A Markovian Model for Process Setup and Improvement

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Abstract: We examine the setup and improvement policies for a production process with multiple performance states. Assume that the production process deteriorates randomly over time, following a Markovian process with known transition probabilities. In order to reduce the production cost incurred because of process deterioration, the process is inspected at the end of each period. Then one of three actions may be taken: do nothing, perform routine process setup, or perform routine setup and process improvement. The routine setup operation returns the process to its best performance state, whereas the process improvement action may reduce future production and setup costs and improve the process-state transition probabilities. A discounted Markovian model is formulated to find the strategy that minimizes the total cost of operating the production process. © 1997 John Wiley & Sons, Inc. Naval Research Logistics **44:** 383–400, 1997

1. INTRODUCTION

Process improvement has been an important theme of quality management. Several models have been developed to examine process setup, process improvement, and product quality issues in manufacturing environments. Porteus [13] introduced a model with three quality-improvement actions for a simple EOQ environment and demonstrated the relationship between production setup policy and product quality. The three actions are reducing setup cost, reducing the probability of out-of-control states, and a combination of both. Fine [5, 6] presented a quality-based learning model that linked quality-control and learning curve research to study the process control policy. He assumed that production cost can be reduced by quality-based learning and showed that ignoring the effects of learning may lead to underinvesting in process improvement activities. Tapiero [16] studied the learning effects resulting from quality-control policies under the assumption that future product quality is an explicit function of production level and quality-related activities up to the present. Fine and Porteus [7] used a Markovian model to study the economics of investing in gradual process improvement activities. They developed a steady process state model, assuming that the improvement of the process is a function of the investment amounts determined by the producer. Their results illustrate how uncertainties in the benefit of process improvement discourage firms from pursuing setup time reduction and quality improvement.

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Note that the production processes considered in the above articles are generally assumed to have two performance states: in control and out of control. As pointed out by Marcellus and Dada [9], under this assumption, when the process needs correction, the decision maker has only one option; that is, to simultaneously perform a routine setup (restore the process to the in-control state) and improve the process. Therefore, instead of using the two-state process, Marcellus and Dada [9] studied a low-volume production system in which the process is always in control. They consider the action of quality improvement produced by that response to the defective item found by inspection. By focusing their attention on the timing of process improvement, they derived some properties regarding the choice of improvement actions. Dada and Marcellus [3] extended Fine's [6] and their own model by considering the situation in which the decision is based on partial information of the process condition and on the resulting transition probabilities between states. In their model, they consider three actions: continue the process, perform a routine setup, or perform both a routine setup and a process improvement.

Without considering the issue of process improvement, the issue of process setup is very similar to that of machine maintenance and replacement. For example, Derman [4] used a Markovian process with multiple performance states to describe the machine deterioration process. It was assumed that the process is periodically inspected to determine whether the machine should be replaced under the fixed replacement cost. A decision rule is developed to minimize the expected cost of operating the machine. Kolesar [8] extended Derman's model by assuming the cost of operating the machine is dependent on machine states. More recently, Chand, McClurg, and Ward [1] considered the possibility of replacing an old machine with a newer (better) machine that has a lower operating cost. They assumed that the setup cost associated with replacing a machine decreases as the number of replacements increases. The cost components considered are purchasing price and salvage value of the machine, maintenance cost, labor and operating costs of the machine, and cost of replacing a machine. Other recent maintenance models include those developed by Tijms and van der Duyn Schouten [17], Chand and Sethi [2], and Meredith and Camm [11]. Readers are referred to Meredith [10], Mine and Osaki [12], and Ross [14] for extensive lists of studies on the topic.

The objective of this article is to examine the process setup and improvement issues under a more general set in performance states, in particular, with respect to those models by Porteus [13], Fine [6], and Marcellus and Dada [9]. It is assumed in this article that machine deterioration follows a Markovian process with multiple performance states. Furthermore, the machine is inspected at the end of each period, and, based on the observed machine state, one of the following three alternatives is chosen: (1) leave the process as it is, (2) perform a routine setup, or (3) perform a routine setup and process improvement. It is assumed that the process state is adjusted to the best state after each routine setup, and that a process improvement action improves future machine performance in terms of lower production costs, setup costs, and/or slower machine deterioration speeds. We formulate the problem as a discounted Markovian decision model with the objective of developing an optimal decision rule to minimize the expected total operating cost of the machine.

The model developed in this article can also be viewed as a direct extension of Kolesar's [8] machine-replacement model by considering possible process improvement actions. Unlike a process with a single state or two states, the decision in process with multiple performance states is much more complicated because we have to consider the factors of time, process states, and their interaction in selecting the optimal action.

The article is organized as follows. In the next section, the mathematical model is formu-

lated. Sections 3 and 4 report several important properties related to the optimal decision rule and methods of finding the optimal decision rule. In Section 5, an upper bound for the optimal number of process improvement actions is derived. Numerical examples and a sensitivity analysis are presented in Section 6. In Section 7, possible extensions of the model are discussed.

2. MODEL FORMULATION

Consider a production process with n performance states, $1, 2, \ldots, n$, with better performance at state i than at state $i + 1$. In other words, production costs for state i are lower than those for state $i + 1$. It is assumed that the production process deteriorates randomly over time, following a Markovian process with known transition probabilities.

Let $\mathbf{Q}(0) = (q_{ij}(0))_{n\times n}$ be the initial (at time zero) process state transition matrix, where $q_{ij}(0)$ is the probability that the process may shift from state i to state j during one time period in which no process setup and/or improvement action are taken. It is assumed that, without a process setup, an unmodified production process can only remain in the same state or deteriorate to a worse state. Consequently, $q_{ij}(0) = 0$, for $j < i$.

To avoid incurring excessive production cost resulting from process deterioration, the process is inspected at the end of each period and, based on the process state observed, one of the following three actions may be taken:

Action 1: Continue the process. Leave both the process state and deterioration process unchanged.

Action 2: Perform a routine setup. A routine setup includes, for example, equipment recalibration and replacement of machine tools. Process setups result in returning the process to state 1.

Action 3: Perform a routine setup and process improvement. The result is that, after returning the process to state 1, the process improvement action improves overall process performance by reducing future production costs and/or future setup costs, and/or by improving future state transition probabilities.

Let k be the number of times that Action 3 has been taken since the beginning of production, and let i be the process state observed at the end of period t. The production cost, setup cost, and process improvement cost are related to process condition, and will therefore be affected by the process improvement actions.

Given observed process state i and the number of improvement actions k since the beginning, we use $c_i(k)$, $s_i(k)$, and $u_i(k)$ to represent the cost of production, setup, and improvement, respectively. For given k, the one-period cost associated with taking Actions 1, 2, and 3 are $c_i(k)$, $c_i(k) + s_i(k)$, and $c_i(k) + s_i(k) + u_i(k)$, respectively. Note that because we assume that the process is inspected at the end of each period, the definition of production cost $c_i(k)$ implies that the process shifts at the beginning of each period or that the product quality of the entire period is determined by its state at the end of the period. This assumption is used primarily to simplify model formulation and does not change the structure of the problem.

We assume that both $c_i(k)$ and $s_i(k)$ are nonincreasing with respect to k and nondecreasing with respect to i. This implies that (a) neither production costs nor setup costs decrease

as the process deteriorates, and (b) neither production costs nor setup costs increase as a result of a process improvement action.

The transition probabilities may also be improved by process improvement actions. Let $\mathbf{Q}(k)$ denote the process-state transition matrix after Action 3 has been taken k times. As in the initial matrix $\mathbf{Q}(0)$, it is assumed that $q_{ij}(k) = 0$, for $j < i$. The difference between $\mathbf{Q}(k)$ and $\mathbf{Q}(k-1)$ represents the improvement in transition probabilities as a result of taking Action 3. Desired improvements in the transition matrix are described by the relationship $q_{ii}(k+1) \geq q_{ii}(k)$ and $q_{ij}(k+1) \leq q_{ij}(k)$, for $i < j$, which implies that, after a process improvement action has been taken, (a) the chance of shifting from one state to a worse state decreases or remains unchanged, and (b) the chance of remaining at the same state increases or remains the same.

We formulate the problem as an infinite horizon discounted Markovian decision model with the objective of developing a stationary decision rule [12, 14] that minimizes the long-term discounted operating cost. Let β denote the discount rate, and define, at period t, $V_t(k, i)$ as the minimum expected total β-discounted cost from period t to infinity, given that the process is in state i at the end of period t and that k process improvement actions have been taken. Then, for period $t-1$, the expected discounted cost $V_{t-1}(k, i)$ is equal to

$$G_t^1(k, i) = c_i(k) + \beta \sum_{j=i}^{n} q_{ij}(k) V_t(k, j), \qquad \text{if Action 1 is taken,}$$

$$G_t^2(k, i) = c_i(k) + s_i(k) + \beta \sum_{j=1}^{n} q_{1j}(k) V_t(k, j), \qquad \text{if Action 2 is taken,}$$

or

$$G_t^3(k+1, i) = c_i(k) + s_i(k) + u_i(k) + \beta \sum_{j=1}^{n} q_{1j}(k+1) V_t(k+1, j), \qquad \text{if Action 3 is taken.}$$

Consequently, we have

$$V_{t-1}(k, i) = \min\{G_t^1(k, i), G_t^2(k, i), G_t^3(k+1, i)\}. \tag{1}$$

Note that $V_t(k, i)$ is the minimal discounted cost from time t to infinity. $V_t(k, i)$ depends only on the state (k, i) at time t because all the costs are state-dependent rather than time-dependent. Hence, the subscript t can be omitted from the expression; that is, $V(k, i) = V_t(k, i)$. Equation (1) can therefore be rewritten as

$$V(k, i) = c_i(k) + \min\left\{ \beta \sum_{j=i}^{n} q_{ij}(k) V(k, j),\ s_i(k) + \beta \sum_{j=1}^{n} q_{1j}(k) V(k, j),\ s_i(k) + u_i(k) + \beta \sum_{j=1}^{n} q_{1j}(k+1) V(k+1, j) \right\}. \tag{2}$$

3. PROPERTIES OF THE OPTIMAL DECISION RULE

In this section, we discuss several properties related to the optimal solution to the model given in (2). These properties allow us to have a better understanding of the model's cost structure and the optimal decision rule.

We first consider the finite-horizon stochastic dynamic programming problem described as follows. Let T be the index representing the number of periods since the beginning of the production process. Furthermore, let $V_t^T(k, i)$ be the minimum expected total β-discounted cost from the current period t to T given that the current process state is i and the number of process improvement actions taken so far is k. Let $V_T^T(k, i) = 0$, for all k and i. Then, for any $t < T$, $V_t^T(k, i)$ satisfies the following relationship:

$$V_{t-1}^T(k, i) = c_i(k) + \min\left\{\beta \sum_{j=i}^{n} q_{ij}(k)V_t^T(k, j),\ s_i(k) + \beta \sum_{j=1}^{n} q_{1j}(k)V_t^T(k, j),\right.$$

$$\left. s_i(k) + u_i(k) + \beta \sum_{j=1}^{n} q_{1j}(k+1)V_t^T(k+1, j)\right\}. \quad (3)$$

For any fixed t, when T goes to infinity, $V_t^T(k, i)$ converges to $V(k, i)$ defined in (2). Based on (3), we can use backward induction to prove the following lemma (a proof is given in the appendix).

LEMMA 1: If $c_i(k)$, $s_i(k)$, $s_i(k) + u_i(k)$, and $\Sigma_{j=h}^n q_{ij}(k)$, for $h \le n$, are nondecreasing in i and nonincreasing in k, then $V_t^T(k, i)$ is nondecreasing in i and nonincreasing in k.

We obtain the next proposition because $V_t^T(k, i)$ converges to $V(k, i)$ when T goes to infinity.

PROPOSITION 1: If $c_i(k)$, $s_i(k)$, $s_i(k) + u_i(k)$ and $\Sigma_{j=h}^n q_{ij}(k)$, for $h \le n$, are nondecreasing in i and nonincreasing in k, then $V(k, i)$ is nondecreasing in i and nonincreasing in k.

$V(k, i) \le V(k, i + 1)$ implies that the expected total cost is smaller if the process begins in a better process state. The result is intuitively true. On the other hand, because a larger k implies lower future production or setup costs, and/or a slower deterioration process, it should result in a lower total future discounted cost. This is a reasonable property expected in most production situations. The condition on transition probabilities in Proposition 1 is well known in maintenance models (Condition B in Derman [4].) The condition is also called the condition of increasing failure rate (IFR) of the system.

One special example of $\mathbf{Q}(k)$ satisfying the IFR property is as follows: When $k = 0$, $q_{ij}(0) \le q_{(i+1)j}(0)$, for $j = i + 2, \ldots, n$ and $i = 1, 2, \ldots, n$. When $k > 0$,

$$q_{ii}(k + 1) = \lambda_k q_{ii}(k) + (1 - \lambda_k),$$

$$q_{ij}(k + 1) = \lambda_k q_{ij}(k), \qquad \text{for } i \ne j,$$

where $0 \le \lambda_k \le 1$. In this example, the process deterioration speed increases as the process shifts to a worse state, and the transition probabilities from the current state to its worse states decrease proportionally after an improvement action has been taken.

By finding the difference between the third and the second terms within the braces in the right side of (2), the cost increment of taking Action 3 over Action 2 is given by

$$u_i(k) - \beta \sum_{j=1}^{n} [q_{1j}(k)V(k, j) - q_{1j}(k + 1)V(k + 1, j)]. \tag{4}$$

Consequently, Action 2 is a more economical choice than Action 3 if and only if (4) is greater than or equal to zero. That is,

$$u_i(k) \geq \beta \sum_{j=1}^{n} [q_{1j}(k)V(k, j) - q_{1j}(k + 1)V(k + 1, j)]. \tag{5}$$

Because the right side of (5) is independent of process state i, the choice between Actions 2 and 3 is purely dependent on $u_i(k)$. We can then easily show the following proposition (a proof is given in the appendix).

PROPOSITION 2: Assume that the assumptions of Lemma 1 hold, and that there exists a constant $\alpha > 0$, such that $u_i(k) \geq \alpha$, for all i and k. Then

(a) for any given k, there is an integer $i(k)^*$ such that Action 2 is preferred over Action 3 if $i \geq i(k)^*$; and
(b) There is an integer k^* such that, for all $k > k^*$, taking Action 3 is not economical.

If $u_i(k) = u(k)$ for all i, we can use (5) to obtain the following corollary.

COROLLARY: If the assumptions of Proposition 2 hold and $u_i(k) = u(k) \geq \alpha$, for all k, then the choice between Actions 2 and 3 is determined only by the number of process improvements that has been taken so far, not by the current process state.

Intuitively, the existence of maximum k is due to the assumption on decreasing marginal benefit of process improvement. After k^* improvement actions have been taken, the future saving of process improvement is less than the cost required to perform the improvement action.

From Proposition 1, $\Sigma_{j=i}^{n} q_{ij}(k)V_t(k, j)$ is nondecreasing with respect to i. Using this property and the corollary, we obtain the following proposition.

PROPOSITION 3: If the assumptions of Proposition 2 hold, $u_i(k) = u(k) > \alpha$, $\alpha > 0$, and $s_i(k) = s(k)$, for all k, then there exists an index k^* such that the following decision is optimal: For any number of improvement actions k, there exists an index $i(k)$ such that

(a) when $k < k^*$, Action 1 is optimal for $i \in [1, i(k)]$; otherwise, take Action 3.
(b) when $k = k^*$, Action 1 is optimal when $i \in [1, i(k^*)]$; otherwise, take Action 2.

Proposition 3 indicates that Action 2 should not be taken for all process states until the number of the process improvement actions reaches k^*. This result is similar to that in the

two-process states model by Marcellus and Dada [9] and in some other inspection and maintenance models.

4. OPTIMAL DECISION RULE

In this section we discuss the issues related to finding the optimal decision rule, including the determination of the optimal number of process improvement actions. It should be pointed out that the results associated with the well-known Markovian decision model with finite state space cannot be directly applied to our model because of the addition of process improvement actions. Because the process improvement actions affect future costs as well as the process-state transition matrix, the decision rule is dependent on the process state i and the number of improvement actions k as well.

It can be verified that our model is equivalent to a Markovian decision model with infinite state space $S = \{(k, i), i = 1, 2, \ldots, n, k = 1, 2, \ldots, \infty\}$ and finite action space $\{a_1, a_2, a_3\}$ corresponding to Actions 1, 2, and 3. From existing results in Markovian decision models, we know that it is difficult to directly find the exact solution for an infinite state space model. Fortunately, by using Proposition 2, we are able to truncate the number of states in the state space of the model until it is finite, under which exact solution procedures are available. Let k^* be the optimal number of process improvement actions of the model. Action 3 should not be taken for any number $k \geq k^*$ at any process state. Hence, for any integer $K \geq k^*$, the Markovian decision model with state space $S_K = \{(k, i), i = 1, 2, \ldots, n, k = 1, 2, \ldots, K\}$ is equivalent to the above Markovian decision model, but with infinite state space. Note that the action space corresponding to state (K, i), for all i, should be modified and restricted to Actions 1 and 2 only. The transition probabilities from state $s = (k, i)$ to state $s' = (k', j)$ under different actions are summarized as follows:

$$p_{ss'}(a_1) = \begin{cases} q_{ij}(k), & \text{if } k' = k, \\ 0, & \text{otherwise,} \end{cases}$$

$$p_{ss'}(a_2) = \begin{cases} q_{1j}(k), & \text{if } k' = k, \\ 0, & \text{otherwise,} \end{cases}$$

$$p_{ss'}(a_3) = \begin{cases} q_{1j}(k'), & \text{if } k' = k + 1, \\ 0, & \text{otherwise.} \end{cases}$$

The costs corresponding to Actions 1, 2, and 3 are $c_i(k)$, $c_i(k) + s_i(k)$, and $c_i(k) + s_i(k) + u_i(k)$, respectively, for $(k, i) \in S_K$.

For the above Markovian decision model with finite state space and finite action space, its optimal solution can be obtained numerically. One commonly used solution approach involves solving a linear programming formulation. Let $\{y_{sa_i}\}$ be the randomized decision policy. The linear programming corresponding to our model is as follows:

$$\text{LP}(K)\text{: Minimize } \sum_{s\in S_K} [c_i(k)y_{sa_1} + (c_i(k) + s_i(k))y_{sa_2} + (c_i(k) + s_i(k) + u_i(k))y_{sa_3}],$$

$$\text{s.t.} \quad y_{(k,i)a_1} - \beta \sum_{j=1}^{n} q_{ji}(k)y_{(k,j)a_1} + [1 - \beta q_{1i}(k)]y_{(k,1)a_2}$$

$$+ [1 - \beta q_{1i}(k+1)]y_{(k,1)a_3} = \theta_{(k,i)}, \qquad \text{for } k \neq K, \qquad \text{for all } i,$$

$$y_{(K,i)a_1} - \beta \sum_{j=1}^{n} q_{ji}(K)y_{(K,j)a_1} + [1 - \beta q_{1i}(K)]y_{(K,1)a_2} = \theta_{(K,i)}, \qquad \text{for all } i,$$

$$y_{sa_1} \geq 0, \qquad y_{sa_2} \geq 0, \qquad y_{sa_3} \geq 0, \qquad s \in S_K. \tag{6}$$

where $\theta_s \geq 0$ and where $\Sigma_{s\in S_K} \theta_s = 1$ is the probability distribution of the initial process states.

It is easy to verify that the constraint set of the linear programming formulation has nK constraints and $(3nK - n)$ variables. From existing Markovian decision process results, we know that, when $\theta_s > 0$, for all $s \in S_K$, the optimal basic solution of LP(K) has a one-to-one correspondence with the nonrandomized optimal stationary policy of the Markovian decision model. That is, for any state $s = (k, i)$, Action j, $j = 1, 2, 3$, is taken if and only if $y^*_{sa_j} > 0$. From the discussion in the previous section, we know that $y^*_{sa_3} = 0$, if $k \geq k^*$, and, when $k < k^*$, there exists an index i_k, such that $y^*_{s'a_3} > 0$, for $s' = (k, i)$ with $i > i_k$ under the assumptions of Proposition 3.

From the above discussion, if an upper bound K of k^* can be found, we can obtain both the optimal number of process improvement actions k^* and the optimal decision corresponding to each k and state i. Therefore, finding K is critical to ensure that the problem is computationally solvable. With the use of the upper bound, we can find the optimal decision rule and optimal number of improvements. The following proposition gives an upper bound for K (a proof is given in the appendix).

PROPOSITION 4: Let K be an integer. K is an upper bound of k^*, if, for $k \geq K$,

$$u_i(k-1) \geq \frac{\beta}{1-\beta}(\Delta_c(k) + \Delta_s(k) + \Delta_Q(k)),$$

where

$$\Delta_c(k) = \max_{1\leq i\leq n} \{c_i(k-1) - c_i(k)\}, \qquad \Delta_s(k) = \max_{1\leq i\leq n} \{s_i(k-1) - s_i(k)\},$$

and

$$\Delta_Q(k) = \max_{1\leq i\leq n} \{|(\mathbf{Q}(k-1) - \mathbf{Q}(k))V(k)|_i\}.$$

From (2), we know that the future decision rule depends only on the current process state and the number of process improvement actions taken so far. Therefore, the optimal decision can be found by successively solving a series of fixed-size linear programming problems, each with updated initial process conditions. This significantly reduces the computational efforts required to obtain the optimal solution. We conjecture that the total cost has

a monotone decreasing marginal saving with respect to k. That is, $V(k-1, i) - V(k, i)$ is a decreasing function of k. When this is true, we can show that $K < k^*$ if and only if, for any $k < K$, there exists at least one $s = (k, i_k)$ such that the solution of LP(K), $y^*_{sa_3} > 0$. In other words, if we can find a K value such that the optimal solution of LP(K) has $y^*_{sa_3} = 0$, for some $k < K$ and for all i, then we conclude $k^* < K$.

5. NUMERICAL RESULTS

In this section, an example is used to report our computational experience with the model and the proposed solution procedure. Based on this example, sensitivity analysis is also carried out to investigate the effects of (1) the discount rate and (2) the rate of improvement on the optimal policy. We will focus on the cost benefits of process improvement actions for three situations:

1. The improvement action reduces only the production cost.
2. The improvement action reduces only the setup cost.
3. The improvement action improves only the process-state transition matrix.

It should be pointed out that although the sensitivity analysis is based on one example, similar results were observed in other examples that have been tested.

5.1. Example

The example that will be used for illustration is assumed to have the following initial settings: (1) a production process with five possible process states: state 1 to state 5; (2) at each process state, initial production cost $c_i(0)$, setup cost $s_i(0)$ and process improvement cost $u_i(0)$ defined by the process state table shown below:

Process state	1	2	3	4	5
c_i (0)	2.0	3.0	7.0	11.0	30.0
s_i (0)	10.0	10.0	10.0	10.0	10.0
u_i (0)	6.0	6.0	6.0	6.0	6.0

(3) initial transition matrix $\mathbf{Q}(0)$ given as

$$\mathbf{Q}(0) = \begin{pmatrix} 0.6 & 0.2 & 0.1 & 0.1 & 0.0 \\ 0.0 & 0.6 & 0.2 & 0.1 & 0.1 \\ 0.0 & 0.0 & 0.6 & 0.25 & 0.15 \\ 0.0 & 0.0 & 0.0 & 0.6 & 0.4 \\ 0.0 & 0.0 & 0.0 & 0.0 & 1.0 \end{pmatrix},$$

and, (4) discount rate β set at 0.985.

CASE 1 (production cost improvement): We examine the cost benefit of making a process improvement that affects only the future production cost; that is, $s_i(k) = s_i(0)$, $u_i(k) = u_i(0)$, for $i = 1, 2, \ldots, 5$. Furthermore, the transition matrix is not affected by k; that is, $\mathbf{Q}(k) = \mathbf{Q}(k-1)$. The production cost is assumed to be proportionally decreasing

in k. That is, $c_i(k) = \delta_i i(k-1)$, for $i = 1, 2, \ldots, 5$, where $\delta_1 = 0.9$, $\delta_2 = 0.91$, $\delta_3 = 0.92$, $\delta_4 = 0.93$, and $\delta_5 = 0.94$.

With the use of the solution procedure described in the previous section, the optimal number of improvement actions is found to be $k^* = 10$. The optimal decision rule can be summarized by the following table:

	Process state				
k	1	2	3	4	5
0–3	a_1	a_3	a_3	a_3	a_3
4–9	a_1	a_1	a_3	a_3	a_3
10	a_1	a_1	a_2	a_2	a_2

The above table is read as follows: If the number of times that Action 3 has been taken so far is 0–3, then Action 1 should be taken if the process state is 1. Otherwise, take Action 3. If the number of times that Action 3 has been taken is 4–9, Action 1 should be taken if the process is at state 1 or 2. Otherwise, Action 3 should be taken. Note that no more 10 total process improvement actions should be taken. Once k reaches 10, Action 3 should not be taken again. Instead, take Action 1 if the process is at state 1 or 2, and take Action 2 otherwise.

Figure 1 shows the relationship between the total cost and the number of improvement actions k. It is found that, in this example, more than 20% of the cost may be saved by

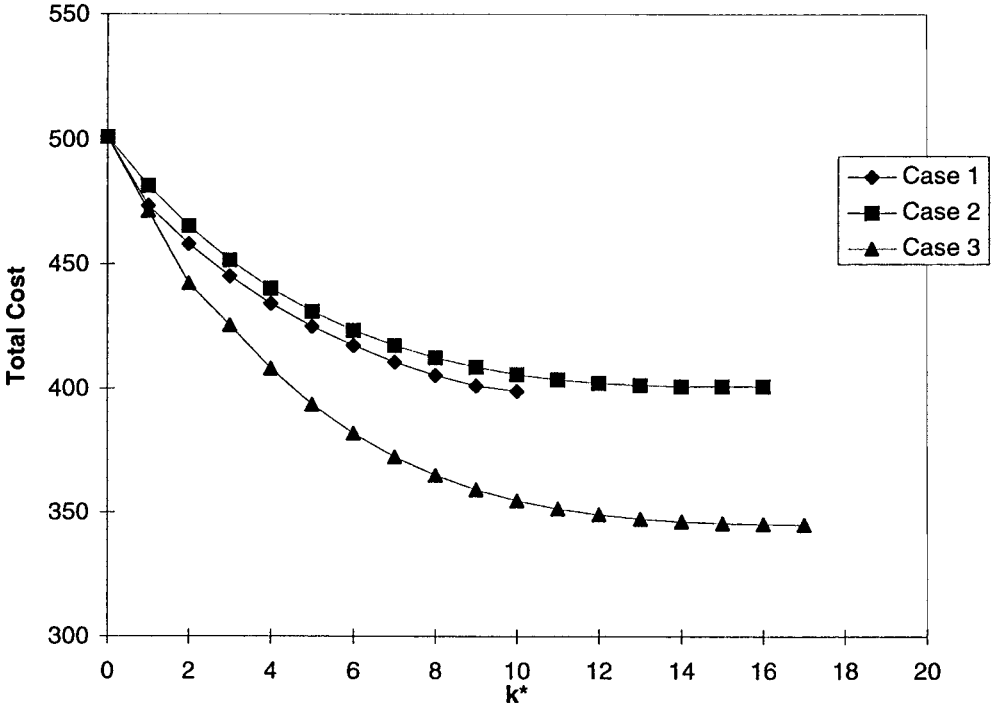

Figure 1. Total cost as a function of the number of improvement actions.

properly choosing the number of improvement actions. In addition, a decreasing marginal cost benefit is observed.

For comparison, we also evaluate the situation in which the number of improvement actions is larger than k^*. As expected, when $k > k^*$, the overall system cost increases as k increases. However, we find that the marginal increase in the total cost is much smaller than that when k is less than k^*, which suggests that the cost penalty of overimproving the process is less than that of underimproving the process.

Furthermore, it is found that the changeover state from Action 1 to Action 3 or Action 2 increases as the number of improvement actions increases; that is, $i(k)$ in Proposition 3 is generally a nondecreasing function of k. This implies that the setup or the improvement action should be delayed as the number of improvement action increases.

CASE 2 (setup cost improvement): We assume that the process improvement actions reduce only future setup costs. The relationship between the setup cost and the number of process improvement actions is given by $s_i(k) = 0.9 * s_i(k - 1)$. It was found that the optimal number of improvement actions is 15. The corresponding decision rule is as follows: As long as $k < 15$, take Action 1 if the process is in state 1. Otherwise, take Action 3. When $k = 15$, take Action 1 when the process is in state 1, and Action 2 for other process states. We can see the similar relation (see Figure 1) between the total cost and the number of improvement k. From our computational experience, it is also found that, as k becomes larger, the change from Action 1 to Action 2 or to Action 3 occurs at the same or a better process state. This is different from what we observed in Case 1. The result is intuitive. Because k is larger, future setup costs are smaller. As a result, it becomes more attractive to use frequent setups to reduce the production cost because the process can be kept in better process states.

CASE 3 (process deterioration speed improvement): We examine the cost benefit of making process improvements that can improve only the process-state transition matrix. We assume that $\mathbf{Q}(k) = 0.9 * \mathbf{Q}(k - 1) + 0.1 * \mathbf{I}$. The optimal number of improvement actions is found to be 17. The relationship between the total cost and the number of improvement actions is also given in Figure 1.

5.2. Sensitivity Analysis

In this subsection, we conduct a sensitivity analysis to study the effects of (1) discount rate, and (2) improvement rate on the optimal solution. The example presented in Section 5.1 is used as the baseline for the analysis.

Discount Rate β. Because an increase in the discount factor β implies essentially that future process improvement returns have a greater impact on the total cost, as β increases, a larger number of improvement actions are desired to achieve a long-term system performance. Our computational results are consistent with this conclusion. Figure 2 gives the total cost for selected discount rates ranging from 0.95 to 0.986, with an increment of 0.04. The three curves in Figure 2 correspond to the three cases discussed in the last subsection, in which the improvement action improves the production cost, setup cost, or process deterioration probabilities, respectively. In the same figure, we also include the costs when improvement actions are not considered. We can easily find that the cost benefit of considering improvement actions increases as the discount factor β increases. This indicates that it is even more profitable to invest in process improvement while the inflation rate is lower.

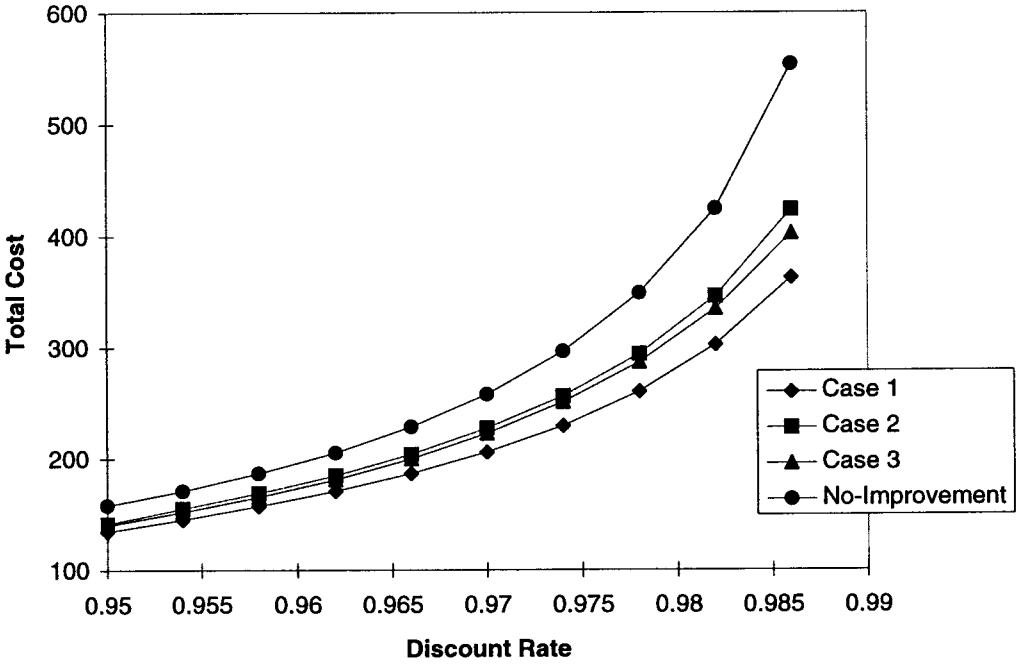

Figure 2. Total cost per unit time as a function of defective cost.

The optimal k^* associated with selected values of β are reported in Figure 3. As expected, the result shows that the number of optimal process improvement actions is positively related to β. We find that k^* has approximately a linear relationship with β when β is not very close to 1. It is also found that, among the three cases considered in the study, k^* corresponding to Case 3 increases at the fastest rate as β increases; therefore, for a large discount rate, there is an incentive to perform a much larger number of improvements on the process deterioration speed.

Rate of Improvement. It is evident that the rate of improvement for process improvement action affects the total cost of the model. We perform the sensitivity analysis of the rate of improvement for three different cases discussed in the previous section. We use the same initial parameters as given in Example 5.1. In Case 1, when improvement action improves production cost, we consider that the production costs after taking k improvement actions become $c_i(k) = \delta_i c_i(k-1)$ for each process state i, where δ_i is determined by $\delta_1 = \delta_2 = 0.9 + \Delta\delta$, $\delta_3 = 0.92 + \Delta\delta$, $\delta_4 = 0.93 + \Delta\delta$, and $\delta_5 = 0.94 + \Delta\delta$. In the sensitivity analysis, $\Delta\delta$ ranges from 0.00 to 0.045 with an increment of 0.005. The costs associated with the optimal solutions for selected values of $\Delta\delta$ are reported in Figure 4.

Similarly, for Cases 2 and 3, we assume the setup cost as a function of k in Case 2 is $s_i(k) = (0.9 + \Delta\delta) * s_i(k-1)$ and the transition probabilities as a function of Case 3 is $\mathbf{Q}(k) = (0.9 + \Delta\delta)\mathbf{Q}(k-1) + (1 - 0.9 - \Delta\delta)\mathbf{I}$, respectively. Figure 5 contains the results for selected values of $\Delta\delta$ ranging from 0.00 to 0.1 with an increment 0.01 each time. Notice that the X axis in Figures 4 and 5 is $\Delta\delta$ rather than the improvement rate. The figures show that the relationship between the total cost and $\Delta\delta$ is almost linear. It appears that the payoffs of improving process production cost and process deterioration probabilities always increase significantly as the improvement rates increase. However, the increase in

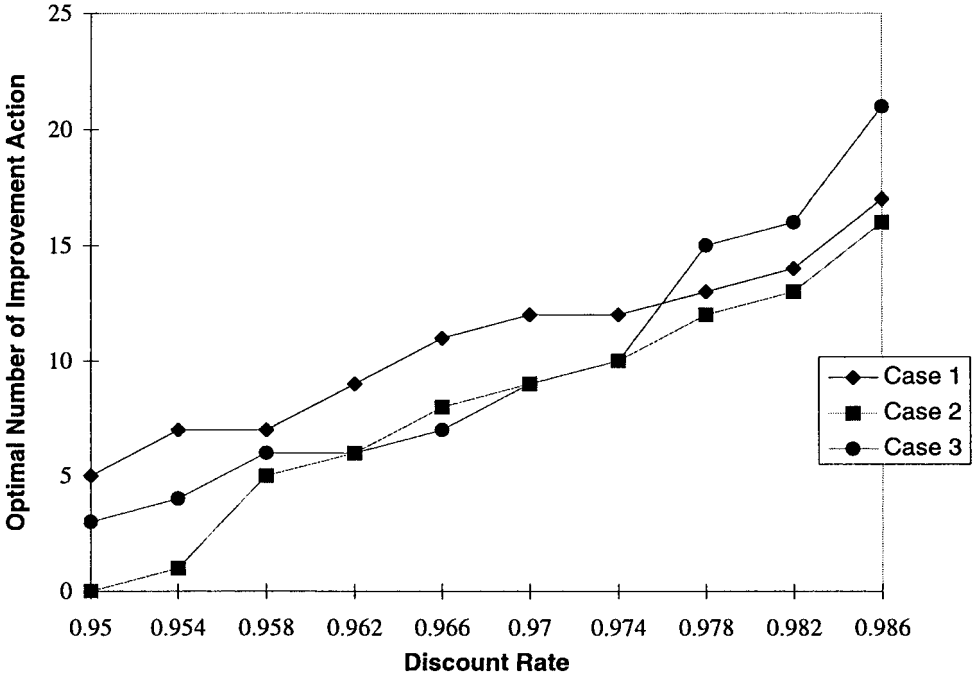

Figure 3. Relationship between discount rate and the optimal number of improvement actions.

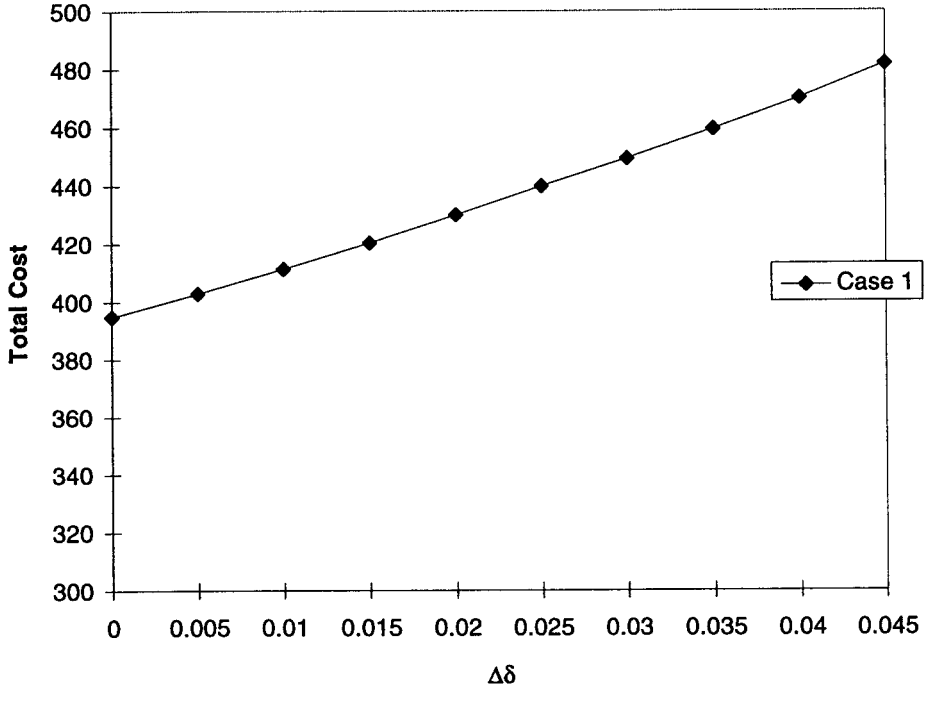

Figure 4. Effects of improvement rate on production cost.

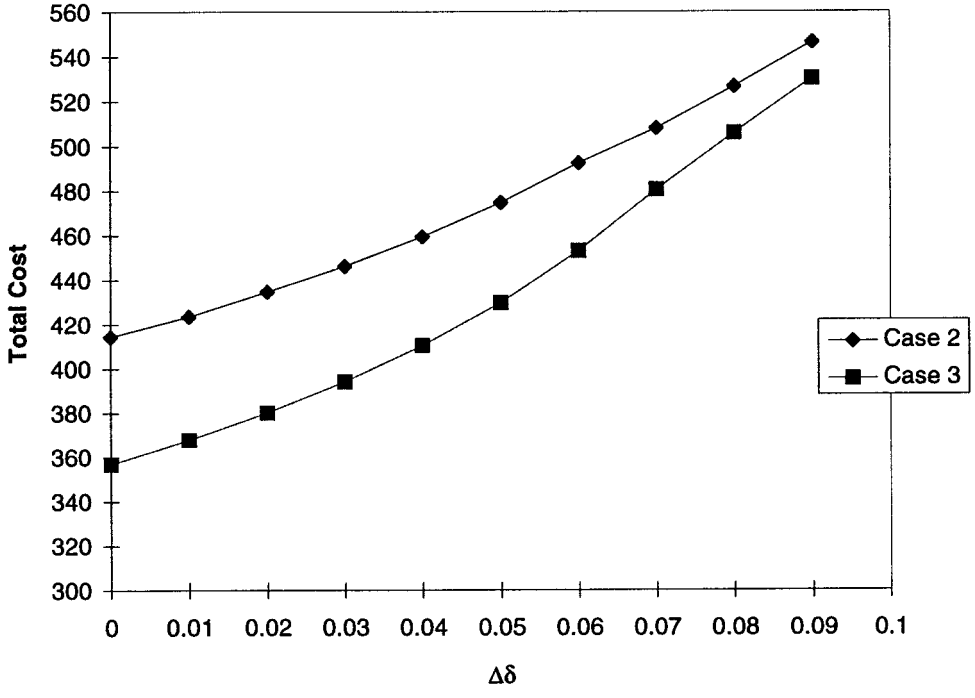

Figure 5. Effects of improvement rate on setup cost and process deterioration.

the improvement rate of the setup cost improves the total cost significantly when the rate of improvement is large. The effect decreases rapidly as the rate of improvement declines.

6. CONCLUSION

In this article we examine the process setup and improvement issues under a more general set of assumptions, in particular, with respect to those considered in the models by Porteus [13], Fine [6], and Marcellus and Dada [9]. The production process under study has multiple performance states and deteriorates randomly over time, following a Markovian process with known transition probabilities. The model allows three possible actions after the process is inspected, including (a) do nothing, (b) perform routine process setup, and (c) perform routine setup and process improvement. The routine setup operation returns the process to its most recent best performance state, whereas the process improvement action may reduce future production and setup costs and improve the process-state transition probabilities.

It is possible to extend the model to allow additional types of corrective and process improvement actions. For example, it is possible to assume that, after a process improvement action, the process may return to any state or a better-than-ever state (which can be interpreted as adopting a new technology) with a given probability. In these generalized situations, the analysis methods in this article can also be performed, and results are expected to be similar. Furthermore, it may also be interesting to extend the model to consider a continuous deterioration process or an infinite number of process states. It is expected that the model becomes more complicated because of the dependence of time and state. In this situation,

however, it may be possible to formulate the problem by using a continuous-time Markovian process when the state space is finite.

APPENDIX

PROOF (of Lemma 1): The proof is based on backward induction. It is clear that the lemma is true when $t = T$. Assume that it is true for t. For $t - 1$, the statement that $V^T_{t-1}(k, i)$ is nondecreasing in i and nonincreasing in k implies that $V^T_{t-1}(k, i) - V^T_{t-1}(k', i')$ is nonnegative when $(k', i') = (k, i - 1)$, and when $(k', i') = (k + 1, i)$. Note that the difference of the minima of the two items is greater than or equal to the minimum of the difference of these two items. From (3), we obtain the following inequality:

$$V^T_{t-1}(k, i) - V^T_{t-1}(k', i') \geq c_i(k) - c_{i'}(k') + \min\left\{\beta\left[\sum_{j=i}^{n} q_{ij}(k)V^T_t(k, j) - \sum_{j=i'}^{n} q_{i'j}(k')V^T_t(k', j)\right],\right.$$

$$s_i(k) - s_{i'}(k') + \beta\sum_{j=1}^{n}[q_{1j}(k)V^T_t(k, j) - q_{1j}(k')V^T_t(k', j)],\ s_i(k) + u_i(k) - s_{i'}(k') - u_{i'}(k')$$

$$\left. + \beta\sum_{j=1}^{n}[q_{1j}(k + 1)V^T_t(k + 1, j) - q_{1j}(k' + 1)V^T_t(k' + 1, j)]\right\}. \tag{7}$$

We first evaluate the right side of (7). Let $(k', i') = (k, i - 1)$. From the Lemma's assumption, $c_i(k) - c_{i'}(k') \geq 0$. Because $q_{i(i-1)} = 0$, the first term within the braces of (7) is

$$\sum_{j=i}^{n} q_{ij}(k)V^T_t(k, j) - \sum_{j=i'}^{n} q_{i'j}(k')V^T_t(k', j) = \sum_{j=i-1}^{n}[q_{ij}(k) - q_{(i-1)j}(k)]V^T_t(k, j)$$

$$= \sum_{j=i-1}^{n}\left[\sum_{h=j}^{n}(q_{ih}(k) - q_{(i-1)h}(k))\right]U_t(k, j), \tag{8}$$

where $U_t(k, i - 1) = V^T_t(k, i - 1)$, and $U_t(k, j) = V^T_t(k, j) - V^T_t(k, j - 1)$, for $j = i, i + 1, \ldots, n$. Because $U_t(k, j) \geq 0$ (as assumed by induction) and $[\Sigma^n_{h=j} q_{ih}(k) - q_{(i-1)h}(k)] \geq 0$, for all j, expression (8) is nonnegative.

Because $k' = k$, the second and the third terms within the braces in (7) become $s_i(k) - s_{i-1}(k)$ and $s_i(k) + u_i(k) - s_{i-1}(k) - u_{i-1}(k)$, respectively. From the lemma's assumption on $c_i(k)$, $s_i(k)$, and $s_i(k) + u_i(k)$, we know that both terms are nonnegative. Hence, $V^T_{t-1}(k, i) - V^T_{t-1}(k', i') \geq 0$ for $(k', i') = (k, i - 1)$.

When $(k', i') = (k + 1, i)$, the first term within the braces in (7) is

$$\sum_{j=i}^{n}[q_{ij}(k)V^T_t(k, j) - q_{ij}(k')V^T_t(k', j)]$$

$$= \sum_{j=i}^{n} q_{ij}(k)[V^T_t(k, j) - V^T_t(k + 1, j)] + \sum_{j=i}^{n}[q_{ij}(k) - q_{ij}(k + 1)]V^T_t(k + 1, j). \tag{9}$$

The first term on the right side of the above equation is nonnegative because of the induction assumption. The second term on the right side of the above equation can be expressed as

$$\sum_{j=i}^{n}[q_{ij}(k) - q_{ij}(k + 1)]V^T_t(k + 1, j) = \sum_{j=i}^{n}\left[\sum_{h=j}^{n} q_{ih}(k) - q_{ih}(k + 1)\right]W_t(k + 1, j),$$

where $W_t(k + 1, i) = V^T_t(k + 1, i)$ and $W_t(k + 1, j) = V^T_t(k + 1, j) - V^T_t(k + 1, j - 1)$, for $j = i + 1, i + 2, \ldots, n$. Because $W_t(k, j) \geq 0$, $\Sigma^n_{h=j} q_{ih}(k) - q_{ih}(k') \geq 0$, for all j, expression (9) is nonnegative. For the second and third terms, note that

$$\sum_{j=1}^{n} [q_{1j}(k)V_t^T(k, j) - q_{1j}(k+1)V_t^T(k+1, j)]$$
$$= \sum_{j=1}^{n} q_{1j}(k)[V_t^T(k, j) - V_t^T(k+1, j)] + \sum_{j=1}^{n} [q_{1j}(k) - q_{1j}(k+1)]V_t^T(k+1, j)].$$

With the use of the same argument to show that the second and the third terms within the braces in (7) are nonnegative, we conclude that both terms on the right side of the last equation are nonnegative. Hence, $V_{t-1}^T(k, i) - V_{t-1}^T(k', i') \geq 0$ for $(k', i') = (k+1, i)$.

PROOF (of Proposition 2): Note that the right side of (5) is independent of i and $u_i(k)$ is nondecreasing in i. If Action 2 is better than Action 3 for some state i_0 [i.e., inequality (5) holds], then it is true for all $i \geq i_0$. Furthermore, from Proposition 1, $V(k, i) \geq V(k+1, i)$ and $V(k, i) \geq 0$. Therefore, we know that $V(k, i)$, for $i = 1, 2, \ldots, n$, converges to a finite value, say $V(i)$, as k increases. Hence, the right side of (5) is

$$\beta \sum_{j=1}^{n} [q_{1j}(k)V(k, j) - q_{1j}(k+1)V(k+1, j)]$$
$$= \beta \left[\sum_{j=1}^{n} q_{1j}(k)(V(k, j) - V(j)) + \sum_{j=1}^{n} (q_{1j}(k) - q_{1j}(k+1))V(j) \right.$$
$$\left. + \sum_{j=1}^{n} q_{1j}(k+1)(V(j) - V(k+1, j)) \right].$$

From the assumption on transition probabilities, we know that $q_{1j}(k) - q_{1j}(k+1)$ converges to zero as k increases. Hence, we conclude that there exists a k^* such that, when $k > k^*$, the right side of (5) is always less than α, and therefore part (b) is proved.

PROOF (of Proposition 4): To simplify our notation, let $\mathbf{c}(k) = (c_1(k), c_2(k), \ldots, c_n(k))^T$, $\mathbf{s}(k) = (s_1(k), s_2(k), \ldots, s_n(k))^T$, and $\mathbf{V}(k) = (V(k, 1), V(k, 2), \ldots, V(k, j))^T$, respectively. Note that k^* is the optimal number of process improvement, and that only Action 1 or Action 2 should be taken when $k = k^*$. Equation (2) thereby becomes

$$V(k, i) = c_i(k^*) + \min\left\{ \beta \sum_{j=i}^{n} q_{ij}(k^*)V(k^*, j),\ s_i(k^*) + \beta \sum_{j=1}^{n} q_{1j}(k^*)V(k^*, j) \right\}. \tag{10}$$

Furthermore, assuming that we have obtained the optimal decision, then we can find two matrices $\mathbf{P}_1$, $\mathbf{P}_2$ such that (10) can be rewritten as

$$\mathbf{V}(k^*) = \mathbf{c}(k^*) + \mathbf{P}_1\mathbf{s}(k^*) + \beta\mathbf{P}_2\mathbf{Q}(k^*)\mathbf{V}(k^*).$$

$\mathbf{P}_1$ is a diagonal matrix, with (i, i)th component equaling 1 if Action 2 is preferred to Action 1 in (10) at process state i, and zero otherwise. Let $e_i = (0, 0, \ldots, 1, \ldots, 0)$, where 1 is at the ith position of the vector. The ith row of $\mathbf{P}_2$ is e_1 if Action 2 is preferred to Action 1 in (10) at process state i, and equals e_i otherwise. Because $\mathbf{P}_1$ and $\mathbf{P}_2$ are matrices corresponding to the optimal decision rule and when $k < k^*$, Action 3 is preferred to Action 2 (Proposition 2). We obtain for $k = k^* - 1$

$$\mathbf{V}(k^* - 1) \leq \mathbf{c}(k^* - 1) + \mathbf{P}_1\mathbf{s}(k^* - 1) + \beta\mathbf{P}_2\mathbf{Q}(k^* - 1)\mathbf{V}(k^* - 1).$$

The difference between $\mathbf{V}(k^* - 1)$ and $\mathbf{V}(k^*)$ is

$$\begin{aligned}\mathbf{V}(k^* - 1) - \mathbf{V}(k^*) &\leq \mathbf{c}(k^* - 1) - \mathbf{c}(k^*) + \mathbf{P}_1[\mathbf{s}(k^* - 1) - \mathbf{s}(k^*)] \\ &\quad + \beta\mathbf{P}_2[\mathbf{Q}(k^* - 1)\mathbf{V}(k^* - 1) - \mathbf{Q}(k^*)\mathbf{V}(k^*)] \\ &= \mathbf{c}(k^* - 1) - \mathbf{c}(k^*) + \mathbf{P}_1[\mathbf{s}(k^* - 1) - \mathbf{s}(k^*)] \\ &\quad + \beta\mathbf{P}_2\mathbf{Q}(k^* - 1)[\mathbf{V}(k^* - 1) - \mathbf{V}(k^*)] + \beta\mathbf{P}_2[\mathbf{Q}(k^* - 1) - \mathbf{Q}(k^*)]\mathbf{V}(k^*).\end{aligned}$$

Hence

$$[\mathbf{I} - \beta\mathbf{P}_2\mathbf{Q}(k^* - 1)][\mathbf{V}(k^* - 1) - \mathbf{V}(k^*)] \le \mathbf{c}(k^* - 1) - \mathbf{c}(k^*) + \mathbf{P}_1[\mathbf{s}(k^* - 1) - \mathbf{s}(k^*)] + \beta\mathbf{P}_2[\mathbf{Q}(k^* - 1) - \mathbf{Q}(k^*)]\mathbf{V}(k^*).$$

Let

$$\Delta_c(k^*) = \max_{1\le i\le n}\{c_i(k^* - 1) - c_i(k^*)\}, \qquad \Delta_s(k^*) = \max_{1\le i\le n}\{s_i(k^* - 1) - s_i(k^*)\}, \quad \text{and}$$

$$\Delta_Q(k^*) = \max_{1\le i\le n}\{[(\mathbf{Q}(k^* - 1) - \mathbf{Q}(k^*))\mathbf{V}(k^*)]_i\} < \max_{1\le i\le n}\{[|\mathbf{Q}(k^* - 1) - \mathbf{Q}(k^*)|\mathbf{V}(0)]_i\}.$$

Then let $\mathbf{1} = (1, 1, \ldots, 1)^T$. We obtain

$$[\mathbf{I} - \beta\mathbf{P}_2\mathbf{Q}(k^* - 1)](\mathbf{V}(k^* - 1) - \mathbf{V}(k^*)) \le (\Delta_c(k^*) + \Delta_s(k^*) + \Delta_Q(k^*))\mathbf{1}.$$

Notice that, because each component of $\mathbf{Q}(k)$ is nonnegative and the sum of each row of $\mathbf{Q}(k^* - 1)$ is one, we obtain the following inequality:

$$(\mathbf{V}(k^* - 1) - \mathbf{V}(k^*)) \le \frac{1}{1 - \beta}(\Delta_c(k^*) + \Delta_s(k^*) + \Delta_Q(k^*))\mathbf{1}.$$

From (5), we know that Action 2 is preferred to Action 3 if

$$\begin{aligned} u_i(k) &\ge \beta\sum_{j=1}^{n}[q_{1j}(k)V(k, j) - q_{1j}(k + 1)V(k + 1, j)] \\ &= \beta\sum_{j=1}^{n} q_{1j}(k)[V(k, j) - V(k + 1, j)] + \sum_{j=1}^{n}[q_{1j}(k) - q_{1j}(k + 1)]V(k + 1, j). \end{aligned}$$

With the use of the assumption on $q_{ij}(k)$ and the result in Proposition 1 that $V(k + 1, j)$ is an increasing function of j, we obtain that

$$u_i(k) \ge \beta\sum_{j=1}^{n} q_{1j}(k)[V(k, j) - V(k + 1, j)].$$

Therefore, K is an upper bound of the number of the improvement actions if the following inequality holds for $k \ge K$:

$$u_i(k - 1) \ge \frac{\beta}{1 - \beta}(\Delta_c(k) + \Delta_s(k) + \Delta_Q(k)).$$

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