THE VALUATION OF RESET OPTIONS WITH MULTIPLE STRIKE RESETS AND RESET DATES

SZU-LANG LIAO*
CHOU-WEN WANG

This article makes two contributions to the literature. The first contribution is to provide the closed-form pricing formulas of reset options with strike resets and predecided reset dates. The exact closed-form pricing formulas of reset options with strike resets and continuous reset period are also derived. The second contribution is the finding that the reset options not only have the phenomena of Delta jump and Gamma jump across reset dates, but also have the properties of Delta waviness and Gamma waviness, especially near the time before reset dates. Furthermore, Delta and Gamma can be negative when the stock price is near the strike resets at times close to the reset dates. © 2003 Wiley Periodicals, Inc. Jrl Fut Mark 23:87–107, 2003

We are grateful to an anonymous referee for his valuable comments; the remaining errors are ours. The usual caveat applies.
*Correspondence author, Department of Money and Banking, National Chengchi University 64, Chih-nan Rd., Sec. 2, Taipei 116, Taiwan; e-mail: liaosl@nccu.edu.tw

Received August 2001; Accepted April 2002

■ *Szu-Lang Liao is an associate professor of finance in the Department of Money and Banking at National Chengchi University in Taipei, Taiwan.*

■ *Chou-Wen Wang is an assistant professor of finance in the Department of Finance at National Kaohsiung First University of Science and Technology, Kaohsiung, Taiwan.*

The Journal of Futures Markets, Vol. 23, No. 1, 87–107 (2003)
Published online in Wiley InterScience (www.interscience.wiley.com). DOI: 10.1002/fut.10055

INTRODUCTION

Path-dependent options, whose payoffs are influenced by the path of the prices of underlying assets, have become increasingly popular in recent years. One of the path-dependent options is the look-back option, whose payoff depends, in particular, on the minimum or maximum price of the underlying asset during the option's lifetime. There is another kind of path-dependent option known as a reset option. Unlike the look-back option, the strike price of a reset option will be reset to a new strike price only on the prespecified reset dates if the price of the underlying asset is lower than one of the strike resets.

Reset options have been issued in practice for many years. The Chicago Board Options Exchange (CBOE) and the New York Stock Exchange (NYSE) both introduced S&P 500 index put warrants with a 3-month reset period in late 1996. Morgan Stanley issued a reset warrant with an initial strike price of $44.73 in July 1997. The strike price would be adjusted to $39.76 on August 5, 1997, if the price of its underlying asset fell below $39.76. A more recent example comes from Taiwan, where Grand Cathay Securities had six reset options listed on the Taiwan Stock Exchange (TSE; codes in the TSE are 0517, 0522, 0523, 0527, 0528, and 0538) from 1998 to 1999. Most reset options, including all of the reset options listed in the TSE, are options with multiple strike resets and reset dates. For example, the reset condition of 0522 of the TSE is that the strike price would be adjusted if the 6-day average closing price of 2323 on the TSE fell below 98%, 96%, 94%, 92%, 90% of the initial strike price of $81 during the first 3 months after the warrant was issued.

Because the reset warrants are new derivative products in financial markets, few studies have been done on their pricing problems. Gray and Whaley (1997) examined the pricing of the put warrant with periodic reset and the warrant's risk characteristics. They further provided a closed-from solution for reset options with a single reset date in a later paper (Gray & Whaley, 1999). Cheng and Zhang (2000) studied reset options whose strike price will be reset to the prevailing stock price if the option is out of money. A closed-form pricing formula in terms of a multivariate normal distribution is derived under the risk-neutral framework. However, the reset conditions of reset options investigated by Cheng and Zhang (2000) are not the general cases of reset products in practice. Let the underlying asset price at time t be denoted by $S(t)$. The terminal payoff of a reset option with n reset dates and initial strike price K_0, which was studied by Cheng and Zhang (2000), is as follows:

$$C(T) = \mathrm{Max}[S(T) - \mathrm{Min}[K_0, S(t_1), \ldots, S(t_n)], 0] \qquad (1)$$

In practice, however, the terminal payoff of the reset option is more often set as

$$C(T) = \text{Max}[S(T) - K^*, 0] \equiv [S(T) - K^*]^+ \tag{2}$$

where

$$K^* = \begin{cases} K_0 & \text{if Min}[S(t_1), \ldots, S(t_n)] > D_1 \\ K_i & \text{if } D_i \geq \text{Min}[S(t_1), \ldots, S(t_n)] > D_{i+1}, \quad i = 1, \ldots, m-1 \\ K_m & \text{if } D_m \geq \text{Min}[S(t_1), \ldots, S(t_n)] \end{cases} \tag{3}$$

and K_i, $i = 1, \ldots, m$, are the reset strike prices; D_i, $i = 1, \ldots, m$, are the strike resets.

Our first contribution in this article is to derive the exact closed-form solution for reset options with strike resets and predecided reset dates, as specified in (2) and (3), under the risk-neutral framework. Furthermore, we also provide the closed-form solution for reset options with strike resets and continuous reset dates, which is the limiting case of the former.

Some previous studies, such as Cheng and Zhang (2000), have pointed out the phenomenon of Delta jump across reset dates. The second contribution of this article is the finding that, in addition to Delta jump, a reset option with strike resets also has the phenomena of Gamma jump, Delta waviness, and Gamma waviness as well. The waviness of delta and gamma means that the delta and gamma of reset options will oscillate when the stock price passes across the strike resets. When the time is approaching the reset dates and the stock price is near the strike resets, delta and gamma may change their values from positive to negative. The phenomena of Delta jump and Gamma jump near reset time as well as the properties of Delta waviness and Gamma waviness will make the risk management more difficult.

PRICING RESET OPTIONS WITH *m* STRIKE RESETS AND *n* RESET DATES

We assume the dynamics of underlying asset price are described by the following stochastic differential equation:

$$dS(t) = uS(t)\,dt + \sigma S(t)\,dW_t \tag{4}$$

where u and $\sigma > 0$ are constants, and W_t is a one-dimensional standard Brownian motion defined in a filtered probability space (Ω, F, P). The money market account, $B(t)$, corresponds to the wealth accumulated from an initial \$1 investment at spot interest rate r in each subsequent period. Therefore,

$$dB(t) = rB(t)\,dt \tag{5}$$

or equivalently,

$$B(T) = B(t)e^{r(T-t)} \tag{6}$$

Let Q be the spot martingale measure with Radon-Nikodym derivative

$$\frac{dQ}{dP} = \exp\left(\frac{r-u}{\sigma}W_T - \frac{1}{2}\left(\frac{r-u}{\sigma}\right)^2 T\right) \tag{7}$$

Under the spot martingale measure or risk neutral probability measure, Q, the dynamics of the underlying asset price, $S(t)$, become

$$dS(t) = rS(t)\,dt + \sigma S(t)\,dW_t^Q \tag{8}$$

where the process W_t^Q is defined by

$$dW_t^Q = dW_t - \frac{r-u}{\sigma}dt \tag{9}$$

In view of (2) and (3), the payoff at expiry of the reset option with m strike resets and n predecided reset dates can be written as

$$\begin{aligned} C(T) = {} & [S(T) - K_0]^+ I(\underset{1\le j\le n}{\text{Min}} S(t_j) > D_1) \\ & + [S(T) - K_1]^+ \{I(\underset{1\le j\le n}{\text{Min}} S(t_j) > D_2) - I(\underset{1\le j\le n}{\text{Min}} S(t_j) > D_1)\} + \cdots \\ & + [S(T) - K_{m-1}]^+\{I(\underset{1\le j\le n}{\text{Min}} S(t_j) > D_m) - I(\underset{1\le j\le n}{\text{Min}} S(t_j) > D_{m-1})\} \\ & + [S(T) - K_m]^+\{1 - I(\underset{1\le j\le n}{\text{Min}} S(t_j) > D_m)\} \end{aligned} \tag{10}$$

where $I(\cdot)$ is an indicator function. Under the risk-neutral probability measure, Q, the arbitrage-free price of reset option $C(t)$ at time t is

$$\begin{aligned} C(t) &= e^{-r(T-t)} E_Q[C(T) \mid F_t] \\ &= e^{-r(T-t)} \sum_{l=1}^{m} E_Q\{[S(T) - K_{l-1}]^+ I(\underset{1\le j\le n}{\text{Min}} S(t_j) > D_l) \mid F_t\} \end{aligned}$$

$$- e^{-r(T-t)} \sum_{l=1}^{m} E_Q\{[S(T) - K_l]^+ I(\underset{1\le j\le n}{\text{Min}} S(t_j) > D_l) \mid F_t\}$$
$$+ e^{-r(T-t)} E_Q\{[S(T) - K_m]^+ \mid F_t\} \quad (11)$$

From (11), we know that the key to the solution is to compute the following expression:

$$e^{-r(T-t)} E_Q\{[S(T) - K_h]^+ I(\underset{1\le j\le n}{\text{Min}} S(t_j) > D_i) \mid F_t\} \quad (12)$$

We present the result in the following theorem.

Theorem: The explicit solution to (12) is as follows:

$$e^{-r(T-t)} E_Q\{[S(T) - K_h]^+ I(\underset{1\le j\le n}{\text{Min}} S(t_j) > D_i) \mid F_t\}$$
$$= \sum_{g=1}^{n} [S(t) N_{n+1}(D_g^{i,h}; \Sigma_g) - K_h e^{-r(T-t)} N_{n+1}(\hat{D}_g^{i,h}; \Sigma_g)] \quad (13)$$

where $N_{n+1}(\cdot; \Sigma)$ is the cumulative probability of an $(n+1)$-dimensional multivariate normal distribution with mean vector 0 and covariance matrix Σ. For $i,h = 1,\ldots, m$, the parameters in (13) are defined as follows:

$$D^{i,h} = \begin{bmatrix} d_{i,1} & e_{1,2} & \ldots & e_{1,n} & y_h \\ e_{2,1} & d_{i,2} & \ldots & e_{2,n} & y_h \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ e_{n,1} & e_{n,2} & \ldots & d_{i,n} & y_h \end{bmatrix} \quad (14)$$

and $D_j^{i,h}$ stands for the jth row of $D^{i,h}$;

$$d_{i,j} = \frac{\ln\left(\frac{S(t)}{D_i}\right) + \left(r + \frac{1}{2}\sigma^2\right)(t_j - t)}{\sigma\sqrt{t_j - t}} \quad (15)$$

$$e_{i,j} = \frac{\left(r + \frac{1}{2}\sigma^2\right)(t_j - t_i)}{\sigma\sqrt{|t_j - t_i|}} \quad (16)$$

$$y_h = \frac{\ln\left(\frac{S(t)}{K_h}\right) + \left(r + \frac{1}{2}\sigma^2\right)(T - t)}{\sigma\sqrt{T - t}} \quad (17)$$

$\hat{D}^{i,h}$ is similarly defined as $D^{i,h}$ with the parameters $d_{i,j}$, $e_{i,j}$, and y_h replaced by $\hat{d}_{i,j}$, $\hat{e}_{i,j}$, and $\hat{y}_h$, respectively:

$$\hat{d}_{i,j} = d_{i,j} - \sigma\sqrt{t_j - t} \tag{18}$$

$$\hat{e}_{i,j} = \frac{\left(r - \frac{1}{2}\sigma^2\right)(t_j - t_i)}{\sigma\sqrt{|t_j - t_i|}} \tag{19}$$

$$\hat{y}_h = y_h - \sigma\sqrt{T - t} \tag{20}$$

and the correlation matrix

$$\Sigma_g = \langle \rho_{ij}^g \rangle_{(n+1)\times(n+1)} \quad i, j = 1, \ldots, n + 1 \tag{21}$$

where ρ_{ij}^g is given by[1]

$$\rho_{ij}^g = \rho_{ji}^g = \begin{cases} 1, & i = j \\ \sqrt{\left|\dfrac{t_g - t_j}{t_g - t_i}\right|}, & 1 \le i < j \le g - 1 \text{ or } g + 1 \le i < j \le n \\ -\sqrt{\dfrac{t_g - t_i}{t_g - t}}, & 1 \le i \le g - 1, j = g \\ -\sqrt{\dfrac{t_g - t_i}{T - t}}, & 1 \le i \le g - 1, j = n + 1 \\ \sqrt{\dfrac{t_i - t_g}{T - t}}, & g + 1 \le i \le n, j = n + 1 \\ \sqrt{\dfrac{t_g - t}{T - t}}, & i = g, j = n + 1 \\ 0, & \text{otherwise} \end{cases} \tag{22}$$

We prove the theorem in Appendix A.

Accordingly, the closed-form solution for a reset option with m strike resets and n predecided reset dates $C(t)$ is

$$C(t) = S(t)\left\{N(y_m) + \sum_{l=1}^{m}\sum_{g=1}^{n}[N_{n+1}(D_g^{l,l-1};\ \Sigma_g) - N_{n+1}(D_g^{l,l};\ \Sigma_g)]\right\}$$

[1]Here we define $T = t_{n+1}$.

$$
\begin{aligned}
&-\sum_{l=1}^{m-1} K_l e^{-r(T-t)}\left\{\sum_{g=1}^{n}[N_{n+1}(\hat{D}_g^{l+1,l};\ \Sigma_g) - N_{n+1}(\hat{D}_g^{l,l};\ \Sigma_g)]\right\}\\
&- K_0 e^{-r(T-t)}\sum_{g=1}^{n} N_{n+1}(\hat{D}_g^{1,0};\ \Sigma_g)\\
&- K_m e^{-r(T-t)}\left[N(\hat{y}_m) - \sum_{g=1}^{n} N_{n+1}(\hat{D}_g^{m,m};\ \Sigma_g)\right] \qquad (23)
\end{aligned}
$$

where $N(\cdot)$ is the cumulative probability of the standard normal distribution.

In view of (23), we can replicate the reset option by borrowing M dollars and purchasing A shares of stock at price $S(t)$. The amount Δ and M are as follows:

$$
\Delta = N(y_m) + \sum_{l=1}^{m}\sum_{g=1}^{n}[N_{n+1}(D_g^{l,l-1},\ \Sigma_g) + N_{n+1}(D_g^{l,l},\ \Sigma_g)] \qquad (24)
$$

$$
\begin{aligned}
M = &\sum_{l=1}^{m-1} K_l e^{-r(T-t)}\left\{\sum_{g=1}^{n}[N_{n+1}(\hat{D}_g^{l,l-1},\ \Sigma_g) - N_{n+1}(\hat{D}_g^{l,l},\ \Sigma_g)]\right\}\\
&+ K_0 e^{-r(T-t)}\sum_{g=1}^{n} N_{n+1}(\hat{D}_g^{1,0},\ \Sigma_g)\\
&+ K_m e^{-r(T-t)}\left[N(\hat{y}_m) - \sum_{g=1}^{n} N_{n+1}(\hat{D}_g^{m,m},\ \Sigma_g)\right] \qquad (25)
\end{aligned}
$$

Similar to the closed-form valuations of exotic options, such as options on the maximum or minimum of several assets (Johnson, 1987), discrete partial barrier options (Heynen & Kat, 1996), reset options (Cheng & Zhang, 2000), or economic models with limited dependent variables, including multinomial probit, panel studies, spatial analysis, and time series analysis, the closed-form solutions for reset options involve the multivariate normal distribution functions.

Among the methods of evaluating multivariate normal cumulative probabilities, as pointed out by Gollwitzer and Rackwitz (1987), Deàk (1988), and Vijverberg (1997), Monte Carlo simulator methods seem to be the most promising for higher order probabilities, preferable over analytical approximations or numerical integration methods. Hajivassiliou, McFadden, and Ruud (1996) surveyed eleven Monte Carlo techniques of evaluating multivariate normal probabilities; they found that the Geweke-Hajivassiliou-Keane (GHK) simulator is the most reliable method overall. Consequently, for the closed-form solution for reset options with a large number of reset dates, we suggest using the GHK simulator to compute the multivariate normal cumulative probabilities.

In practice, reset derivatives are usually related to the arithmetic averages of stock prices in most financial markets. Consequently, we denote as the arithmetic average of stock prices at time t_j. Then, for an arithmetic average reset option with strike resets and predecided reset dates, the terminal payoff becomes

$$C(T) = [S(T) - K^*]^+ \tag{26}$$

where

$$K^* = \begin{cases} K_0 & \text{if Min } [A(t_1), \ldots, A(t_n)] > D_1 \\ K_i & \text{if } D_i \geq \text{Min } [A(t_1), \ldots, A(t_n)] > D_{i+1}, i = 1, \ldots, m-1 \\ K_m & \text{if } D_m \geq \text{Min } [A(t_1), \ldots, A(t_n)] \end{cases} \tag{27}$$

Because the sum of lognormal variables is not lognormal, and there is no recognizable probability distribution for it, there are no closed-form pricing formulas for the options based on the arithmetic average of asset values.

However, we can derive an approximated closed-form formula for the arithmetic average reset options by assuming that the arithmetic averages, $A(t_j)$, are approximately lognormally distributed. Using Wilkinson approximation, which is also used by Levy (1992) in pricing Asian options, we may estimate the mean and standard deviation of log $A(t_j)$ through the true first 2 moments of $A(t_j)$. Then, following the similar procedure in Appendix A, we can derive the closed-form formulas straightforwardly.[2]

ANALYSES OF RESET OPTIONS

Characteristics of Reset Options

First, we discuss some properties of reset options. Consider a 1-year maturity reset option with an initial strike price at 100. The strike price will be adjusted if the closing price of the underlying stock falls below 90% or 80% of the initial strike price. We will compare the prices of the

[2]The approximated closed-form formulas of arithmetic average reset options are available upon request.

reset options with two strike resets and one, two, and three reset dates to the plain vanilla call option. The results are presented in Table I.[3]

From Table I, we can see that some characteristics of reset options are similar to the standard European call option. For example, the values of reset options are increasing functions of stock price, risk-free interest rate, and the volatility of stock returns. In addition, there are four properties that uniquely exist in reset options. First, the values of reset options increase with the number of reset dates. Second, under the same strike resets, D_j, lower reset strike prices, K_j, will result in higher values of reset options. Third, due to more protection toward the holders of reset options, the values of reset options are always greater than that of standard European call option. Finally, in the case of higher values of stock prices than strike resets, and smaller volatility of stock returns, the difference between the prices of reset options and plain vanilla call options is insignificant. Take a stock price of 115 and a volatility of stock returns of 10% as an example. In this case, the price of the reset option and the plain vanilla call option are almost the same.

Reset Options with Continuous Reset Dates

When n approaches infinity with a remaining time to maturity $T - t$, the set of discrete reset dates becomes a continuous reset period. The terminal payoff of a reset option with continuous reset period is as follows:

$$C(T) = C_T^m + \sum_{l=1}^{m} C_T^{l-1} I(\underset{o \le t \le T}{\text{Min}} S(t) > D_l) - \sum_{l=1}^{m} C_T^{l} I(\underset{o \le t \le T}{\text{Min}} S(t) > D_l) \tag{28}$$

where

$$C_T^i = (S(T) - K_i)^+$$

In view of (28), we can replicate the reset option with the following strategy:

1. Purchase one European call option with strike price K_m.
2. Purchase one European down-and-out call option with strike price K_{i-1} and barrier D_i, for each $i = 1, \ldots, m$.
3. Short sell one European down-and-out call option with strike price K_i and barrier D_i, for each $i = 1, \ldots, m$.

[3]The computer codes in Matlab for computing the values of reset options in Table I and drawing Figures 1 and 2 are available on our Web site (http://140.119.79.103/liaosl/index.htm) under the filenames reset_p_new.m, Dleta_fig1.m, and Gamma_fig2.m, respectively.

TABLE I

Prices of Plain Vanilla Call Option ($n = 0$) and Reset Options with Two Strike Resets and Multiple Reset Dates ($n = 1, 2, 3$)

			$r = 0.05$							$r = 0.07$			
			n							n			
σ	$S(t)$	(K_1, K_2)	0	1	2	3	σ	$S(t)$	(K_1, K_2)	0	1	2	3
10%	85	(85,75)	0.58642	5.58703	5.88328	5.97312	10%	85	(85,75)	0.85289	6.65175	6.93881	7.18147
		(90,80)		3.00121	3.16958	3.31565			(90,80)		3.79178	3.96583	4.11063
		(95,85)		1.41284	1.48840	1.55139			(95,85)		1.90907	8.15692	2.05914
	100	(85,75)	6.80496	6.80555	6.82544	6.88005		100	(85,75)	8.13929	8.13981	8.15692	8.20321
		(90,80)		6.80527	6.81558	6.84304			(90,80)		8.13958	8.14881	8.17308
		(95,85)		6.80508	6.80881	6.81835			(95,85)		8.13941	8.14290	8.15173
	115	(85,75)	19.99328	19.99328	19.99328	19.99328		115	(85,75)	21.82797	21.82797	21.82797	21.82797
		(90,80)		19.99328	19.99328	19.99328			(90,80)		21.82797	21.82797	21.82797
		(95,85)		19.99328	19.99328	19.99328			(95,85)		21.82797	21.82797	21.82797
30%	85	(85,75)	6.41706	11.12159	12.11464	12.64265	30%	85	(85,75)	7.00369	11.86480	12.87902	13.41577
		(90,80)		9.45983	10.17486	10.55884			(90,80)		10.16190	10.89624	11.28895
		(95,85)		8.07483	8.55448	8.81710			(95,85)		8.73030	9.22543	9.49559
	100	(85,75)	14.23125	14.90918	15.70977	16.36859		100	(85,75)	15.21050	15.89146	16.69015	17.34352
		(90,80)		14.64791	15.15729	15.58753			(90,80)		15.63089	16.14127	16.56984
		(95,85)		14.42704	14.69268	14.93300			(95,85)		15.40883	15.67586	15.91605
	115	(85,75)	24.86422	24.87852	25.00677	25.23149		115	(85,75)	26.17698	26.19093	26.31543	26.53248
		(90,80)		24.87295	24.95177	25.09203			(90,80)		26.18553	26.26240	26.39846
		(95,85)		24.86822	24.90563	24.97589			(95,85)		26.18091	26.21754	26.28593
50%	85	(85,75)	13.15626	17.40265	18.40044	18.91034	50%	85	(85,75)	13.76753	18.07425	19.08303	19.59765
		(90,80)		16.11624	16.86578	17.25232			(90,80)		16.77442	17.53364	17.92452
		(95,85)		14.97846	15.50826	15.78613			(95,85)		15.62093	16.15847	16.43997
	100	(85,75)	21.79260	23.28866	24.40864	25.17150		100	(85,75)	22.63693	24.13794	25.25793	26.01920
		(90,80)		22.79338	23.58630	24.13648			(90,80)		23.64253	24.43672	24.98668
		(95,85)		22.35212	22.85544	23.21837			(95,85)		23.19976	23.70450	24.06779
	115	(85,75)	32.11914	32.40670	33.02638	33.63755		115	(85,75)	33.17280	33.45846	34.07209	34.37589
		(90,80)		32.30639	32.72852	33.15495			(90,80)		33.35910	33.77774	34.19969
		(95,85)		32.21672	32.46378	32.72754			(95,85)		33.27001	33.51529	33.77658

Note. Here, $S(t) = 100$, $K_0 = 100$, $D_1 = 90$, $D_2 = 80$, $t = 0$, $T = 1$. The reset dates are the last day of each month. n represents the number of reset dates, for example, $n = 3$ means that the reset dates are 1/12, 2/12, and 3/12.

Consequently, we can derive the pricing formulas of reset options with a continuous reset period by discovering the prices of down-and-out call options. Based on the closed-form solutions of European single-barrier options provided by Rubinstein and Reiner (1991),[4] we have

$$e^{-r(T-t)}E_Q\{[S(T)-K_j]^+ I(\min_{t\le u\le T} S(u) > D_{j+1}) \mid F_t\}$$
$$= \left\{S(t)\left[N(y_j) - \left(\frac{D_{j+1}}{S(t)}\right)^{2(r+0.5\,\sigma^2)/\sigma^2} N(f_1^{j+1,j})\right]\right.$$
$$\left. - K_h e^{-r(T-t)}\left[N(\hat{y}_j) - \left(\frac{D_{j+1}}{S(t)}\right)^{2(r-0.5\sigma^2)/\sigma^2} N(f_2^{j+1,j})\right]\right\} \qquad (29)$$

where

$$f_{1\,2}^{i,j} = \frac{\ln\left(\frac{D_i^2}{S(t)K_j}\right) + \left(r \pm \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}} \qquad (30)$$

Therefore, the price of a reset option with a continuous reset period is

$$C(t) = S(t)\left\{N(y_0) + \sum_{l=1}^{m}\left(\frac{D_l}{S(t)}\right)^{2(r+0.5\sigma^2)/\sigma^2}[N(f_1^{l,l}) + N(f_1^{l,l-1})]\right\}$$
$$- \sum_{l=1}^{m-1} K_l \frac{K_l}{S(t)} e^{-r(T-t)}\left[\left(\frac{D_l}{S(t)}\right)^{2(r-0.5\sigma^2)/\sigma^2} N(f_2^{l,l})\right.$$
$$\left. - \left(\frac{D_{l+1}}{S(t)}\right)^{2(r-0.5\sigma^2)/\sigma^2} N(f_2^{l+1,l})\right]$$
$$+ K_0 e^{-r(T-t)}\left(\frac{D_1}{S(t)}\right)^{2(r-0.5\sigma^2)/\sigma^2} N(f_2^{1,0})$$
$$- K_m e^{-r(T-t)}\left[N(\hat{y}_m) + \left(\frac{D_m}{S(t)}\right)^{2(r-0.5\sigma^2)/\sigma^2} N(f_2^{m,m})\right] \qquad (31)$$

Delta Jump and Gamma Jump

We now consider some important properties of reset options, such as Delta jump and Gamma jump. When reset options are issued, the issuers must hedge the risk exposure induced by the reset options. We provide the delta and gamma of reset options in Appendix B.

[4]See also Musiela and Rutkowski (1997), pp. 211–214.

To describe the phenomena of Delta jump and Gamma jump, without loss of generality, we simplify the reset options with only one reset date. Let us define the following expressions:

$$X(i, j) \equiv \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-d_{j,1}^2}{2}\right) N(G_{i,j}) \tag{32}$$

$$Y(i, j) = \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-y_i^2}{2}\right) N(Z_{i,j}) \tag{33}$$

$$DX(i, j) \equiv \frac{\exp\left(\frac{-d_{j,1}^2}{2}\right)}{\sigma S(t)\sqrt{2\pi}} \left[-\frac{N(G_{i,j})d_{j,1}}{\sqrt{t_1 - t}} + \frac{\exp\left(\frac{-G_{i,j}^2}{2}\right)}{\sqrt{2\pi(T-t)(1-\rho^2)}} - \frac{\rho \exp\left(\frac{-G_{i,j}^2}{2}\right)}{\sqrt{2\pi(t_1 - t)(1-\rho^2)}} \right] \tag{34}$$

$$DY(i, j) \equiv \frac{\exp\left(\frac{-y_i^2}{2}\right)}{\sigma S(t)\sqrt{2\pi}} \left[-\frac{N(Z_{i,j})y_i}{\sqrt{t_1 - t}} + \frac{\exp\left(\frac{-Z_{i,j}^2}{2}\right)}{\sqrt{2\pi(t_1-t)(1-\rho^2)}} - \frac{\rho \exp\left(\frac{-Z_{i,j}^2}{2}\right)}{\sqrt{2\pi(T - t)(1-\rho^2)}} \right] \tag{35}$$

where

$$G_{i,j} = \frac{y_i - \rho d_{j,1}}{\sqrt{1-\rho^2}}, \quad Z_{i,j} = \frac{d_{j,1} - \rho y_i}{\sqrt{1-\rho^2}}; \hat{X}, \hat{Y}, D\hat{X}, \text{ and } D\hat{Y}$$

are the expressions with $d_{i,j}$ and y_i replaced by $\hat{d}_{i,j}$ and $\hat{y}_i$, respectively. Thus, the Delta and Gamma of reset options with one reset date are as follows:

$$\begin{aligned}
&\text{Delta}(t, S(t)) \\
&= N(y_m) + \sum_{l=1}^{m} \Big\{ [N_2(d_{l,1}, y_{l-1}, \Sigma_1) - N_2(d_{l,1}, y_l, \Sigma_1)] \\
&\quad + \frac{1}{\sigma\sqrt{t_1 - t}}[X(l-1, l) - X(l, l)] + \frac{1}{\sigma\sqrt{T-t}}[Y(l-1, l) - Y(l, l)] \Big\} \\
&\quad - \sum_{l=1}^{m-1} \frac{K_l e^{-r(T-t)}}{S(t)} \left\{ \frac{1}{\sigma\sqrt{t_1 - t}}[\hat{X}(l, l+1) - \hat{X}(l, l)] + \frac{1}{\sigma\sqrt{T-t}}[\hat{Y}(l, l+1) - \hat{Y}(l, l)] \right\} \\
&\quad - \frac{K_0 e^{-r(T-t)}}{S(t)} \left[\frac{1}{\sigma\sqrt{t_1 - t}} \hat{X}(0, 1) + \frac{1}{\sigma\sqrt{T-t}} \hat{Y}(0, 1) \right] \\
&\quad + \frac{K_m e^{-r(T-t)}}{S(t)} \left[\frac{1}{\sigma\sqrt{t_1 - t}} \hat{X}(m, m) + \frac{1}{\sigma\sqrt{T-t}} \hat{Y}(m, m) \right]
\end{aligned} \tag{36}$$

$$
\begin{aligned}
&\text{Gamma}(t, S(t)) \\
&= \frac{\exp\left(\frac{-y_m^2}{2}\right)}{S(t)\sigma\sqrt{2\pi(T-t)}} \\
&\quad + \sum_{l=1}^{m} \frac{1}{S(t)\sigma}\left\{\frac{1}{\sqrt{t_1-t}}[X(l-1,l) - X(l,l)] + \frac{1}{\sqrt{T-t}}[Y(l-1,l) - Y(l,l)]\right\} \\
&\quad + \sum_{l=1}^{m} \frac{1}{\sigma}\left\{\frac{1}{\sqrt{t_1-t}}[DX(l-1,l) - DX(l,l)] + \frac{1}{\sqrt{T-t}}[DY(l-1,l) - DY(l,l)]\right\} \\
&\quad + \sum_{l=1}^{m-1} \frac{K_l e^{-r(T-t)}}{\sigma S(t)^2}\left\{\frac{1}{\sqrt{t_1-t}}[\hat{X}(l,l+1) - \hat{X}(l,l)] + \frac{1}{\sqrt{T-t}}[\hat{Y}(l,l+1) - \hat{Y}(l,l)]\right. \\
&\quad \left. - \frac{S(t)}{\sqrt{t_1-t}}[D\hat{X}(l,l+1) - D\hat{X}(l,l)] - \frac{S(t)}{\sqrt{T-t}}[D\hat{Y}(l,l+1) - D\hat{Y}(l,l)]\right\} \\
&\quad + \frac{K_0 e^{-r(T-t)}}{\sigma S(t)^2}\left[\frac{1}{\sqrt{t_1-t}}\hat{X}(0,1) + \frac{1}{\sqrt{T-t}}\hat{Y}(0,1) - \frac{S(t)}{\sqrt{t_1-t}}D\hat{X}(0,1)\right. \\
&\quad \left. - \frac{S(t)}{\sqrt{T-t}}D\hat{Y}(0,1)\right] - \frac{K_m e^{-r(T-t)}}{\sigma S(t)^2}\left[\frac{1}{\sqrt{t_1-t}}\hat{X}(m,m)\right. \\
&\quad \left. + \frac{1}{\sqrt{T-t}}\hat{Y}(m,m) - \frac{S(t)}{\sqrt{t_1-t}}D\hat{X}(m,m) - \frac{S(t)}{\sqrt{T-t}}D\hat{Y}(m,m)\right] \qquad (37)
\end{aligned}
$$

where t_1 is the reset date.

When $t \to t_1$, then

$$
N_2(d_{l,1}, y_{l-1}, \Sigma_1) \to N(y_{l-1}),\ X(i,j) \to 0, \quad DX(i,j) \to 0, \quad Y(i,j) \to \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-y_j^2}{2}\right)\Bigg|_{t=t_1}
$$

and

$$
DY(i,j) \to \frac{1}{\sigma S(t)\sqrt{2\pi(T-t_1)}} \exp\left(\frac{-y_j^2}{2}\right)\Bigg|_{t=t_1}
$$

Consequently, the Delta and Gamma at time t_1 are as follows:

$$
\text{Delta}(t_1, S(t_1)) = N(y_0)|_{t=t_1} \qquad (38)
$$

$$
\text{Gamma}(t_1, S(t_1)) = \frac{\exp\left(\frac{-y_0^2}{2}\right)}{S(t)\sigma\sqrt{2\pi(T-t)}}\Bigg|_{t=t_1} \qquad (39)
$$

However, the delta and gamma at $t > t_1$ are given by the following expressions:

$$\text{Delta}(t, S(t)) = N(y_0)I[S(t_1) > D_1] + \sum_{g=1}^{m-1} N(y_g)I[D_g \geq S(t_1) > D_{g+1}] + N(y_m)I[D_m \geq S(t_1)] \tag{40}$$

$$\text{Gamma}(t, S(t)) = \frac{\exp\left(\frac{-y_0^2}{2}\right)}{S(t)\sigma\sqrt{2\pi(T-t)}} I[S(t_1) > D_1] + \sum_{g=1}^{m-1} \frac{\exp\left(\frac{-y_g^2}{2}\right)}{S(t)\sigma\sqrt{2\pi(T-t)}} I[D_g \geq S(t_1) > D_{g+1}] + \frac{\exp\left(\frac{-y_m^2}{2}\right)}{S(t)\sigma\sqrt{2\pi(T-t)}} I[D_m \geq S(t_1)] \tag{41}$$

From (38) to (41), we can see that the Delta and Gamma at t_1 are continuous only when the condition $S(t_1) \geq D_1$ holds. Therefore, Delta jump and Gamma jump exist when the stock price at t_1 is below D_1. In other words, we should carefully implement the Delta and Gamma hedges on the reset dates when the stock price is below the highest strike reset.

Delta Waviness and Gamma Waviness

In addition to the properties of Delta jump and Gamma jump on the reset dates, there exist the phenomena of Delta waviness and Gamma waviness before the reset dates, especially near the reset dates. Consider the following example. The stock price is currently \$100, and the strike price of the reset option will be adjusted if the stock price falls below 80%, 70%, 60%, 50%, and 40% of the initial strike price of \$100 three months later. Assume the risk-free interest rate is 5% and the volatility of stock returns is 30%. We illustrate the properties of Delta waviness and Gamma waviness in Figures 1 and 2, respectively. As shown in the figures, unlike the Delta and Gamma of the plain vanilla call options, which are definitely non-negative, the Delta and Gamma of reset options will fluctuate dramatically, and can be negative as the time approaches the reset dates. When the stock prices are away from the neighborhoods of strike resets, the behaviors of Delta and Gamma are the same as that of plain vanilla

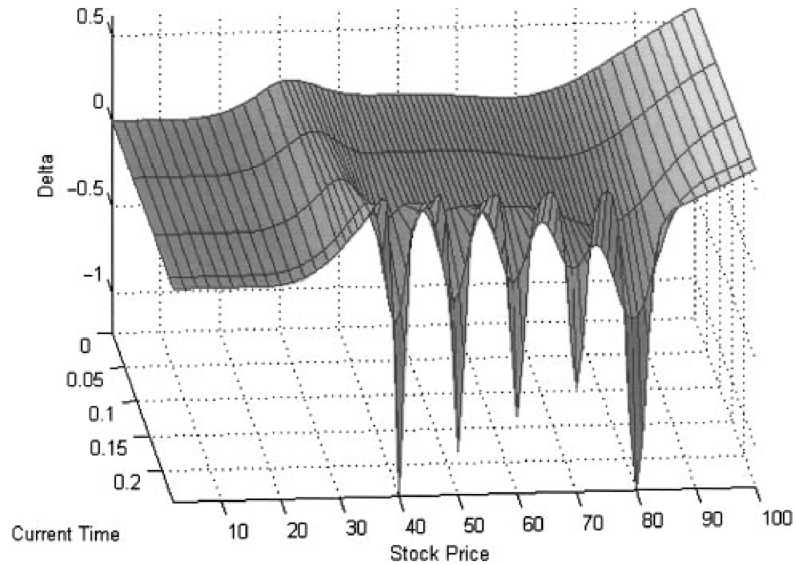

FIGURE 1

Delta of reset option with five strike resets. Here, $S(t) = 100$, $K_0 = 100$, $[K_1, \dots K_5] = [80, 70, 60, 50, 40]$, $[D_1, \dots D_5] = [80, 70, 60, 50, 40]$, $r = 0.05$, and $\nu = 0.03$. Unlike the Delta of the plain vanilla call option, which is definitely non-negative, the Delta of the reset call option will fluctuate dramatically and may be negative as time approaches the reset dates. The Deltas are local minimums when the stock price touches strike resets, but the Deltas are local maximums when the stock price is at about the middle of two adjacent strike resets. The phenomenon of Delta waviness is more significant as time approaches the reset dates.

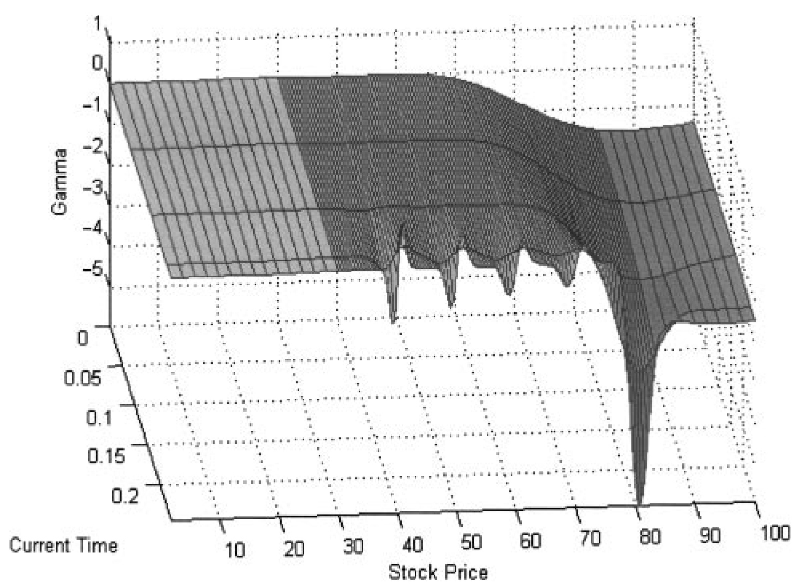

FIGURE 2

Gamma of reset option with five strike resets. Here, $S(t) = 100$, $K_0 = 100$, $[K_1, \dots K_5] = [80, 70, 60, 50, 40]$, $[D_1, \dots D_5] = [80, 70, 60, 50, 40]$, $r = 0.05$, and $\nu = 0.03$. When the stock price is away from the neighborhoods of strike resets, the behavior of Gamma is the same as that of plain vanilla call options. However, if the stock price is near strike resets, the Gamma oscillates across the strike resets. The phenomenon of Gamma waviness is more significant when time approaches the reset dates.

call options. However, if the stock prices are near strike resets, the Delta and Gamma will oscillate. The phenomena are more significant when the time approaches the reset dates. From Figure 1, if the time approaches the reset dates, the Delta is a local minimum when the stock price touches strike reset, but the Delta is a local maximum when the stock price is at about the middle of two adjacent strike resets. The dramatic change of Delta between two adjacent strike resets also increases the difficulty of risk management. The wavinesses of Delta and Gamma are as important as Delta jump and Gamma jump in hedging reset options.

CONCLUSION

We have provided the closed-form pricing formula for reset options with strike resets and predecided reset dates. In addition to Delta jump and Gamma jump across the reset dates, we have also discovered the phenomena of Delta waviness and Gamma waviness near the reset dates. For future research, it would be interesting to investigate the hedging strategies of reset options due to the phenomena of Delta jump and Delta waviness across reset dates.

APPENDIX A

Proof of the Theorem

To carry out the proof of Theorem 1, we divide (12) into two parts:

$$e^{-r(T-t)}E_Q\{[S(T) - K_h]^+ I(\min_{j=1,n} S(t_j) > D_i) \mid F_t\} = A - B \qquad \text{(A.1)}$$

where[5]

$$A = e^{-r(T-t)} \sum_{g=1}^{n} E_Q\{S(T)I(S(t_g) > D_i, S(T) \geq K_h, S(t_j) \geq S(t_g), j \neq g, j = 1,\ldots, n) \mid F_t\} \qquad \text{(A.2)}$$

$$B = K_h e^{-r(T-t)} \sum_{g=1}^{n} E_Q\{I(S(t_g) > D_i, S(T) \geq K_h, S(t_j) \geq S(t_g), j \neq g, j = 1,\ldots, n) \mid F_t\} \qquad \text{(A.3)}$$

Under the spot martingale measure Q, the stock price at time t_j equals

$$S(t_j) = S(t)\, \exp\left[\left(r - \frac{1}{2}\sigma^2\right)(t_j - t) + \sigma(W_{t_j}^Q - W_t^Q)\right] \qquad \text{(A.4)}$$

[5]We can represent the minimum of several assets with the expression in (A.2). For details, see Johnson (1987).

It is convenient to introduce an auxiliary probability measure, P_R, on (Ω, F) by setting its Radon-Nikodym derivative as follows:

$$\frac{dP_R}{dQ} = \exp\left[\sigma W_T^Q - \frac{1}{2}\sigma^2 T\right] \tag{A.5}$$

By Girsanov's theorem, W_t^R, defined by

$$dW_t^R = dW_t^Q - \sigma\, dt \tag{A.6}$$

is a standard Brownian motion under the measure P_R. Then we can rewrite (A.2) as follows:

$$\begin{aligned}
A &= S(t)\sum_{g=1}^{n} P_R\{(S(t_g) > D_i,\ S(T) \geq K_h,\ S(t_j) \geq S(t_g),\ j \neq g,\ j = 1,\ldots,n)|F_t\} \\
&= S(t)\sum_{g=1}^{n} P_R\left[\frac{\ln S(t) - \ln D_i + \left(r + \frac{1}{2}\sigma^2\right)(t_g - t)}{\sigma\sqrt{t_g - t}} \geq \frac{-(W_{t_g}^R - W_t^R)}{\sqrt{t_g - t}},\right. \\
&\qquad \frac{\ln S(t) - \ln K_h + \left(r + \frac{1}{2}\sigma^2\right)(T - t)}{\sigma\sqrt{T - t}} \geq \frac{-(W_T^R - W_t^R)}{\sqrt{T - t}}, \\
&\qquad \left.\frac{\left(r + \frac{1}{2}\sigma^2\right)(t_j - t_g)}{\sigma\sqrt{|t_j - t_g|}} \geq \frac{-(W_{t_j}^R - W_{t_g}^R)}{\sqrt{|t_j - t_g|}},\ j \neq g,\ j = 1,\ldots,n\right]
\end{aligned} \tag{A.7}$$

Here, we use the fact that $W_s^R - W_t^R$ is normally distributed with mean 0 and variance $(s - t)$, and is independent of F_t. Therefore, we have

$$\begin{aligned}
A = S(t)\sum_{g=1}^{n} P_R[&e_{g,1} \geq Z_1,\ldots, e_{g,g-1} \geq Z_{g-1},\ d_{i,g} \geq Z_g,\ e_{g,g+1} \\
&\geq Z_{g+1},\ldots, e_{g,n} \geq Z_n,\ y_h \geq Z_{n+1}]
\end{aligned} \tag{A.8}$$

where Z_is are

$$\begin{aligned}
&[Z_1,\ldots, Z_{n+1}] \\
&= \left[\frac{(W_{t_g}^R - W_{t_1}^R)}{\sqrt{t_g - t_1}},\ldots, -\frac{(W_{t_g}^R - W_t^R)}{\sqrt{t_g - t}},\ldots, -\frac{(W_{t_n}^R - W_{t_g}^R)}{\sqrt{t_n - t_g}}, -\frac{(W_T^R - W_t^R)}{\sqrt{T - t}}\right]
\end{aligned} \tag{A. 9}$$

Consequently, taking $1 \leq i < j \leq g$ as an example, we have

$$\rho_{i,j}^g = E[Z_i Z_j] = E\left[\frac{(W_{t_g}^R - W_{t_i}^R)}{\sqrt{t_g - t_i}}\frac{(W_{t_g}^R - W_{t_j}^R)}{\sqrt{t_g - t_j}}\right] = \sqrt{\frac{t_g - t_j}{t_g - t_i}} \tag{A.10}$$

We can repeat the above method to obtain covariance matrix Σ_g as in (22). Therefore, the solution for A is

$$\begin{aligned} & S(t) \sum_{g=1}^{n} N_{n+1}[e_{g,1}, \ldots, e_{g,g-1}, d_{i,g}, e_{g,g+1}, \ldots, e_{g,n}, y_h, \Sigma_g] \\ & = S(t) \sum_{g=1}^{n} N_{n+1}[D_g^{i,h}, \Sigma_g] \qquad \text{(A.11)} \end{aligned}$$

Similarly, (A.3) can be computed by the same technique. This completes the proof of the theorem.

APPENDIX B

Delta and Gamma of Reset Options

To derive the Delta and Gamma of reset options, we apply the chain rule of differentiation:

$$\frac{\partial N_{n+1}(D_g^{i,h},\ \Sigma_g)}{\partial S(t)} = \frac{\partial N_{n+1}(D_g^{i,h},\ \Sigma_g)}{\partial d_{i,g}} \frac{\partial d_{i,g}}{\partial S(t)} + \frac{\partial N_{n+1}(D_g^{i,h},\ \Sigma_g)}{\partial y_h} \frac{\partial y_h}{\partial S(t)} \qquad \text{(B.1)}$$

where

$$\frac{\partial d_{i,g}}{\partial S(t)} = \frac{1}{\sigma S(t)\sqrt{t_g - t}}, \quad \frac{\partial y_h}{\partial S(t)} = \frac{1}{\sigma S(t)\sqrt{T - t}} \qquad \text{(B.2)}$$

We then have the Delta and Gamma of reset options with m strike resets and n predecided reset dates as follows:

$$\begin{aligned} & \text{Delta}(t, S(t)) \\ & = N(y_m) + \sum_{l=1}^{m} \sum_{g=1}^{n} [N_{n+1}(D_g^{l,l-1}, \Sigma_g) - N_{n+1}(D_g^{l,l}, \Sigma_g)] \\ & \quad + \sum_{l=1}^{m} \sum_{g=1}^{n} \left\{ \frac{1}{\sigma\sqrt{t_g - t}} \left[\frac{\partial N_{n+1}(D_g^{l,l-1}, \Sigma_g)}{\partial d_{l,g}} - \frac{\partial N_{n+1}(D_g^{l,l}, \Sigma_g)}{\partial d_{l,g}} \right] \right. \\ & \quad \left. + \frac{1}{\sigma\sqrt{T - t}} \left[\frac{\partial N_{n+1}(D_g^{l,l-1}, \Sigma_g)}{\partial y_{l-1}} - \frac{\partial N_{n+1}(D_g^{l,l}, \Sigma_g)}{\partial y_l} \right] \right\} \\ & \quad - \sum_{l=1}^{m-1} \sum_{g=1}^{n} \frac{K_l e^{-r(T-t)}}{S(t)} \left\{ \frac{1}{\sigma\sqrt{t_g - t}} \left[\frac{\partial N_{n+1}(\hat{D}_g^{l+1,l}, \Sigma_g)}{\partial \hat{d}_{l+1,g}} - \frac{\partial N_{n+1}(\hat{D}_g^{l,l}, \Sigma_g)}{\partial \hat{d}_{l,g}} \right] \right. \end{aligned}$$

$$+\frac{1}{\sigma\sqrt{T-t}}\left[\frac{\partial N_{n+1}(\hat{D}_g^{l+1,l},\Sigma_g)}{\partial\hat{y}_l}-\frac{\partial N_{n+1}(\hat{D}_g^{l,l},\Sigma_g)}{\partial\hat{y}_l}\right]\Bigg\}$$

$$-\frac{K_0e^{-r(T-t)}}{S(t)}\left\{\sum_{g=1}^{n}\left[\frac{1}{\sigma\sqrt{t_g-t}}\frac{\partial N_{n+1}(\hat{D}_g^{1,0},\ \Sigma_g)}{\partial\hat{d}_{1,g}}+\frac{1}{\sigma\sqrt{T-t}}\frac{\partial N_{n+1}(\hat{D}_g^{1,0},\ \Sigma_g)}{\partial\hat{y}_0}\right]\right\}$$

$$+\frac{K_me^{-r(T-t)}}{S(t)}\left\{\sum_{g=1}^{n}\left[\frac{1}{\sigma\sqrt{t_g-t}}\frac{\partial N_{n+1}(\hat{D}_g^{m,m},\Sigma_g)}{\partial\hat{d}_{m,g}}+\frac{1}{\sigma\sqrt{T-t}}\frac{\partial N_{n+1}(\hat{D}_g^{m,m},\Sigma_g)}{\partial\hat{y}_m}\right]\right\}\tag{B.3}$$

$\text{Gamma}(t, S(t))$

$$=\frac{1}{\sigma\sqrt{2\pi(T-t)}}e^{-(1/2)y_m^2}$$

$$+\frac{1}{\sigma S(t)}\sum_{l=1}^{m}\sum_{g=1}^{n}\left\{\frac{1}{\sqrt{t_g-t}}\left[\frac{\partial N_{n+1}(D_g^{l,l-1},\ \Sigma_g)}{\partial d_{l,g}}-\frac{\partial N_{n+1}(D_g^{l,l},\ \Sigma_g)}{\partial d_{l,g}}\right]\right.$$

$$\left.+\frac{1}{\sqrt{T-t}}\left[\frac{\partial N_{n+1}(D_g^{l,l-1},\ \Sigma_g)}{\partial y_{l-1}}-\frac{\partial N_{n+1}(D_g^{l,l},\ \Sigma_g)}{\partial y_l}\right]\right\}$$

$$+\sum_{l=1}^{m}\sum_{g=1}^{n}[A(l,l-1,g)+B(l,l-1,g)]$$

$$-\sum_{l=1}^{m-1}\sum_{g=1}^{n}\frac{K_le^{-r(T-t)}}{S(t)}\sum_{l=1}^{m}\sum_{g=1}^{n}[\hat{A}(l+1,l,g)+\hat{B}(l+1,l,g)]$$

$$-\frac{K_0e^{-r(T-t)}}{\sigma^2S(t)^2}\sum_{g=1}^{n}\left\{\frac{1}{\sqrt{t_g-t}}\left[\frac{\partial^2N_{n+1}(\hat{D}_g^{1,0},\Sigma_g)}{\partial\hat{d}_{1,g}^2}\frac{1}{\sqrt{t_g-t}}+\frac{\partial^2N_{n+1}(\hat{D}_g^{1,0},\Sigma_g)}{\partial\hat{d}_{1,g}\partial\hat{y}_0}\frac{1}{\sqrt{T-t}}\right]\right.$$

$$\left.+\frac{1}{\sqrt{T-t}}\left[\frac{\partial^2N_{n+1}(\hat{D}_g^{1,0},\Sigma_g)}{\partial\hat{d}_{1,g}\partial\hat{y}_0}\frac{1}{\sqrt{t_g-t}}+\frac{\partial^2N_{n+1}(\hat{D}_g^{1,0},\Sigma_g)}{\partial\hat{y}_0^2}\frac{1}{\sqrt{T-t}}\right]\right\}$$

$$+\frac{K_me^{-r(T-t)}}{\sigma^2S(t)^2}\sum_{g=1}^{n}\left\{\frac{1}{\sqrt{t_g-t}}\left[\frac{\partial^2N_{n+1}(\hat{D}_g^{m,m},\Sigma_g)}{\partial\hat{d}_{m,g}^2}\frac{1}{\sqrt{t_g-t}}+\frac{\partial^2N_{n+1}(\hat{D}_g^{m,m},\Sigma_g)}{\partial\hat{d}_{m,g}\partial\hat{y}_m}\frac{1}{\sqrt{T-t}}\right],\right.$$

$$\left.+\frac{1}{\sqrt{T-t}}\left[\frac{\partial^2N_{n+1}(\hat{D}_g^{m,m},\Sigma_g)}{\partial\hat{d}_{m,g}\partial\hat{y}_m}\frac{1}{\sqrt{t_g-t}}+\frac{\partial^2N_{n+1}(\hat{D}_g^{m,m},\Sigma_g)}{\partial\hat{y}_m^2}\frac{1}{\sqrt{T-t}}\right]\right\}$$

$$+\sum_{l=1}^{m-1}\sum_{g=1}^{n}\frac{K_le^{-r(T-t)}}{S(t)^2}\left\{\frac{1}{\sigma\sqrt{t_g-t}}\left[\frac{\partial N_{n+1}(\hat{D}_g^{l+1,l},\Sigma_g)}{\partial\hat{d}_{l+1,g}}-\frac{\partial N_{n+1}(\hat{D}_g^{l,l},\Sigma_g)}{\partial\hat{d}_{l,g}}\right]\right.$$

$$\left.+\frac{1}{\sigma\sqrt{T-t}}\left[\frac{\partial N_{n+1}(\hat{D}_g^{l+1,l},\Sigma_g)}{\partial\hat{y}_l}-\frac{\partial N_{n+1}(\hat{D}_g^{l,l},\Sigma_g)}{\partial\hat{y}_l}\right]\right\}$$

$$+\frac{K_0 e^{-r(T-t)}}{S(t)^2}\left\{\sum_{g=1}^{n}\left[\frac{1}{\sigma\sqrt{t_g-t}}\frac{\partial N_{n+1}(\hat{D}_g^{1,0},\Sigma_g)}{\partial \hat{d}_{1,g}}+\frac{1}{\sigma\sqrt{T-t}}\frac{\partial N_{n+1}(\hat{D}_g^{1,0},\Sigma_g)}{\partial \hat{y}_0}\right]\right\}$$
$$-\frac{K_m e^{-r(T-t)}}{S(t)^2}\left\{\sum_{g=1}^{n}\left[\frac{1}{\sigma\sqrt{t_g-t}}\frac{\partial N_{n+1}(\hat{D}_g^{m,m},\Sigma_g)}{\partial \hat{d}_{m,g}}+\frac{1}{\sigma\sqrt{T-t}}\frac{\partial N_{n+1}(\hat{D}_g^{m,m},\Sigma_g)}{\partial \hat{y}_m}\right]\right\} \tag{B.4}$$

where

$$A(i,j,k)=\frac{1}{S(t)\sigma^2\sqrt{t_k-t}}\left\{\frac{1}{\sqrt{t_k-t}}\left[\frac{\partial^2 N_{n+1}(D_k^{i,i-1},\Sigma_k)}{\partial \hat{d}_{i,k}^2}-\frac{\partial^2 N_{n+1}(D_k^{j,j},\Sigma_k)}{\partial \hat{d}_{j,k}^2}\right]\right.$$
$$\left.+\frac{1}{\sqrt{T-t}}\left[\frac{\partial^2 N_{n+1}(D_k^{i,i-1},\Sigma_k)}{\partial d_{i,k}\,\partial \hat{y}_{i-1}}-\frac{\partial^2 N_{n+1}(D_k^{j,j},\Sigma_k)}{\partial d_{j,k}\partial \hat{y}_j}\right]\right\} \tag{B.5}$$

$$B(i,j,k)=\frac{1}{S(t)\sigma^2\sqrt{T-t}}\left\{\frac{1}{\sqrt{t_k-t}}\left[\frac{\partial^2 N_{n+1}(D_k^{i,i-1},\Sigma_k)}{\partial d_{i,k}\,\partial y_{i-1}}-\frac{\partial^2 N_{n+1}(D_k^{j,j},\Sigma_k)}{\partial y_j\,\partial d_{j,k}}\right]\right.$$
$$\left.+\frac{1}{\sigma\sqrt{T-t}}\left[\frac{\partial^2 N_{n+1}(D_k^{i,i-1},\Sigma_k)}{\partial \hat{y}_{i-1}^2}-\frac{\partial^2 N_{n+1}(D_k^{j,j},\Sigma_k)}{\partial \hat{y}_j^2}\right]\right\} \tag{B.6}$$

and $\hat{A}(i,j,k)$ and $\hat{B}(i,j,k)$ are similar to $A(i,j,k)$ and $B(i,k,j)$ with the parameters $d_{i,j}$ and y_i replaced by $\hat{d}_{i,j}$ and $\hat{y}_i$, respectively.

By observing (B.3) and (B.4), we see the key elements for computing the hedge ratio are $\partial N_{n+1}(D_g^{i,h},\Sigma_g)/\partial d_{i,g}$, $\partial^2 N_{n+1}(D_g^{i,h},\Sigma_g)/\partial d_{i,g}\partial y_h$ and $\partial^2 N_{n+1}(D_g^{i,h},\Sigma_g)/\partial d_{i,g}^2$. To derive the derivatives, as Curnow and Dunnett (1961) pointed out,[6] we have

$$N_{n+1}(D_g^{i,h},\ \Sigma_g)$$
$$=\int_{-\infty}^{d_{i,g}} N_n\left[\frac{e_{g,j}-\rho_{jg}^g x}{\sqrt{1-(\rho_{jg}^g)^2}},\ 1\le j\le n, j\ne g,\ \frac{y_h-\rho_{n+1g}^g x}{\sqrt{1-(\rho_{n+1g}^g)^2}},\ \langle\rho_{qk\cdot g}^g\rangle_{n\times n}\right] f(x)\,dx \tag{B.7}$$

where

$$\rho_{qk\cdot g}^g=\frac{\rho_{qk}^g-\rho_{qg}^g\,\rho_{kg}^g}{\sqrt{1-(\rho_{qg}^g)^2}\sqrt{1-(\rho_{kg}^g)^2}},\ 1\le q,k\le n+1,\ q,k\ne g \tag{B.8}$$

[6]A similar technique is also used to study the hedge ratio of discrete barrier options by Wei (1998).

and $f(\cdot)$ is the standard normal probability density function. Hence, $\partial N_{n+1}(D_g^{i,h}, \Sigma_g)/\partial d_{i,g}$ can be calculated as follows:

$$\frac{\partial N_{n+1}(D_g^{i,h},\ \Sigma_g)}{\partial d_{i,g}}$$

$$= N_n\left[\frac{e_{g,j} - \rho_{jg}^g d_{i,g}}{\sqrt{1 - (\rho_{jg}^g)^2}},\ 1 \leq j \leq n, j \neq g,\ \frac{y_h - \rho_{n+1g}^g d_{i,g}}{\sqrt{1 - (\rho_{n+1g}^g)^2}},\ \langle \rho_{qk\cdot g}^g \rangle_{n\times n}\right] f(d_{i,g}) \quad \text{(B.9)}$$

Then, following a similar procedure, we can straightforwardly obtain the derivatives $\partial^2 N_{n+1}(D_g^{i,h}, \Sigma_g)/\partial d_{i,g}\partial y_h$ and $\partial^2 N_{n+1}(D_g^{i,h}, \Sigma_g)/\partial d_{i,g}^2$.

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