Insurance: Mathematics and Economics 61 (2015) 110–124

Contents lists available at ScienceDirect

Insurance: Mathematics and Economics

journal homepage: www.elsevier.com/locate/ime

Age-specific copula-AR-GARCH mortality models

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ARTICLE INFO

Article history:
Received March 2014
Received in revised form
October 2014
Accepted 18 December 2014
Available online 31 December 2014

Keywords:
Stochastic mortality model
AR-GARCH
Copula
Mortality dependence
Lee–Carter model

ABSTRACT

In this paper, we propose AR-GARCH (autoregression-generalized autoregressive conditional heteroskedasticity) models to fit and forecast mortality rates for a given age by two alternative approaches. Specifically, one approach is to fit a time series of mortality rates for some age to an AR(n)-GARCH(1, 1) model, and project the mortality rate for that age in the next nth year; the other is to fit an AR(1)-GARCH(1, 1) model, and project the mortality rates recursively for the age in the next consecutive years. Further, we employ the copula method to capture the inter-age mortality dependence. Adopting mortality data of Japan, the UK, and the USA, we demonstrate that it is indispensable to consider the conditional heteroskedasticity in our mortality models which provide better performances in out-of-sample projection and prediction intervals with a higher degree of coverage than the Lee–Carter model. Finally, we numerically illustrate with mortality data of Japan that VaR (Value at Risk) values for longevity risk, regarded as additional reserves for annuity or pension providers, will be overestimated if the conditional heteroskedasticity or/and the inter-age mortality dependence structure are ignored.

1. Introduction

Mortality improvement, a downward time trend in mortality rates, has been a substantial issue for annuity and pension products in recent decades. An inappropriate measure in mortality improvement would underestimate premiums and reserves of annuity and pension products and then expose annuity and pension providers to risks of financial distress.

Some methods of managing longevity risk have been proposed in the literature. Li and Luo (2012), Cairns et al. (2014), and Cairns (2013) apply mortality-linked securities to hedging longevity and mortality risks, and measure the hedge effectiveness. Wang et al. (2010), Tsai and Chung (2013), Lin and Tsai (2013, 2014), Wang et al. (2013a), and Cox et al. (2013) hedge longevity/mortality risks with strategies of natural hedges or mortality immunization. Hári et al. (2008), Olivieri and Pitacco (2009), Lin and Tzeng (2010), and Plat (2011) present risk-based capitals for longevity risk.

Projecting mortality rates and modeling its randomness also have aroused much attention. The Lee and Carter (1992) model is the most widely cited and used method in mortality fitting and forecasting. It assumes that the dynamics of the logarithm of central death rates are driven by an age-specific constant plus the speed of change at each age multiplied by an overall time trend of mortality rates. The CBD model proposed by Cairns et al. (2006) is also widely used where the logit function of mortality rates is captured by an overall time trend plus a time trend related to age. There are many extensions of the two models. Renshaw and Haberman (2006) and Haberman and Renshaw (2009) provide modifications of the Lee–Carter age-period model and add extra bilinear terms (non-age specific factors) to the structure of the traditional Lee–Carter model. Li et al. (2009) consider individual heterogeneity in each age-period cell in the Lee–Carter model, and provide better goodness of fit. Plat (2009) proposes a model combining the good factors in the Lee–Carter and CBD models. Each of these models is useful for mortality prediction. However, all of them model the mean level of mortality rates, and ignore the variance level of mortality rates and the temporal dependence structure between inter-age mortality rates.

The assumption of mortality independence traditionally adopted by stochastic mortality models has been challenged. In reality, we can observe that mortality deterioration, such as deadly infectious diseases (e.g., the influenza pandemic in 1918) and natural

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http://dx.doi.org/10.1016/j.insmatheco.2014.12.007
0167-6687/

disasters (e.g., the tsunami in December 2004), and gradual mortality improvement, such as medical breakthroughs, and nutrition and public health enhancements, extensively influence social groups across periods, ages, genders, and populations. Loisel and Serant (2007) are the first to take inter-age and inter-period correlations into account and propose a multi-dimensional extension of the Lee–Carter model; they find that a positive correlation exists in practice and cannot be neglected. Yang et al. (2008) demonstrate that based on the mortality data of France and Switzerland, the residuals of the Lee–Carter model are not independent across ages and periods. Wills and Sherris (2010) demonstrate the effect of age dependence on pricing longevity bonds. Li and Hardy (2011) show that using female populations of the United States and Canada simultaneously, an overall time trend plus a population specific time trend can reduce population basis risks produced by two independent time trends under the Lee–Carter model. As a result, it is indispensable to incorporate mortality dependence structures into stochastic mortality models.

Mitchell et al. (2013) propose a model which is analogous to the Lee–Carter model and expresses the change in the logarithm of mortality rates for each age group, rather than the level of the logarithm of mortality rates, as an age-dependent linear transformation of mortality index. The mortality rates for future consecutive years are projected conditioning on the mortality rates for the current year. However, they employ SVD (singular value decomposition) to calibrate the parameters and to capture the dependence structure between mortality rates for ages, which in turn leads to a limited dependence structure. Lin et al. (2013) propose a stochastic model which incorporates normalized multivariate correlations between multi-country mortality indexes. Wang and Yang (2013) incorporate the mortality dependence into the Lee–Carter model by considering age-specific mortality correlations under the multivariate Gaussian distribution. However, it is well known that the multivariate normality assumption is usually not supported empirically by the multivariate data. The copula method proposed by Sklar (1959) provides a fabulous alternative to the multivariate normal distribution and constructs a highly flexible and non-standard multivariate dependence structure. Consequently, the first goal of this paper is to employ the copula method to capture the inter-age mortality dependence structure, which is still absent in the literature to the best of our knowledge.

Observing that mortality data display a linear relationship and volatility clustering phenomenon, the second goal of this paper is to employ AR-GARCH (autoregression-generalized autoregressive conditional heteroscedasticity) models to capture the marginal dynamics of mortality rates. We propose two alternative approaches to fitting and forecasting in our AR-GARCH models. Specifically, for the first approach we fit the mortality data for an age to an AR(n)-GARCH(1,1) model and then employ it to project the mortality rate for that age for the next nth year; for the second one, we fit the AR(1)-GARCH(1,1) to mortality rates for an age and predict recursively the mortality rates for that age for the next consecutive years. Our AR structure is analogous to the Lee–Carter model under which the logarithm of mortality level is a linear function. However, under our models the trend in the mortality data for an age comes from the previous mortality rate for the same age rather than the overall hidden mortality index. Consequently, conditioning on the mortality rate for a specific age in the current year, we can predict the future mortality rates for that age. Even though we fit and forecast the mortality rates age by age, we can still consider the dependence structure of the residuals across ages in our models by adopting the copula method.

In empirical testing, adopting mortality data for both genders of Japan, the UK, and the USA, we demonstrate that (G)ARCH error structures produce better in-sample goodness of fit than Gaussian one based on the criteria of Akaike Information Criterion (AIC) and Bayes Information Criterion (BIC). Compared to the Lee–Carter model, our models also provide better performances in out-of-sample projection according to the MAPE (mean absolute percentage error) criterion and prediction intervals with a higher degree of confidence. In addition, our models of the two approaches produce similar mean forecasts but variance ones; the AR(n) can produce reliable coverage of forecasted mortality rates with narrower confidence intervals. Finally, using the mortality data for both genders of Japan, we empirically test the dependence structure between mortality rates for different ages. For different mortality projection periods, we demonstrate that the Gaussian copula, the simplest one, usually provides better goodness of fit than the time-varying Gaussian copula, and the static and time-varying Student's t copulas. In addition, using the copula and AR models with the best goodness-of-fit error structures, we show that VaR (Value at Risk) and CVaR (Conditional Value at Risk) values which can be regarded as additional reserves of life annuities and pension annuities against longevity risks may be overestimated if the inter-age mortality dependence and (G)ARCH error structures are neglected.

The remainder of this paper is organized as follows. Section 2 describes the models. Sections 3 and 4 compare the goodness of fit and the forecasting performances between our models and the Lee–Carter model, respectively. In Section 5, we further consider a copula structure between mortality rates for different ages and apply it to an estimation of VaR values for longevity in life annuities and pension annuities, respectively. The conclusions are presented in Section 6.

2. The models

2.1. AR-GARCH mortality models

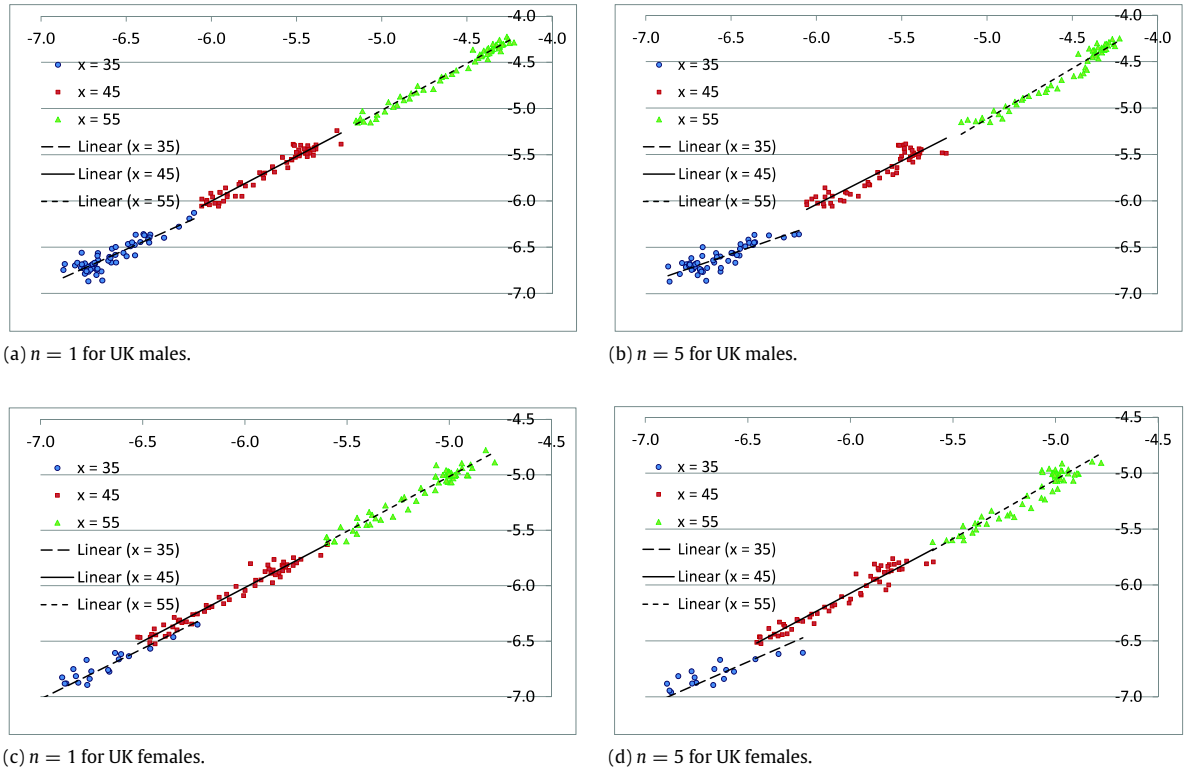
Denote $p_{x,t}$ and $q_{x,t} = 1 - p_{x,t}$ the probabilities that a person aged x in year t will survive one year and die within one year, respectively, and $\mu_{x,t}$ the associated force of mortality. The assumption that the force of mortality $\mu_{x,t}$ is constant within each integer age x and year t (that is, $\mu_{x+r,t+s} = \mu_{x,t}$ for $r, s \in [0, 1)$) implies that $\mu_{x,t} = -\ln(p_{x,t})$; moreover, the central death rate for age x in year t under the assumption is equal to the force of mortality, that is, $m_{x,t} = \mu_{x,t}$. The assumption of a constant force of mortality has been made by many mortality models for mortality data transformation between $q_{x,t}$ and $m_{x,t}$ or $\mu_{x,t}$.

The classical Lee–Carter model expresses the logarithm of central death rates as follows:

$$\ln(m_{x,t}) = a_x + b_x \times k_t + e_{x,t}, \quad x = x_L, \ldots, x_U, \tag{2.1}$$

where a_x represents the average age-specific mortality, k_t is the general mortality level, b_x is the age-specific reaction to the time-varying factor, and $e_{x,t}$ is the error term capturing the age-specific effects not reflected in the model. As suggested in Lee and Carter (1992), the parameters in (2.1) can be estimated by the singular value decomposition (SVD) or a close approximation to SVD, and the time trend k_t follows a random walk with drift as $k_t = \theta + k_{t-1} + \varepsilon_t$, where $\varepsilon_t | I_{t-1} \sim N(0, \sigma_\varepsilon^2)$ and I_{t-1} represents the information up to time $t-1$. The estimated variance $(\hat{\sigma}_\varepsilon^2)$ of ε_t is used to calculate the uncertainty of the forecasted k_t over any given horizon, and then the confidence intervals on the forecasted k_t can be used in the same way to calculate the confidence intervals on the logarithm of the forecasted central death rates (see Lee and Miller (2001)).

Lee–Carter model is vulnerable in that it models the level of the logarithm of central death rates as a linear function of mortality index, and hence misrepresents the age-specific dependence structure or variance of mortality rates. In order to reform that drawback, Mitchell et al. (2013) express the logarithm of central death rates for the next year as that for the current year, rather



(a) $n = 1$ for UK males.

(b) $n = 5$ for UK males.

(c) $n = 1$ for UK females.

(d) $n = 5$ for UK females.

Fig. 1. $\ln(m_{x,\,1950+\tau+n})$ against $\ln(m_{x,\,1950+\tau})$, $\tau = 1, \ldots, 60 - n$, for $x = 35$, 45 and 55.

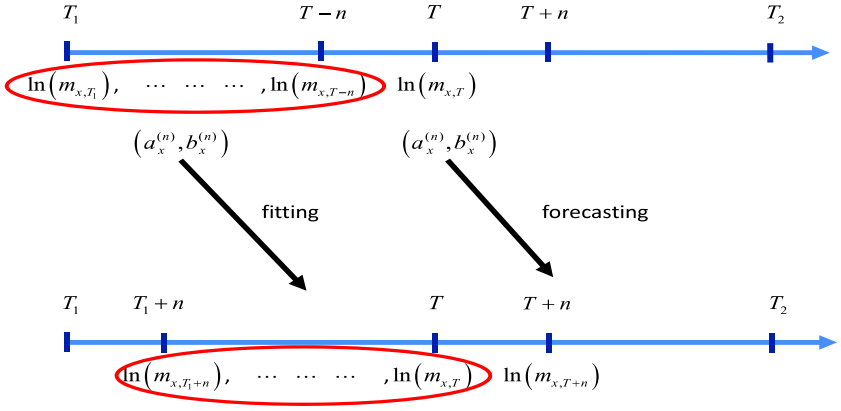


Fig. 2. Diagram for mortality fitting and forecasting.

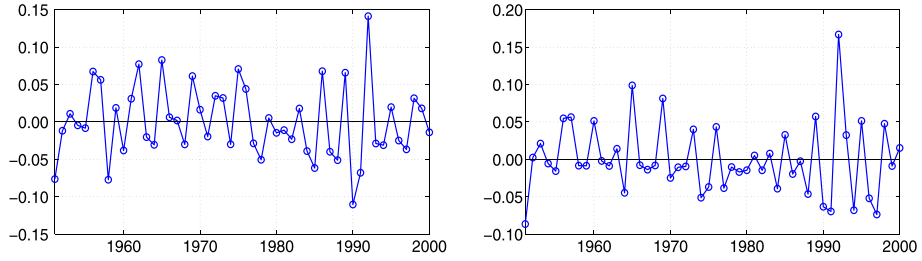
than a temporal hidden mortality index, plus a random vector for each age group so that the trend in the mortality data comes from mortality rates for the previous years rather than a trend in the hidden mortality index. Mitchell et al. (2013) also demonstrate that this modification of conditioning on the mortality rates for the current year leads to the predictions of very high quality for the future years.

Along this line, we propose an AR(n)-GARCH(1,1) model to describe the evolution of the logarithm of central death rates for a specific age x, that is,

$$\ln(m_{x,\,t}) = a_x^{(n)} + b_x^{(n)} \times \ln(m_{x,\,t-n}) + \epsilon_{x,\,t}^{(n)}, \tag{2.2}$$

where n is a predetermined projection time period and $\epsilon_{x,\,t}^{(n)}$ is the error term. Fig. 1(a)–(d) display the relationship between a sequence of the logarithm of central death rates and the sequence of corresponding ones lagged by n years ($n = 1$ and 5) for individuals aged 35, 45, and 55 for both genders of the UK, where the mortality data come from the Human Mortality Database (HMD, www.mortality.org). From the figures, we observe that two sequences of the logarithm of central death rates with a time lag of n years show a linear relationship, and the intercept term and the slope coefficient are adjusted to reflect the rates of mortality improvement or deterioration over n years. Based on the observation, we replace the temporal mortality index with the mortality rates for the same ages in the previous years as a unique explanatory variable, which is different from Mitchell et al. (2013).

In this paper, we make out-of-sample projections for future consecutive years by the AR models with empirical data. As exhibited in Fig. 2, given a study period $[T_1, T_2]$ for which mortality rates are available, we assume that we currently stand at the end of year T and would like to project the mortality rates and evaluate the forecasting performance for years $T + 1, \ldots, T_2$ with the mortality data for $[T_1, T]$. Conditioning on $m_{x,\,T}$, the logarithm of central death rate for year $T + n$, $\ln(m_{x,\,T+n})$, can be forecasted by (2.2) with $(\hat{a}_x^{(n)}, \hat{b}_x^{(n)})$, the estimate of $(a_x^{(n)}, b_x^{(n)})$, for each $n = 1, \ldots, T_2 - T$. It is called the AR(n) model hereafter (in fact, a



(a) Japan males. (b) Japan females.

Fig. 3. $\ln(m_{45,t}) - \ln(\hat{m}_{45,t})$ for $n = 1$ over $t = 1951$–2000.

special case of the regular AR(n) model). Alternatively, $\ln(m_{x,T+k})$ can also be predicted recursively with the initial value $\ln(m_{x,T})$ by $(\hat{a}_x^{(1)}, \hat{b}_x^{(1)})$ and (2.2) with $n = 1$ for $k = 1, \ldots, T_2 - T$, and we call it the AR(1) model hereafter. Specifically, for the AR(n) model,

$$\ln(\hat{m}_{x,T+n}) = \hat{a}_x^{(n)} + \hat{b}_x^{(n)} \times \ln(m_{x,T}), \quad n = 1, \ldots, T_2 - T.$$

For the AR(1) model,

$$\ln(\hat{m}_{x,T+1}) = \hat{a}_x^{(1)} + \hat{b}_x^{(1)} \times \ln(m_{x,T})$$

and

$$\begin{aligned}\ln(\hat{m}_{x,T+k}) &= \hat{a}_x^{(1)} + \hat{b}_x^{(1)} \times \ln(\hat{m}_{x,T+k-1}) \\ &= \hat{a}_x^{(1)} \times \frac{1 - (\hat{b}_x^{(1)})^k}{1 - \hat{b}_x^{(1)}} + (\hat{b}_x^{(1)})^k \ln(m_{x,T}),\end{aligned}$$

$k = 2, \ldots, T_2 - T$.

Note that the AR(n) model needs $T_2 - T$ pairs of estimated parameters $(\hat{a}_x^{(n)}, \hat{b}_x^{(n)})$, $n = 1, \ldots, T_2 - T$, whereas the AR(1) model needs only one parameter pair $(\hat{a}_x^{(1)}, \hat{b}_x^{(1)})$. Moreover, $\ln(m_{x,T+n})$ has the variance $\mathrm{Var}[\epsilon_{x,T+n}^{(n)}]$ for the former, and $\mathrm{Var}[\sum_{h=0}^{n-1}(\hat{b}_x^{(1)})^h \times \epsilon_{x,T+n-h}^{(1)}]$ for the latter. If the sequence $\{\epsilon_{x,T+n-h}^{(1)} : h = 0, \ldots, n-1\}$ is positively correlated, then $\mathrm{Var}[\sum_{h=0}^{n-1}(\hat{b}_x^{(1)})^h \times \epsilon_{x,T+n-h}^{(1)}]$ for the AR(1) model is increasing in n since $\hat{b}_x^{(1)}$ is generally close to one.

Moreover, observing the errors between the logarithm of central death rates and the fitted ones by (2.2) with $n = 1$ which are assumed to be Gaussian white noises for Japan males and females aged 45, shown in Fig. 3, we find a typical volatility clustering phenomenon that large (small) changes tend to be followed by large (small) ones. Thus, we further assume that the sequence of error terms, $\epsilon_{x,t}^{(n)}$, for age x follows a GARCH(1,1) model, that is,

$$\epsilon_{x,t}^{(n)} = \sigma_{x,t}^{(n)} \cdot \varepsilon_{x,t}^{(n)} \tag{2.3}$$

and

$$(\sigma_{x,t+1}^{(n)})^2 = \alpha_{x,0}^{(n)} + \alpha_{x,1}^{(n)} \cdot (\epsilon_{x,t}^{(n)})^2 + \beta_{x,1}^{(n)} \cdot (\sigma_{x,t}^{(n)})^2 \tag{2.4}$$

where $\alpha_{x,0}^{(n)} > 0$, $\alpha_{x,1}^{(n)} \geq 0$, $\beta_{x,1}^{(n)} \geq 0$, $\epsilon_{x,t+1}^{(n)} | I_t$ is normally distributed with mean zero and variance $(\sigma_{x,t+1}^{(n)})^2$, and $\varepsilon_{x,t}^{(n)}$ is a sequence of independent and identical standard normal random variables. If $\beta_{x,1}^{(n)} = 0$, then the GARCH(1,1) model reduces to the ARCH(1) model; if $\alpha_{x,1}^{(n)} = 0$ as well, then $\epsilon_{x,t}^{(n)}$ becomes a sequence of independent and identically normally distributed random variables with mean zero and variance $\alpha_{x,0}^{(n)}$, and is called Gaussian white noise (WN). In this paper, we fit the AR(1) and AR(n) models associated with GARCH(1,1), ARCH(1) and Gaussian white noise (WN) as error structures to the mortality rates for a specific age, which are called the AR-GARCH, AR-ARCH and AR-WN models, respectively.

2.2. Conditional copula methods for AR-GARCH mortality models

We model the mortality rates age by age with (2.2)–(2.4) in the preceding subsection, and then introduce the copula method to capture the inter-age mortality dependence structure in the current and next subsections. Copulas, first introduced by Sklar (1959), are tools for modeling the dependence between random variables. Copula model, a competitive alternative to the Gaussian dependence structure, meets the need of constructing flexible and non-standard multivariate distributions. Besides, many studies demonstrate that the specification of time-varying correlation gives better results than unconditional copula models (e.g., Alexandra and Paul (2010)), so we also employ the time-varying copulas.

Recall that a copula function is a multivariate cumulative distribution function (cdf) of standard uniform random variables (see Nelsen (1999)). Let F_i and f_i be the conditional cumulative distribution function (cdf) and conditional probability density function (pdf) of a continuous random variable $Y_{i,t}$ at time t given I_{t-1} with parameter set θ_i, $i = 1, \ldots, m$, where I_{t-1} represents the information up to time $t-1$, and F and f be the m-dimensional conditional cumulative distribution function and conditional probability density distribution of the random vector $\tilde{Y}_t = (Y_{1,t}, \ldots, Y_{m,t})$ given I_{t-1}. To capture the dependence between inter-age mortality rates, we rewrite F in terms of a conditional copula and m univariate conditional marginal cdfs $F_1, \ldots, F_m$ as follows:

$$\begin{aligned}&F(y_{1,t}, \ldots, y_{m,t}; \theta, \vartheta | I_{t-1}) \\ &\quad = C(F_1(y_{1,t}; \theta_1 | I_{t-1}), \ldots, F_m(y_{m,t}; \theta_m | I_{t-1}); \vartheta | I_{t-1})\end{aligned}$$

where $y_{i,t}$ is the realized value of $Y_{i,t}$, $i = 1, \ldots, m$, C is an m-dimensional conditional copula function with parameter set ϑ and $\theta = \bigcup_{i=1}^m \theta_i$. As proved by Sklar (1959), the copula function uniquely exists when $\tilde{Y}_t$ is a continuous random vector. For the continuous conditional random vector $\tilde{Y}_t$ given I_{t-1}, its conditional copula density is related to f with the following canonical representation (Patton, 2006):

$$\begin{aligned}&f(y_{1,t}, \ldots, y_{m,t}; \theta, \vartheta | I_{t-1}) \\ &\quad = c(u_{1,t}, \ldots, u_{m,t}; \theta, \vartheta | I_{t-1}) \prod_{i=1}^m f_i(y_{i,t}; \theta_i | I_{t-1}),\end{aligned} \tag{2.5}$$

where c is a conditional copula density function and $u_{i,t} = F_i(y_{i,t}; \theta_i | I_{t-1})$, $i = 1, \ldots, m$.

In this paper, we employ the time-varying Student's t copula to capture the co-movement between inter-age mortality rates. First, let $T_{R,\nu}$ be a multivariate Student's t distribution with $m \times m$ correlation matrix R of ν degrees of freedom. An m-dimensional Student's t copula function is of the form:

$$C(u_{1,t}, \ldots, u_{m,t}; R, \nu) = T_{R,\nu}(T_\nu^{-1}(u_{1,t}), \ldots, T_\nu^{-1}(u_{m,t})),$$

where T_ν^{-1} is the inverse function of a univariate standard Student's t cdf with ν degrees of freedom. Its associated copula density

function is defined as

$$c(u_{1,t},\ldots,u_{m,t};R,\nu) = |R|^{-\frac{1}{2}}\frac{\Gamma\left(\frac{\nu+m}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)}\left[\frac{\Gamma\left(\frac{\nu}{2}\right)}{\Gamma\left(\frac{\nu+1}{2}\right)}\right]^m \times \frac{\left(1+\frac{1}{\nu}\zeta_t' R^{-1}\zeta_t\right)^{-\frac{\nu+m}{2}}}{\prod_{i=1}^m\left(1+\frac{\zeta_{i,t}^2}{\nu}\right)^{-\frac{\nu+1}{2}}},$$

where $\zeta_t = (\zeta_{1,t},\ldots,\zeta_{m,t})'$ and $\zeta_{i,t} = T_\nu^{-1}(u_{i,t})$, $i = 1,\ldots,m$. Note that the Student's t copula degenerates to a Gaussian copula as ν goes to infinity.

Empirical evidence tends to support the time-varying dependence structure between multivariate financial asset returns. Patton (2006) first introduces the concept of time-varying copulas by using transformations of the past observations and an autoregressive term to construct time-varying dependence parameters. Vogiatzoglou (2010) incorporates the dynamic conditional correlation (DCC) proposed by Engle (2002) into a multivariate Student's t copula. The DCC matrix R_t of the time-varying Student's t copula is of the form:

$$R_t = \text{diag}(\Sigma_t)^{-\frac{1}{2}}\cdot\Sigma_t\cdot\text{diag}(\Sigma_t)^{-\frac{1}{2}}, \tag{2.6}$$

where

$$\Sigma_t = (1-\gamma-\eta)\cdot\overline{Q}+\gamma\cdot\zeta_{t-1}\zeta_{t-1}'+\eta\cdot\Sigma_{t-1},$$

$\overline{Q}$ is an unconditional correlation matrix of ζ_t, and γ and η are nonnegative parameters satisfying $\gamma+\eta<1$. Eq. (2.6) guarantees that R_t is a correlation matrix as long as Σ_t is positive definite. Note that if $\gamma=\eta=0$ then $\Sigma_t=\overline{Q}$ and $R_t=\text{diag}(\overline{Q})^{-\frac{1}{2}}\cdot\overline{Q}\cdot\text{diag}(\overline{Q})^{-\frac{1}{2}}$, the correlation matrix of the (static) Student's t copula. Therefore, the (static) Student's t copula ($\gamma=\eta=0$), the Gaussian copula ($\gamma=\eta=0$ and $\nu\to\infty$) and the time-varying Gaussian copula ($\nu\to\infty$) are special cases of the time-varying Student's t copula.

2.3. Calibration of copula-AR-GARCH mortality models

Eq. (2.5) is for the sole random vector $\tilde{Y}_t = (Y_{1,t},\ldots,Y_{m,t})$ given I_{t-1}. To apply the time-varying dependence structure to our case (see Fig. 2), consider $T-(T_1+n)+1$ random vectors $\tilde{Y}_t$ given I_{t-1}, $t = T_1+n,\ldots,T$. Let m be the length of the age span, and $Y_{i,t} = \varepsilon_{x_i,t}^{(n)} = \epsilon_{x_i,t}^{(n)}/\sigma_{x_i,t}^{(n)}$ by (2.3), $i = 1,\ldots,m$ and $t = T_1+n,\ldots,T$, for each of $n = 1,\ldots,T_2-T$ where x_i is the ith age in the age span, $\epsilon_{x_i,t}^{(n)} = \ln(m_{x_i,t})-[a_{x_i}^{(n)}+b_{x_i}^{(n)}\times\ln(m_{x_i,t-n})]$ from (2.2), and $\sigma_{x_i,t}^{(n)}$ follows (2.4). Since $\tilde{Y}_t|I_{t-1}$, $t = T_1+n,\ldots,T$, are independent, the log-likelihood function of $\prod_{t=T_1+n}^T f(y_{1,t},\ldots,y_{m,t};\theta,\vartheta|I_{t-1})$ by (2.5) is

$$\begin{aligned}&\sum_{t=T_1+n}^T \ln[f(y_{1,t},\ldots,y_{m,t};\theta,\vartheta|I_{t-1})]\\ &= \sum_{t=T_1+n}^T\sum_{i=1}^m \ln[f_i(y_{i,t};\theta_i|I_{t-1})]\\ &\quad+\sum_{t=T_1+n}^T \ln[c(u_{1,t},\ldots,u_{m,t};\theta,\vartheta|I_{t-1})].\end{aligned} \tag{2.7}$$

The first term on the right-hand side involves the marginal parameter sets $\theta_1,\ldots,\theta_m$, and the second term involves both the marginal parameter set $\theta = \bigcup_{i=1}^m\theta_i$ and the copula parameter set ϑ; the input of conditional copula density is obtained by the integral transform which needs the calibrated marginal parameters. It is difficult to estimate the marginal parameter set θ and the copula parameter set ϑ simultaneously via the maximum likelihood estimation from (2.7). Instead, we employ the inference functions for margins (IFM) method proposed by Joe and Xu (1996) to estimate the parameters of a copula-AR-GARCH mortality model. The IFM method is highly efficient and easy to implement, and is widely used in the literature; see, for example, Patton (2006), Okimoto (2008), Guégan and Zang (2010), Garcia and Tsafack (2011), Kenourgios et al. (2011), and Kumar and Okimoto (2011). The IFM estimation technique involves two steps:

Step 1: For the first term on the right-hand side of (2.7), we calibrate the relevant parameter set θ_j for the conditional marginal distribution f_j via the maximum likelihood estimation (MLE) by

$$\hat{\theta}_j = \arg\max_{\theta_j}\sum_{t=T_1+n}^T\sum_{i=1}^m \ln[f_i(y_{i,t};\theta_i|I_{t-1})],\quad j=1,\ldots,m.$$

In view of the equation above, we can estimate the marginal parameter set θ_j separately as $\hat{\theta}_j = \arg\max_{\theta_j}\sum_{t=T_1+n}^T \ln[f_j(y_{j,t};\theta_j|I_{t-1})]$ for each of $j = 1,\ldots,m$.

Step 2: Using $\hat{\theta} = \bigcup_{i=1}^m\hat{\theta}_i$ with $\hat{\theta}_j$, $j = 1,\ldots,m$, obtained from Step 1, we can estimate the copula parameter set ϑ by MLE from the second term on the right-hand side of (2.7). Specifically, given the calibrated marginal parameter set $\hat{\theta} = \bigcup_{j=1}^m\hat{\theta}_j$, the copula parameter set ϑ is obtained by

$$\hat{\vartheta} = \arg\max_{\vartheta}\sum_{t=T_1+n}^T \ln[c(u_{1,t},\ldots,u_{m,t};\hat{\theta},\vartheta|I_{t-1})].$$

Finally, the IFM estimator is defined as $\hat{\Theta}_{IFM} = (\hat{\theta},\hat{\vartheta})$.

Table 1
The percentages of the total number of the smallest AICs (BICs) over 101 for AR(1).

Gender		Male			Female		
Country	AIC/BIC	GARCH	ARCH	WN	GARCH	ARCH	WN
Japan	AIC	33%	29%	38%	23%	41%	36%
	BIC	17%	27%	56%	12%	32%	56%
UK	AIC	5%	47%	48%	4%	41%	55%
	BIC	2%	38%	60%	0%	26%	74%
USA	AIC	6%	28%	66%	10%	20%	70%
	BIC	1%	21%	78%	3%	11%	86%

Note: $101 = 101\cdot 1$ from 101 ages ($x = 0$–100) and $n = 1$.

3. Fitting the models

In the section, we fit our models using mortality data for both genders of Japan, the UK, and the USA available from the Human Mortality Database. Sixty yearly central death rates for each gender and country are obtained by dividing the yearly observations of age-specific death numbers by the matching population sizes exposed to the risk of death from 1950 to 2009.

We first estimate the parameters $a_x^{(n)}$ and $b_x^{(n)}$ in (2.2) with the logarithm of central death rates for the period 1950–2000 and the age span 0–100, and then forecast the mortality rates over the period 2001–2009 in the next section. That is, $T_1 = 1950$, $T = 2000$, $T_2 = 2009$, and $n = 1,\ldots,9$. Two most common criteria, Akaike Information Criterion (AIC) and Bayes Information Criterion (BIC), are adopted for each of (x,n), $x = 0,\ldots,100$ and $n = 1,\ldots,9$, to find the best goodness of fit among the AR-GARCH, AR-ARCH and AR-WN models. Thus, for each model, there are totally $101(=101\cdot 1)$ values under the AR(1) and $909(=101\cdot 9)$ values under the AR(n) for each of AIC and BIC. The smallest AIC or BIC means the best goodness of fit. For each model, we display the percentages of the total number of the smallest AICs (BICs) over 101 for the AR(1) and over 909 for the AR(n) in Tables 1 and 2, respectively.

Table 2
The percentages of the total number of the smallest AICs (BICs) over 909 for AR(n).

Gender		Male			Female		
Country	AIC/BIC	GARCH	ARCH	WN	GARCH	ARCH	WN
Japan	AIC	25%	34%	41%	18%	24%	58%
	BIC	15%	27%	58%	8%	18%	74%
UK	AIC	6%	24%	70%	7%	23%	70%
	BIC	2%	17%	81%	1%	16%	83%
USA	AIC	11%	58%	31%	10%	57%	33%
	BIC	5%	53%	42%	3%	51%	46%

Note: $909 = 101 \cdot 9$ from 101 ages ($x = 0$–100) and $n = 1, \ldots, 9$.

In terms of AIC for the AR(1) in Table 1, both the GARCH and ARCH models totally take 62% for Japan males and 64% for Japan females, 52% for the UK males and 45% for the UK females, and 34% for the USA males and 30% for the USA females; in terms of AIC for the AR(n) in Table 2, both the GARCH and ARCH models take 59% for Japan males and 42% for Japan females, 30% for both genders of the UK, and 69% for the USA males and 67% for the USA females. In terms of BIC, although the GARCH and ARCH models get much lower percentages, they still have 44% for both genders of Japan and 40% for the UK males under the AR(1), and 42% for Japan males and 58% (54%) for the USA males (females) under the AR(n). Thus, considering the conditional heteroskedasticity in (2.2) is crucial in fitting mortality data for both genders of the three countries. Besides, except for the AIC case for Japan males under AR(1), the percentages in Tables 1 and 2 for the ARCH model are higher than those for the GARCH one.

Fig. 4 illustrates the real and fitted mortality rates for males of Japan, the UK, and the USA with $x = 45$ and 85 under the AR-GARCH, AR-ARCH and AR-WN models with $n = 1$ and 5. We find some features of the models as follows. First, the fitted mortality rates for each model with $n = 1$ largely imitate the true ones; the shape of the fitted mortality rates for each model with $n = 5$ still copies that of the true ones but induces a longer time lag. The similarity between the real and fitted mortality rates reasonably decreases in n. Moreover, the differences between the models become apparent with $n = 5$ for USA males aged 45 and 85 (see Fig. 4(i) and (l)); even so, the shapes and the trends of the fitted mortality rates among the three models are still very similar in these cases.

4. Forecasting performance

In this section, we examine the forecasting performances between the Lee–Carter model and our models with and without conditional heteroskedasticity. To measure the error between the true and forecasted mortality rates, we adopt the MAPE (mean absolute percentage error), a common statistical quantity as used in D'Amato et al. (2012) and Wang et al. (2013b); specifically,

$$MAPE^{T+1}_{x,T_2-T} = \frac{1}{T_2 - T}\sum_{n=1}^{T_2-T}\left|\frac{\hat{q}_{x,T+n} - q_{x,T+n}}{q_{x,T+n}}\right| \times 100\%.$$

To evaluate the forecasting performances of our models associated with the GARCH, ARCH and WN, we set $n = 1$ in (2.2)–(2.4) for the AR(1) model and $n = 1, \ldots, 9$ in (2.2)–(2.4) for the AR(n) model, respectively. We forecast the mortality rates for the years 2001–2009 ($T = 2000$ and $T_2 = 2009$) for both genders of Japan, the UK, and the USA with the parameters estimated by the mortality data for the years 1950–2000 ($T_1 = 1950$) under our models and the Lee–Carter model. Since the performance of the Lee–Carter model in mortality fitting and forecasting depends on the choice of age span, we choose two age spans. One is the age span 21–85 (from the age of young adults to a life expectancy for some well-developed countries) which is most adopted to evaluate the forecasting performances of mortality models in the literature such as Haberman and Renshaw (2011); that is, $[x_L, x_U] = [21, 85]$. For comparisons, we also extend the age span to 21–100 ($[x_L, x_U] = [21, 100]$).

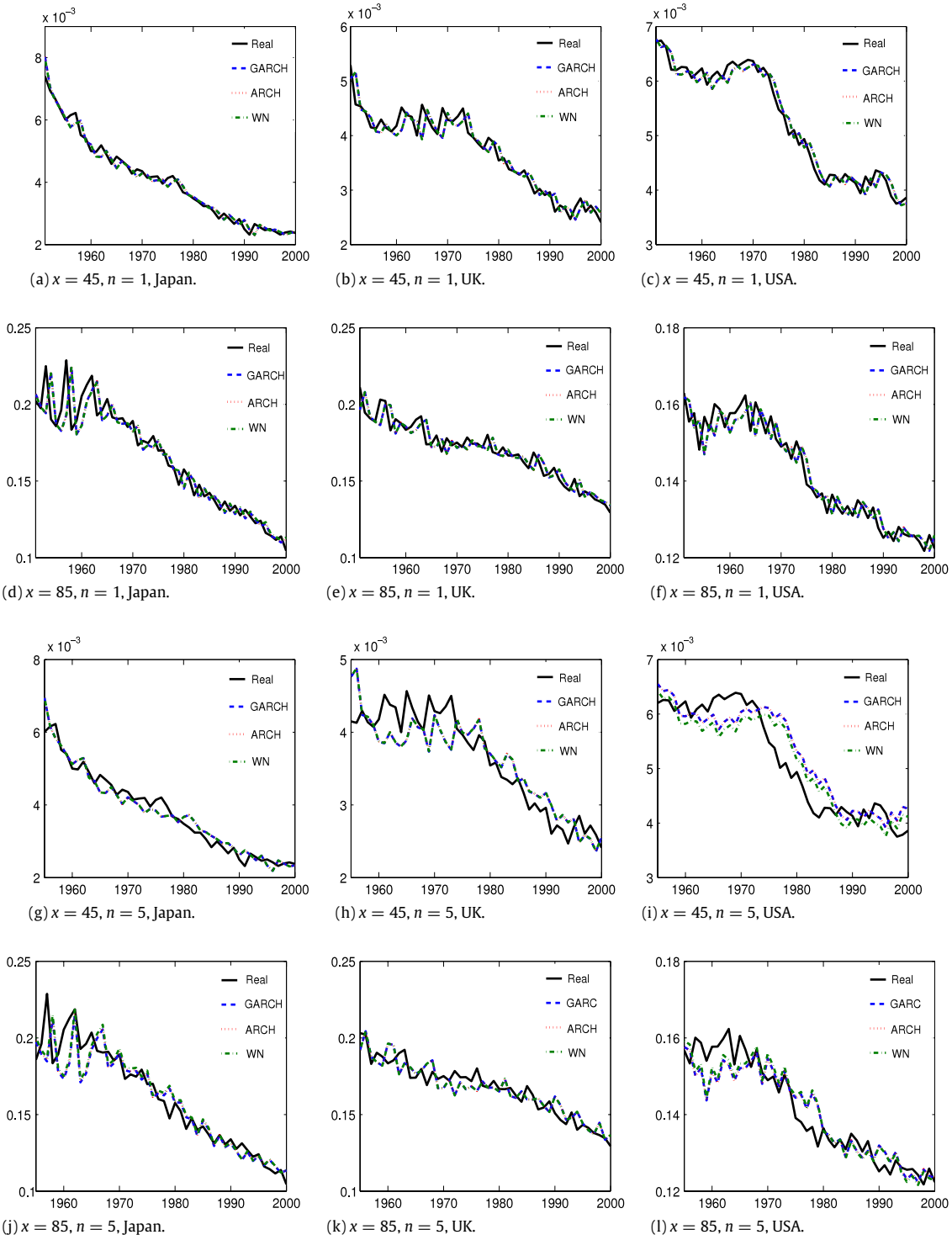
Tables 3 and 4 display the means and standard deviations of MAPEs over the age span 21–85, respectively. In the tables, the GARCH, ARCH and WN columns report the statistics under these three error structures. Both the means and standard deviations of MAPEs over six combinations of both genders and three countries for the GARCH, ARCH and WN under the AR(1) and AR(n) models are lower than those for the Lee–Carter model except for the AR(n) model for the USA males. As a result, the averages of the means and standard deviations of MAPEs over these six combinations are much smaller than those (12.52% for the mean and 7.11% for the standard deviation) for the Lee–Carter model. Among the three error structures, the GARCH under the AR(1) and the WN under the AR(n) yield the lowest averages of both the means and standard deviations of MAPEs. Next, Tables 5 and 6 give the results when

Table 3
Means of MAPEs over ages 21, . . . , 85.

Model		AR(1) (%)				AR(n) (%)			
Country	Gender	GARCH	ARCH	WN	LC	GARCH	ARCH	WN	LC
Japan	Male	**6.01**	6.07	6.13	12.05	5.55	5.71	**5.17**	12.05
	Female	**5.84**	5.92	5.89	20.50	**5.98**	6.26	6.44	20.50
UK	Male	9.25	9.19	**9.00**	14.35	**7.94**	7.96	8.03	14.35
	Female	10.62	**10.56**	11.23	12.90	8.84	8.82	**7.69**	12.90
USA	Male	6.88	6.97	**6.48**	8.75	12.27	12.11	11.21	**8.75**
	Female	**6.43**	6.47	6.53	6.57	6.83	6.82	**6.29**	6.57
Average		**7.51**	7.53	7.54	12.52	7.90	7.95	**7.47**	12.52

Table 4
Standard deviations of MAPEs over ages 21, . . . , 85.

Model		AR(1) (%)				AR(n) (%)			
Country	Gender	GARCH	ARCH	WN	LC	GARCH	ARCH	WN	LC
Japan	Male	3.07	**3.00**	3.11	8.28	2.73	2.75	**2.16**	8.28
	Female	2.97	3.00	**2.73**	11.46	**2.93**	2.96	3.15	11.46
UK	Male	**6.49**	6.58	6.70	7.66	5.77	5.68	**5.53**	7.66
	Female	**6.00**	6.03	6.40	6.78	4.54	4.52	**4.48**	6.78
USA	Male	4.01	4.08	**3.86**	4.79	12.48	12.57	11.81	**4.79**
	Female	2.78	2.85	**2.72**	3.69	2.10	**2.08**	2.47	3.69
Average		**4.22**	4.26	4.25	7.11	5.09	5.09	**4.93**	7.11



(a) $x = 45$, $n = 1$, Japan. (b) $x = 45$, $n = 1$, UK. (c) $x = 45$, $n = 1$, USA.

(d) $x = 85$, $n = 1$, Japan. (e) $x = 85$, $n = 1$, UK. (f) $x = 85$, $n = 1$, USA.

(g) $x = 45$, $n = 5$, Japan. (h) $x = 45$, $n = 5$, UK. (i) $x = 45$, $n = 5$, USA.

(j) $x = 85$, $n = 5$, Japan. (k) $x = 85$, $n = 5$, UK. (l) $x = 85$, $n = 5$, USA.

Fig. 4. Real vs. fitted $q_{x,t}$ for males of Japan, the UK and the USA.

the age span is extended to 21–100. Similarly, all of the GARCH, ARCH, and WN under the AR(1) and AR(n) models produce smaller means and standard deviations than the Lee–Carter model except for the AR(n) model for the USA males. Thus, we conclude that the forecasting performances of the AR(1) and AR(n) models with the three error structures are quite satisfactory, and both the models outperform the Lee–Carter model.

As shown in Tables 3–6, the averages of the means and standard deviations of MAPEs over the six combinations under both the AR(1) and AR(n) models do not differ too much no matter whether the conditional heteroskedasticity is involved (GARCH and ARCH) or not (WN). Since the mortality rates for the same age x in the next nth year can be predicted by the AR(n) from (2.2) for that age, the structure of error terms in (2.4) for each age x and each n can be different. Thus, even if we adopt the AR(1) or the AR(n) with the best fitted error structure producing the lowest AIC or BIC among GARCH, ARCH and NW for each age x and each n, which is labeled as MIX (a mixture of GARCH, ARCH, and WN), the means and standard deviations of MAPEs also do not differ too much among the three error structures and their mixture. The forecasting performances

Table 5
Means of MAPEs over ages 21, . . . , 100.

Model		AR(1) (%)				AR(n) (%)			
Country	Gender	GARCH	ARCH	WN	LC	GARCH	ARCH	WN	LC
Japan	Male	**6.41**	7.06	7.86	11.93	**5.49**	5.90	5.68	11.93
	Female	**5.84**	6.24	6.95	21.33	**6.04**	6.47	6.67	21.33
UK	Male	**10.27**	10.32	10.53	13.39	**8.47**	8.58	8.66	13.39
	Female	10.05	**9.99**	10.91	11.20	8.11	8.11	**7.22**	11.20
USA	Male	6.83	6.91	**6.56**	8.05	11.05	10.92	10.17	**8.05**
	Female	**6.10**	6.12	6.16	6.14	6.24	6.24	**5.81**	6.14
Average		**7.58**	7.77	8.16	12.01	7.57	7.70	**7.37**	12.01

Table 6
Standard deviations of MAPEs over ages 21, . . . , 100.

Model		AR(1) (%)				AR(n) (%)			
Country	Gender	GARCH	ARCH	WN	LC	GARCH	ARCH	WN	LC
Japan	Male	**3.35**	4.38	5.17	8.29	**2.92**	3.38	3.60	8.29
	Female	**3.26**	4.17	5.47	11.57	**2.84**	2.99	3.20	11.57
UK	Male	**6.39**	6.54	6.95	7.96	5.39	5.33	**5.22**	7.96
	Female	**5.60**	5.61	5.88	7.06	4.39	4.36	**4.19**	7.06
USA	Male	3.84	3.91	**3.75**	4.52	11.59	11.65	10.92	**4.52**
	Female	2.68	2.73	**2.63**	3.04	2.30	**2.27**	2.47	3.04
Average		**4.19**	4.56	4.97	7.08	**4.91**	5.00	4.93	7.08

between the AR(1) and AR(n) depend on the data; for example, the AR(1) is more suitable for the USA, and the AR(n) outperforms the AR(1) for the UK.

Figs. 5, 6 and 7 give plots of the real mortality rates and the forecasted ones associated with 90% confidence intervals (CIs) over nine years (2001–2009) for males aged 45, 55, 65, and 75 of Japan, the UK and the USA, respectively. The forecasted mortality rates and CIs are produced by the Lee–Carter model based on the age span 21–85 and by the AR(1) model with the GARCH and WN error structures (left column) and the AR(n) model with the MIX (the error structure is selected with the lowest AIC among GARCH, ARCH and WN) and WN error structures (right column). First, we find that the forecasted mortality rates under the AR(n) model, especially the MIX one, tend to imitate the trend and volatility of the true ones, whereas those under the AR(1) and Lee–Carter models are always approximately linearly decreasing. For example, for the Japan males aged 65 (shown in Fig. 5(f)), the predicted mortality rates under the AR(n) with the MIX capture the trend of the real ones.

Second, the upper and lower bounds of 90% CIs under the Lee–Carter model also linearly decrease as the forecasted mortality rates do, and its CIs become wide with time. The CIs under the AR(1) model with the GARCH and WN also become wide since the variance of the forecasted mortality rate for each of the next n years is recursively obtained by $\mathrm{Var}[\sum_{h=0}^{n-1}(\hat{b}_x^{(1)})^h \times \epsilon_{x,T+n-h}^{(1)}]$ which is increasing in n where $\epsilon_{x,T+n-h}^{(1)}$ follows (2.3) and (2.4) with $n = 1$, whereas those under the AR(n) model with the MIX and WN do not necessarily widen with time because the variance of the forecasted mortality rate for the next nth year can be directly obtained as $\mathrm{Var}[\epsilon_{x,T+n}^{(n)}]$ by applying (2.3) and (2.4) one time. Therefore, the CIs under the AR(1) are reasonably wider than those under the AR(n). Notwithstanding the narrower CIs for AR(n) than AR(1), the AR(n) produces quite satisfactory coverage of the forecasted mortality rates. Besides, the error structures under the AR(1) and AR(n) also contribute to confidence intervals with different width. However, which kind of error structure can produce narrower or wider CIs depends on the mortality data; for example, the GARCH produces wider CIs than the WN for Japan males aged 55 but narrower ones for age 65 under the AR(1) model (see Fig. 5(c) and (e), respectively).

Since the forecasted mortality rates are closer to the real ones under our AR(1) or AR(n) models, the forecasted mortality rates can be almost fully covered by corresponding CIs except for UK males aged 55 (see Fig. 6(d)), and USA males aged 55 (see Fig. 7(c) and (d)). On the contrary, whether the forecasted mortality rates under the Lee–Carter model can be covered varies case by case due to that the forecasted mortality rates are sometimes far away from the real ones. For example, the CIs under the Lee–Carter model cannot cover the forecasted mortality rates for the UK males aged 65 and 75 (see Fig. 6(e) and (g)) and for the USA males aged 45 (see Fig. 7(a)), whereas the CIs can give the full coverage of the forecasted mortality rates for the USA males aged 65 (see Fig. 7(e)).

We make a summary of the results from Tables 3–6 and Figs. 5–7. First, the AR(1) and AR(n) both yield more accurate forecasted mortality rates with more satisfactory confidence intervals than the Lee–Carter model. Second, the confidence intervals under the AR(1) are obviously much wider than those under the AR(n), but the coverage of the forecasted mortality rates for the AR(n) is not necessarily worse than that for the AR(1); the confidence intervals also vary with the error structures. Since the confidence intervals with different width can produce different risk-based capitals required to avoid the financial distress resulted from longevity risk, we will analyze Value at Risk (VaR) for longevity risk under the AR(1) and AR(n) with error structures and inter-age mortality dependence in the following section.

5. Value at risk for longevity risk

Since life insurers price products according to their estimated mortality rates, it is important to care about how much loss they may suffer from the difference between the real mortality rates and the estimated ones. In particular, the modern society is facing the problem of a rapidly aging population, which poses a big threat to annuity providers and pension funds. The goal of the following subsections is to study and quantify the possible losses incurred by longevity risks under our models with and without conditional heteroskedasticity, respectively, in life annuities and pension annuities. Furthermore, we take a dependence structure between mortality rates for different ages into consideration with the copula method. Because the estimated losses can be regarded

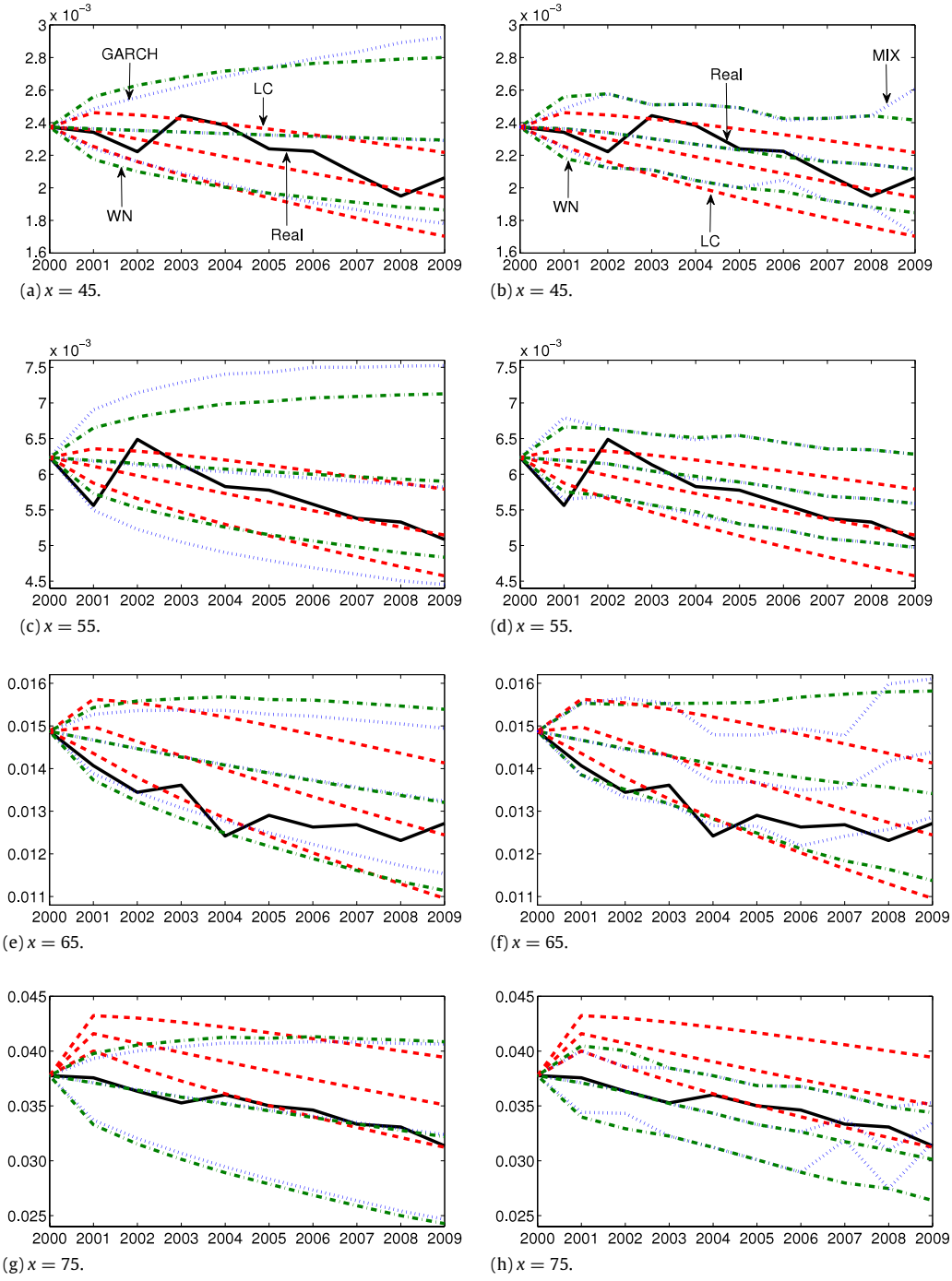


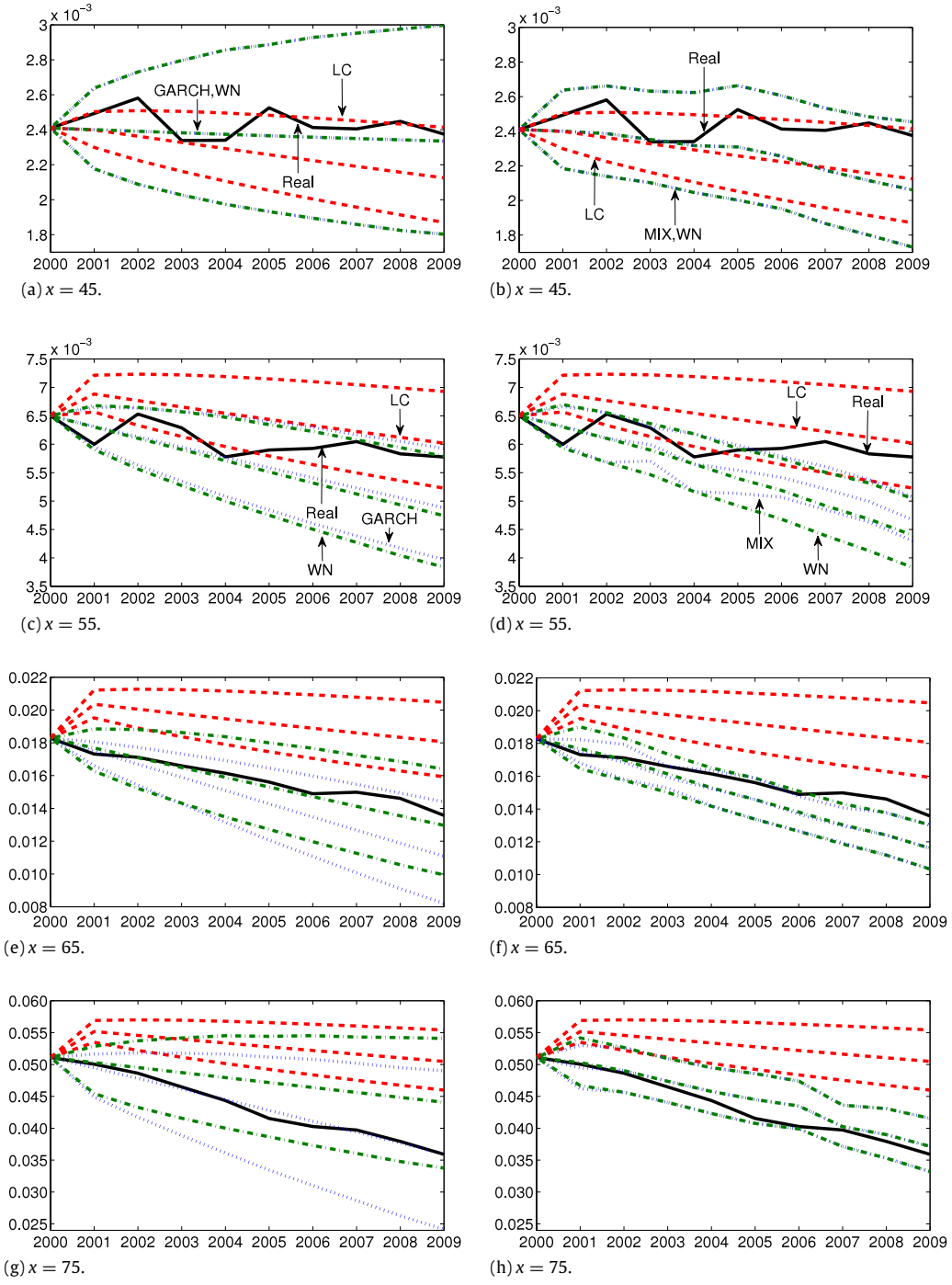
Fig. 5. Real vs. forecasted $q_{x,t}$ with 90% confidence intervals for Japan males AR(1) (left column) and AR(n) (right column).

as the additional reserves for life insurers to remain solvent when they expose to longevity risks, we quantify these losses by employing VaR (Value at Risk) and CVaR (Conditional Value at Risk), and show the effects of conditional heteroskedasticity and the mortality dependence on them.

5.1. Testing mortality dependence

In the section, we use the mortality data for both genders aged 65–89 of Japan as an example. First, we estimate the parameters using the longest sample period 1947–2009 available from the HMD, and then forecast the mortality rates for 2010–2034; that is, we reset $T_1 = 1947$, $T = 2009$, $T_2 = 2034$, and $n = 1, \ldots, 25$. Moreover, we adopt the fitting and forecasting results with the sole WN and a mixture of GARCH, ARCH, and WN where the error structure is determined by the AIC.

Second, we test the best goodness of fit among four mortality dependence structures: the static Gaussian copula, the time-varying Gaussian copula, the Student's t copula, and the time-varying Student's t copula for each n. Table A.1 in the Appendix reports the testing results of LLF (log-likelihood function), AIC and BIC. Table 7 summarizes the percentages of the total number of the best goodness of fit over 25 for each dependence structure based on AIC and BIC, respectively. We find that the static Gaussian is usually the best choice because it takes the largest portions in AIC and BIC (44% in AIC and 56% in BIC for males; 56% in AIC and



(a) $x = 45$. (b) $x = 45$.

(c) $x = 55$. (d) $x = 55$.

(e) $x = 65$. (f) $x = 65$.

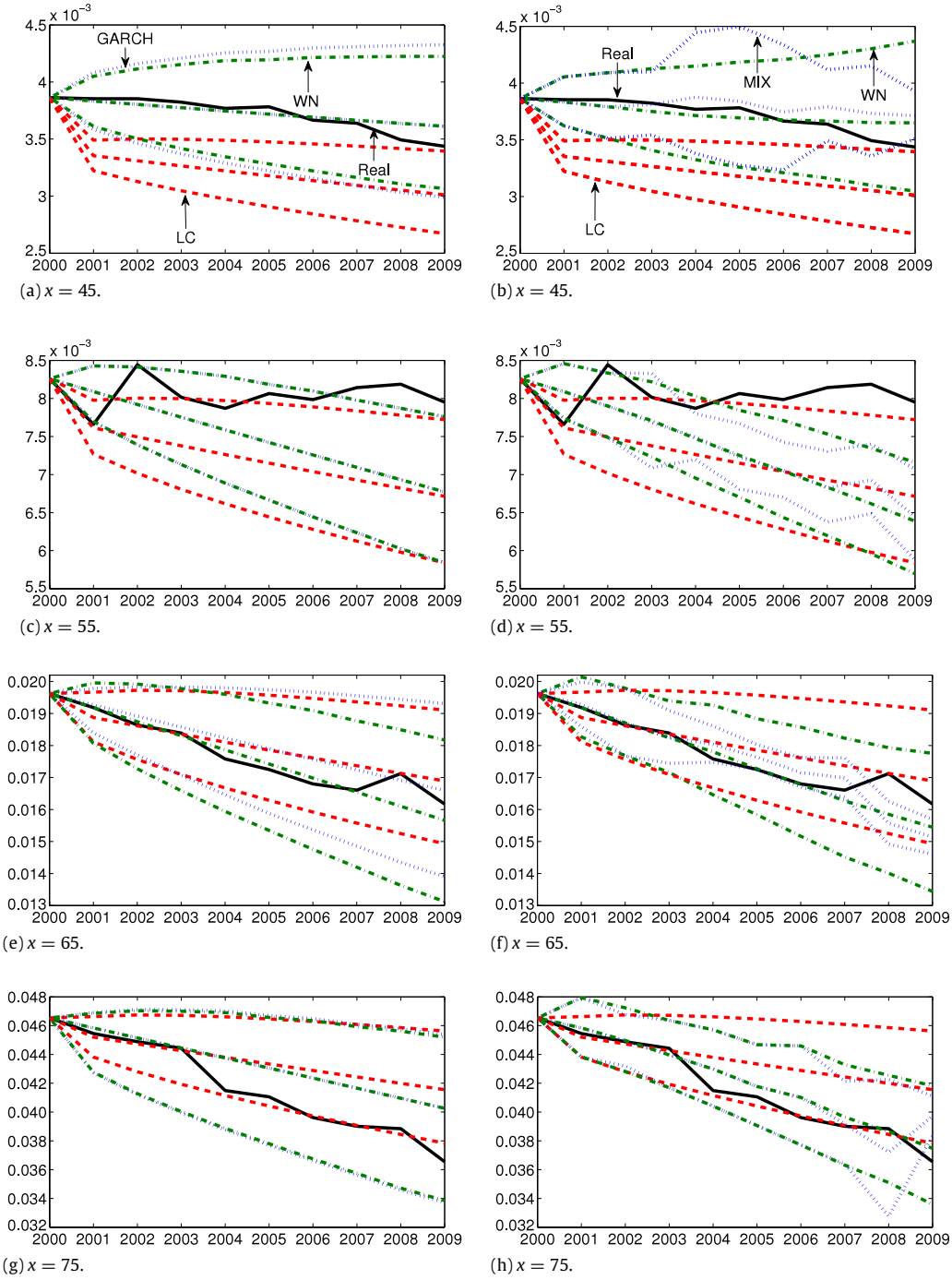
(g) $x = 75$. (h) $x = 75$.

Fig. 6. Real vs. forecasted $q_{x,t}$ with 90% confidence intervals for UK males AR(1) (left column) and AR(n) (right column).

Table 7
The percentages of the total number of the best goodness of fit over 25 for each dependence structure.

Copula	Static Gaussian	Time-varying Gaussian	Student's t	Time-varying Student's t
Panel A: male				
AIC	**44**%	20%	12%	24%
BIC	**56**%	20%	20%	4%
Panel B: female				
AIC	**56**%	12%	24%	8%
BIC	**68**%	4%	24%	4%

68% in BIC for females). In order to further enhance the statistical comparisons, the likelihood ratio test is conducted for two nested models to determine if the null model (static Gaussian copula) is more appropriate than each of the three alternative models (time-varying Gaussian copula, Student's t copula, and time-varying Student's t copula) since the static Gaussian copula is a special case of the other three structures; the results are displayed in Table A.2 in the Appendix. We find that the likelihood ratio test fails to reject the static Gaussian copula at a 1% significance level for most n values. As a result, using the static Gaussian copula to capture the inter-age mortality dependence structure is appropriate. For the simplicity, we adopt the static Gaussian copula for the AR(n) in the applications of calculating VaR values for longevity risk in



(a) $x = 45$. (b) $x = 45$.

(c) $x = 55$. (d) $x = 55$.

(e) $x = 65$. (f) $x = 65$.

(g) $x = 75$. (h) $x = 75$.

Fig. 7. Real vs. forecasted $q_{x,t}$ with 90% confidence intervals for USA males AR(1) (left column) and AR(n) (right column).

Sections 5.2 and 5.3. Since Table A.1 in the Appendix shows that the time-varying Gaussian copula is the best choice for $n = 1$, we take the time-varying Gaussian copula for the AR(1) in the same applications.

5.2. Life annuities

Consider a life annuity immediate that pays \$1 at the end of each year as long as the insured is alive. To examine the effect of longevity risk, we ignore the interest rate risk by assuming that the pricing interest rate is equal to the actual market interest rate throughout the section. Moreover, we assume that both mortality risk and interest rate risk are independent under the measure. Then, using the above forecasted mortality rates, we calculate the price (net single premium, NSP) of a life annuity immediate for an insured aged x at the beginning of year 2010 (i.e., $T + 1$) as

$$a_{x,2010} = \sum_{k=1}^{\omega-x} {}_k\hat{p}_{x,\,2010} \cdot v_{2010}^{k}, \tag{5.1}$$

where v_{2010}^{k} denotes the discount factor of \$1 paid in year $2010+k$, ${}_k\hat{p}_{x,\,2010}$, the estimate of ${}_kp_{x,\,2010}$ (the k-year survival probability for the insured aged x in year 2010), is given by

$$\begin{aligned} {}_k\hat{p}_{x,\,2010} &= \hat{p}_{x,\,2010} \times \hat{p}_{x+1,\,2011} \times \cdots \times \hat{p}_{x+k-1,\,2010+k-1}, \\ & k = 1, 2, \ldots, \omega - x, \end{aligned} \tag{5.2}$$

Table 8
Prices of life annuities, and α-VaRs and α-CVaRs of the additional reserves for $\alpha = 95\%$ for AR(1).

Age	HMD	MIX with copula			MIX			WN		
x	$a_{x,2010}$	$a_{x,2010}$	$Q_x(\alpha)$	$CQ_x(\alpha)$	$a_{x,2010}$	$Q_x(\alpha)$	$CQ_x(\alpha)$	$a_{x,2010}$	$Q_x(\alpha)$	$CQ_x(\alpha)$
Panel A: male										
65	12.90	13.36	0.3211	0.4076	13.36	0.3959	0.4917	13.25	0.6318	0.7808
75	8.19	8.39	0.2245	0.2757	8.39	0.2685	0.3316	8.31	0.4743	0.5877
85	3.34	3.37	0.0649	0.0799	3.37	0.0760	0.0943	3.34	0.1401	0.1713
Panel B: female										
65	14.99	15.60	0.1721	0.2088	15.60	0.2148	0.2655	15.62	0.3548	0.4310
75	9.82	10.14	0.1277	0.1583	10.14	0.1562	0.1929	10.13	0.2925	0.3574
85	3.81	3.85	0.0423	0.0529	3.85	0.0491	0.0609	3.85	0.0814	0.0997

Table 9
Prices of life annuities, and α-VaRs and α-CVaRs of the additional reserves for $\alpha = 95\%$ for AR(n).

Age	HMD	MIX with copula			MIX			WN		
x	$a_{x,2010}$	$a_{x,2010}$	$Q_x(\alpha)$	$CQ_x(\alpha)$	$a_{x,2010}$	$Q_x(\alpha)$	$CQ_x(\alpha)$	$a_{x,2010}$	$Q_x(\alpha)$	$CQ_x(\alpha)$
Panel A: male										
65	12.90	13.50	0.0502	0.0644	13.50	0.0517	0.0656	13.70	0.0727	0.0934
75	8.19	8.52	0.0490	0.0606	8.52	0.0504	0.0636	8.57	0.0764	0.0985
85	3.34	3.38	0.0277	0.0344	3.38	0.0287	0.0357	3.37	0.0543	0.0683
Panel B: female										
65	14.99	15.77	0.0249	0.0318	15.77	0.0249	0.0314	15.93	0.0349	0.0452
75	9.82	10.31	0.0567	0.0720	10.31	0.0563	0.0726	10.31	0.0429	0.0553
85	3.81	3.86	0.0245	0.0302	3.86	0.0246	0.0309	3.86	0.0300	0.0378

Table 10
α-VaRs and α-CVaRs of the pension fund for AR(1).

	MIX with copula		MIX		WN	
α	99%	95%	99%	95%	99%	95%
t_f	Panel A: VaR					
1	1,648,667	1,171,035	2,877,425	2,068,662	2,931,470	2,112,530
3	1,749,071	1,242,351	3,052,660	2,194,643	3,109,996	2,241,183
5	1,855,589	1,318,010	3,238,567	2,328,297	3,299,395	2,377,671
t_f	Panel B: CVaRs					
1	1,919,344	1,465,200	3,238,315	2,543,341	3,288,930	2,588,492
3	2,036,232	1,554,431	3,435,529	2,698,230	3,489,226	2,746,131
5	2,160,238	1,649,096	3,644,752	2,862,553	3,701,720	2,913,370

and $\hat{p}_{z,y}$ is the predicted one-year survival probability for age z in year y. We adopt the predicted cohort mortality rate sequence $\{\hat{p}_{x+i,\,2010+i} : i = 0, 1, \ldots, \omega - x - 1\}$ for calculating ${}_k\hat{p}_{x,\,2010}$. We also assume that the interest rate is constant for simplicity, i.e., $i = 3\%$ and $v_{2010}^k = [1/(1+i)]^k$, and set the limiting age ω equal to 90 such that the mortality rate for age 90, q_{90}, is equal to 1.

Recall that economic capital in finance is the amount of risk capital required by a firm to secure survival or solvent in a worst case scenario for some risks. Economic capital is often calculated as VaR (Value at Risk) or CVaR (Conditional Value at Risk). By definition, α-CVaR is the expected loss exceeding α-VaR. We use our AR(1)/AR(n) models associated with the MIX (plus time-varying/static Gaussian copula) and WN to obtain the distribution of the forecasted price of life annuity. Let DP_x denote the distribution of the predicted price for x and $DP_x(\alpha)$ be the percentile of order α (α is the confidence level) of DP_x. The α-VaR of the additional reserve for longevity risk is given by

$$Q_x(\alpha) = DP_x(\alpha) - a_{x,2010},$$

and the α-CVaR is given as

$$CQ_x(\alpha) = \frac{1}{1-\alpha}\int_\alpha^1 Q_x(\gamma)d\gamma.$$

Tables 8 and 9 show the prices of life annuities, 95%-VaRs and 95%-CVaRs of the additional reserves for both genders aged 65, 75 and 85 under the AR(1) and the AR(n), respectively. They also display the prices of life annuities based on the real period mortality rate sequence for age x in year 2009, $\{p_{x+i,\,2009} : i = 0, 1, \ldots, \omega - x - 1\}$, from the HMD as a benchmark for comparisons. The prices of the annuities under the MIX with and without the copula are equal, and are not that much different from those under the WN for both of the AR(1) and AR(n); the prices under our models are all higher than those using the real period mortality rate sequence. However, due to the confidence intervals with different width, the VaR and CVaR values for the AR(1) are much larger than those for the AR(n). The VaR and CVaR values under the MIX are lower than those under the WN, and the MIX with the copula generally reduces them further.

5.3. Pension annuities

In this subsection, we build a representative closed pension fund which includes N_0 retirees for both genders aged 65–89, whose age and gender composition is the portrayal of the Japan elderly population for year 2009 (the beginning of year 2010 (=T

Table 11
α-VaRs and α-CVaRs of the pension fund for AR(n).

	MIX with copula		MIX		WN	
α	99%	95%	99%	95%	99%	95%
t_f	Panel A: VaR					
1	429,894	294,100	560,806	382,980	698,196	484,052
3	456,074	312,011	594,959	406,303	740,717	513,531
5	483,849	331,013	631,192	431,047	785,826	544,805
t_f	Panel B: CVaRs					
1	484,667	378,455	641,946	490,572	802,277	613,545
3	514,183	401,503	681,040	520,447	851,135	650,910
5	545,497	425,955	722,516	552,143	902,970	690,550

+ 1)); thus, $N_0 = \sum_{g\in\{M,F\}} \sum_{x=65}^{89} n^g_{x,2010}$ where $n^g_{x,2010}$ is the population size of Japan for the gender g (M for males and F for females) and age x in year 2010. The pension annuity is also a life annuity immediate. There are no new entrants into the pension fund and the premiums (calculated as in Section 5.1) are collected from these retirees at time 0 when the closed fund commences (denoted as $t_f = 0$ at the beginning of year 2010). The initial value of the pension fund's total assets (premiums collected) at time 0, A_0, equals the present value of the future expected liabilities ($1 \times N_{t_f}$, the population size in year $2010 + t_f$, $t_f = 1, 2, \ldots$), L_0, that is, $L_0 = A_0 = \sum_{g\in\{M,F\}} \sum_{x=65}^{89} n^g_{x,2010} \cdot a^g_{x,2010}$ where $a^g_{x,2010} = \sum_{k=1}^{\omega-x} {}_k\hat{p}^g_{x,2010} \cdot v^k_{2010}$ by (5.1).

Table A.1
LLFs, AICs, and BICs for testing mortality dependence.

Copula	Static Gaussian			Time-varying Gaussian			Student's t			Time-varying Student's t		
	LLF	AIC	BIC	LLF	AIC	BIC	LLF	AIC	BIC	LLF	AIC	BIC
n	Panel A: males											
1	1106	−1096	−1086	1116	**−1104**	**−1091**	1107	−1096	−1084	**1116**	−1103	−1089
2	1083	**−1073**	**−1063**	1083	−1071	−1059	1083	−1072	−1061	**1083**	−1070	−1057
3	1084	**−1074**	**−1064**	1084	−1072	−1060	1084	−1073	−1062	**1084**	−1071	−1058
4	1120	**−1110**	**−1100**	1120	−1108	−1096	1120	−1109	−1098	**1120**	−1107	−1094
5	1082	**−1072**	**−1062**	1082	−1070	−1058	1082	−1071	−1060	**1082**	−1069	−1056
6	1163	**−1153**	**−1143**	1163	−1151	−1139	1163	−1152	−1141	**1163**	−1150	−1137
7	1115	**−1105**	**−1095**	1115	−1103	−1091	1115	−1104	−1093	**1115**	−1102	−1089
8	1127	−1117	−1107	1132	**−1120**	**−1108**	1127	−1116	−1105	**1132**	−1119	−1106
9	1120	−1110	−1100	1131	**−1119**	**−1107**	1120	−1109	−1098	**1131**	−1118	−1105
10	1059	**−1049**	**−1039**	1059	−1047	−1036	1059	−1048	−1037	**1059**	−1046	−1034
11	1150	−1140	−1130	1156	**−1144**	**−1132**	1150	−1139	−1128	**1156**	−1143	−1130
12	1041	−1031	−1022	1047	**−1035**	**−1024**	1041	−1030	−1020	**1047**	−1034	−1022
13	1078	**−1068**	**−1059**	1078	−1066	−1055	1078	−1067	−1057	**1078**	−1065	−1053
14	1025	**−1015**	**−1005**	1025	−1013	−1001	1025	−1014	−1003	**1025**	−1012	−1000
15	903	−893	**−883**	905	−893	−882	904	−893	−883	**906**	**−893**	−881
16	1002	**−992**	**−983**	1002	−990	−979	1002	−991	−981	**1002**	−989	−977
17	885	−875	−866	885	−873	−862	888	**−877**	**−867**	**889**	−876	−864
18	869	−859	−850	871	−859	−848	872	−861	**−851**	**874**	**−861**	−849
19	808	−798	−789	808	−796	−785	813	−802	**−792**	**816**	**−803**	−792
20	845	−835	−826	845	−833	−822	849	−838	−829	**861**	**−848**	**−836**
21	869	−859	**−851**	869	−857	−847	871	**−860**	−851	**871**	−858	−847
22	958	−948	−939	958	−946	−936	961	−950	**−940**	**964**	**−951**	−940
23	829	−819	**−810**	832	−820	−810	830	−819	−810	**834**	**−821**	−810
24	871	−861	−853	871	−859	−850	873	**−862**	**−853**	**873**	−860	−850
25	840	**−830**	**−821**	840	−828	−818	840	−829	−820	**840**	−827	−816
No.	0	11	14	0	5	5	0	3	5	25	6	1
n	Panel B: females											
1	990	−980	−969	1012	**−1000**	**−987**	990	−979	−967	**1012**	−999	−985
2	1021	**−1011**	**−1000**	1021	−1009	−997	1021	−1010	−998	**1021**	−1008	−995
3	1055	**−1045**	**−1035**	1055	−1043	−1031	1055	−1044	−1033	**1055**	−1042	−1029
4	1059	**−1049**	**−1039**	1059	−1047	−1035	1059	−1048	−1037	**1059**	−1046	−1033
5	1049	**−1039**	**−1029**	1049	−1037	−1025	1049	−1038	−1027	**1049**	−1036	−1023
6	1117	**−1107**	**−1097**	1117	−1105	−1093	1117	−1106	−1095	**1117**	−1104	−1091
7	1085	**−1075**	**−1065**	1086	−1074	−1062	1085	−1074	−1063	**1086**	−1073	−1060
8	1063	**−1053**	**−1043**	1064	−1052	−1040	1063	−1052	−1041	**1064**	−1051	−1038
9	1123	**−1113**	**−1103**	1123	−1111	−1099	1123	−1112	−1101	**1123**	−1110	−1097
10	952	**−942**	**−932**	953	−941	−929	953	−942	−931	**954**	−941	−928
11	1050	−1040	**−1030**	1052	−1040	−1028	1052	−1041	−1030	**1054**	**−1041**	−1028
12	961	−951	**−942**	965	**−953**	−942	962	−951	−940	**965**	−952	−940
13	966	**−956**	**−947**	966	−954	−943	966	−955	−945	**966**	−953	−941
14	987	**−977**	**−968**	987	−975	−964	987	−976	−966	**987**	−974	−962
15	847	**−837**	**−828**	847	−835	−824	847	−836	−826	**847**	−834	−822
16	921	−911	**−902**	925	**−913**	−901	921	−910	−900	**925**	−912	−900
17	847	**−837**	**−828**	849	−837	−826	848	−837	−827	**850**	−837	−825
18	844	**−834**	**−825**	846	−834	−823	845	−834	−824	**847**	−834	−822
19	916	−906	−897	916	−904	−893	925	**−914**	**−904**	**925**	−912	−900
20	936	−926	−918	936	−924	−914	954	**−943**	**−933**	**954**	−941	−930
21	896	−886	−877	896	−884	−873	905	**−894**	**−885**	**905**	−892	−881
22	929	−919	−911	929	−917	−907	935	**−924**	**−915**	**936**	−923	−912
23	872	−862	−854	872	−860	−850	876	**−865**	**−856**	**876**	−863	−852
24	905	−895	−886	905	−893	−883	908	**−897**	**−888**	**908**	−895	−884
25	827	−817	−808	830	−818	−808	830	−819	−810	**834**	**−821**	**−810**
No.	0	14	17	0	3	1	0	6	6	25	2	1

For $t_f = 1, 2, \ldots, T_f$, the fund value in year $2010 + t_f$ equals

$$A_{t_f} = A_{t_f-1}(1 + R_{t_f}) - \sum_{g\in\{M,F\}} \sum_{x=65+t_f}^{90} \tilde{n}^g_{x,2010+t_f}, \tag{5.3}$$

where R_{t_f} is the return between time $t_f - 1$ and time t_f, $\tilde{n}^g_{x,2010+t_f} = \tilde{n}^g_{x-1,2010+t_f-1} \cdot \tilde{p}^g_{x-1,2010+t_f-1}$ is the simulated population size for gender g and age x in year $2010+t_f$ with $\tilde{n}^g_{x-1,2010+t_f-1} = n^g_{x-1,2010}$ for $t_f = 1$, and $\tilde{p}^g_{x-1,2010+t_f-1}$ is the simulated one-year survival probability for gender g and age $x-1$ in year $2010 + t_f - 1$. The pension fund's liabilities in year $2010 + t_f$ involving two uncertain variables $\tilde{n}$ and $\tilde{a}$ are given by

$$L_{t_f} = \sum_{g\in\{M,F\}} \sum_{x=65+t_f}^{89} \tilde{n}^g_{x,2010+t_f} \cdot \tilde{a}^g_{x,2010+t_f}, \tag{5.4}$$

where

$$\tilde{a}^g_{x,2010+t_f} = \sum_{k=1}^{\omega-x} {}_k\tilde{p}^g_{x,\,2010+t_f} \cdot v^k_{2010+t_f}.$$

Simulation of A_{t_f} and L_{t_f} involves simulation of $\tilde{p}^g_{x,2010+t_f}$ for ages 65 and higher and for $t_f = 1, \ldots, T_f$ under our AR(1) and AR(n) models with the MIX plus the copula, the MIX and the WN, respectively. Therefore, α-VaRs of the pension fund in year $2010 + t_f$ is the α percentile of $(L_{t_f} - A_{t_f})$, and α-CVaRs of the pension fund in year $2010+t_f$ is the expected loss beyond α percentile of $(L_{t_f} - A_{t_f})$. We ignore the interest rate risk again and assume the interest rate is constant as in Section 5.2 (that is, $R_{t_f} = 3\%$ for $t_f = 1, 2, \ldots, T_f$), and set N_0 equal to 10^7.

The simulation results for $t_f = 1, 3, 5$ are displayed in Tables 10 and 11. As in Tables 8 and 9, the VaR and CVaR values for the AR(1) are much larger than (about four times on average) those for the AR(n). The VaR and CVaR values reasonably increase with t_f. Implementing our model with the MIX reduces the VaR and CVaR values of this pension fund compared to our model with the sole WN only, and the MIX with the copula further reduces these values to a great extent. As a result, the order (from the highest to the lowest) in the VaR and CVaR values is the WN, the MIX, and the MIX with copula. Therefore, from Tables 10 and 11, we can infer that due to incorporation of conditional heteroskedasticity and/or mortality dependence structure into our AR(1) and AR(n) models, a pension funder in Japan can set aside less cash buffer for longevity risk, and adopting the AR(n) even dramatically curtails the risk-based capitals.

6. Conclusions

Mortality rates have been improving dramatically, which is a threat to the financial soundness of annuity providers, retirement programs and social security systems. In this paper, we propose age-specific copula-AR-GARCH mortality models. Our AR structure is analogous to the Lee–Carter model; the prediction under our model is conditioned on the mortality rate at the current year for a specific age, and that under the Lee–Carter model is based on the overall mortality index. We also propose two alternatives of using AR(n): one is fitting the AR(n) to mortality rates and predicting the mortality rates for the next nth year; the other is fitting the AR(1) to mortality rates and predicting the mortality rates recursively for the next consecutive years. Further, we consider conditional heteroskedasticity and inter-age mortality dependence structures.

Using mortality data for both genders of Japan, the UK, and the USA, we demonstrate that the non-Gaussian error structures (GARCH and ARCH) can improve in-sample goodness of fit of the Gaussian one (WN); compared to the Lee–Carter model, our models also give better performances in out-of-sample projection and more satisfactory coverage of the forecasted mortality rates with confidence intervals. Finally, in order to study the effect of conditional heteroskedasticity and mortality dependence, the VaR and CVaR measures are employed on the prices of a life annuity immediate and a demography-based pension fund. Using the mortality data for both genders of Japan, we show that the static Gaussian copula usually gives better goodness of fit than the time-varying Gaussian copula and the static/time-varying Student's t copulas. In addition, we illustrate that VaR and CVaR values can be reduced further by incorporating the time-varying/static Gaussian copula into the AR(1)/AR(n) models associated with the MIX (a mixture of error structures from GARCH, ARCH and WN).

To sum up, we contribute to stochastic mortality models in several major ways. First, our models can be fitted by mortality data of a single age and then applied to forecasting for that age, so our models have more flexibility in implementing mortality fitting and forecasting than the Lee–Carter or CBD based models. More specifically, we do not re-run the fitting to obtain the model's parameters for forecasting mortality rates when the age span is changed, whereas the estimated parameters and forecasting results under the Lee–Carter model in (2.1) depend on the age span $[x_L, x_U]$. Second, the variance level and the error structures are taken into account. We propose two approaches, AR(1) and AR(n), to fitting and forecasting, which produce similar mean forecasts but quite different variance ones. Li and Chan (2011) state that a well-qualified mortality model not only has good mean predictions but also prediction intervals with a high degree of coverage. Our AR(1) and AR(n) models both contribute to good coverage on forecasted mortality rates and the AR(n) even produces satisfactory coverage with narrower prediction intervals; the non-Gaussian error structures (GARCH and ARCH) can further improve goodness of fit of our models. Third, although the parameters under our

Table A.2
Statistics of likelihood ratio (LR) test.

n	Male			Female		
	LR(tvG)	LR(sSt)	LR(tvSt)	LR(tvG)	LR(sSt)	LR(tvSt)
1	**19**	1	**19**	**43**	0	**43**
2	0	0	0	2	0	2
3	0	0	0	0	0	0
4	0	0	0	0	0	0
5	0	0	0	0	0	0
6	0	0	0	0	0	0
7	0	0	0	1	0	1
8	**11**	0	**11**	2	0	2
9	**21**	0	**21**	0	1	1
10	0	0	0	2	2	3
11	**12**	0	**12**	4	3	8
12	**12**	0	**12**	**7**	0	7
13	0	0	0	0	0	0
14	0	0	0	0	0	0
15	4	2	7	0	0	0
16	0	0	0	**7**	1	8
17	0	5	7	4	2	6
18	4	6	**11**	4	2	5
19	0	**11**	**17**	0	**18**	**18**
20	0	**9**	**32**	0	**35**	**35**
21	0	4	4	0	**19**	**19**
22	0	5	**13**	0	**12**	**14**
23	**7**	3	**11**	0	**9**	9
24	0	4	4	0	**7**	7
25	0	1	1	**7**	6	**15**
No. of significance	6	2	10	4	6	6

* LR(A) = 2 × [ln(the likelihood for the alternative model A) − ln(the likelihood for the null model)] where A = tvG, sSt and tvSt denote time-varying Gaussian, static Student's t, and time-varying Student's t, respectively, and the null model is static Gaussian.
** A bold value means that the null model is rejected at a 1% significance level.

AR-GARCH models are estimated separately for each age, we still capture the correlations between inter-age mortality rates with the copula methods. In light of the article and the recent literature (see, for example, Yang and Wang (2013), and Wang and Yang (2013)), we plan to incorporate multi-country mortality data or cohort-based concepts into econometrical models such as this article for further development in the future.

Acknowledgments

The authors would like to thank the anonymous referee for his/her valuable comments and suggestions. Support from the National Science Council of Taiwan (NSC 102-2410-H-194-014), the Global Risk Institute of Canada, the Natural Sciences and Engineering Research Council of Canada (NSERC 611441), is gratefully acknowledged.

Appendix

Table A.1 reports the testing results of log-likelihood function (LLF), Akaike Information Criterion (AIC) and Bayes Information Criterion (BIC) for each n under the static Gaussian copula, the time-varying Gaussian copula, the Student's t copula, and the time-varying Student's t copula. Table A.2 displays the results of likelihood ratio (LR) test between the null model (the static Gaussian copula) and one of the alternative models (the time-varying Gaussian copula, the Student's t copula, and the time-varying Student's t copula).

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