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REVISIT THE CROSS-COUNTRY ASSET ALLOCATION IN PORTFOLIO CHOICE

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ABSTRACT

In this study, we review the investment choice problem in international portfolio management for long-term investors (i.e., institutional investors, asset managers, financial planners, and wealthy individuals) where, in particular, the exchange rate risk and the interest rate risk are incorporated. While the theoretical literature has made significant development, the case with exact solution are still relatively few. Starting with the new perspective in Lioui and Poncet (2003), they show that the optimal portfolio can be divided into three parts: the international speculative portfolio, the domestic interest rate hedging portfolio and the cross-country interest rate differential hedging portfolio.

Since the second hedging component presented in Lioui and Poncet (2003) is an indirect solution, we adopt a specific case that all diffusion coefficients in the dynamics of the state variables is constant to clarify the hedging implications. The results show that the optimal strategy follows a four-fund separation theorem and the number of the funds is irrelevant to the amount of the assets. For non-myopic investors, the currency risk-hedging component will not vanish due to the Purchase Power Parity (PPP) deviation and the hedging demand becomes smaller when the investors shorten his time horizon.

Key words and phrases: currency rate; interest rate; hedging; separation theorem; Pur-

chase Power Parity.

JEL classification: G15, E43, G11, D91, E31.

1. Introduction

The trend of globalization and the rising importance of international financial markets inspire an extension of the portfolio theory in considering foreign investments. Popular foreign investments include stocks, bonds, real estate, mutual funds, and pooled trusts. Foreign investments are not just diversification components to domestic portfolios; they might help to mitigate interest rate risk. Campbell, Viceira, and White (2003) argue that domestic fixed income securities are risky for long-term investors because real interest rates vary over time and the investments need to be rolled over with uncertain future interest rates. They illustrate that the interest rate risk can be hedged by holding foreign currency if the domestic currency tends to depreciate when the domestic real interest rate falls. Hence the major issue in our analysis has been the optimal investment behaviors for the long-term investors (i.e., institutional investors, asset managers, financial planners, and wealthy individuals) regarding the international portfolio selection. International assets bring currency exposure and risk with them, and so the discussion of speculative, hedging issues and strategic asset allocation become crucial.

In spite of the evidence on the gains from diversifying internationally, researches have shown that investor's portfolios have a disproportionately high share invested in domestic assets, see French and Porterba (1991). Solnik (1974), Stulz (1981, 1983) and Adler and Dumas (1983) suggest that the desire to hedge against home inflation may increase the demand for domestic assets relative to foreign assets. For a review on international portfolio choice, see Uppal (1993). Within this international economy, the changes of real exchange rates, real interest rates and stock prices follow the diffusion processes. A country-specific representative individual trades on available assets to maximize the expected utility of his final wealth. The traditional solution to this problem is derived by using the stochastic dynamic programming technique pioneered in finance by Merton (1969, 1971). The investor's optimal portfolio strategy is known to contain a speculative element and as many hedge components as the number of state variables.

Instead of using stochastic control methods, the so-called martingale approach has been alternatively used by Pliska (1986), Karatzas et al. (1987) and Cox and Huang (1989, 1991) to study intertemporal consumption and portfolio policies when markets are complete, which was also the case in the earlier dynamic programming literature. The martingale technology describes the feasible investment strategy set by an intertemporal budget equation and then solves the static investment problem in an infinite dimensional Arrow-Debreu economy. As mentioned in Vila and Zariphopoulou (1997), the martingale approach is appealing for two reasons. First, it can be used to solve for the asset demand under very general investment decisions regarding the stochastic opportunity set. Second, and consequently, it can be applied in a general setting to solve for the equilibrium investment opportunity set (see Duffie and Huang (1985)).

1.1 Hedging Issues

In addition to the speculative component, two hedging components are obtained in Lioui and Poncet (2003). The first hedging component is associated with domestic interest rate risk and the second one with the risk brought about by the co-movements of the interest rates and the market price of risk, which turns out to depend on interest rate differentials across countries and to encompass hedging against purchasing power parity (PPP). (see Lioui and Poncet, 2003).

PPP is a theory which states that exchange rates between currencies are in equilibrium when the purchasing power of the two countries are the same. This means that the exchange rate between two countries should equal the ratio of the two countries' price level of a fixed basket of goods and services. When a country's domestic price level is increasing (i.e., a country experiences inflation), that country's exchange rate must depreciate in order to return to PPP. The basis for PPP is the law of one price. In the absence of transportation and other transaction costs, competitive markets will equalize the price of an identical good in two countries when the prices are expressed in the same currency.

Our model involves estimating the characteristics of the yield curve and the market prices of risk only. We consider the economy which consists of two major currencies: a foreign currency and the domestic one, together with their bond funds and stock portfolios. Then the parameters describing the current financial market, the investment time horizon and the risk aversion parameter of the investor are fully investigated. Finally, we have obtained optimal solution in order to clarify the hedging demands under certain market structure.

1.2 Long-Term Issues

Campbell and Viceira (2002) built rigorous theoretical models to show that the optimal portfolio selections for the long-term investors are not the same as for the short-term investors. If an investors anticipates that he will learn more by observing financial market to update his preference parameters in response to asset returns, this introduces a new type of intertemporal hedging demand into the portfolio selection. In order to fully explore the proposed optimal problem, we work in a continuous-time stochastic framework and use the tools of martingale method. Most financial planning of the investors adopt static portfolio optimization models, such as single-period mean variance allocation in Markowitz (1959), which are short-sighted and when rolled forward lead to myopic portfolio rebalancing unless severely constrained by the portfolio manager's intuition. The Markowitz's models are static (i.e., single period) and these investment strategies are referred to as short-term investors' asset allocation (or tactical asset allocation). The tactical asset allocation is under the assumption that an investor has a mean-variance criterion in making his financial decisions.

Campbell and Viceira (2002) argues that time variation in the opportunity set generate large differences between optimal portfolios for long-term investors, who concern themselves expected returns and risks change over time, and short-term investors. Balduzzi and Lynch (1999) and Barberis (2000) have recently shown that the utility costs of behaving myopically and ignoring predictability can be substantial. Long-term financial planning seems preferable for the fund managers with a liability benchmark. Merton (1971, 1973) explored the optimal solution of the dynamic portfolio in a multi-period framework given that the investment opportunity sets do not vary over time. In our study, the single period short-term theory is extended to the long-term framework that the opportunity set is time-varying.

Starting with the new perspective in Lioui and Poncet (2003), they show that the optimal portfolio can be divided into the international speculative portfolio, the domestic interest rate hedging portfolio and the cross-country interest rate differential hedging portfolio. In this study, we revisit the portfolio allocation problem where currency rate risk and interest rate risk are present. In our model, continuous trading is assumed in the international financial market and the state variables are the currencies traded, a major foreign currency and the domestic currency. The decision variables are the weights of the assets in our opportunities, i.e., the stock indices, the traded currencies and the bonds in each country that are involved. We construct the wealth constraint using the martingale methodology to obtain the optimal international portfolio. The features of this study are summarized in the following

1. We review and investigate the speculative and hedging implication of time-varying risk. Five sources of uncertainty in the model economy are considered: interest rate risks represented by the innovations for the domestic and foreign markets, market risks from the domestic and foreign markets, and the currency rate risk.

2. Lioui and Poncet (2003) obtain an indirect currency risk hedging component to covariances of assets with exogenous variables. The development of our approach adding to their works in obtaining an explicit strategy with certain market structure to clarify the hedge effects in financial decision allowing for global investors.

3. We show that the optimal international portfolio follow a four-fund separation theorem in maximizing the expected utility. Since the asset prices in the financial market change continuously, the international portfolio must be rebalanced to obtain his optimal solution.

The rest of this paper is organized as follows. Section 2 describes the financial market and the proposed model, starting from the basic framework and followed by the dynamics of invested opportunity set and the martingale constraints. Section 3 explores its explicit characteristics regarding the fund wealth on the optimal investment decision incorporating the currency rate and interest rate risks. Section 4 presents the closed-form solution for the model with constant parameters. An example with simplified assumptions is fully explored in Section 5. Conclusions are presented in Section 6.

2. The Market Framework and the Model

2.1 The Market Framework

We consider an economy in which the investor allocate his wealth between a domestic money market account B_d, a foreign money market account B_f, a domestic discount bond P_d maturing at date T_d, a foreign discount bond P_f maturing at T_f, a domestic stock index S_d and a foreign stock index S_f. These assets comprise a complete market from the domestic investor's viewpoint. There are five sources of uncertainty across the two economies in terms of five independent Wiener processes $Z(t)' = \left[\begin{array}{ccccc} Z_1(t) & Z_2(t) & Z_3(t) & Z_4(t) & Z_5(t) \end{array} \right]$. (here $'$ denotes transposition). The independence hypothesis on these Brownian motions implies no loss of generality since we can always shift from uncorrelated to correlated Wiener processes (and vice versa) via the Cholesky decomposition of the correlation matrix.

First we assume that the currency rate e between the domestic and the foreign market satisfies

$$\frac{de(t)}{e(t)} = \mu_e(t)dt + \sum_{i=1}^{5} \sigma_{ei}(t)dZ_i(t),$$

where $\mu_e(t), \sigma_{ei}(t)$, $1 \leq i \leq 5$ are prescribed deterministic functions.

To fully describe the stochastic model for the whole forward-rate curve, the domestic instantaneous forward interest rate f_d is assumed to satisfy

$$df_d(t,T) = \mu_d(t,T)dt + \sum_{i=1}^{5} \sigma_{di}(t,T)dZ_i(t),$$

where $\mu_d(t,T)$ and $\sigma_{di}(t,T)$, $1 \leq i \leq 5$ are prescribed deterministic functions.

According to Heath et al. (1992), we simplify HJM model and get the forward rate $f_d(t,T)$ at time t for the period $(T, T+dt)$ and the short term spot rate process $r_d(t)$ at time t follows the diffusion process. The domestic spot rate $r_d(t)$ is simply given by the forward-rate for maturity equal to the current date, i.e. $r_d(t) = f_d(t,t)$. The domestic money market account $B_d(t)$, starting at $B_d(0) = 1$, is

$$B_d(t) = \exp\left\{\int_0^t r_d(\tau)d\tau\right\}.$$

Upon integration, one finds that

$$r_d(t) = f_d(0,t) + \int_0^t \mu_d(\tau,t)d\tau + \sum_{i=1}^{5}\int_0^t \sigma_{di}(\tau,t)Z_i(\tau).$$

Moreover, for the HJM model it makes the motion of the spot rate non-Markov. The price of the domestic discount bond P_d maturing at date T_d satisfies

$$P_d(t,T_d) = \exp\left\{-\int_t^{T_d} f_d(t,\tau)d\tau\right\},$$

and, with Itô's lemma, the differential of P_d satisfies

$$\frac{dP_d(t,T_d)}{P_d(t,T_d)} = \left(r_d(t) + h_d(t,T_d)\right)dt + \sum_{i=1}^{5} k_{di}(t,T_d)dZ_i(t),$$

where the deterministic function $h_d(t,T_d)$ is

$$h_d(t,T_d) = \frac{1}{2}\sum_{i=1}^{5}\left(\int_t^{T_d}\sigma_{di}(t,\tau)d\tau\right)^2 - \int_t^{T_d}\mu_d(t,\tau)d\tau,$$

and

$$k_{di}(t,T_d) = \int_t^{T_d}\sigma_{di}(t,\tau)d\tau, \quad 1 \le i \le 5.$$

Following Sorensen (1999), we assume further that the price of the domestic stock index S_d satisfies

$$\frac{dS_d(t)}{S_d(t)} = \left(\overline{\mu_d}(t) + r_d(t)\right)dt + \sum_{i=1}^{5}\overline{\sigma_{di}}(t)dZ_i(t),$$

where $\overline{\mu_d}(t), \overline{\sigma_{di}}(t)$, $1 \le i \le 5$ are deterministic functions.

We adopt the convention that when no confusion arises, all the relations satisfied by foreign assets are identical to the corresponding domestic ones, with modified subscript f. Then we have

$$df_f(t,T) = \mu_f(t,T)dt + \sum_{i=1}^{5} \sigma_{fi}(t,T)dZ_i(t),$$

$$r_f(t) = f_f(0,t) + \int_0^t \mu_f(\tau,t)d\tau + \sum_{i=1}^{5} \int_0^t \sigma_{fi}(\tau,t)Z_i(\tau),$$

$$\frac{dP_f(t,T_f)}{P_f(t,T_f)} = (r_f(t) + h_f(t,T_f))\,dt + \sum_{i=1}^{5} k_{fi}(t,T_f)dZ_i(t),$$

where

$$\begin{aligned} h_f(t,T_f) &= \frac{1}{2}\sum_{i=1}^{5}\left(\int_t^{T_f} \sigma_{fi}(t,\tau)d\tau\right)^2 - \int_t^{T_f} \mu_f(t,\tau)d\tau, \\ k_{fi}(t,T_f) &= \int_t^{T_f} \sigma_{fi}(t,\tau)d\tau, \quad 1 \le i \le 5, \end{aligned}$$

and

$$\frac{dS_f(t)}{S_f(t)} = (\overline{\mu_f}(t) + r_f(t))\,dt + \sum_{i=1}^{5} \overline{\sigma_{fi}}(t)dZ_i(t).$$

According to the domestic viewpoint, all prices of foreign assets should be converted by the real currency rate e. All converted prices are denoted by the symbol $\widehat{\ }$. With Itô's lemma, the converted foreign money market $\widehat{B_f} := B_f \cdot e$ satisfies

$$\frac{d\widehat{B_f}(t)}{\widehat{B_f}(t)} = (\mu_e(t) + r_f(t))\,dt + \sum_{i=1}^{5} \sigma_{ei}(t)dZ_i(t).$$

The converted price of foreign instantaneous stock index $\widehat{S_f} := S_f \cdot e$ (see Lioui and Poncet (2003))

$$\frac{d\widehat{S_f}(t)}{\widehat{S_f}(t)} = \{\xi_f(t) + r_f(t)\}\,dt + \sum_{i=1}^{5} \chi_{fi}(t)dZ_i(t),$$

where

$$\xi_f(t) = \mu_e(t) + \overline{\mu_f}(t) + \sum_{i=1}^{5} \sigma_{ei}(t)\overline{\sigma_{fi}}(t),$$

and

$$\chi_{fi}(t) = \sigma_{ei}(t) + \overline{\sigma_{fi}}(t), \quad 1 \le i \le 5.$$

The converted foreign discount bond price $\widehat{P_f} := P_f \cdot e$ satisfies

$$\frac{d\widehat{P_f}(t,T_f)}{\widehat{P_f}(t,T_f)} = \{\zeta_f(t,T_f) + r_f(t)\}\,dt + \sum_{i=1}^{5} \eta_{fi}(t,T_f)dZ_i(t),$$

where

$$\zeta_f(t,T_f) = \mu_e(t) + h_f(t,T_f) + \sum_{i=1}^{5} \sigma_{ei}(t)k_{fi}(t,T_f),$$

and

$$\eta_{fi}(t,T_f) = \sigma_{ei}(t) + k_{fi}(t,T_f), \quad 1 \le i \le 5.$$

2.2 The Martingale Method

The international financial market is assumed to be free of frictions and arbitrage opportunities, so there exists a probability measure which is equivalent to the historical probability measure P with respect to a given numéraire such that the prices expressed in terms of this numéraire are martingales.

We select the numéraire as the riskless asset yielding $r_d(t)$ and the corresponding probability measure Q is the so-called risk neutral probability. The Radon-Nikodym derivative dQ/dP is given by

$$\frac{dQ}{dP} = \delta(t) = \exp\left\{-\int_0^t \Phi(\tau)'dZ(\tau) - \frac{1}{2}\int_0^t \Phi\,(\tau)'\,\Phi(\tau)d\tau\right\},$$

and $\Phi(t)$, the market prices of risk, is defined by means of $\Theta(t)$, which is

$$\Theta(t) = \begin{bmatrix} \sigma_{e1}(t) & \sigma_{e2}(t) & \sigma_{e3}(t) & \sigma_{e4}(t) & \sigma_{e5}(t) \\ k_{d1}(t,T_d) & k_{d2}(t,T_d) & k_{d3}(t,T_d) & k_{d4}(t,T_d) & k_{d5}(t,T_d) \\ \eta_{f1}(t,T_f) & \eta_{f2}(t,T_f) & \eta_{f3}(t,T_f) & \eta_{f4}(t,T_f) & \eta_{f5}(t,T_f) \\ \overline{\sigma_{d1}}(t) & \overline{\sigma_{d2}}(t) & \overline{\sigma_{d3}}(t) & \overline{\sigma_{d4}}(t) & \overline{\sigma_{d5}}(t) \\ \chi_{f1}(t) & \chi_{f2}(t) & \chi_{f3}(t) & \chi_{f4}(t) & \chi_{f5}(t) \end{bmatrix}_{5\times 5},$$

and

$$
\begin{aligned}
\Phi(t) &= \Theta(t)^{-1} \begin{bmatrix} \mu_e(t) + r_f(t) - r_d(t) \\ h_d(t, T_d) \\ \zeta_f(t, T_f) + r_f(t) - r_d(t) \\ \overline{\mu_d}(t) \\ \xi_f(t) + r_f(t) - r_d(t) \end{bmatrix}, \\
&= \Theta(t)^{-1} \begin{bmatrix} \mu_e(t) \\ h_d(t, T_d) \\ \zeta_f(t, T_f) \\ \overline{\mu_d}(t) \\ \xi_f(t) \end{bmatrix} + \Theta(t)^{-1} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \left(r_f(t) - r_d(t)\right), \\
&= \Phi_1(t) + \Phi_2(t) \left(r_f(t) - r_d(t)\right),
\end{aligned}
$$

where $\Phi_1(t), \Phi_2(t)$ are 5×1 deterministic functions.

In a complete market, all the risks brought about by the economic factors must be embedded in the stochastic discount factor (the pricing kernel), so that the market price of risk sums up all the relevant information available on the market.

3. The Optimization Program

Our problem is the selection of an optimal, self-financing portfolio allocation strategy which maximizes the expected utility. We assume further that the insurer's horizon T is shorter than the maturing dates of the domestic and foreign bonds, which ensures that all bonds are long-lived assets from the insurer's viewpoint. Here we choose the $CRRA$ utility function $U(W)$ such as

$$
\begin{aligned}
U(W) &= \frac{1}{\gamma} W^{\gamma}, \quad 0 < \gamma < 1, \\
&= \ln W, \qquad \gamma = 0.
\end{aligned}
$$

The power utility is chosen for two reasons. First, the investors are in general large companies which define their strategies with respect to the amount of money they are managing, more or less in a scaling way. This feature is well captured by the use of the power utility function. Second, pension funds are regulated in such a way that they can not reach negative values. This is true also in the power utility case, thanks to the infinite marginal utility at zero.

The wealth $W(t)$ of the investors at each time t is

$$\begin{aligned} W(t) &= \Gamma_{B_d}(t)B_d(t) + \Gamma_{\widehat{B_f}}(t)\widehat{B_f}(t) \\ &\quad + \Gamma_{p_d}(t)P_d(t) + \Gamma_{\widehat{P_f}}(t)\widehat{P_f}(t) + \Gamma_{S_d}(t)S_d(t) + \Gamma_{\widehat{S_f}}(t)\widehat{S_f}(t), \end{aligned}$$

where $(\Gamma_i(t) : i \in \{B_d, \widehat{B_f}, P_d, \widehat{P_f}, S_d, \widehat{S_f}\})$ stand for the numbers of units of each asset. Applying Itô's lemma under the consideration of self-financing strategy and noting that the domestic money market account is a riskless asset from the insurer's viewpoint, we have (c.f. Merton (1971))

$$\frac{dW(t)}{W(t)} = (\cdot)dt + \pi(t)'\Theta(t)dZ(t), \tag{1}$$

where

$$\pi(t)' = \left[\pi^{\widehat{B_f}}(t) \;\; \pi^{P_d}(t) \;\; \pi^{\widehat{P_f}}(t) \;\; \pi^{S_d}(t) \;\; \pi^{\widehat{S_f}}(t) \right],$$

is the portfolio weight vector of the risky assets and $(\cdot)$ denotes an irrelevant function, a notation which will be frequently used in the sequel.

Define the optimal growth portfolio $\rho(t)$ as (also see Merton (1992) and Long (1990))

$$\rho(t) = B_d(t)\delta(t)^{-1},$$

then

$$\rho(t) = \exp\left\{ \int_0^t \Phi(\tau)'dZ(\tau) + \int_0^t \left(r_d(\tau) + \frac{1}{2}\Phi(\tau)'\Phi(\tau) \right) d\tau \right\}.$$

The investor's international portfolio selection problem is written as

$$\max \;\; E\left[U\left(W(T)\right)\right], \quad 0 < \gamma < 1$$

with the martingale constraint

$$E\left[\frac{W(T)}{\rho(T)}\right] = W(0).$$

Here $E[\cdot]$ is the expectation operator under the historical probability measure P. Following Lioui and Poncet (2003) and according to Cox and Huang (1989, 1991), the first order condition of the optimization problem is

$$W(T) = \lambda^{\frac{1}{\gamma-1}}\rho(T)^{\frac{1}{1-\gamma}},$$

where the Lagrange multiplier λ is characterized by

$$W(0) = \lambda^{\frac{1}{\gamma-1}} E\left[\rho(T)^{\frac{\gamma}{1-\gamma}}\right].$$

The optimal wealth $V(t)$ at time t is equal to

$$\begin{aligned} V(t) &= \lambda^{\frac{1}{\gamma-1}} \rho(t) E_t\left[\rho(T)^{\frac{\gamma}{1-\gamma}}\right] \\ &= \lambda^{\frac{1}{\gamma-1}} \rho(t)^{\frac{1}{1-\gamma}} P_d(t,T)^{\frac{\gamma}{\gamma-1}} E_t\left[\theta(t,T)^{\frac{\gamma}{\gamma-1}}\right], \end{aligned} \tag{2}$$

where

$$\theta(t,T) = \frac{P_d(T,T)\rho(t)}{P_d(t,T)\rho(T)} = \frac{\rho(t)}{P_d(t,T)\rho(T)}, \tag{3}$$

and $E_t[\cdot]$ is the expectation operator under the probability measure P and conditional with respect to $\digamma_t$, the filtration at time t. Defining $E_t\left[\theta(t,T)^{\frac{\gamma}{\gamma-1}}\right]$ as $J(\gamma;t,T)$ and invoking Itô's lemma, we have formally

$$\frac{dJ(\gamma;t,T)}{J(\gamma;t,T)} = (\cdot)dt + \sigma_J(\gamma,t,T)'dZ(t),$$

where $\sigma_J(\gamma;t,T)$ is the 5×1 diffusion vector of the process $dJ(\gamma;t,T)/J(\gamma;t,T)$.

Applying Itô's lemma to (2), we have

$$\frac{dV(t)}{V(t)} = (\cdot)dt + \left[\frac{1}{1-\gamma}\Phi(t)' - \frac{\gamma}{1-\gamma}\sigma_{P_d}(t,T)' + \sigma_J(\gamma,t,T)'\right] dZ(t), \tag{4}$$

where

$$\sigma_{P_d}(t,T)' = \left[\ k_{d1}(t,T_d)\ \ k_{d2}(t,T_d)\ \ k_{d3}(t,T_d)\ \ k_{d4}(t,T_d)\ \ k_{d5}(t,T_d)\ \right]. \tag{5}$$

Identifying the diffusion terms of (1) and (4), we obtain the expression of optimal allocation strategy $\pi(t)$ of risky assets as

$$\pi(t) = \Theta(t)^{-1}\left\{\frac{1}{1-\gamma}\Phi(t) - \frac{\gamma}{1-\gamma}\sigma_{P_d}(t,T) + \sigma_J(\gamma;t,T)\right\}. \tag{6}$$

Lastly, turning to the benchmark case of the logarithmic utility, $\Theta(t)^{-1}(\frac{1}{1-\gamma}\Phi(t))$ in equation (6) readily reveals the investor's myopic behavior, i.e., the speculative component. While $\Theta(t)^{-1}(-\frac{\gamma}{1-\gamma}\sigma_{P_d}(t,T) + \sigma_J(\gamma;t,T))$ are the hedge terms in the optimal solution. Since prices in the financial market change continuously, the optimal portfolio must be rebalanced continuously in order to maintain the proposed weights.

4. Constant Parameter Models

In this section, we adopt the foregoing model and the methodology to a specific case, in which all diffusion coefficients appeared in the dynamics of the state variables are constants instead of deterministic functions. The following proposition is the summary of the optimal asset allocation strategy in this constant case, and note that all coefficients without argument notation are all constants.

Proposition 1 (An International Investment Model - a four-fund theorem) *Given the dynamics of the investment opportunity set follow the diffusion process in equation (13), (14), (17), (18) and (20), the domestic CRRA investor's optimal allocation strategy $\pi(t)$ of risky assets is divided into three parts: the international myopic portfolio π_1, the domestic interest rate hedging portfolio π_2 and the cross-country interest rate differential hedging portfolio π_3. It constitutes a four-fund theorem in optimal investment strategy. In four-fund theorem, the international portfolio invests in the following four funds to maximize the expected utility: the international myopic portfolio $\mathbf{w}_M$ with level $\frac{a}{1-\gamma}$; the domestic interest rate hedge portfolio $\mathbf{w}_Y$ with level $\frac{-b\gamma}{1-\gamma}$; the cross country interest rate differential hedge portfolio $\mathbf{w}_E$ with level $\frac{c\gamma}{1-\gamma}$ and finally, the domestic riskless asset with level $1-\frac{a}{1-\gamma}+\frac{b\gamma}{1-\gamma}-\frac{c\gamma}{1-\gamma}$. The optimal allocation strategy $\pi(t)$ of risky assets is given by*

$$\begin{aligned}
\pi(t) &= \pi_1+\pi_2+\pi_3, \qquad (7)\\
&= \frac{1}{1-\gamma}\Theta(t)^{-1}\Phi(t)\\
&\quad -\frac{\gamma}{1-\gamma}\Theta(t)^{-1}\left[\begin{array}{lllll} k_{d1}(t,T) & k_{d2}(t,T) & k_{d3}(t,T) & k_{d4}(t,T) & k_{d5}(t,T)\end{array}\right]'\\
&\quad +\frac{\gamma}{1-\gamma}\Lambda(t,T)\Theta(t)^{-1}\left[\begin{array}{lllll} l_1 & l_2 & l_3 & l_4 & l_5\end{array}\right]',\\
&= \frac{a}{1-\gamma}\cdot\mathbf{w}_M-\frac{b\gamma}{1-\gamma}\cdot\mathbf{w}_Y+\frac{c\gamma}{1-\gamma}\cdot\mathbf{w}_E.
\end{aligned}$$

where

$$\Phi(t)=\Phi_1(t)+\Phi_2(t)\left(r_f(t)-r_d(t)\right)$$

$$\Lambda(t,T)=\left\{\left(\Psi(T)-\Psi(t)\right)\left(r_f(t)-r_d(t)\right)+\left(\widetilde{\Psi}(T)-\widetilde{\Psi}(t)-\Upsilon(T)+\Upsilon(t)\right)\right\}$$

and

$$\begin{aligned}\Psi(T) &= \int_0^t \Phi_2(\tau)' \Phi_2(\tau) d\tau, \\ \widetilde{\Psi}(t) &= \int_0^t \Phi_1(\tau)' \Phi_2(\tau) d\tau, \\ \Upsilon(t) &= \int_0^t \Psi(\tau) q^*(\tau) d\tau.\end{aligned}$$

$$\begin{aligned}\pi_1 &= \frac{1}{1-\gamma}\Theta(t)^{-1}\Phi(t) = \frac{a}{1-\gamma}\cdot \mathbf{w}_M, \\ \mathbf{w}_M &= \frac{\Theta(t)^{-1}\Phi(t)}{1_5'\Theta(t)^{-1}\Phi(t)}, \\ a &= 1_5'\Theta(t)^{-1}\Phi(t), \\ \pi_2 &= -\frac{\gamma}{1-\gamma}\Theta(t)^{-1}\begin{bmatrix} k_{d1}(t,T) & k_{d2}(t,T) & k_{d3}(t,T) & k_{d4}(t,T) & k_{d5}(t,T)\end{bmatrix}' = \frac{-b\gamma}{1-\gamma}\cdot \mathbf{w}_Y, \\ \mathbf{w}_Y &= \frac{\Theta(t)^{-1}\begin{bmatrix} k_{d1}(t,T) & k_{d2}(t,T) & k_{d3}(t,T) & k_{d4}(t,T) & k_{d5}(t,T)\end{bmatrix}'}{1_5'\Theta(t)^{-1}\begin{bmatrix} k_{d1}(t,T) & k_{d2}(t,T) & k_{d3}(t,T) & k_{d4}(t,T) & k_{d5}(t,T)\end{bmatrix}'}, \\ b &= 1_5'\Theta(t)^{-1}\begin{bmatrix} k_{d1}(t,T) & k_{d2}(t,T) & k_{d3}(t,T) & k_{d4}(t,T) & k_{d5}(t,T)\end{bmatrix}', \\ \pi_3 &= \frac{\gamma}{1-\gamma}\Lambda(t,T)\Theta(t)^{-1}\begin{bmatrix} l_1 & l_2 & l_3 & l_4 & l_5\end{bmatrix}' = \frac{c\gamma}{1-\gamma}\cdot \mathbf{w}_E, \\ \mathbf{w}_E &= \frac{\Lambda(t,T)\Theta(t)^{-1}\begin{bmatrix} l_1 & l_2 & l_3 & l_4 & l_5\end{bmatrix}'}{1_5'\Lambda(t,T)\Theta(t)^{-1}\begin{bmatrix} l_1 & l_2 & l_3 & l_4 & l_5\end{bmatrix}'}, \\ c &= 1_5'\Lambda(t,T)\Theta(t)^{-1}\begin{bmatrix} l_1 & l_2 & l_3 & l_4 & l_5\end{bmatrix}'.\end{aligned}$$

given a, b, and c are real constants. (see the Appendix for the definitions of the notations and the related lemmas).

The explicit expression of the optimal allocation strategy (7) is a revision to the Proposition 2 appeared in Lioui and Poncet (2003). Here we reproduce the last paragraph in page 2227 of their paper: "Sheer inspection of (A.14) shows that $E_t^p\left[\widehat{\theta}(t,\tau)^{\alpha/(\alpha-1)}\right]$ will be random at time t only because $\phi(t)$ is stochastic. As shown in (A.12), the latter is random because of the interest rate differential $(r_f(t) - r_d(t))$. In a Gaussian framework, any conditional expectation of the exponential function of this differential will be the exponential of an *affine* function of the instantaneous differential. It follows that

$$\widehat{J}(\alpha;t,\tau) \equiv E_t^p\left[\widehat{\theta}(t,\tau)^{\frac{\alpha}{\alpha-1}}\right] = e^{A(\alpha;t,\tau)+B(\alpha;t,\tau)(r_f(t)-r_d(t))},$$

where $A(\cdot)$ and $B(\cdot)$ are deterministic functions." But in their previous paper, Lioui and Poncet (2001), also under (nearly) identical assumptions, they claim that the expectation is in the form of $\exp\left(A(\alpha;t,\tau)+B(\alpha;t,\tau)(r_f(t)-r_d(t))^2\right)$, i.e. a *quadratic* function of the instantaneous differential; see (13) of this paper. The latter observation is correct (however, the formula (13) of Lioui and Poncet (2001) requires revisions). In fact, by a simple example and the standard device of stochastic analysis, one can easily refute the argument appeared in Lioui and Poncet (2003) as follows. Take $r(t)=Z(t)$, where $Z(t)$ is the standard one-dimensional Wiener process. According to (10), (11) and (12) in Appendix, we have

$$E_t\left\{\exp\left(-k\int_t^T r(\tau)^2 d\tau\right)\right\}=\exp\left\{-\psi(t)r(t)^2-\varphi(t)\right\},$$

where

$$\begin{aligned}\psi(t) &= \sqrt{\frac{k}{2}}\tanh\sqrt{2k}(T-t),\\ \varphi(t) &= \frac{1}{2}\ln\left[\cosh\sqrt{2k}(T-t)\right].\end{aligned}$$

In the appendix of Lioui and Poncet (2001), the authors put considerable effort on the evaluation of the conditional expectation but, some revisions are required on the last term in (A.5) of their paper by taking the term $\nu_1 b(t,\tau_D)/(\nu_2\sigma_s(\tau_D-t)\nu_2)$ out of the integral.

5. Illustrative Example

With the explicit expression for the hedging demands, we now analyze the analytical results with respect to the key parameters, namely the currency risk and interest rate risk defined by the parameters. We begin with several assumptions and then state formally the result in the proposition. The market assumptions are as follows

$$\Theta(t)=\begin{bmatrix}\sigma_{e1} & 0 & 0 & 0 & 0\\ 0 & \sigma_{d2}(T_d-t) & 0 & 0 & 0\\ 0 & 0 & \sigma_{f3}(T_f-t) & 0 & 0\\ 0 & 0 & 0 & \overline{\sigma_{d4}} & 0\\ 0 & 0 & 0 & 0 & \chi_{f5}\end{bmatrix}_{5\times 5}$$

$$
\begin{aligned}
\Phi(t) &= \Theta(t)^{-1}\begin{bmatrix} \mu_e + r_f(t) - r_d(t) \\ h_d \\ \zeta_f + r_f(t) - r_d(t) \\ \overline{\mu_d} \\ \xi_f + r_f(t) - r_d(t) \end{bmatrix}, \\
&= \begin{bmatrix} \mu_e/\sigma_{e1} \\ h_d/\sigma_{d2}(T_d - t) \\ \zeta_f/\sigma_{f3}(T_f - t) \\ \overline{\mu_d}/\overline{\sigma_{d4}} \\ \xi_f/\chi_{f5} \end{bmatrix} + \begin{bmatrix} 1/\sigma_{e1} \\ 0 \\ 1/\sigma_{f3}(T_f - t) \\ 0 \\ 1/\chi_{f5} \end{bmatrix} (r_f(t) - r_d(t)), \\
&= \Phi_1(t) + \Phi_2(t)\,(r_f(t) - r_d(t)),
\end{aligned}
$$

$$
\begin{aligned}
d\,(r_f(\tau) - r_d(\tau)) &= q^*(\tau)dt + \sum_{i=1}^{5} l_i dZ_i(\tau), \\
q^*(\tau) &= q, \\
(d\,(r_f(\tau) - r_d(\tau)))^2 &= \sum_{i=1}^{5} l_i^2 d\tau,
\end{aligned}
$$

$$\Psi(t) = \int_o^t \Phi_2(\tau)^\top \Phi_2(\tau)d\tau = (\frac{1}{\sigma_{e1}^2} + \frac{1}{\chi_{f5}^2})t + \frac{1}{\sigma_{f3}}(\frac{1}{T_f - t} - \frac{1}{T_f}),$$

$$\Upsilon(t) = \int_0^t \Psi(\tau)q^*(\tau)d\tau = (\frac{1}{\sigma_{e1}^2} + \frac{1}{\chi_{f5}^2})\frac{q\cdot t^2}{2} + \frac{q}{\sigma_{f3}}(\ln(\frac{T_f}{T_f - t}) - \frac{t}{T_f}),$$

$$\widetilde{\Psi}(t) = \int_0^t \Phi_1(\tau)^\top \Phi_2(\tau)d\tau = (\frac{\mu_e}{\sigma_{e1}^2} + \frac{\xi_f}{\chi_{f5}^2})t + \frac{\zeta_f}{\sigma_{f3}}(\frac{1}{T_f - t} - \frac{1}{T_f}),$$

$$
\begin{aligned}
&\sigma_J(\gamma; t, T)^\top \\
=\;& \frac{\gamma}{1-\gamma}\left\{(\Psi(T) - \Psi(t))\,(r_f(t) - r_d(t)) + \left(\widetilde{\Psi}(T) - \widetilde{\Psi}(t) - \Upsilon(T) + \Upsilon(t)\right)\right\} \\
&\times \begin{bmatrix} l_1 & l_2 & l_3 & l_4 & l_5 \end{bmatrix}, \\
=\;& \frac{\gamma}{1-\gamma}\left\{\begin{array}{c} [(\frac{1}{\sigma_{e1}^2} + \frac{1}{\chi_{f5}^2})(T-t) + \frac{1}{\sigma_{f3}}(\frac{1}{T_f - T} - \frac{1}{T_f - t}]\,(r_f(t) - r_d(t)) \\ +(\frac{\mu_e}{\sigma_{e1}^2} + \frac{\xi_f}{\chi_{f5}^2})(T-t) + \frac{\zeta_f}{\sigma_{f3}}(\frac{1}{T_f - T} - \frac{1}{T_f - t}) \\ -(\frac{1}{\sigma_{e1}^2} + \frac{1}{\chi_{f5}^2})\frac{q\cdot(T^2 - t^2)}{2} + \frac{q}{\sigma_{f3}}(\ln(\frac{T_f - T}{T_f - t}) + \frac{T-t}{T_f})) \end{array}\right\} \\
&\times \begin{bmatrix} l_1 & l_2 & l_3 & l_4 & l_5 \end{bmatrix}.
\end{aligned}
$$

$$\sigma_{P_d}(t, T)^\top = \begin{bmatrix} 0 & \sigma_{d2}(T-t) & 0 & 0 & 0 \end{bmatrix}.$$

$$
\begin{aligned}
\pi(t) &= \Theta(t)^{-1}\left\{\frac{1}{1-\gamma}\Phi(t) - \frac{\gamma}{1-\gamma}\sigma_{P_d}(t,T) + \sigma_J(\gamma;t,T)\right\} \qquad (8) \\
&= \frac{1}{1-\gamma}\begin{bmatrix} \frac{\mu_e + r_f(t) - r_d(t)}{\sigma_{e1}^2} \\ \frac{h_d}{\sigma_{d2}(T_d-t)^2} \\ \frac{\zeta_f + r_f(t) - r_d(t)}{\sigma_{f3}(T_f-t)^2} \\ \frac{\mu_d}{\sigma_{d4}^2} \\ \frac{\xi_f + r_f(t) - r_d(t)}{\chi_{f5}^2} \end{bmatrix} - \frac{\gamma}{1-\gamma}\begin{bmatrix} 0 \\ \frac{\sigma_{d2}(T-t)}{\sigma_{d2}(T_d-t)} \\ 0 \\ 0 \\ 0 \end{bmatrix} \\
&\quad + \frac{\gamma}{1-\gamma}\left\{\begin{array}{c} [(\frac{1}{\sigma_{e1}^2} + \frac{1}{\chi_{f5}^2})(T-t) \\ +\frac{1}{\sigma_{f3}}(\frac{1}{T_f-T} - \frac{1}{T_f-t}]\,(r_f(t) - r_d(t)) \\ +(\frac{\mu_e}{\sigma_{e1}^2} + \frac{\xi_f}{\chi_{f5}^2})(T-t) \\ +\frac{\zeta_f}{\sigma_{f3}}(\frac{1}{T_f-T} - \frac{1}{T_f-t}) \\ -(\frac{1}{\sigma_{e1}^2} + \frac{1}{\chi_{f5}^2})\frac{q\cdot(T^2-t^2)}{2} \\ +\frac{q}{\sigma_{f3}}(\ln(\frac{T_f-T}{T_f-t}) + \frac{T-t}{T_f})) \end{array}\right\}\begin{bmatrix} \frac{l_1}{\sigma_{e1}} \\ \frac{l_2}{\sigma_{d2}(T_d-t)} \\ \frac{l_3}{\sigma_{f3}(T_f-t)} \\ \frac{l_4}{\sigma_{d4}} \\ \frac{l_5}{\chi_{f5}} \end{bmatrix}.
\end{aligned}
$$

As there is no reason to assume that any of these hypothetical cases will occur, it is likely that empirical tests using them will in general underestimate the size of the currency risk premia. In the general case where investors are not myopic, however, the market price of currency risk will not vanish. This is because the expected rates of return on all assets embedded in $J(\gamma;t,T)$ will, in particular, be influenced by $\sigma_J(\gamma;t,T)$, i.e. by currency-related risk. The latter, which is tantamount to PPP deviation risk, will be hedged at equilibrium, and hence priced. Since deviations from PPP imply that the national real spot rates will differ, currency risk is related to the risk involved in the random fluctuations of real interest rate spreads across countries which is discussed in Lioui and Poncet (2003).

As evidenced by equation (8), in the special case where investors exhibit logarithmic utility, the hedging demand becomes smaller when the investor shortens his time horizon. Hence, equilibrium rates of return are consistent with the market evidence.

6. Concluding Remarks

The benefits of international diversification have been known for many decades, but it is only recently that investors have started allocating a significant portion of their portfolio holdings in foreign assets. To manage the risk of international portfolios, investors need to know the speculative and hedging demands in the cross-country variation in global return uncertainty.

This study investigates the international asset allocation for global investors, which incorporates the hedge demands in controlling the stochastic variation due to PPP deviation. The development of our approach adding to the previous works of Lioui and Poncet (2003) is that we compare the obtained optimal strategies with certain market structure in order to clarify the hedge effects in financial decision allowing for global investors. Finally, hypothetical mutual funds are constructed in our work to fulfill the proposed demands. The optimal investment strategies are a leveraged growth optimal portfolio, but with contingent leverages as time goes by.

Following the four-fund theorem stated in Rudof and Ziemba (2004), the optimal portfolio consists of into four components: the international myopic portfolio, the domestic interest rate hedge portfolio, the cross country interest rate differential hedge portfolio and the domestic riskless asset. With respect to the most common approach used in the literature, the market structure and the certain utility employed to describe the investor's attitude toward risk allow us to find the general pattern of the optimal strategy for investors through dynamic fund separation methodology.

Appendix

Evaluation of a Certain Conditional Expectation

Theorem 2 (Feynman-Kac Formula, c.f. Lamberton et al (1991), Theorem 5.1.7) *Let u be a well-behaved function defined on $[0,T] \times R^n$. If u satisfies*

$$\frac{\partial u}{\partial t} + A_t u - ru = 0, \forall (t,x) \in [0,T] \times R^n$$

and

$$u(T,x) = f(x),$$

then

$$u(t,x) = E\left\{ f(X_T^{t,x}) \exp\left(-\int_t^T r(\tau, X_\tau^{t,x}) d\tau \right) \right\},$$

where the A_t is the infinitesimal operator of the n dimensional diffusion process $dX_t = b(t, X_t)dt + \sigma(t, X_t)dW_t$. The conditional expectation is taken with respect to t, where $X_t = x$.

We consider a conditional expectation $u(t, x)$ as the following

$$u(t, x)) = E\left\{\exp\left(-k\int_t^T Z(\tau)^2 d\tau\right)\right\} \tag{9}$$

which is conditioned at t and $X_t = Z(t) = x$, where $Z(t)$ is a standard one-dimensional Wiener process and k is a constant. Note that, the conditional expectation (9) is akin to (30) modulo a deterministic factor and the evaluation of the more general (30) may benefit from the following approach. By Feynman-Kac formula, we immediately write down the PDE satisfied by u, which is

$$\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2} - kx^2 u$$

and subject to the boundary condition

$$u(T, x) = 1.$$

Assume that u satisfies the form

$$u(t, x) = \exp\left\{-\psi(t)x^2 - \varphi(t)\right\}, \tag{10}$$

then the boundary condition becomes

$$\psi(T) = 0, \varphi(T) = 0.$$

After the separation of variables, we have

$$x^2\left(\psi'(t) + 2\psi(t)^2 - k\right) = 0$$

and

$$\psi(t) = \varphi'(t).$$

Solution of the above two ODEs yields

$$\psi(t) = \sqrt{\frac{k}{2}}\tanh\sqrt{2k}(T - t) \tag{11}$$

and

$$\varphi(t) = \frac{1}{2} \ln \left[\cosh \sqrt{2k}(T-t) \right]. \tag{12}$$

Evaluation of Constant Parameter Models The following list is the summary of the underlying dynamics in this constant case, and note that all coefficients without argument notation are all constants.

$$\frac{de(t)}{e(t)} = \mu_e dt + \sum_{i=1}^{5} \sigma_{ei} dZ_i(t), \tag{13}$$

$$df_d(t,T) = \mu_d(t,T)dt + \sum_{i=1}^{5} \sigma_{di} dZ_i(t), \tag{14}$$

$$r_d(t) = f_d(0,t) + \int_0^t \mu_d(\tau,t)d\tau + \sum_{i=1}^{5} \sigma_{di} Z_i(t), \tag{15}$$

$$B_d(t) = \exp\left\{ \int_0^t r_d(\tau)d\tau \right\},$$

$$\frac{dP_d(t,T_d)}{P_d(t,T_d)} = \left(h_d(t,T_d) + r_d(t)\right) dt + \sum_{i=1}^{5} k_{di}(t,T_d) dZ_i(t),$$

where

$$\begin{aligned} h_d(t,T_d) &= \frac{1}{2}\left(T_d - t\right)^2 \sum_{i=1}^{5} \sigma_{di}^2 - \int_t^{T_d} \mu_d(t,\tau)d\tau, \\ k_{di}(t,T_d) &= -\sigma_{di}(T_d - t), \quad 1 \le i \le 5, \end{aligned} \tag{16}$$

$$\frac{dS_d(t)}{S_d(t)} = \left(\overline{\mu_d} + r_d(t)\right) dt + \sum_{i=1}^{5} \overline{\sigma_{di}} dZ_i(t), \tag{17}$$

$$df_f(t,T) = \mu_f(t,T)dt + \sum_{i=1}^{5} \sigma_{fi} dZ_i(t), \tag{18}$$

$$r_f(t) = f_f(0,t) + \int_0^t \mu_f(\tau,t)d\tau + \sum_{i=1}^{5} \sigma_{fi} Z_i(t), \tag{19}$$

$$B_d(t) = \exp\left\{ \int_0^t r_d(\tau)d\tau \right\},$$

$$\frac{dP_f(t,T_f)}{P_f(t,T_f)} = \left(r_f(t) + h_f(t,T_f)\right) dt + \sum_{i=1}^{5} k_{fi}(t,T_f) dZ_i(t),$$

where

$$\begin{aligned}
h_f(t,T_f) &= \frac{1}{2}(T_f-t)^2\sum_{i=1}^{5}\sigma_{fi}^2 - \int_t^{T_f}\mu_f(t,\tau)d\tau,\\
k_{fi}(t,T_f) &= -\sigma_{fi}(T_f-t), \quad 1\le i\le 5,
\end{aligned}$$

$$\frac{dS_f(t)}{S_f(t)} = (\overline{\mu_f}+r_f(t))\,dt + \sum_{i=1}^{5}\overline{\sigma_{fi}}dZ_i(t), \tag{20}$$

$$\frac{d\widehat{B_f}(t)}{\widehat{B_f}(t)} = (\mu_e+r_f(t))\,dt + \sum_{i=1}^{5}\sigma_{ei}dZ_i(t),$$

$$\frac{d\widehat{S_f}(t)}{\widehat{S_f}(t)} = \{\xi_f+r_f(t)\}\,dt + \sum_{i=1}^{5}\chi_{fi}dZ_i(t),$$

where

$$\begin{aligned}
\xi_f &= \mu_e+\overline{\mu_f}+\sum_{i=1}^{5}\sigma_{ei}\overline{\sigma_{fi}},\\
\chi_{fi} &= \sigma_{ei}+\overline{\sigma_{fi}}, \quad 1\le i\le 5,
\end{aligned}$$

$$\frac{d\widehat{P_f}(t,T_f)}{\widehat{P_f}(t,T_f)} = \{\zeta_f(t,T_f)+r_f(t)\}\,dt + \sum_{i=1}^{5}\eta_{fi}(t,T_f)dZ_i(t),$$

where

$$\begin{aligned}
\zeta_f(t,T_f) &= \mu_e+h_f(t,T_f)+\sum_{i=1}^{5}\sigma_{ei}k_{fi}(t,T_f),\\
\eta_{fi}(t,T_f) &= \sigma_{ei}+k_{fi}(t,T_f), \quad 1\le i\le 5,
\end{aligned}$$

and

$$\Theta(t) = \begin{bmatrix}
\sigma_{e1} & \sigma_{e2} & \sigma_{e3} & \sigma_{e4} & \sigma_{e5}\\
k_{d1}(t,T_d) & k_{d2}(t,T_d) & k_{d3}(t,T_d) & k_{d4}(t,T_d) & k_{d5}(t,T_d)\\
\eta_{f1}(t,T_f) & \eta_{f2}(t,T_f) & \eta_{f3}(t,T_f) & \eta_{f4}(t,T_f) & \eta_{f5}(t,T_f)\\
\overline{\sigma_{d1}} & \overline{\sigma_{d2}} & \overline{\sigma_{d3}} & \overline{\sigma_{d4}} & \overline{\sigma_{d5}}\\
\chi_{f1} & \chi_{f2} & \chi_{f3} & \chi_{f4} & \chi_{f5}
\end{bmatrix},$$

and

$$\begin{aligned}
\Phi(t) &= \Theta(t)^{-1}\begin{bmatrix}\mu_e\\ h_d(t,T_d)\\ \zeta_f(t,T_f)\\ \overline{\mu_d}\\ \xi_f\end{bmatrix} + \Theta(t)^{-1}\begin{bmatrix}1\\0\\1\\0\\1\end{bmatrix}(r_f(t)-r_d(t))\\
&= \Phi_1(t)+\Phi_2(t)\,(r_f(t)-r_d(t))\,.
\end{aligned}\tag{21}$$

From the definition of $P_d(t,T)$, we have

$$\begin{aligned} P_d(t,T) &= \exp\left\{-\int_t^T f_d(t,\tau)d\tau\right\} \\ &= \exp\left\{-\int_t^T f_d(0,\tau)d\tau - \int_t^T\int_0^t \mu_d(u,\tau)dud\tau - \sum_{i=1}^5 \sigma_{di}\,(T-t)\,Z_i(t)\right\} \end{aligned}$$

and since

$$f_d(t,T) = f_d(0,T) + \int_0^t \mu_d(u,T)du + r_d(t) - f_d(0,t) - \int_0^t \mu_d(u,t)du$$

and (15), the expression of $r_d(t)$

$$r_d(t) = f_d(0,t) + \int_0^t \mu_d(\tau,t)d\tau + \sum_{i=1}^5 \sigma_{di}Z_i(t),$$

we thus obtain

$$\begin{aligned} P_d(t,T) &= \exp\left\{-\int_t^T \left(f_d(0,\tau) - f_d(0,t)\right)d\tau - \int_t^T\int_0^t \mu_d(u,\tau)dud\tau\right\} \\ &\quad\times\exp\left\{-(T-t)r_d(t) + (T-t)\int_0^t \mu_d(u,t)du\right\}. \end{aligned} \tag{22}$$

We have also

$$\int_t^T r_d(\tau)d\tau = \int_t^T \left(f_d(0,\tau) + \int_0^\tau \mu_d(u,\tau)du + \sum_{i=1}^5 \sigma_{di}Z_i(\tau)\right)d\tau.$$

From

$$\int_t^T Z_i(\tau)d\tau = \int_t^T (T-\tau)dZ_i(\tau) + (T-t)Z_i(t),$$

it follows that

$$\begin{aligned} \int_t^T r_d(\tau)d\tau &= \int_t^T \left(f_d(0,\tau) + \int_0^\tau \mu_d(u,\tau)du\right)d\tau \\ &\quad+\sum_{i=1}^5 \sigma_{di}\left(\int_t^T (T-\tau)dZ_i(\tau) + (T-t)Z_i(t)\right) \\ &= \int_t^T \left(f_d(0,\tau) + \int_0^\tau \mu_d(u,\tau)du\right)d\tau + \sum_{i=1}^5 \sigma_{di}\int_t^T (T-\tau)dZ_i(\tau) \\ &\quad+(T-t)r_d(t) - (T-t)\,f_d(0,t) - (T-t)\int_0^t \mu_d(u,t)du. \end{aligned} \tag{23}$$

Thus, by substituting (22) and (23) into (3), the definition of θ, we have

$$\begin{aligned}
\theta(t,T) &= \exp\left\{-\int_t^T \Phi(\tau)'dZ(\tau) - \int_t^T \left(r_d(\tau) + \frac{1}{2}\Phi\left(\tau\right)'\Phi(\tau)\right)d\tau\right\} P_d(t,T)^{-1} \\
&= \exp\left\{-\int_t^T \Phi(\tau)'dZ(\tau) - \int_t^T \frac{1}{2}\Phi\left(\tau\right)'\Phi(\tau)d\tau\right\} \\
&\quad \times \exp\left\{-\sum_{i=1}^{5}\int_t^T \sigma_{di}(T-\tau)dZ_i(\tau) + 2(T-t)f_d(0,t)\right\}.
\end{aligned}$$

Upon inspection, only the first term in the last equality would generate stochastic components after taking conditional expectations. We proceed to carry out the calculation.

Applying the decomposition of $\Phi(t)$ in (21) to the integral

$$\exp\left\{\int_t^T \Phi(\tau)'dZ(\tau) + \int_t^T \frac{1}{2}\Phi\left(\tau\right)'\Phi(\tau)d\tau\right\}$$

we have

$$\begin{aligned}
&\exp\left\{\int_t^T \Phi(\tau)'dZ(\tau) + \int_t^T \frac{1}{2}\Phi\left(\tau\right)'\Phi(\tau)d\tau\right\} \qquad (24)\\
= \ &\exp\left\{\int_t^T \Phi_1(\tau)'dZ(\tau) + \int_t^T \left(r_f(\tau) - r_d(\tau)\right)\Phi_2(\tau)'dZ(\tau)\right\} \\
&\times \exp\left\{\frac{1}{2}\int_t^T \Phi_1\left(\tau\right)'\Phi_1(\tau)d\tau + \int_t^T \Phi_1\left(\tau\right)'\Phi_2(\tau)\left(r_f(\tau) - r_d(\tau)\right)d\tau\right\} \\
&\times \exp\left\{\frac{1}{2}\int_t^T \Phi_2\left(\tau\right)'\Phi_2(\tau)\left(r_f(\tau) - r_d(\tau)\right)^2 d\tau\right\}.
\end{aligned}$$

We neglect the integrals $\frac{1}{2}\int_t^T \Phi_1\left(\tau\right)'\Phi_1(\tau)d\tau$, $\int_t^T \Phi_1(\tau)'dZ(\tau)$on the right-hand side of (24) because of the deterministic contributions after taking the conditional expectations. There are three stochastic integrals left, namely

$$\begin{aligned}
&\frac{1}{2}\int_t^T \Phi_2\left(\tau\right)'\Phi_2(\tau)\left(r_f(\tau) - r_d(\tau)\right)^2 d\tau, \\
&\int_t^T \Phi_1\left(\tau\right)'\Phi_2(\tau)\left(r_f(\tau) - r_d(\tau)\right)d\tau \quad \text{and} \\
&\int_t^T \left(r_f(\tau) - r_d(\tau)\right)\Phi_2(\tau)'dZ(\tau).
\end{aligned}$$

Note that, from (15) and (19) the dynamics of $r_f(t) - r_d(t)$ is

$$r_f\left(t\right) - r_d(t) = q(t) + \sum_{i=1}^{5} l_i Z_i(t), \qquad (25)$$

where

$$\begin{aligned} q(t) &= f_f(0,t) + \int_0^t \mu_f(\tau,t)d\tau - f_d(0,t) - \int_0^t \mu_d(\tau,t)d\tau, \\ l_i &= \sigma_{fi} - \sigma_{di}, \quad i = 1 \text{ to } 5. \end{aligned}$$

Applying (25), we have

$$\begin{aligned} d\left(r_f(\tau) - r_d(\tau)\right) &= q^*(\tau)dt + \sum_{i=1}^{5} l_i dZ_i(\tau), \qquad (26) \\ \left(d\left(r_f(\tau) - r_d(\tau)\right)\right)^2 &= \sum_{i=1}^{5} l_i^2 d\tau, \end{aligned}$$

where $q^*(\tau) = dq(\tau)/d\tau$.

Define $\Psi(t)$ such that

$$\Psi(t) = \int_o^t \Phi_2(\tau)' \Phi_2(\tau) d\tau. \qquad (27)$$

Integration by parts and the application of Itô's lemma with $\left(r_f(\tau) - r_d(\tau)\right)^2$render the integral

$$\frac{1}{2}\int_t^T \Phi_2(\tau)' \Phi_2(\tau) \left(r_f(\tau) - r_d(\tau)\right)^2 d\tau,$$

into

$$\begin{aligned} & \frac{1}{2}\int_t^T \Phi_2(\tau)' \Phi_2(\tau) \left(r_f(\tau) - r_d(\tau)\right)^2 d\tau, \qquad (28) \\ = \; & \frac{1}{2}\left(r_f(T) - r_d(T)\right)^2 \Psi(T) - \frac{1}{2}\left(r_f(t) - r_d(t)\right)^2 \Psi(t) \\ & - \int_t^T \Psi(\tau)\left(r_f(\tau) - r_d(\tau)\right) d\left(r_f(\tau) - r_d(\tau)\right) \\ & - \frac{1}{2}\int_t^T \Psi(\tau)\left(d\left(r_f(\tau) - r_d(\tau)\right)\right)^2. \end{aligned}$$

After substituting (26) into (28), it is clear that the only term we need to specify is

$$\int_t^T \Psi(\tau)\left(r_f(\tau) - r_d(\tau)\right) d\left(r_f(\tau) - r_d(\tau)\right),$$

and

$$\begin{aligned} & \int_t^T \Psi(\tau)\left(r_f(\tau) - r_d(\tau)\right) d\left(r_f(\tau) - r_d(\tau)\right) \\ = \; & \int_t^T \Psi(\tau)\left(r_f(\tau) - r_d(\tau)\right) q^*(\tau)d\tau + \sum_i \int_t^T q_i \Psi(\tau) dZ_i(\tau) \\ = \; & \left(r_f(T) - r_d(T)\right)\Upsilon(T) - \left(r_f(t) - r_d(t)\right)\Upsilon(t) \\ & - \int_t^T \Upsilon(\tau) q^*(\tau)d\tau - \sum_i \int_t^T l_i \Upsilon(\tau) dZ_i(\tau) + \sum_i \int_t^T l_i \Psi(\tau) dZ_i(\tau), \end{aligned}$$

through repeated integration by parts, where

$$\Upsilon(t) = \int_0^t \Psi(\tau) q^*(\tau) d\tau. \tag{29}$$

Thus, we may summarize our results in the following lemmas

Lemma 3 *With the assumptions of our financial model, there exist two deterministic functions $\Psi(T)$ in equation (27) and $\Upsilon(T)$ in equation (29) such that*

$$\begin{aligned} & \frac{1}{2}\int_t^T \Phi_2(\tau)' \Phi_2(\tau) \left(r_f(\tau) - r_d(\tau)\right)^2 d\tau \qquad (30) \\ = \; & \frac{1}{2}\left(r_f(T) - r_d(T)\right)^2 \Psi(T) - \frac{1}{2}\left(r_f(t) - r_d(t)\right)^2 \Psi(t) \\ & - \left(r_f(T) - r_d(T)\right) \Upsilon(T) + \left(r_f(t) - r_d(t)\right) \Upsilon(t) + \sum_i \int_t^T (\cdot) dZ_i(\tau) + (\cdot). \end{aligned}$$

Lemma 4 *With the assumptions of our financial model, there exist two deterministic functions $\widetilde{\Psi}(T)$ such that the integral*

$$\int_t^T \Phi_1(\tau)' \Phi_2(\tau) \left(r_f(\tau) - r_d(\tau)\right) d\tau$$

may be treated in a similar fashion. The final result is

$$\begin{aligned} & \int_t^T \Phi_1(\tau)' \Phi_2(\tau) \left(r_f(\tau) - r_d(\tau)\right) d\tau \qquad (31) \\ = \; & \left(r_f(T) - r_d(T)\right) \widetilde{\Psi}(T) - \left(r_f(t) - r_d(t)\right) \widetilde{\Psi}(t) + \sum_i \int_t^T (\cdot) dZ_i(\tau) + (\cdot), \end{aligned}$$

where

$$\widetilde{\Psi}(t) = \int_0^t \Phi_1(\tau)' \Phi_2(\tau) d\tau. \tag{32}$$

Lemma 5 *With the assumptions of our financial model, substituting the expression (25) into the stochastic integral*

$$\int_t^T \left(r_f(\tau) - r_d(\tau)\right) \Phi_2(\tau)' dZ(\tau),$$

we have

$$\begin{aligned} & \int_t^T \left(r_f(\tau) - r_d(\tau)\right) \Phi_2(\tau)' dZ(\tau) \\ = \; & \int_t^T \left(q(\tau) + \sum_{i=1}^5 l_i Z_i(\tau)\right) \Phi_2(\tau)' dZ(\tau) \\ = \; & \sum_i \int_t^T (\cdot) dZ_i(\tau) + \sum_{i,j} \int_t^T l_i \Phi_{2j}(\tau) Z_i(\tau) dZ_j(\tau), \qquad (33) \end{aligned}$$

where $\Phi_{2j}(\tau)$, $1 \le j \le 5$ *denotes the* j *th component of the* 5×1 *function* $\Phi_2(\tau)$.

Collecting all the results of (30),(31) and (33) obtained above, we compute $J(\gamma; t, T)$ as

$$\begin{aligned} J(\gamma; t, T) &= E_t\left[\theta(t,T)^{\frac{\gamma}{\gamma-1}}\right] \\ &= A(\gamma; t, T) \exp\left\{\frac{\gamma}{2(1-\gamma)}\left(r_f(t) - r_d(t)\right)^2 \left(\Psi(T) - \Psi(t)\right)\right\} \\ &\quad \times \exp\left\{\frac{\gamma}{1-\gamma}\left(r_f(t) - r_d(t)\right)\left(\widetilde{\Psi}(T) - \widetilde{\Psi}(t) - \Upsilon(T) + \Upsilon(t)\right)\right\}, \end{aligned}$$

where $A(\gamma; t, T)$ is a deterministic function. Here we utilize the independence property of $(r_f(T) - r_d(T))-$ $(r_f(t) - r_d(t))$ with respect to the conditional expectation operator $E_t[\cdot]$ because of the expression (25), and the fact such that the expression $\int_t^T l_i \Phi_{2j}(\tau) Z_i(\tau) dZ_j(\tau)$ is independent with respect to $E_t[\cdot]$ is also used. Applying Itô's lemma, we have

$$\begin{aligned} &\frac{dJ(\gamma; t, T)}{J(\gamma; t, T)} \\ &= \frac{\gamma}{1-\gamma}\left\{\left(\Psi(T) - \Psi(t)\right)\left(r_f(t) - r_d(t)\right) + \left(\widetilde{\Psi}(T) - \widetilde{\Psi}(t) - \Upsilon(T) + \Upsilon(t)\right)\right\} \\ &\quad \times d\left(r_f(t) - r_d(t)\right) + (\cdot)\, dt \\ &= \frac{\gamma}{1-\gamma}\left\{\left(\Psi(T) - \Psi(t)\right)\left(r_f(t) - r_d(t)\right) + \left(\widetilde{\Psi}(T) - \widetilde{\Psi}(t) - \Upsilon(T) + \Upsilon(t)\right)\right\} \\ &\quad \times \sum_{i=1}^{5} l_i dZ_i(t) + (\cdot)\, dt. \end{aligned}$$

We immediately obtain the following proposition:

Proposition 6 *The instantaneous conditional* $(\frac{\gamma}{\gamma-1})$ *moment of the Arrow-Debreu prices of the reference country bond of maturity* T *is given by*

$$\begin{aligned} J(\gamma; t, T) &= E_t\left[\theta(t,T)^{\frac{\gamma}{\gamma-1}}\right] \\ &= A(\gamma; t, T) \exp\left\{\frac{\gamma}{2(1-\gamma)}\left(r_f(t) - r_d(t)\right)^2 \left(\Psi(T) - \Psi(t)\right)\right\} \\ &\quad \times \exp\left\{\frac{\gamma}{1-\gamma}\left(r_f(t) - r_d(t)\right)\left(\widetilde{\Psi}(T) - \widetilde{\Psi}(t) - \Upsilon(T) + \Upsilon(t)\right)\right\}, \end{aligned}$$

where $A(\gamma;t,T)$ is a deterministic function. The diffusion vector $\sigma_J(\gamma;t,T)^\top$ of the process of $\frac{dJ(\gamma;t,T)}{J(\gamma;t,T)}$ is given by

$$\begin{aligned}&\sigma_J(\gamma;t,T)' \\ = \ &\frac{\gamma}{1-\gamma}\left\{(\Psi(T)-\Psi(t))\,(r_f(t)-r_d(t))+\left(\widetilde{\Psi}(T)-\widetilde{\Psi}(t)-\Upsilon(T)+\Upsilon(t)\right)\right\} \\ &\times\left[\begin{array}{ccccc} l_1 & l_2 & l_3 & l_4 & l_5\end{array}\right].\end{aligned} \tag{34}$$

Substituting the expressions of $\Psi(t), \Upsilon(t)$ and $\widetilde{\Psi}(T)$ in (27), (29) and (32), respectively and (5), (16), (21) and (34) into (6), we obtain the expression of optimal allocation strategy $\pi(t)$ of risky assets.

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