\delta}\right)\cdot\sigma\sigma_S\times\sum_{t=n}^{T}\int_{n-1}^{t-1}e^{-\lambda(t-1-v)}\rho dv$$
$$=\sigma\left(\frac{1-e^{-\lambda\delta}}{\lambda\delta}\right)\sum_{t=n}^{T}\left[\left(\frac{1-e^{-\lambda(t-n)}}{\lambda}\right)\left(\frac{\sigma}{\lambda}+\rho\sigma_S\right)\right.$$
$$+\frac{\sigma}{2\lambda^2}\left(e^{-\lambda(T+t-1)}+e^{-\lambda(t-n)}\right.$$
$$\left.\left.-e^{-\lambda(t+n-2)}-e^{-\lambda(T-t+1)}\right)\right].$$

Appendix E

We derive the volatility term $\widehat{\sigma}_I^{(n)}\sqrt{T}$ of the composite asset $U_T=\frac{A_{1,T}^{(n)}}{A_{2,T}^{(n)}}$. Because we reduce the valuation problem of a type-I guarantee to the valuation of a standard European call option on a single lognormally distributed underlying asset $U_\bullet$ with unit strike price, the value of $\widehat{\sigma}_I^{(n)}$ represents a crucial input to our option pricing formula.

Because $\widehat{\sigma}_I^{(n)}\sqrt{T}$ is the volatility of the composite asset U_T, $U_T=\frac{A_{1,T}^{(n)}}{A_{2,T}^{(n)}}$, by Ito's lemma, the variance of $\ln U_T$ is given by

$$\left(\widehat{\sigma}_I^{(n)}\right)^2T=\mathrm{Var}_0^Q\left(\ln A_{1,T}^{(n)}\right)+\mathrm{Var}_0^Q\left(\ln A_{2,T}^{(n)}\right)$$
$$-2\mathrm{Cov}_0^Q\left(\ln A_{1,T}^{(n)},\ln A_{2,T}^{(n)}\right)$$
$$=(T-n+1)\left(\frac{\sigma_f^2}{3}+\sigma_S^2-\rho\sigma_f\sigma_S\right).$$

Appendix F

In this appendix, we derive the initial market value of the first term of $\pi_T^{(n)}(II)$ for the special case of $\delta=1$.

Specifically, $E_0^Q[\prod_{t=n}^{T}\max(\exp(\overline{R}(t-1,t)),\exp(R_t))\times B_T^{-1}]$, the first term of $E_0^Q[\frac{\pi_T^{(n)}(II)}{B_T}]$, is calculated as

$$E_0^Q\left[e^{-\int_0^T r_u du}\times\prod_{t=n}^{T}\max\left(\exp\left(\overline{R}(t-1,t)\right),\exp\left(R_t\right)\right)\right]$$
$$=E_0^Q\left[e^{-\int_0^T r_u du}\times\prod_{t=n}^{T}\max\left(e^{\overline{R}(t-1,t)},\frac{S_t}{S_{t-1}}\right)\right]$$
$$=E_0^Q\left[e^{-\int_0^{n-1}r_u du}\times\prod_{t=n}^{T}e^{-\int_{t-1}^{t}r_u du}\right.$$
$$\times\prod_{t=n}^{T}\max\left(e^{g_1(t)+\left(\frac{1-e^{-\lambda}}{\lambda}\right)\sigma\int_0^{t-1}e^{-\lambda(t-1-v)}dW_v},\right.$$
$$\left.\left.e^{\int_{t-1}^{t}r_u du-\frac{1}{2}\sigma_s^2+\sigma_s(Z_t-Z_{t-1})}\right)\right]$$
$$=E_0^Q\left[e^{-\int_0^{n-1}r_u du}\right.$$
$$\times\prod_{t=n}^{T}\max\left(e^{-\int_{t-1}^{t}r_u du+g_1(t)+\left(\frac{1-e^{-\lambda}}{\lambda}\right)\sigma\int_0^{t-1}e^{-\lambda(t-1-v)}dW_v},\right.$$
$$\left.\left.e^{-\frac{1}{2}\sigma_s^2+\sigma_s(Z_t-Z_{t-1})}\right)\right],$$

where

$$-\int_{t-1}^{t}r_u du+g_1(t)+\left(\frac{1-e^{-\lambda}}{\lambda}\right)\sigma\int_0^{t-1}e^{-\lambda(t-1-v)}dW_v$$
$$=-\int_{t-1}^{t}\left(f(0,u)+\frac{\sigma^2}{2\lambda^2}\left(1-e^{-\lambda u}\right)^2\right.$$
$$\left.+\ \sigma\int_0^u e^{-\lambda(u-v)}dW_v\right)du$$
$$+g_1(t)+\frac{\sigma\left(e^{-\lambda(t-1)}-e^{-\lambda t}\right)}{\lambda}\int_0^{t-1}e^{\lambda v}dW_v$$
$$=g_3(t)-\sigma\int_{t-1}^{t}\int_0^u e^{-\lambda(u-v)}dW_v du$$
$$+\frac{\sigma\left(e^{-\lambda(t-1)}-e^{-\lambda t}\right)}{\lambda}\int_0^{t-1}e^{\lambda v}dW_v$$
$$=g_3(t)-\frac{\sigma}{\lambda}\int_{t-1}^{t}\left(1-e^{-\lambda(t-v)}\right)dW_v, \tag{11}$$

and $g_3(t)$ is defined as

$$g_3(t)=-\int_{t-1}^{t}f(0,u)du$$
$$-\int_{t-1}^{t}\frac{\sigma^2}{2\lambda^2}\left(1-e^{-\lambda u}\right)^2du+g_1(t)$$
$$=-\int_{t-1}^{t}f(0,u)du-\frac{\sigma^2}{2\lambda^2}\left[1+\frac{2}{\lambda}\left(e^{-\lambda t}-e^{-\lambda(t-1)}\right)\right.$$
$$\left.-\frac{1}{2\lambda}\left(e^{-2\lambda t}-e^{-2\lambda(t-1)}\right)\right]+g_1(t).$$

The last equality sign of Eq. (11) holds as

$$\sigma\int_{t-1}^{t}\int_0^u e^{-\lambda(u-v)}dW_v du$$
$$=\sigma\left(\int_0^{t-1}\int_{t-1}^{t}e^{-\lambda(u-v)}du dW_v\right.$$
$$\left.+\int_{t-1}^{t}\int_v^t e^{-\lambda(u-v)}du dW_v\right)$$
$$=\sigma\left(\int_0^{t-1}\frac{\left(e^{-\lambda(t-1)}-e^{-\lambda t}\right)e^{\lambda v}}{\lambda}dW_v\right.$$

$$+\int_{t-1}^{t}\frac{\left(\mathrm{e}^{-\lambda v}-\mathrm{e}^{-\lambda t}\right)\mathrm{e}^{\lambda v}}{\lambda}\mathrm{d}W_v\Bigg)$$
$$=\frac{\sigma\left(\mathrm{e}^{-\lambda(t-1)}-\mathrm{e}^{-\lambda t}\right)}{\lambda}\int_0^{t-1}\mathrm{e}^{\lambda v}\mathrm{d}W_v$$
$$+\frac{\sigma}{\lambda}\int_{t-1}^{t}\left(1-\mathrm{e}^{-\lambda(t-v)}\right)\mathrm{d}W_v.$$

Therefore, the first term of $E_0^Q\left[\frac{\pi_T^{(n)}(II)}{B_T}\right]$ becomes

$$E_0^Q\Bigg[\mathrm{e}^{-\int_0^{n-1}r_u\mathrm{d}u}\times\prod_{t=n}^{T}\max\Bigg(\mathrm{e}^{-\int_{t-1}^{t}r_u\mathrm{d}u+g_1(t)+\left(\frac{1-\mathrm{e}^{-\lambda}}{\lambda}\right)\sigma\int_0^{t-1}\mathrm{e}^{-\lambda(t-1-v)}\mathrm{d}W_v},\ \mathrm{e}^{-\frac{1}{2}\sigma_s^2+\sigma_s(Z_t-Z_{t-1})}\Bigg)\Bigg]$$
$$=E_0^Q\Bigg[\mathrm{e}^{-\int_0^{n-1}r_u\mathrm{d}u}\times\prod_{t=n}^{T}\max\left(\mathrm{e}^{g_3(t)-\frac{\sigma}{\lambda}\int_{t-1}^{t}\left(1-\mathrm{e}^{-\lambda(t-v)}\right)\mathrm{d}W_v},\mathrm{e}^{-\frac{1}{2}\sigma_s^2+\sigma_s(Z_t-Z_{t-1})}\right)\Bigg]$$
$$=E_0^Q\left[\mathrm{e}^{-\int_0^{n-1}r_u\mathrm{d}u}\right]\times\prod_{t=n}^{T}E_0^Q\Bigg[\max\Bigg(\mathrm{e}^{g_3(t)-\frac{\sigma}{\lambda}\int_{t-1}^{t}\left(1-\mathrm{e}^{-\lambda(t-v)}\right)\mathrm{d}W_v},\ \mathrm{e}^{-\frac{1}{2}\sigma_s^2+\sigma_s(Z_t-Z_{t-1})}\Bigg)\Bigg]. \quad (12)$$

It is easy to determine that $E_0^Q\left[\mathrm{e}^{-\int_0^{n-1}r_u\mathrm{d}u}\right]=\mathrm{e}^{-\int_0^{n-1}f(0,u)\mathrm{d}u}$. The payoff structure inside the second expectation of Eq. (12), $\max\left(\mathrm{e}^{g_3(t)-\frac{\sigma}{\lambda}\int_{t-1}^{t}\left(1-\mathrm{e}^{-\lambda(t-v)}\right)\mathrm{d}W_v},\mathrm{e}^{-\frac{1}{2}\sigma_s^2+\sigma_s(Z_t-Z_{t-1})}\right)$, can be viewed as that of an exchange option. If both underlying assets of the exchange option are lognormally distributed, we can employ the formula derived by Margrabe (1978) and follow the same line of proof as in Proposition 1 to derive the initial market value of the multi-period rate of return guarantee.

Let

$$X_t=g_3(t)-\frac{\sigma}{\lambda}\int_{t-1}^{t}\left(1-\mathrm{e}^{-\lambda(t-v)}\right)\mathrm{d}W_v$$

and

$$Y_t=-\frac{1}{2}\sigma_S^2+\sigma_S\left(Z_t-Z_{t-1}\right).$$

Because both X_t and Y_t are normally distributed, e^{X_t} and e^{Y_t} are lognormally distributed, and the value of $E_0^Q\left[\max\left(e^{g_3(t)-\frac{\sigma}{\lambda}\int_{t-1}^{t}\left(1-\mathrm{e}^{-\lambda(t-v)}\right)\mathrm{d}W_v},\ \mathrm{e}^{-\frac{1}{2}\sigma_s^2+\sigma_s(Z_t-Z_{t-1})}\right)\right]$ can be derived by using Margrabe (1978) formula. We first compute the variances of X_t and Y_t and the covariance between them as follows:

$$\begin{aligned}\mathrm{Var}_0^Q(X_t)&=\mathrm{Var}_0^Q\left(-\frac{\sigma}{\lambda}\int_{t-1}^{t}\left(1-\mathrm{e}^{-\lambda(t-v)}\right)\mathrm{d}W_v\right)\\&=\frac{\sigma^2}{\lambda^2}\int_{t-1}^{t}\left(1-\mathrm{e}^{-\lambda(t-v)}\right)^2\mathrm{d}v\\&=\frac{\sigma^2}{\lambda^2}\left(1-\frac{2}{\lambda}\left(1-\mathrm{e}^{-\lambda}\right)+\frac{1}{2\lambda}\left(1-\mathrm{e}^{-2\lambda}\right)\right)\end{aligned}$$
$$\mathrm{Var}_0^Q(Y_t)=\mathrm{Var}_0^Q\left(\sigma_S\left(Z_t-Z_{t-1}\right)\right)=\sigma_S^2$$

and

$$\begin{aligned}&\mathrm{Cov}_0^Q(X_t,Y_t)\\&=\mathrm{Cov}_0^Q\left(-\frac{\sigma}{\lambda}\int_{t-1}^{t}\left(1-\mathrm{e}^{-\lambda(t-v)}\right)\mathrm{d}W_v,\sigma_S\int_{t-1}^{t}\mathrm{d}Z_v\right)\\&=-\frac{\sigma\sigma_S}{\lambda}\mathrm{Cov}_0^Q\left(\int_{t-1}^{t}\left(1-\mathrm{e}^{-\lambda(t-v)}\right)\mathrm{d}W_v,\int_{t-1}^{t}\rho\mathrm{d}W_v+\sqrt{1-\rho^2}\mathrm{d}\widehat{W}_v\right)\\&=-\frac{\rho\sigma\sigma_S}{\lambda}\int_{t-1}^{t}\left(1-\mathrm{e}^{-\lambda(t-v)}\right)\mathrm{d}v\\&=-\frac{\rho\sigma\sigma_S}{\lambda}\left(1-\frac{1-\mathrm{e}^{-\lambda}}{\lambda}\right).\end{aligned}$$

Next, we define two new assets, $A_{3,t}=\mathrm{e}^{X_t}B_t$ and $A_{4,t}=\mathrm{e}^{Y_t}B_t$, and follow Margrabe (1978) to obtain

$$\begin{aligned}E_0^Q\left[\max\left(\mathrm{e}^{X_t},\mathrm{e}^{Y_t}\right)\right]&=E_0^Q\left[B_t^{-1}\max\left(A_{3,t},A_{4,t}\right)\right]\\&=E_0^Q\left[B_t^{-1}\max\left(A_{3,t}-A_{4,t},0\right)\right]+E_0^Q\left[B_t^{-1}A_{4,t}\right]\\&=A_{3,0}N\left(d_3(t)\right)-A_{4,0}N\left(d_4(t)\right)+A_{4,0},\end{aligned}$$

where

$$d_3(t)=\frac{\ln\left(A_{3,0}/A_{4,0}\right)+\widehat{\sigma}_{II}^2t/2}{\widehat{\sigma}_{II}\sqrt{t}}$$
$$d_4(t)=d_3(t)-\widehat{\sigma}_{II}\sqrt{t},$$

and $\widehat{\sigma}_{II}\sqrt{t}$ is the volatility term of the composite asset $\frac{A_{3,t}^{(n)}}{A_{4,t}^{(n)}}$, calculated as

$$\begin{aligned}\widehat{\sigma}_{II}^2t&=\mathrm{Var}_0^Q\left[\ln\left(\frac{A_{3,t}}{A_{4,t}}\right)\right]\\&=\mathrm{Var}_0^Q\left[\ln\left(\frac{\mathrm{e}^{X_t}}{\mathrm{e}^{Y_t}}\right)\right]\\&=\mathrm{Var}_0^Q\left(X_t\right)+\mathrm{Var}_0^Q\left(Y_t\right)-2\mathrm{Cov}_0^Q\left(X_t,Y_t\right)\\&=\frac{\sigma^2}{\lambda^2}\left(1-\frac{2}{\lambda}\left(1-\mathrm{e}^{-\lambda}\right)+\frac{1}{2\lambda}\left(1-\mathrm{e}^{-2\lambda}\right)\right)\\&\quad+\sigma_S^2+\frac{2\rho\sigma\sigma_S}{\lambda}\left(1-\frac{1-\mathrm{e}^{-\lambda}}{\lambda}\right).\end{aligned}$$

Because $A_{3,t}$ and $A_{4,t}$ are not tradable assets in the market, we derive their initial market values, $A_{3,0}$ and $A_{4,0}$, as follows:

$$\begin{aligned} A_{3,0} &= E_0^Q\left[\frac{A_{3,t}}{B_t}\right] = E_0^Q\left[e^{X_t}\right] \\ &= E_0^Q\left[\exp\left(g_3(t) - \frac{\sigma}{\lambda}\int_{t-1}^{t}\left(1-\mathrm{e}^{-\lambda(t-v)}\right)\mathrm{d}W_v\right)\right] \\ &= \exp\left[g_3(t) + \frac{\sigma^2}{2\lambda^2}\mathrm{Var}_0^Q\left(\int_{t-1}^{t}\left(1-\mathrm{e}^{-\lambda(t-v)}\right)\mathrm{d}W_v\right)\right] \\ &= \exp\left[g_3(t) + \frac{\sigma^2}{2\lambda^2}\int_{t-1}^{t}\left(1-\mathrm{e}^{-\lambda(t-v)}\right)^2\mathrm{d}v\right] \\ &= \exp\left[g_3(t) + \frac{\sigma^2}{2\lambda^2}\left(1 - \frac{2}{\lambda}\left(1-\mathrm{e}^{-\lambda}\right)\right.\right. \\ &\quad \left.\left. + \frac{1}{2\lambda}\left(1-\mathrm{e}^{-2\lambda}\right)\right)\right] \end{aligned}$$

$$A_{4,0} = E_0^Q\left[\frac{A_{4,t}}{B_t}\right] = E_0^Q\left[e^{Y_t}\right] = \mathrm{e}^0 = 1.$$

Finally, as Eq. (12) shows, the initial market value of the multi-period rate of return guarantee $\pi_0^{(n)}(II)$ equals $\mathrm{e}^{-\int_0^{n-1} f(0,u)\mathrm{d}u} \times \prod_{t=n}^{T} E_0^Q\left[\max\left(e^{X_t}, e^{Y_t}\right)\right] - \mathrm{e}^{-\int_0^{n-1} f(0,u)\mathrm{d}u}$, and we have derived that $E_0^Q\left[\max\left(e^{X_t}, e^{Y_t}\right)\right] = A_{3,0}N(d_3(t)) - A_{4,0}N(d_4(t)) + A_{4,0}$. Therefore, it follows that

$$\pi_0^{(n)}(II) = \exp\left(-\int_0^{n-1} f(0,u)\,\mathrm{d}u\right) \times \left\{\prod_{t=n}^{T}\left[A_{3,0}N(d_3(t)) - A_{4,0}N(d_4(t)) + A_{4,0}\right] - 1\right\}.$$

Appendix G

In the numerical analysis, we analyze the effect of mortality using the UK standard tables for annuitant and pensioner populations for the period 1991–1994 proposed by CMI Bureau (1999). We describe the methodology for constructing ${}_T p_x$, the survival probability that an employee aged x remains alive after T years, as follows.

CMI Bureau calculates the base mortality rates (q_x) first, where q_x is the probability of dying in one year for an employee aged x. By definition, $q_x = 1 - \exp\left(-\int_0^1 \mu_{x+s}\mathrm{d}s\right)$, where μ_{x+s} is the hazard rate. According to the report issued by CMI Bureau (1999), μ_{x+s} is expressed as

$$\mu_{x+s} = a_1 + a_2\gamma + \exp\left\{b_1 + b_2\gamma + b_3\left(2\gamma^2 - 1\right)\right\},$$

where $\gamma = (x+s-70)/50$. In our numerical results, we consider two parameter sets: (i) $a_1 = 0.0003$, $a_2 = 0$, $b_1 = -5.265363$, $b_2 = 6.683129$, and $b_3 = -0.9$ for female pensioners; (ii) $a_1 = 0.00014429$, $a_2 = -0.00040629$, $b_1 = -4.399861$, $b_2 = 5.568973$, and $b_3 = -0.654909$ for male pensioners. These parameter estimates are firstly obtained from CMI Bureau (1999).

The projected mortality rates ($q_{x,t}$) is obtained by considering the reduction factor ($RF(x,t)$), which is $q_{x,t} = q_x \cdot RF(x,t)$. The projected mortality improvement factor for age x at time t by CMI Bureau (1999) is expressed as

$$RF(x,t) = \alpha(x) + [1-\alpha(x)]\cdot[1-\beta(x)]^{t/20}$$

$$\alpha(x) = \begin{cases} 0.13 & x < 60 \\ 1 + 0.87 \cdot \dfrac{x-110}{50} & 60 \le x < 110 \\ 1 & x \ge 110 \end{cases}$$

$$\beta(x) = \begin{cases} 0.55 & x < 60 \\ \dfrac{(110-x)\cdot 0.55 + (x-60)\cdot 0.29}{50} & 60 \le x < 110 \\ 0.29 & x \ge 110. \end{cases}$$

Therefore, the survival probability can be calculated as ${}_T p_x = (1-q_x)(1-q_{x+1,1})\cdots(1-q_{x+T-1,T-1})$.

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