Pension Fund Management Using the Markov Chain Approximation

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Abstract

Funding policy and portfolio selection are two crucial issues in pension fund management. Merton (1969, 1971) initially explores these problems in a continuous time framework by constructing the Hamilton-Jacobi-Bellman (HJB) equations. This type of approach becomes complicated when control constraints are incorporated under an incomplete market. In this paper, we suggest using the Markov chain approximation methods proposed by Kushner and Dupuis (1992) to obtain the optimal solutions numerically. Monitoring mechanism linking plausible scenarios and numerical solutions are employed to scrutinize the contributions and asset allocations for defined benefit pension schemes. In the numerical illustration, we estimate the optimal strategies within a simplified two-asset opportunity set. The results show that the plan turnovers, the initial fund levels, and the time horizon heavily influence the optimal strategies.

Keywords: Stochastic control; Markov chain approximation; Pension fund management

1. Introduction

There are two types of pension schemes: defined benefit (hereafter DB) and defined contribution (hereafter DC) plans. The benefits in a DB plan are fixed in advance and the plan sponsor adjusts the contributions annually, while the benefits in a DC plan are determined by the performance of the invested portfolio and the contributions are fixed. Although DB plans have a longer history than DC plans and plan members prefer DB plans, the global trend caused by the retirement related regulations is moving from DB to DC. Such a trend lets plan sponsors transfer the financial risks to plan members through members' individual retirement accounts. DC plans can be extremely risky relative to a DB benchmark (see Blake, Cairns and Dowd, 2001). In this study, we propose a computational method to assist the decision making process for pension plans and this method can be applied to DB plans as well as DC plans.

Most conventional pension models are one-period models that employ the mean-variance approach. The pension plan manager searches for an optimal investment decision for the next period, considering the plan's current positions and expectations about future funding, investment returns, and risks. Such a mechanism has two drawbacks. First, the aggregation of single-period optimal decisions across periods might not be optimal for multiple periods as a whole. Second, single-period decisions cannot simultaneously deal with the investment and funding sides of the pension plan because the linkage between investment and

funding appears only in the multi-period setting. Sharpe (1991) describes the mean-variance approach as a way that characterizes investors' goals parsimoniously, employs a myopic view (i.e., one period at a time), and focuses on only two aspects of the probability distribution of returns over that period. Since most pension fund holders are long-term investors and the financial strength of a pension plan depends on both sides of the balance sheet, pension plan management should be considered within a multi-period framework and take both funding and investment into account.

An important tool that can be used to assist pension fund managers in developing optimal investment and funding policies over time is the stochastic optimal control theory. This theory can be used to solve long-term financial planning problems through global optimization across periods. It can also deal with the liability risk of a pension fund resulted from demographic uncertainties that are outside the financial markets and are often referred to as background risks in the finance literature.[1] The availability of inexpensive but high-speed computers has aided the popularity of the stochastic control method. Many papers emerge to tackle the pension fund management problem using this method, e.g., O'Brien (1986, 1987), Haberman and Sung (1994), Haberman (1997), Runggaldier (1998), Schäl (1998), Chang (1999), Cairns (2000), Chang (2000, 2000), Chang and Chen (2002), Chang et al. (2002), Menoncin (2002), and Chang, Tzeng and Miao (2003).

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1. Examples of background variables are the investor's wage process, the contributions to and withdrawals from a pension fund, and the indemnities paid by the insurer to the insured.

Although scholars can formulate the optimal strategy of a pension plan as a stochastic control problem in a continuous-time framework and obtain the HJB equation, solving the HJB equation analytically is rather difficult. The HJB equation has been solved in few particular cases only and the non-negative constraints make the task more difficult (see Merton (1969, 1971) and Josa-Fombellidal and Rinc-Zapatero (2001)). The stochastic duality theory of Bismut (1973) might ease the problem a little bit, but it is still awkward for practical use because of the complicated and long-winded measure-theoretical technicalities (see Pliska (1997); Cvitanić and Karatzas (1992), He and Pagé (1993), Cuoco (1997), and the references therein).

Therefore, we employ the Markov chain approximation method proposed by Kushner and Dupuis (1992) to solve the control problems numerically. As stated in the papers and monographs of Kushner (Kushner and Dupois (1986); Kushner (1990, 1997, 1998)), the Markov chain approximation method is a far-reaching approach in solving continuous-time stochastic control problems numerically. Hindy, Huang and Zhu (1997) describe a numerical analysis technique based on this methodology. Munk (2000) uses Markov chain approximation method in optimal consumption/investment problem with unidentifiable income risk and liquidity constraints. Monoyious (2004) uses the same method in pricing options with transaction costs. The main merit of the Markov chain approximation methods lies in, as the preface of the book "Numerical Methods for Stochastic Optimal Control Problems in Continuous Time" authored by Kushner and Dupois (1992) goes, the less demanding convergence proof, the general applicability, and the relatively practical implementations. Within the framework of the Markov chain approximating methods, we are able to reformulate the constrained problem of a pension fund with the consideration for demographic uncertainties.

Our proposed framework involves constructing an approximating control process to minimize the risk measurements according to a probabilistic experience set of actuarial assumptions. A brief summary of the advantages of this approach is listed below.

1. The non-myopic optimal solution given short-sale constraints in the incomplete market can be obtained through numerical approximation. Although the feedback controls of the constrained problem can be constructed from the solution of the dual problem, it is awkward for practical uses due to the recondite measure-theoretical technicalities.

2. The numerical approach can generate optimal solutions and is capable of evaluating certain sort of managerial interventions such as the investment time frame, the background risks, and the associated factors in measuring the contribution rate risk and the solvency risk. The approach employed in this study can be implemented onto DC plans in which contributions are fixed and the benefits depend on the return on the fund's portfolio.

The rest of the paper is structured as follows. Section 2 establishes the model as a controlled diffusion problem and outlines the dynamic programming approach. The optimal controls in the feedback form are obtained as well. Section 3 outlines the approach of Markov chain approximation method. The collection of more detailed formulae is in Appendix. Section 4 computes the optimal portfolio composition and contributions for a DB pension plan and presents the approximated solution results of the control problem. Section 5 consists of conclusions.

2. The Model

2.1 The Controlled Diffusion Problem

Following the earlier works done by O'Brien (1986, 1987) and Cairns (2000), we formulate the funding and investment decisions of pension funds as a stochastic control problem. These decisions are modeled using continuous-time stochastic processes over a specific time horizon. The financial market variables are modeled in a filtered probability space (Ω_1, F^1, P_1) with filtrations F_t^1, $t \in [0,T]$, and the pension plan's turnover and demographic variables are modeled in a filtered probability space (Ω_2, F^2, P_2) with filtrations F_t^2, $t \in [0,T]$. The sample space for the dynamics of the pension plan with regard to both assets and liabilities are then the product space:

$$(\Omega, F, P^*) = (\Omega_1 \times \Omega_2, F^1 \times F^2, P_1 \times P_2), \quad F_t = F_t^1 \times F_t^2.$$

For simplicity, we assume that the probability measures on the financial market and the plan's demographics are independent. Since the uncertainties resulted from the plan turnovers are not traded in the financial market, our model is an incomplete one.

Funding policies and investment strategies for the pension plan are defined by a stochastic process u(t)=((C(t), p(t)), t∈ [0,T]), i.e., a feedback control, with values in R² adapted to the natural filtration (F_t) of the Brownian motion, where C(t) denotes the contribution rates (contributions to the pension plan per unit of time) at time t and p(t) is the proportion vector of assets held in the fund portfolio at time t. Note that u is an adapted process, i.e. C(t) is F_t^1 measurable and p(t) is F_t^2 measurable. [2]

A continuous-time framework of the controlled stochastic process for funding levels F incorporating the demographic feature of the pension plan is described by the following stochastic differential equation:

$$dF_t = F_t d\delta(t, F_t) + C_t dt - B_t dt + \sigma_b dZ_b(t),$$
$$F_{t_0} = F_0, \ t \geq t_0, \tag{1}$$

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[2] In the following, the function notation may be abbreviated by dropping (t) or subscripts t when no confusion arises.

where F_t = the fund size at time t,

$d\delta(t,F_t)$ = the return rate on the assets between time t and t+dt,

C_t = the contribution rate at time t,

B_t = the projected benefit outgoing rate at time t,

σ_b = the volatility of B_t, and

$Z_b(t)$ = the Brownian motion process at time t.

The controlled process F_t follows a time homogeneous Markov process for a fixed time horizon T. $F_t d\delta(t,F_t) + C_t dt - B_t dt$ is the controlled drift function and $\sigma_b dZ_b(t)$ is the controlled diffusion function.

We assume that the pension board decides to invest in N mutual funds with the prices-per-unit $S_1(\cdot),\cdots,S_N(\cdot)$ being continuous, strictly positive, and the returns following:

$$dS_i = S_i[\mu_i dt + \sum_{j=1}^{d}\sigma_{ij}dZ_j],$$

where $i = 1,...,N$ and $Z_1,\cdots,Z_d$ are d independent Wiener processes. The independence hypothesis poses no loss of generality since we can transform the uncorrelated Wiener processes to correlated ones (and vice versa) via the Cholesky decomposition of the correlation matrix.

In the following, we focus on a two-asset setting that has been seen in the literature (e.g., Cairns (2000)) and can be justified by the so-called mutual fund theorem (see example in Merton (1973) and Magill (1976)). More specifically, we simplify equation (2) as:

$$\begin{cases} \dfrac{dS_1}{S_1} = \mu_1 dt + \sigma_{11}dZ_1 + \sigma_{12}dZ_2, S_1(0)=1, \\ \dfrac{dS_2}{S_2} = \mu_2 dt + \sigma_{21}dZ_1 + \sigma_{22}dZ_2, S_2(0)=1, \end{cases} \quad (2)$$

where the constants μ_1 and μ_2 measure the expected growth in assets, and σ_{11}, σ_{12}, σ_{21}, and σ_{22} together describe the instantaneous volatilities. If we set $\sigma_{22} = \sigma_{21} = \sigma_{12} = 0$, then asset S_2 can be regarded as the risk-free asset in the financial market.

The proportions of the funds invested in asset S_1 and S_2 are denoted by p and $1-p$, respectively. $d\delta$, the return on assets between time t and $t+dt$, is formulated as:

$$d\delta = [p\mu_1 + (1-p)\mu_2]dt + [p \quad 1-p]\begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix}[dZ_1 \quad dZ_2]^T.$$

Since pension fund managers are not allowed to take short positions, we impose the short-sale constraint on p and the non-negativity of F_t and C_t:

$$a \le p \le b, [a,b] \subset [0,1], F_t \ge 0, C_t \ge 0. \quad (3)$$

2.2 Finite Time Control Problem

Under the above setting, the problem of choosing the optimal asset allocation and funding strategy over the admissible control space Ω can be formulated as the following minimization problem:

$$\inf_{u\in\Omega} E_t^u \left\{ \int_t^T \exp(-\beta s)L(F_s, u(F_s,s))ds + \exp(-\beta T)G(F_T) | F_t \right\}, \quad (4)$$

where $E_t^u\{\}$ denotes the expectation operator at time t, β is the discount rate applied to the risk performance, and $G(\cdot)$ represents the boundary condition indicating the terminal management requirement. $L(\cdot)$ is the risk measurement of the pension plan given the admissible policy $\{u(F_t,t),t \ge 0\}$. We assume that $L(\cdot)$ and $G(\cdot)$ are strictly positive, continuously differentiable, and satisfying certain regular conditions (see Karatzas et al., (1997)).

Two types of risks concerning the stability and security of funding are employed to monitor the stability and security of funding: the contribution rate risk and the solvency risk. These two types of risks characterize the trade-offs in the decision making process. We follow Chang (1999, 2000) to construct a ratio-induced measure associated with these two risks to derive the optimal contributions and investments subject to specific constraints through dynamic optimization. The ratio-induced measure $L(\cdot)$ is defined as:

$$L(F,C(s,F_s),F_s) = (1-\frac{C}{NC})^2 + k(1-\frac{F}{\eta AL})^2, \quad (5)$$

where NC is the projected normal cost rate, AL is the projected accrued liability, k and η are constants in measuring the trade-offs. $(1-\frac{C}{NC})^2$ measures the deviations of the contributions from the normal costs relative to the size and is related to the stability of the plan; $(1-\frac{F}{\eta AL})^2$ measures the relative deviation of the fund level and the actuarial liability and serves as an indicator of the security of the plan.

Following Merton (1969, 1971), we define

$$V(t,F) = \inf_{C,p} E_t \left\{ \int_t^T \exp(-\beta s)L(s,C(s,F_s),F_s)ds + \exp(-\beta T)G(F_T,T) \right\},$$

and obtain the formal Bellman equation of optimality as (see Fleming and Rishel (1975)):

$$0 = \inf_{C,p} \left\{ \exp(-\beta t)L(t,C,F) + V_t + (\mu_2 F + \lambda pF + C - B)V_F \right. \quad (6)$$

$$\left. + \frac{1}{2}V_{FF}[F^2(\varepsilon_2 p^2 + \varepsilon_1 p + \varepsilon_0) + \sigma_b^2] \right\},$$

where $\lambda = \mu_1 - \mu_2$,

$\varepsilon_2 = (\sigma_{11} - \sigma_{21})^2 + (\sigma_{12} - \sigma_{22})^2$,

$\varepsilon_1 = 2[(\sigma_{11} - \sigma_{21})\sigma_{21} + (\sigma_{12} - \sigma_{22})\sigma_{22}]$,

$\varepsilon_0 = \sigma_{21}^2 + \sigma_{22}^2$,

$V_t = \dfrac{\partial V}{\partial t}$, $V_F = \dfrac{\partial V}{\partial F}$, and $V_{FF} = \dfrac{\partial^2 V}{\partial F^2}$.

The first order condition in equation (6) without short-selling constraints implies that the optimal contribution and the risky asset proportion is (also see Chang, 2000):

$$C^* = NC - \frac{NC}{2}e^{\beta t}V_F, \tag{7}$$

$$p^* = \frac{-V_F}{FV_{FF}}\frac{\lambda}{\varepsilon_2} - \frac{1}{2}\frac{\varepsilon_1}{\varepsilon_2} \tag{8}$$

We thus have a proposition characterizing the plan manager's optimal strategy under the two-asset model.

Proposition 1: The unconstrained solutions to the optimal control problem formed in equations (1), (3), and (4) under the two-asset assumption with the ratio-induced measure defined in equation (5) are equations (7) and (8).

The optimal plan contributions are expressed in equation (7) as the combination of the projected NC and an adjustment factor reflecting the marginal effect on the value function from the fund level, the discount factor, and the projected normal cost. Given that $V_F > 0$, i.e., the marginal changes of the indirect utility function due to the fund level is positive, the optimal contribution decreases with the discount factor β. In the special case that the marginal change of the indirect utility function due to the fund level is zero, the optimal contribution is the projected normal cost.

Equation (8) expresses the optimal portfolio as the usual mean-variance optimal portfolio. The optimal proportion of the first risky asset is inversely related to the Arrow-Pratt risk aversion index and is proportional to the excess premium of the first risky asset over the second one. Furthermore, the optimal proportion of the funds allocated to the first risky asset increases with the negative correlation between assets, i.e., $\varepsilon_1 < 0$.

Since the risky assets are correlated, $-\frac{1}{2}\frac{\varepsilon_1}{\varepsilon_2}$ is required to adjust the co-variation effect. When the risky assets are independent of each other, our solution is the same as the one recognized in the classic paper of Merton (1973). Similarly, if one of the two assets is risk free, the optimal proportion p^* for the risky asset reduces to the Merton-type ratio. The optimal allocation to the first risky asset may also be explained by a single asset with the hypothetical mean λ and variance ε_2, after adjusted by $-\frac{1}{2}\frac{\varepsilon_1}{\varepsilon_2}$.

Since the value of the risky asset varies with time, the pension fund manager has to rebalance the portfolio continuously to achieve the optimal allocation. The optimal strategy looks like that the fund manager set normal cost estimates first and then makes adjustments based on the marginal changes in value function. Increases in the marginal changes of the value function due to the fund levels will result in decreases of the contribution.

3. Markov Chain Approximation Method

An efficient approach to obtain the solutions of the optimal cost function $V(F,t)$ that satisfy the formal dynamic programming equation is the Markov chain approximation method. The basic idea of the Markov chain approximation method is to approximate the original controlled process by an appropriate controlled Markov chain on a finite state space and discrete time grids. The convergence proofs for the Markov chain approximation method are purely probabilistic and have been discussed in details by Kushner and Dupois (1992). The following passages are some sketches of this method in the one-dimensional control process setting.

Suppose that a controlled diffusion process F obeys the stochastic differential equations:

$$dF = b(F, u)dt + \sigma(F, u)dW,$$

where μ denotes the associated process that serves as the controller and dW denotes the increment of the underlying Wiener process. Let u_1 indicate the proportion of the fund invested in asset S_1 and u_2 indicate the contribution rate at time t under the admissible controller set, we have the admissible control space as

$$\Omega = \{u \mid b \ge u_1 \ge a, u_2 \ge 0\},$$
$$b(F,u) = \mu_2 F + \lambda p F + u_2 - B, \text{ and}$$
$$\sigma^2(F,u) = F^2(\varepsilon_2 u_1^2 + \varepsilon_1 u_1 + \varepsilon_0) + \sigma_b^2.$$

Unless explicitly stated, we tacitly assume all controllers appeared in the subsequent discussion to reside in Ω.

Let A^u denote the **Dynkin** operator of the controlled process F with respect to the controller u and consider the formal expression:

$$A^u V(F,t) + L(F,t) = \frac{1}{2}\sigma^2(F,u)V_{FF}(F) + b(F,u)V_F(F) + L(F,u) = 0 \tag{9}$$

Applying the standard finite difference approximations leads us to obtain:

$$V_{FF}(F) := \frac{V(F+h) + V(F-h) - 2V(F)}{h^2}, \text{ and} \tag{10}$$

$$V_F(F) := \frac{V(F+h) - V(F-h)}{2h} \tag{11}$$

Substituting equations (10) and (11) into equation (9), we obtain the approximation scheme after some simplification:

$$V(F) = \frac{\sigma^2(F,u) + b(F,u)h}{2\sigma^2(F,u)}V(F+h) + \frac{\sigma^2(F,u) - b(F,u)h}{2\sigma^2(F,u)}V(F-h)$$
$$+ L(F,u)\frac{h^2}{\sigma^2(F,u)} \tag{12}$$

We interpret the above in the context of Markov chain by introducing the following notations:

$$\frac{\sigma^2(F,u) + b(F,u)h}{2\sigma^2(F,u)} \equiv p(F, F+h \mid u),$$

$$\frac{\sigma^2(F,u) - b(F,u)h}{2\sigma^2(F,u)} \equiv p(F, F-h \mid u), \text{ and}$$

$$\frac{h^2}{\sigma^2(F,u)} \equiv \Delta t(F,u),$$

and rewrite equation (12) in the suggestive form:

$$V(F)=p(F,F+h|u)V(F+h)+p(F,F-h|u)V(F-h)+L(F,u)\Delta t(F,u) \quad (13)$$

If $\inf_u \{\sigma^2(F,u)+|b(F,u)|h\} > 0$ and define $p(F,y|u) \equiv 0$ for $y \neq F+h$, then $\sum_y p(F,y|u)=1$. Thus the totalities $p(F,y|u)$ are the transition probabilities for a Markov chain and $\Delta t(F,u)$ is the interpolation interval.

The above illustration on the Markov chain approximation method is tangential and covers merely the explicit method. The explicit method indicates transitions in state space x only with the interval parameter Δt while the implicit method incorporates both F and t state space transitions. There are also two classical iterating approaches in deriving the final optimal cost function V and the optimal controller u, namely, approximation in the policy space and approximation in the value space.

Once we have the transition probability expressions and the approximation scheme, we select an arbitrary admissible control, say u_0, and insert it into the scheme equation (13). Computing all transition probabilities and solving the corresponding V, we obtain an improved control, say u_1, by

$$u_1(x) = \arg\min_u \left\{ \sum_y p(F,y|u_0)V(y) + L(F,u_0)\Delta t(F,u_0) \right\}$$

Keep iterating and we can obtain the limiting value of u_n and the corresponding V that represent the optimal controller and optimal cost function respectively. This is the approximation in the policy space. The approximation in the value space is to reverse the iteration order of u and V. We will adopt the implicit method with approximations in the policy space in the following sections because of the superior converging speed and numerical accuracy. More detailed formulas are in Appendix.

4. An Illustrative Example

We apply the above proposed dynamic control model to a DB pension scheme of a semi-conductor and electronic company in Taiwan. According to the Labor Standard Law enacted by the Taiwan government in 1984, the employer is required to contribute 2% to 15% of the employees' payroll to a government-managed trust fund. The mandatory pension plan is a DB one since the participant's retirement benefits are based on the length of employed time and the final salary upon retirement. Although the trust fund enjoys minimum returns guaranteed by the government, it is subject to insolvency risk because the contributions coupled with the investment returns may not meet the benefit payments.

This company's pension plan covers 2,768 members with the accrued liability of 380,688,220 NTD. In projecting the dynamics of the plan's workforce, we assume that the group is open but the group size remains intact, i.e., whenever the employee leave, the same number of new employ-

ees comes in. The probability of an employee leaving the workforce at different ages follows an assumed mortality table, while the leaving probability within an age interval is assumed to have a uniform distribution. More detailed descriptions on the dynamics of the plan's workforce can be seen in Chang and Cheng (2002).

The accrued liabilities, normal costs, and benefit outgoes are calculated suing the entry age normal (EAN) cost method (Anderson, 1992). Because we do not have the data to estimate the volatilities of normal costs NC, benefit outgoes B, and withdrawals W, we first simulate 100 sets of NC, B, and W under the assumptions that salaries increase 3% annually and the valuation interest rate is ln(1.03)/52 in the weekly interval. Then we take the simulated standard deviations as the volatilities.

We select subjectively the model parameters as shown in Table 1. The time horizon T is set as 3 years and is further divided into 100 grids/terms. The tried initial fund levels are in the range from 0 to 5×10^8 NTD. Due to computational constraints, only the process of the fund level F is modeled as a stochastic process and the values of other state variables such as NC, B, and AL are projected through simulations. Details of the implementation can be found in Chang et al. (2002). We use a two-dimensional grid with 200 × 100 = 20,000 points to discretize the state space. The algorithm requires about 4 hours solving the problem on a Pentium 4 desktop PC with 990 MB of memory.

To facilitate comparisons, we tried the case without short-sale constraint and the cases with the maximum proportion of the fund invested in asset S_1 being 1, 0.8, 0.6, and 0.4. The contribution rates over the future three-year time period without short-sale constraints are plotted in Figure 1. Figure 1 shows that the contribution rates are around 24% and vary over the years. We further find that imposing short-sale constraints makes insignificant differences in the

Table 1. Parameter Values Used in the Numerical Analysis

Parameter Descriptions	Notation	Parameter Values
The value for the short-term interest rate	r	0.02
The market price of risk	λ_1	0.24
The market price of risk	λ_2	0.25
The expected growth of asset 1	μ_1	0.05
The expected growth of asset 2	μ_2	0.0787
Volatility of the stock index	σ_{11}	0.25
Volatility of the stock index	σ_{12}	-0.12
Volatility of the stock index	σ_{21}	-0.12
Volatility of the stock index	σ_{22}	0.35
Discount factor	β	0.03
Trade-off factor	k	1
Trade-off factor	η	1

optimal contribution rates, which might be due to the plan's specific demographics.

Figures 2 - 6 display the optimal weights in asset S_1 over the future three years and various fund levels under different range constraints on p. We find that the proportion of the pension fund invested in the first asset changes with the range of the fund level. The proportion of the first asset increases with the fund level within the fund range of 2×10^8 NTD and 4×10^8 NTD, but it approaches zero when the fund level is outside of this range. Our speculations about the results are as follows. When the pension fund is seriously under funded (i.e., when the initial level is below 2×10^8 NTD), the manager has to invest all money into the second asset that can generate higher long-term returns (7.87% vs. 5%) at the expense of higher volatility (37% vs. 28%). The manager can hold a "normal" portfolio when the pension is initially adequately funded. More specifically, the optimal investing strategy for the pension plan is to put significant amount of the money on the first asset within the range between 2×10^8 NTD and 4×10^8 NTD. When the fund is "excessively" funded (i.e., the initial fund level is above 4×10^8 NTD), the fund has minor insolvency risk and thus can afford to pursue aggressive investment strategy.
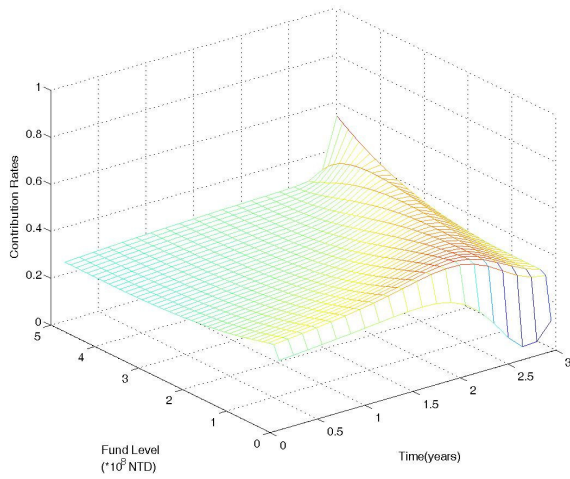

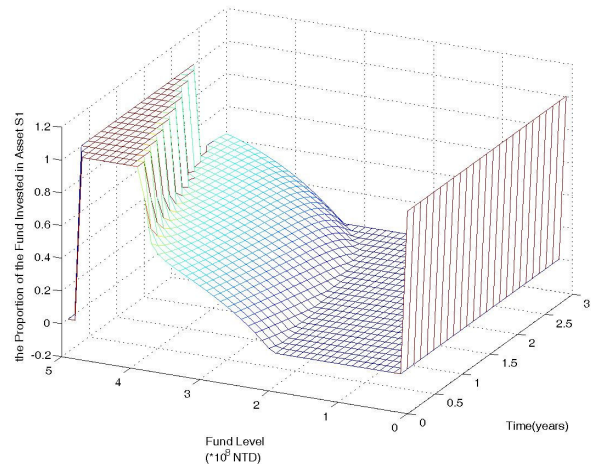**Figure 1.** The Contribution Rates over Future Three Years without Short-sale Constraint

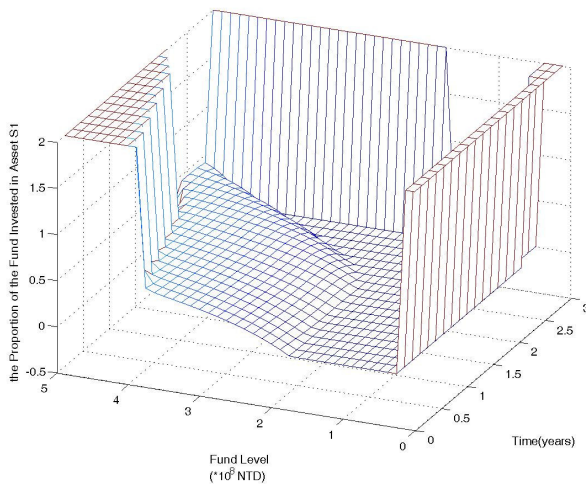**Figure 3.** The Optimal Weights of the First Asset Given p being in [0, 1]

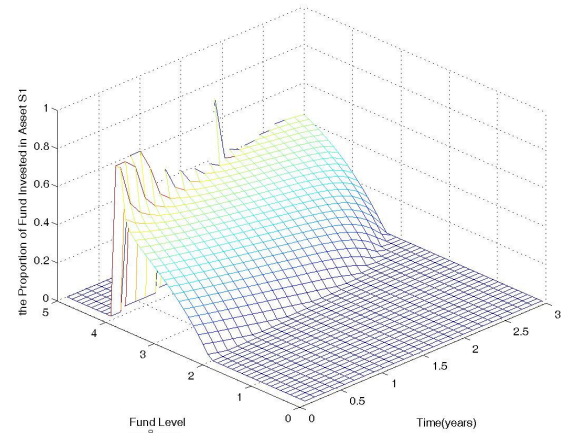**Figure 2.** The Optimal Weights of the First Asset without Short-selling Constraint

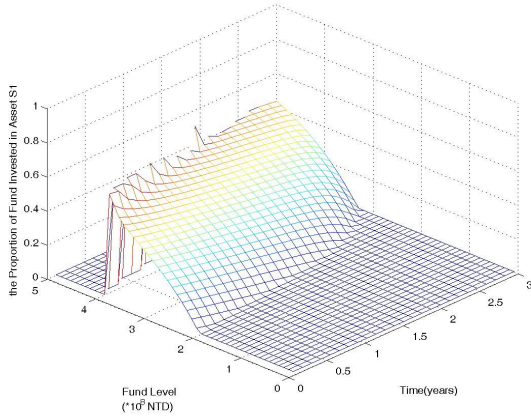**Figure 4.** The Optimal Weights of the First Asset Given *p* being in [0, 0.8]

Figure 5. The Optimal Weights of the First Asset Given p being in [0, 0.6]
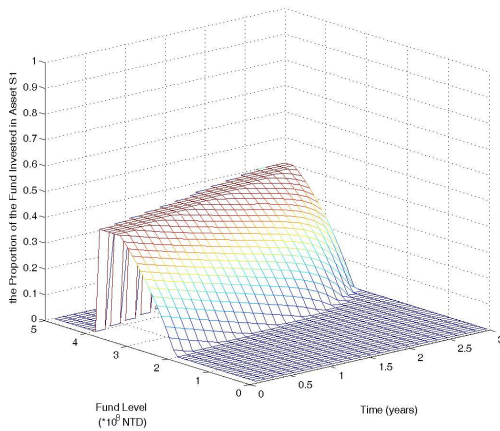

Figure 6. The Optimal Weights of the First Asset Given *p* being in [0, 0.4]

5. Conclusions

In this paper, we introduce a numerical method that relies on the Markov chain approximation to compute the optimal strategies. This method is quite general and market completeness is not required. It links together stochastic simulations and approximating optimal solutions and thus provides an efficient way to construct an optimal control function among a set of admissible solutions that minimize the risk measurement according to a probabilistic experience set of actuarial and economic assumptions. Pension plan managers can use this flexible method to approximate the optimal funding and investment strategies under the considerations for short-sale constraints, actuarial status of the plan, and performance measurements.

Under a simple two-asset model, we illustrate how the Markov chain approximation method is implemented in calculating the optimal contribution and asset weights over the planning horizon. Our results show that the uncertainty due to the plan turnovers, the level of the fund size, and the

time horizon heavily influence the optimal strategies. We also find that short-selling constraints play an important role in deciding the optimal weights.

Appendix

In this section, we list the transitional probability expressions needed in implementing the Markov chain approximation. Define $Q(F, u)$ as:

$$Q(F,u) \equiv \sigma^2(F,u) + |b(F,u)|h$$

If we denote the maximum of $Q(F, u)$ in all admissible states by $\overline{Q}$, we have the following expressions:

$$\Delta t = \frac{\delta h^2}{h^2 + \overline{Q}\delta},$$

$$p(F,n\delta;F,n\delta+\delta \,|\, u) = \frac{h^2}{h^2 + \overline{Q}\delta},$$

$$p(F,n\delta;y,n\delta \,|\, u) = \frac{N(F,y,u)\delta}{h^2 + \overline{Q}\delta}, y \neq F \text{, and}$$

$$p(F,n\delta;F,n\delta \,|\, u) = \frac{\delta(\overline{Q} - Q(F,u))}{h^2 + \overline{Q}\delta},$$

where *h* and δ are the space and time increments respectively, and *N(F, y, u)* is defined by

$$N(F,F \pm h,u) = \frac{\sigma^2(F,u)}{2} + \max(\pm b(F,u),0)h,$$

$$N(F,y,u) = 0 \text{ otherwise.}$$

Broadly speaking, the implicit approximation of policy space consists of three steps: selecting the controller, solving for the cost functions at each *F* and *T*, and working backwards in each time stage. The general iterating schemes are

$$f(F,n\delta)\big|_{u_i} = \sum_y p(F,n\delta;y,n\delta \,|\, u_i) f(y,n\delta) \,|\, u_i +$$

$$p(F,n\delta;F,n\delta+\delta \,|\, u_i) f(F,n\delta+\delta) \,|\, u_i + L(F,u_i)\Delta t \text{, and}$$

$$u_{i+1} = \underset{u}{\arg\min} \left\{ \sum_y p(F,n\delta;y,n\delta \,|\, u) f(y,n\delta) \,|\, u_i + \right.$$

$$p(F,n\delta;F,n\delta+\delta \,|\, u\,) f(F,n\delta+\delta) \,|\, u_i + L(F,u)\Delta t.$$

One must bear in mind that each control u_i is a function of *F* and *t*. The first iterating scheme is in fact a system of equations. In the second scheme, we have to do traditional minimizations with respect to all possible controllers.

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