THE VALUATION OF EUROPEAN OPTIONS WHEN ASSET RETURNS ARE AUTOCORRELATED

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This article derives the closed-form formula for a European option on an asset with returns following a continuous-time type of first-order moving average process, which is called an MA(1)-type option. The pricing formula of these options is similar to that of Black and Scholes, except for the total volatility input. Specifically, the total volatility input of MA(1)-type options is the conditional standard deviation of continuous-compounded returns over the option's remaining life, whereas the total volatility input of Black and Scholes is indeed the diffusion coefficient of a geometric Brownian motion times the square root of an option's time to maturity. Based on the result of numerical analyses, the impact of autocorrelation induced by the MA(1)-type process is significant to option values even when the autocorrelation between asset returns is weak. © 2006 Wiley Periodicals, Inc. Jrl Fut Mark 26:85–102, 2006

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INTRODUCTION

As pointed out in Lo and Wang (1995), there is now a substantial body of evidence that documents the predictability of financial asset returns. In addition to the mean-reverting model, the moving-average process is one popular model to describe predictable financial asset returns. To capture the autocorrelation of financial asset returns, many empirical studies extract the autocorrelation from the asset returns' first moment through the form of a first-order moving average process [MA(1) process], including Hamao, Masulis, and Ng (1990), Bollerslev (1987), and French, Schwert, and Stambaugh (1987), to name a few.

It is also well known that the value of an option may depend on the underlying asset's log-price dynamics. The famous Black-Scholes model assumes that the stock price process is a geometric Brownian motion, which implies stock returns are independent. In distinguishing between the risk-neutral and true distributions of an option's underlying asset return process, Grundy (1991, p. 1049) observes that the Black-Scholes formula still holds even though the underlying asset returns follow an Ornstein-Uhlenbeck process. Along this line of research, Lo and Wang (1995) claim that the unconditional variance of returns is usually fixed for any given set of data irrespective of predictability. Accordingly, when one implements pricing formulas of options on assets with predictable returns, the values of the pricing formulas' parameters should be adjusted to fit the unconditional moments of returns. Based on the preceding assertion and Grundy (1991), Lo and Wang (1995) further price options on an asset with the trending Ornstein-Uhlenbeck process (trending O-U process) by using the Black-Scholes formula with an adjustment for predictability. Apparently, before implementing an option pricing formula with the approach of Lo and Wang (1995), the pricing formula for the used model should be known in advance.

As shown in Lo and Wang (1995), an important result of the arbitrage-free methods for pricing derivatives is: As long as the underlying asset's log-price dynamics are described by an Itô diffusion process with a constant diffusion coefficient, the Black-Scholes formula yields the correct option price regardless of the specification and arguments of the drift. However, no studies exist concerning whether the Black-Scholes formula still holds when the underlying asset returns are described by an MA(1) process. Thus, the main objective here is to fill the gap by introducing a continuous-time MA(1)-type process, which is consistent with the findings in empirical studies, and to price European options on an asset with the process by using the martingale pricing method.

The underlying asset's log-price dynamics in this article are similar to a special case of the discrete-time model used in Jokivuolle (1998). Specifically, Jokivuolle (1998) values European options on autocorrelated indexes, where the *observed* index returns determining the option's terminal payoff are modeled as an infinite-order moving average process, whereas the *true* index returns are specified to be a random walk with drift. However, unlike Jokivuolle (1998), this article assumes that the process of observed returns is an equilibrium price process in continuous time. This setting is based on the common assumption of the martingale pricing method and is more in line with Lo and Wang (1995).

One contribution of the article is to price European options on an asset with continuous-time MA(1)-type dynamics [MA(1)-type options] by the martingale method. As a result, it is found that the pricing formula of MA(1)-type options is not identical to that of Black and Scholes. Accordingly, numerical analyses are conducted herein to gauge the impact of autocorrelation induced by the MA(1)-type process on option values.

The remaining parts of this article are organized as follows. The setting of a continuous-time process for autocorrelated asset returns considered in this article is shown. The pricing formula and the hedge for the MA(1)-type options are illustrated. Results of numerical analyses and a conclusion end the article.

THE SETTING: A CONTINUOUS-TIME PROCESS OF AUTOCORRELATED ASSET RETURNS

Without loss of generality, this article considers the underlying asset to be a stock and denotes the underlying stock price including dividends as S. The current time is t_0, the expiration date of the options considered here is T, and the time to maturity is τ, where $\tau = T - t_0$ and $\tau > 0$. As the stock returns of a first-order moving average process are a common finding in empirical research studies, this article introduces a continuous-time MA(1) process [MA(1)-type process] and assumes the dynamics of the stock price for all $t_0 \leq t \leq T$ as follows:

$$\frac{dS_t}{S_t} = \mu\, dt + \sigma\, dW_t + \beta\sigma\, dW_{t-h} \tag{1}$$

where μ is a constant expected appreciation rate of the stock price, $\sigma > 0$ is a constant volatility coefficient, $dt > 0$ is an infinitesimal time interval, and $h > 0$ is a fixed, but arbitrary, small constant. The coefficient β represents the impact of the past shock, which is assumed to satisfy $|\beta| < 1$.

As the condition of $|\beta| < 1$ is a standard assumption for a discrete-time invertible MA(1) representation, there is no loss of generality when imposing the assumption here. In addition, W_t is a one-dimensional standard Brownian motion defined on a naturally filtered probability space $(\Omega, F, P, (\mathcal{F}_t)_{t\in[0,T]})$ and dW_{t-i}, $i = 0, h$, are the increments of the standard Brownian motion at time $t - i$. In empirical works, h is restricted by the frequency of historical data.

The dynamics of stock prices in (1) are equivalent to the following Itô integral equation:

$$S_t = S_{t_0} + \int_{t_0}^{t} \mu S_u \, du + \int_{t_0}^{t} \sigma S_u \, dW_u + \int_{t_0}^{t} \sigma\beta S_u \, dW_{u-h}, \qquad \forall t \in [t_0, T] \quad (2)$$

where $S_{t_0} \in R_+$ (R_+ denotes the set of all strictly positive real numbers) is the current stock price. In addition, the conditional variance of stock returns at time t conditional on the information set up to time t_0 is

$$Var_{t_0}(R_t) = \sigma^2 dt, \qquad \forall t \in [t_0, t_0 + h)$$

$$Var_{t_0}(R_t) = (1 + \beta^2)\sigma^2 dt, \qquad \forall t \in [t_0 + h, T]$$

and the conditional autocorrelation coefficient is given by

$$\text{Corr}_{t_0}(R_t, R_{t+h}) = \frac{\beta}{\sqrt{1 + \beta^2}}, \qquad \forall t \in [t_0, t_0 + h)$$

$$\text{Corr}_{t_0}(R_t, R_{t+h}) = \frac{\beta}{1 + \beta^2}, \qquad \forall t \in [t_0 + h, T]$$

where R_t denotes the stock return at time t; that is, $R_t \equiv dS_t/S_t$. Based on the conditional variance and autocorrelation coefficient, the main properties of stock returns specified as in (1) can be clearly observed. Obviously, the stock returns are independent and (1) reduces to a geometric Brownian motion when $\beta = 0$. For the case of $\beta \neq 0$, the stock returns specified in (1) exhibit nonzero autocorrelation, which can be positive or negative depending upon the sign of β. Consequently, this process is more flexible than the usual geometric Brownian motion.

It is worth noting that Lo and Wang (1995) assume the log-price dynamics of the underlying stock to be a trending O-U process and hence the autocorrelations in the stock returns are caused by the drift term. In contrast, as shown in Equation (1), the current article proposes a different model for autocorrelated returns, where the autocorrelated behavior comes from the diffusion term.

OPTION PRICING WHEN ASSET RETURNS FOLLOW AN MA(1)-TYPE PROCESS

Consider the problem of pricing a European call option on a specified stock as in Equation (1). Because the underlying stock returns are autocorrelated, it is not easy to value the call by tree methods. This article develops a pricing formula by the martingale methodology.

To price the derivatives, it is more convenient to have a risk-free security. Suppose the short-term interest rate r is constant over the time interval $[t_0, T]$, and the value of a riskless bond denoted by B is assumed to be continuously compounded at the rate r; that is,

$$\frac{dB_t}{B_t} = r\,dt, \qquad \forall t \in [t_0, T] \tag{3}$$

or equivalently $B_t = e^{rt}$, with $B_0 = 1$.

Based on the risk-neutral pricing theory, the current value of a European call option C_{t_0} is

$$\begin{aligned} C_{t_0} &= e^{-r(T-t_0)} E^Q\{(S_T - K)^+ \mid \mathcal{F}_{t_0}\} \\ &= e^{-r(T-t_0)} E^Q\{S_T \cdot 1_{\{S_T > K\}} \mid \mathcal{F}_{t_0}\} - e^{-r(T-t_0)} K \cdot \mathrm{Prob}^Q(S_T > K \mid \mathcal{F}_{t_0}) \end{aligned} \tag{4}$$

Here, Q is a martingale measure corresponding to the use of the riskless bond B as the numeraire, $E^Q\{\cdot \mid \mathcal{F}_{t_0}\}$ denotes the expectation under measure Q conditional on $\mathcal{F}_{t_0}$, K is the strike price of the European call option, $(S_T - K)^+$ is the notation for $\max(S_T - K, 0)$, and $\mathrm{Prob}^Q(S_T > K \mid \mathcal{F}_{t_0})$ denotes the probability that the call is in-the-money at the maturity date under measure Q. In addition, $1_{\{S_T > K\}}$ is an indicator function that takes a value of 1 as $S_T > K$, and 0 otherwise.

According to the martingale pricing method and (4), pricing the MA(1)-type option is done under the martingale probability measure Q that makes the discounted stock price $\tilde{S}_t \equiv S_t/B_t$ into a martingale, which can be represented as

$$E^Q\left\{\frac{S_t}{B_t}\,\middle|\,\mathcal{F}_{t_0}\right\} = \frac{S_{t_0}}{B_{t_0}}, \qquad \forall t \in [t_0, T] \tag{5}$$

According to (3), Equation (5) can be rewritten as:

$$E^Q\left\{\frac{S_t}{S_{t_0}}\,\middle|\,\mathcal{F}_{t_0}\right\} = e^{r(t-t_0)}, \qquad \forall t \in [t_0, T] \tag{6}$$

which shows that the expected stock returns equal the riskless rate r under martingale measure Q. Based on the dynamics of the stock price in (1) and the definition of probability measure Q, the transformation from measure P to Q is shown in the following lemma.

Lemma 1. *Assume that the underlying asset's price process S satisfies Equation (1). Specifically, W is a P-Brownian motion. The transformation from a P-Brownian motion to a Q-Brownian motion* W^Q *is then*

$$dW_z^Q = dW_z + \frac{\sum_{j=0}^{i_z}(-\beta)^j(\mu - r)\,dz}{\sigma} + (-1)^{i_z}\beta^{i_z+1}\,dw_{z-(i_z+1)h}, \quad \forall z \in [t_0, T]$$

where $z = t, t - h$, *and* i_z *is the integer part of* $(z - t_0)/h$.

Proof: See Appendix 1. ❑

Let θ_z denote $dw_{z-(i_z+1)h}$, which in fact are the *realized* past increments of the Brownian motion. The result of Lemma 1 can be represented as:

$$dW_z^Q = dW_z + H_z\,dz$$

where

$$H_z = \frac{\sum_{j=0}^{i_z}(-\beta)^j(\mu - r)}{\sigma} + (-1)^{i_z}\beta^{i_z+1}\theta_z, \qquad \forall z \in [t_0, T]$$

Specifically, H_z is predictable and can be displayed according to time intervals as follows:

$$H_z = \frac{(\mu - r)}{\sigma} + \beta\theta_z, \qquad \forall z \in [t_0, t_0 + h)$$

$$H_z = \frac{(1 - \beta)(\mu - r)}{\sigma} + (-1)\beta^2\theta_z, \qquad \forall z \in [t_0 + h, t_0 + 2h)$$

$$H_z = \frac{(1 - \beta + \beta^2)(\mu - r)}{\sigma} + \beta^3\theta_z, \qquad \forall z \in [t_0 + 2h, t_0 + 3h)$$

$$\vdots \qquad\qquad \vdots$$

$$H_z = \frac{[(1 - \beta + \beta^2 - \cdots + (-\beta)^{i_T})](\mu - r)}{\sigma} + (-1)^{i_T}\beta^{i_T+1}\theta_z, \qquad \forall z \in [t_0 + i_T h, T]$$

Because the existence of measure Q is assured by Girsanov's theorem,[1] the prerequisite for Girsanov's theorem that H_z is a predictable process

[1]Please refer to Klebaner (1998, p. 242) for details.

with $\int_{t_0}^{T} H_u^2\, du < \infty$ should be checked. As the path of a Brownian motion before the current time t_0 is known, the values of $(-1)^{i_z}\beta^{i_z+1}\, dw_{z-(i_z+1)h}$ for all z are bounded under the condition $|\beta| < 1$. In addition, because h, μ, r, σ, and β in (1) are all assumed to be constant, the values of $(z - t_0)/h$ and $[\sum_{j=0}^{i_z}(-\beta)^j(\mu - r)]/\sigma$ are also bounded for all z. Accordingly, the assumption of Girsanov's theorem, $\int_{t_0}^{T} H_u^2\, du < \infty$, is satisfied, and thus the existence of measure Q can be assured.

One modeling issue concerning Equation (1) has not been discussed until now: Does the price process specified in (1) admit arbitrage opportunities? The existence of measure Q provides enough information to answer the preceding question. As shown in Klebaner (1998, p. 258), the sufficient condition for no arbitrage can be stated as follows: "Suppose there exists a probability measure Q, equivalent to P, such that the discounted stock price process $\tilde{S}$ is a martingale under Q. There are then no arbitrage opportunities." Accordingly, an asset with a process specified in (1) allows no arbitrage opportunities, because the existence of measure Q is assured. It also implies that the process defined in (1) can conceivably be used to represent security price fluctuations.[2]

After recognizing that the log-price dynamics defined in (1) allow no arbitrage opportunities, the MA(1)-type options can be valued by the martingale pricing method, which is done under the martingale probability measure Q. By Lemma 1, the process defined in Equation (1) can be transformed to the dynamics of the stock price under measure Q, denoted as S^Q, as follows:

$$\frac{dS_t^Q}{S_t^Q} = r\, dt + \sigma\, dW_t^Q + 1_A \sigma\beta\, dW_{t-h}^Q, \qquad \forall t \in [t_0, T] \tag{7}$$

where $1_A \equiv 1_{\{t_0+h \le t \le T\}}$. Note that the time to maturity $T - t_0$ can be distinguished into two cases: $0 < T - t_0 < h$ and $T - t_0 \ge h$. Trivially, for the case of $t_0 < T < t_0 + h$, the stock price process S^Q under measure Q reduces to a geometric Brownian motion. Accordingly, the Black-Scholes formula is still applicable to the MA(1)-type call option with maturity shorter than h.

For the case of $T \ge t_0 + h$, the price process S^Q is not a geometric Brownian motion. To value the term $\mathrm{Prob}^Q(S_T > K \mid \mathcal{F}_{t_0})$ in (4), the solution for the stock price at time t under measure Q, denoted as S_t^Q, should be at hand. To solve S_t^Q, the dynamics of stock prices under Q displayed in (7) can be viewed as being driven by two Brownian motions,

[2]Please refer to Harrison and Pliska (1981, p. 222) for details.

$W^Q_{1,t-t_0}$ and $W^Q_{2,t-t_0}$, where $\{W^Q_{1,t-t_0}, W^Q_{2,t-t_0}\} \equiv \{W^Q_{t-t_0}, 1_A W^Q_{(t-h)-t_0}\}$. The two Brownian motions have the following properties:

(i) $W^Q_{1,t-t_0}$ and $W^Q_{2,t-t_0}$ are both one-dimensional Brownian motions.

(ii) $W^Q_{1,(t-h)-t_0} \equiv W^Q_{2,t-t_0}$, for $t \in [t_0 + h, T]$, and $dW^Q_{1,t-h} = W^Q_{(t-h)-t_0+dt} - W^Q_{(t-h)-t_0} = dW^Q_{2,t}$ for $t \in [t_0 + h, T]$.

(iii) $dW^Q_{1,t}$ and $dW^Q_{2,t}$ are uncorrelated, that is, $E(dW^Q_{1,t}\, dW^Q_{2,t}) = 0$.

Apparently, dS^Q_t/S^Q_t in (7) can also be represented as

$$\frac{dS^Q_t}{S^Q_t} = r\,dt + \sigma\, d(W^Q_1 + \beta W^Q_2)_t$$

To solve S^Q_t, the quadratic variation of $(W^Q_1 + \beta W^Q_2)_t$, which is denoted as $\langle W^Q_1 + \beta W^Q_2 \rangle_t$, is needed and can be represented as

$$\langle W^Q_1 + \beta W^Q_2 \rangle_t = \int_{t_0}^{t} (1 + 1_{B(u)}\beta)^2\, du \tag{8}$$

where $1_{B(u)} = 1_{\{t_0 \le u \le t-h\}}$. Based on (7) and (8), S^Q_t can be solved by using Itô's lemma as follows:

$$S^Q_t = S_{t_0} \exp\{[r - \tfrac{1}{2}\sigma^2(1+\beta)^2][(t-h) - t_0] + (r - \tfrac{1}{2}\sigma^2)h + \sigma(1+\beta)W^Q_{(t-h)-t_0} + \sigma W^Q_{t-(t-h)}\} \tag{9}$$

It is easy to check that S^Q_t in (9) is the solution such that the discounted stock price $\tilde{S}_t$ is a martingale under measure Q. The term $\mathrm{Prob}^Q(S_T > K \mid \mathcal{F}_{t_0})$ in (4) is then obtained as

$$\begin{aligned} &\mathrm{Prob}^Q(S_T > K \mid \mathcal{F}_{t_0}) \\ &= N\left(\frac{\ln(S_{t_0}/K) + r(T-t_0) - \frac{1}{2}\sigma^2[(1+\beta)^2(T-t_0-h) + h]}{\sigma\sqrt{(1+\beta)^2(T-h-t_0) + h}}\right) \\ &\equiv N(d'_2) \end{aligned} \tag{10}$$

To value the term $E^Q\{S_T \cdot 1_{\{S_T>K\}} \mid \mathcal{F}_{t_0}\}$ in (4), it is convenient to find the probability measure R equivalent to Q such that the following equation is satisfied:

$$E^Q\{S_T \cdot 1_{\{S_T>K\}} \mid \mathcal{F}_{t_0}\} = S_{t_0} e^{r(T-t_0)}\, \mathrm{Prob}^R(S_T > K \mid \mathcal{F}_{t_0}) \tag{11}$$

Define the W^R_z process as

$$dW^R_z = \begin{cases} dW^Q_z - \sigma(1+\beta)\,dz, & \forall z \in [t_0, T-h] \\ dW^Q_z - \sigma\,dz, & \forall z \in (T-h, T] \end{cases} \tag{12}$$

where $z = t, t - h$. Term W^R is then an R-Brownian motion satisfying Equation (11). The existence of such a measure R can be assured by Girsanov's theorem.[3] With the use of (12) and (7), the solution for the stock price at time t under measure R, denoted as S_t^R, can be solved by using Itô's lemma as follows:[4]

$$\begin{aligned}\ln S_t^R = \ln S_{t_0} + [r + \tfrac{1}{2}\sigma^2(1+\beta)^2][t - t_0 - h] + (r + \tfrac{1}{2}\sigma^2)h \\ + \sigma(1+\beta)W_{(t-h)-t_0} + \sigma W_{t-(t-h)} \qquad (13)\end{aligned}$$

The probability that the call is in-the-money at the maturity date under measures R can accordingly be obtained from the solutions S_T^R as

$$\begin{aligned}&\mathrm{Prob}^R(S_T > K \mid \mathcal{F}_{t_0}) \\ &= N\left(\frac{\ln(S_{t_0}/K) + r(T - t_0) + \frac{1}{2}\sigma^2[(1+\beta)^2(T - t_0 - h) + h]}{\sigma\sqrt{(1+\beta)^2(T - h - t_0) + h}}\right) \\ &\equiv N(d_1') \qquad (14)\end{aligned}$$

Therefore, based on (4), (10), (11), and (14), the current value of the MA(1)-type option, C_{t_0}, is priced by the following:

$$C_{t_0} = S_{t_0}N(d_1') - Ke^{-r(T-t_0)}N(d_2')$$

This result is summarized in Proposition 1.

Proposition 1. *Assume that the dynamics of the underlying stock prices are given by (1). The value of a European call option on the preceding underlying stock, which is named as an MA(1)-type option, can be priced by the following:*

(*i*) *When the time to maturity satisfies $T - t_0 < h$, that is, the call option will mature immediately, the Black-Scholes formula still holds for the MA(1)-type option.*
(*ii*) *When the expiration date satisfies $T \geq t_0 + h$, that is, the call will not mature immediately, the value of the MA(1)-type option C_{t_0} is priced by*

$$C_{t_0} = S_{t_0}N(d_1') - Ke^{-r(T-t_0)}N(d_2')$$

[3]Denote Φ_z as follows:

$$\Phi_z = \begin{cases} -\sigma(1+\beta), & \forall z \in [t_0, T-h] \\ -\sigma, & \forall z \in (T-h, T] \end{cases}$$

Because Φ_z is bounded, the prerequisite for Girsanov's theorem that Φ_z is a predictable process with the condition $\int_{t_0}^{T}\Phi_u^2\,du < \infty$ is satisfied.

[4]Please refer to the Appendix for details.

where

$$d_1' = \frac{\ln(S_{t_0}/K) + r(T - t_0) + \frac{1}{2}\sigma^2[(1+\beta)^2(T - t_0 - h) + h]}{\sigma\sqrt{(1+\beta)^2(T - h - t_0) + h}}$$

$$d_2' = \frac{\ln(S_{t_0}/K) + r(T - t_0) - \frac{1}{2}\sigma^2[(1+\beta)^2(T - t_0 - h) + h]}{\sigma\sqrt{(1+\beta)^2(T - h - t_0) + h}}$$

and $N(\cdot)$ *is the distribution function of the standard normal distribution.*

The formula in Proposition 1 obviously indicates that the MA(1)-type option price will eventually converge to the Black-Scholes price when the call closes to maturity. This result is consistent with the assumption of Roll (1977), Duan (1995), and Heston and Nandi (2000), where they assume that the value of an option with one period to expiration obeys the Black-Scholes formula in discrete-time models. Furthermore, the pricing formula in Proposition 1 does not violate the Black-Scholes formula. Apparently, C_{t_0} converges to the Black-Scholes formula when h closes to zero, and it is fully identical to the Black-Scholes formula as $\beta = 0$.

The pricing formula for the MA(1)-type option is analogous to Black and Scholes except for the total volatility input. Denote the continuously compounded τ-period returns as $R(\tau)$, where τ represents the option's time to maturity. When the stock prices satisfy an MA(1)-type process specified as in (1), the conditional variance of $R(\tau)$, conditional on the information up to time t_0, is

$$Var_{t_0}(R(\tau)) = \sigma^2[(1+\beta)^2(T - t_0 - h) + h] \tag{15}$$

and the conditional standard deviation of $R(\tau)$ is the square root of $Var_{t_0}(R(\tau))$, which is an important term of d_1' and d_2' as shown in Proposition 1. As the total volatility input in the standard Black-Scholes formula is indeed the diffusion coefficient of a geometric Brownian motion multiplied by the square root of an option's time to maturity, that is, $\sigma\sqrt{\tau}$, Proposition 1 shows that the total volatility input for an MA(1)-type option is the conditional standard deviation of $R(\tau)$, that is, $\sqrt{Var_{t_0}(R(\tau))}$, where $Var_{t_0}(R(\tau))$ is displayed in (15). According to Grundy (1991, p. 1049), the Black-Scholes formula still holds under the trending O-U specification, Proposition 1 implies that the *pricing formula* for options on an asset with autocorrelated returns depends on the source of the autocorrelation.

To hedge the European call options under a stock-price process specified as in (1), at the current time $t \in [t_0, t_0 + dt)$ one may consider a portfolio V_t that consists of Δ_t shares of stock and ϕ_t units of riskless bonds; that is,

$$V_t = \Delta_t S_t + \phi_t B_t$$

Note that the number of shares in V_t, that is, Δ_t, is also called the hedge ratio. Assume that the portfolio replicates the MA(1)-type call option. The changes in the values of the portfolio and the option, denoted, respectively, as dV_t and dC_t, should then be equal. This implies

$$\begin{aligned}&(\Delta_t \mu S_t + \phi_t r B_t)\,dt + \Delta_t \sigma S_t\,dW_t + \Delta_t \sigma\beta S_t\,dw_{t-h}\\&= \left(\frac{\partial C_t}{\partial t} + \frac{\partial C_t}{\partial S_t}\mu S_t + \frac{1}{2}\frac{\partial^2 C_t}{\partial S_t^2}\sigma^2 S_t^2\right)dt + \frac{\partial C_t}{\partial S_t}\sigma S_t\,dW_t + \frac{\partial C_t}{\partial S_t}\sigma\beta S_t\,dw_{t-h}\end{aligned}$$

where $t \in [t_0, t_0 + dt)$. Note that based on the information up to time t_0, the term dw_{t-h} is known.

In view of the preceding equation and Proposition 1, it is observed that

$$\Delta_t = \frac{\partial C_t}{\partial S_t} = N(d_1')$$

and the number of bonds in the portfolio V_t, that is, ϕ_t, can be decided accordingly. Because portfolio V_t replicates the value of the MA(1)-type option, it is apparent that hedging the MA(1)-type option can be performed just by holding portfolio V_t in the opposite position. Thus, hedging the MA(1)-type option is identical to that of the Black-Scholes model in functional form and is easy to operate.

NUMERICAL ANALYSES

To gauge the impact of autocorrelated stock returns on the option's price, Tables I and II compare the theoretical values of options under an MA(1)-type process to the Black-Scholes prices for various times to maturity $T - t_0$, strike prices K, and autocorrelated parameters β for a hypothetical \$40 stock. The theoretical values of MA(1)-type options are calculated by the result of Proposition 1, and the Black-Scholes values are based on the Black-Scholes formula. It is worth noting that the design of numerical analyses in this article is similar to that of Jokivuolle (1998),

TABLE I
Option Prices Under Negative Autocorrelated Stock Returns
(Daily Frequency)

Strike price	*Black-Scholes price*	*Option price under negative autocorrelated returns, with* $\beta =$						
		−0.025	*−0.050*	*−0.075*	*−0.100*	*−0.250*	*−0.500*	*−0.750*
Panel A. Time-to-maturity $T - t_0 = 7$ *days*								
30	10.014	10.014	10.014	10.014	10.014	10.014	10.014	10.014
35	5.018	5.017	5.017	5.017	5.017	5.017	5.017	5.017
40	0.748	0.732	0.717	0.701	0.685	0.593	0.451	0.337
45	0.004	0.003	0.002	0.002	0.002	0.000	0.000	0.000
50	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
Panel B. Time-to-maturity $T - t_0 = 91$ *days*								
30	10.275	10.264	10.253	10.243	10.234	10.199	10.185	10.185
35	5.915	5.872	5.830	5.789	5.749	5.530	5.281	5.216
40	2.778	2.712	2.647	2.581	2.516	2.124	1.476	0.846
45	1.062	1.006	0.952	0.898	0.845	0.546	0.158	0.004
50	0.338	0.307	0.278	0.250	0.223	0.096	0.006	0.000
Panel C. Time-to-maturity $T - t_0 = 182$ *days*								
30	10.760	10.725	10.691	10.659	10.629	10.482	10.376	10.368
35	6.862	6.791	6.721	6.651	6.582	6.190	5.665	5.436
40	3.986	3.894	3.801	3.709	3.617	3.063	2.142	1.239
45	2.131	2.042	1.954	1.867	1.779	1.271	0.517	0.045
50	1.063	0.994	0.927	0.861	0.797	0.452	0.081	0.000
Panel D. Time-to-maturity $T - t_0 = 273$ *days*								
30	11.284	11.229	11.175	11.124	11.074	10.817	10.582	10.550
35	7.667	7.576	7.485	7.395	7.305	6.788	6.053	5.663
40	4.932	4.820	4.707	4.595	4.482	3.806	2.682	1.577
45	3.031	2.918	2.806	2.694	2.583	1.924	0.897	0.131
50	1.796	1.699	1.603	1.508	1.414	0.891	0.232	0.003
Panel E. Time-to-maturity $T - t_0 = 364$ *days*								
30	11.797	11.725	11.655	11.587	11.521	11.167	10.801	10.732
35	8.377	8.270	8.162	8.056	7.950	7.332	6.426	5.892
40	5.741	5.612	5.482	5.353	5.224	4.448	3.156	1.886
45	3.823	3.691	3.560	3.428	3.297	2.517	1.271	0.248
50	2.492	2.371	2.251	2.133	2.016	1.345	0.428	0.013

Note. This table compares call option prices on a hypothetical $40 stock under a geometric Brownian motion versus autocorrelated MA(1)-type stock returns. The parameter used for the coefficient of dW_t, that is, σ, is 1.75% for daily returns, and the daily continuously compounded risk-free rate is ln(1.025)/364.

in that both articles compare the MA option values to the Black-Scholes prices based on the same value of σ. For both Tables I and II, the coefficient for the current increment of a Brownian motion, that is, σ, is set to be 1.75% per day. Accordingly, unlike Lo and Wang (1995), the variance of daily returns under an MA(1)-type process is different from the geometric Brownian motion's counterpart.

TABLE II

Option Prices Under Positive Autocorrelated Stock Returns (Daily Frequency)

Strike price	*Black-Scholes price*	*Option price under positive autocorrelated returns, with* $\beta =$						
		0.025	*0.050*	*0.075*	*0.100*	*0.250*	*0.500*	*0.750*
Panel A. Time-to-maturity $T - t_0 = 7$ *days*								
30	10.014	10.014	10.014	10.014	10.014	10.014	10.014	10.014
35	5.018	5.018	5.018	5.018	5.019	5.023	5.037	5.064
40	0.748	0.764	0.780	0.796	0.812	0.909	1.072	1.238
45	0.004	0.004	0.005	0.006	0.007	0.016	0.045	0.091
50	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.002
Panel B. Time-to-maturity $T - t_0 = 91$ *days*								
30	10.275	10.288	10.301	10.316	10.331	10.441	10.689	11.001
35	5.915	5.959	6.003	6.048	6.093	6.378	6.885	7.419
40	2.778	2.843	2.908	2.974	3.039	3.432	4.085	4.738
45	1.062	1.117	1.174	1.231	1.288	1.642	2.257	2.893
50	0.338	0.370	0.404	0.438	0.474	0.711	1.177	1.705
Panel C. Time-to-maturity $T - t_0 = 182$ *days*								
30	10.760	10.796	10.834	10.873	10.914	11.180	11.698	12.280
35	6.862	6.934	7.006	7.079	7.153	7.604	8.381	9.177
40	3.986	4.078	4.171	4.263	4.355	4.908	5.828	6.743
45	2.131	2.220	2.309	2.399	2.489	3.035	3.960	4.895
50	1.063	1.133	1.205	1.277	1.351	1.814	2.645	3.524
Panel D. Time-to-maturity $T - t_0 = 273$ *days*								
30	11.284	11.340	11.398	11.457	11.518	11.906	12.623	13.399
35	7.667	7.759	7.852	7.945	8.039	8.609	9.579	10.563
40	4.932	5.045	5.157	5.269	5.382	6.054	7.171	8.279
45	3.031	3.144	3.257	3.370	3.484	4.168	5.317	6.467
50	1.796	1.895	1.995	2.095	2.197	2.825	3.918	5.045
Panel E. Time-to-maturity $T - t_0 = 364$ *days*								
30	11.797	11.870	11.944	12.020	12.098	12.584	13.461	14.389
35	8.377	8.486	8.595	8.704	8.814	9.480	10.606	11.741
40	5.741	5.870	5.999	6.127	6.256	7.027	8.304	9.568
45	3.823	3.955	4.088	4.220	4.353	5.149	6.477	7.800
50	2.492	2.613	2.736	2.860	2.984	3.743	5.044	6.367

Note. This table compares call option prices on a hypothetical $40 stock under a geometric Brownian motion versus autocorrelated MA(1)-type stock returns. The parameter used for the coefficient of dW_t, that is, σ, is 1.75% for daily returns, and the daily continuously compounded risk-free rate is ln(1.025)/364.

Panel A of Tables I and II shows that even extreme autocorrelated parameters do not affect short-maturity in-the-money call option prices very much. To illustrate, as shown in Table I, the value of β has no impact on the $30 7-day call even when β is equal to -0.75. A similar pattern is found as β equals 0.75, where the price under the MA(1)-type process is identical to the standard Black-Scholes price of $10.014.

Tables I and II also exhibit that the MA(1)-type option price converges to the Black-Scholes price when the call closes to maturity, which is consistent with the result of Proposition 1. However, the impact of autocorrelated parameter β grows with the length of time to maturity. As shown in Table I, when β is equal to -0.75, the absolute difference between the Black-Scholes price and the MA(1)-type price for the \$30 364-day call reaches \$1.065 ($|\$10.732 - \$11.797|$), whereas the Black-Scholes price is identical to the MA(1)-type price for the \$30 7-day call. A similar property can also be found in Table II, where the Black-Scholes price undervalues the \$30 364-day call by \$2.592 (\$14.389 − \$11.797) as $\beta = 0.75$.

Given the time to maturity and the autocorrelated parameter β, it is obvious that the differences between the in-the-money Black-Scholes prices and the MA(1)-type prices become large when the strike price increases. However, the pattern is not monotonic. As shown in Tables I and II, the differences eventually decline after the strike price K reaches to \$40 or \$45. To illustrate, in the case of $\beta = 0.75$, the difference in the Black-Scholes price and the MA(1)-type price is \$0.49 (=\$1.238 − \$0.748) for the 7-day at-the-money call (as shown in Panel A of Table II), although the difference is only \$0.087 (=\$0.091 − \$0.004) and \$0.002 (=\$0.002–\$0.000) when the strike prices are \$45 and \$50, respectively.

Table I indicates that ignoring the impact of a *negative* autocorrelation induced by the MA(1)-type process can lead to large *overpricing* of MA(1)-type options. On the contrary, as shown in Table II, ignoring the impact of a *positive* autocorrelation exhibited in stock returns can lead to large *underpricing* of MA(1)-type options. Furthermore, it is also observed that the impact of autocorrelated stock returns on option prices is significant even when the autocorrelated parameter β is small. For example, consider the \$40 91-day call option. The corresponding option price will be overvalued by about 2.43% when $\beta = -0.025$ and will be undervalued by about 2.29% when $\beta = 0.025$. Accordingly, it is also found that the MA(1)-type prices have some degree of asymmetry in the influence of β.

CONCLUSIONS

This article proposes a method of valuing European options on an asset with autocorrelated returns. According to the common findings of empirical research studies, a continuous-time MA(1)-type process is introduced to describe the autocorrelated stock returns, and the closed-form solution for option values under the MA(1)-type process is derived by using martingale pricing theory. The pricing formula of options on stocks with log-price dynamics given in (1) is similar to that of Black and Scholes except for the total volatility input. More specifically, the total volatility input of MA(1)-type options is the conditional standard deviation of continuously compounded τ-period returns, although the total volatility input of Black and Scholes is indeed the diffusion coefficient of a geometric Brownian motion times the square root of the option's time to maturity τ. Because the Black-Scholes formula is also applicable to price options on an asset with trending O-U dynamics, the finding in this article shows that the pricing formula for options on an asset with autocorrelated returns depends on the source of the autocorrelation. Furthermore, as shown in the numerical analyses, the impact of autocorrelation on the option values is significant and asymmetric, even when the autocorrelation of the underlying asset returns is weak.

APPENDIX

The Proof of Lemma 1

To find the transformation from a P-Brownian motion to a Q-Brownian motion such that the discounted stock price $\tilde{S}$ is a martingale under Q, it is useful to divide the option's time to maturity $T - t_0$ into i_T subintervals with the same length h and the $(i_T + 1)$th subinterval with the length of $T - (t_0 + i_T h)$, where i_T denotes the integer part of $(T - t_0)/h$.

To prove Lemma 1, the first time interval $[t_0, t_0 + h)$ is considered in the beginning. Based on (1) and (3), the discounted stock price $\tilde{S}$ satisfies

$$\frac{d\tilde{S}_t}{\tilde{S}_t} = (\mu - r)\, dt + \sigma\beta\, dw_{t-h} + \sigma\, dW_t, \qquad \forall t \in [t_0, t_0 + h) \qquad \text{(A.1)}$$

Note that for this time interval $t \in [t_0, t_0 + h)$, dw_{t-h} is known under $\mathcal{F}_{t_0}$. It means that the nonrandom terms in the above equation are the drift term $(\mu - r)\, dt$ and $\sigma\beta\, dw_{t-h}$. By setting

$$W^Q_{t-t_0} = W_{t-t_0} + \frac{(\mu - r)(t - t_0) + \int_{t_0}^{t} \sigma\beta\, dw_{u-h}}{\sigma}, \qquad \forall t \in [t_0, t_0 + h)$$

the dynamics of discounted stock prices under Q, $\tilde{S}^Q$, are

$$\frac{d\tilde{S}_t^Q}{\tilde{S}_t^Q} = \sigma\, dW_t^Q, \qquad \forall t \in [t_0, t_0 + h)$$

implying that $\tilde{S}$ is a martingale under Q for this time interval. Thus, the transformation from a P-Brownian to a Q-Brownian for this interval is obtained as:

$$dW_t^Q = dW_t + \frac{(\mu - r)}{\sigma} dt + \beta\, dw_{t-h}, \qquad \forall t \in [t_0, t_0 + h) \quad \text{(A.2)}$$

For the next time interval $t \in [t_0 + h, t_0 + 2h)$, note that both dW_{t-h} and dW_t in Equation (1) are stochastic under $\mathcal{F}_{t_0}$. Therefore, the dynamics of $\tilde{S}$ are represented as

$$\frac{d\tilde{S}_t}{\tilde{S}_t} = (\mu - r)\, dt + \sigma\, dW_t + \sigma\beta\, dW_{t-h}, \qquad \forall t \in [t_0 + h, t_0 + 2h) \quad \text{(A.3)}$$

Based on (A.2) and the fact that dW_{t-h} for $t \in [t_0 + h, t_0 + 2h)$ is identical to dW_t for $t \in [t_0, t_0 + h)$, one can set

$$W^Q_{t-(t_0+h)} = W_{t-(t_0+h)} + \frac{(1-\beta)(\mu - r)[t - (t_0 + h)] - \int_{t_0+h}^{t} \sigma\beta^2\, dw_{u-2h}}{\sigma}, \qquad \forall t \in [t_0 + h, t_0 + 2h)$$

and transform (A.3) to the process of $\tilde{S}_t^Q$ as follows:

$$\frac{d\tilde{S}_t^Q}{\tilde{S}_t^Q} = \sigma\, dW_t^Q + \sigma\beta\, dW_{t-h}^Q, \qquad \forall t \in [t_0 + h, t_0 + 2h)$$

This implies that the discounted stock price $\tilde{S}$ is a martingale under Q. Accordingly, the transformation that makes $\tilde{S}$ be a martingale under Q for this time interval is then

$$dW_t^Q = dW_t + \frac{(1-\beta)(\mu - r)}{\sigma}\, dt - \beta^2\, dw_{t-2h}, \qquad \forall t \in [t_0 + h, t_0 + 2h) \quad \text{(A.4)}$$

The transformation from a P-Brownian to a Q-Brownian for the other time subintervals can be similarly obtained recursively and summarized as follows:

$$dW_z^Q = dW_z + \frac{\sum_{j=0}^{i_z} (-\beta)^j (\mu - r)\, dz}{\sigma} + (-1)^{i_z} \beta^{i_z + 1}\, dw_{z-(i_z+1)h}, \qquad \forall z \in [t_0, T]$$

where $z = t, t - h$, and i_z is the integer part of $(z - t_0)/h$. The proof of Lemma 1 is now complete.

The Dynamics of Stock Prices under Measure *R*

Substituting (12) into (7), one can obtain the dynamics of S^R as follows:

$$\frac{dS_t^R}{S_t^R} = r\,dt + 1_C\sigma^2(1 + \beta)\,dt + 1_D\sigma^2(1 + \beta)^2\,dt + 1_E\sigma^2[1 + \beta(1 + \beta)]\,dt + \sigma\,d(W_1^R + \beta W_2^R)_t$$

where

$$1_C \equiv 1_{\{t_0 \le t < t_0 + h\}}, \quad 1_D \equiv 1_{\{t_0 + h \le t \le T - h\}}, \text{ and } 1_E \equiv 1_{\{T - h < t \le T\}}$$

are indicator functions.By applying Itô's lemma, S_t^R can be solved as shown in Equation (13).

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