Valuation and Optimal Strategies of Convertible Bonds

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This article presents a contingent claim valuation of a callable convertible bond with the issuer's credit risk. The optimal call, voluntary conversion, and bankruptcy strategies are jointly determined by shareholders and bondholders to maximize the equity value and the bond value, respectively. This model not only incorporates tax benefits, bankruptcy costs, refunding costs, and a call notice period, but also takes account of the issuer's debt size and structure. The numerical results show that the predicted optimal call policies are generally consistent with recent empirical findings; therefore, calling convertible bonds too late or too early can be rational. © 2006 Wiley Periodicals, Inc. Jrl Fut Mark 26:895–922, 2006

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INTRODUCTION

In spanning the dimensions from common stocks on the one hand to straight bonds on the other, the convertible bond is one of the most popular hybrid financing instruments. Most convertible bonds have call provisions allowing the bond issuer to call the bond back at a predetermined call price, and the bondholder has the right to exchange his convertibles for the firm's common stocks. Similar to a straight bondholder, the convertible bondholder is entitled to receive coupon and principal payments, and thus the potential default risk of the bond issuer is extremely critical. The value of convertibles will be affected not only by the conversion strategy of the bondholders and the call and bankruptcy policies of the bond issuer, but also by each party's expectation of the other's reaction. Therefore the present article intends to provide a structural (firm value) model to analyze the valuation of convertibles along with the optimal call, bankruptcy, and conversion strategies. In particular, the optimal decisions are endogenously determined by the smoothing-pasting conditions, which maximize either the convertible bond value or the equity value. The main reason for applying the structural approach lies in the poor-fitting results of the reduced-form models with respect to bond prices. Rogers (1999) has noted: "The existence of convertible bonds really forces one to consider firm value—so maybe we should go for a structural approach anyway?" Definitely, the structural approach indeed brings some useful insights into corporate financing.

The pioneering structural model of Merton (1974) explains how a risky debt can be viewed as a European contingent claim on the firm's asset value, and derives the closed-form valuation of a risky debt by using the traditional option pricing formula. Ingersoll (1977a), based on the Black-Scholes-Merton methodology and some simplifying assumptions, derives the closed-form pricing formula for convertibles, and shows the conversion will occur only at the time of call or at the maturity of the bond in a setting of perfect market. Meanwhile, Brennan and Schwartz (1977) price a more general convertible bond via a finite-difference method in which they solve a partial-differential equation with some realistic boundary conditions. Both of the above articles generate the implication that the optimal call trigger is equal to the call price divided by the conversion ratio. Subsequently, Ingersoll (1977b) observes convertible bonds are often called too late with respect to the theoretical prediction, and then relaxes some assumptions in an attempt to explain the deviations. Brennan and Schwartz (1980) allow for stochastic interest rates and suggest that for a reasonable range of interest rates, the

errors from adopting the certain interest rate model are likely to be small. Thus for practical purposes, it would be preferable to use a simple model with a constant interest rate for valuing convertible bonds. In addition, Nyborg (1996) provides an excellent survey on the valuation of convertible bonds and reviews the reasons firms issue them.

Recently, Ayache, Forsyth, and Vetzal (2003) price convertible bonds with credit risk by a numerical solution of linear complementarily problems, and present a general and consistent framework for assuming a Poisson default process. Lau and Kwok (2004) construct an effective explicit finite difference algorithm to price a risky convertible bond with consideration of the interaction of the call and conversion features. In order to do so, they further apply a dynamic programming procedure similar to that for pricing an American put. They also consider other intricacies, including the call notice period requirement and the soft-call and hard-call constraints. Their numerical algorithms are capable of capturing the time-dependent characteristic of the critical points at which the convertible bond should be called by the issuer or be converted into common shares by the bondholder. In addition, Sîrbu, Pikovsky, & Shreve (2004) take the interaction of the bond issuer and the bondholder as a two-person zero-sum game where the issuer seeks a call strategy minimizing the convertible bond value, while the bondholder chooses a conversion strategy maximizing the bond value. Given the convertible bond is perpetual, they obtain the game value (the bond value) determined by a nonlinear ordinary differential equation. Nevertheless, neither of the above articles consider the characteristics of the issuer.

Three critical determinants on the valuation and optimal strategies of convertible bonds are usually neglected in the theoretical literature. First of all, the strategic default of the bond issuer and the associated bankruptcy costs are not taken into consideration. The default strategy in the above works is often exogenously determined by limited liability, and therefore the bankruptcy is triggered when the firm's asset value falls below the principal value of the total outstanding debt. However, it is not reasonable to neglect strategic default. As noted by Anderson and Sundaresan (1996) and Mella-Barral and Perraudin (1997), an endogenous strategic default has a significant effect on the bond price. The associated bankruptcy costs are also substantial for the value and strategies of convertible bonds because they are clearly connected with the bond issuer's capital structure. Second, the theoretical literature neglects tax shields enjoyed by the bond issuer, which mainly result from regular coupon payments of convertible bonds. Empirically, Asquith and

Mullins (1991), Campbell, Ederington, and Vankudre (1991), and Asquith (1995) all emphasize tax shields are of vital importance in using the cash-flow advantage hypothesis to explain the empirically observed late calls. The last is the bond issuer's debt structure and debt size, which have a significant real impact on the optimal call strategy of convertibles reported by Altintig and Butler (2005). Undoubtedly the above three elements are essential to the valuation and optimal strategies of the convertible bonds; therefore the present article makes a great effort to incorporate them into a structural framework.

Leland (1994) is the first structural model to take into consideration tax benefits and bankruptcy costs in pricing a perpetual risky straight bond. The optimal bankruptcy policy adopted by the bond issuer is endogenously determined by the smooth-pasting condition maximizing the equity value of the issuer. Following Leland (1994), this article presents a simple but complete structural model to derive the pricing formula for a perpetual callable defaultable convertible bond via the pricing technique of double-barrier options, and shows that a callable convertible bond can be deduced from either a noncallable convertible bond or a call-forcing convertible bond. In particular, the optimal strategies of convertibles are simultaneously determined by the bond issuer to maximize the equity value of the shareholders and by the bondholder to maximize the convertible bond value. The time-independent (infinite maturity) characteristic makes the model tractable to place the convertible bond pricing problem on a firm theoretical foundation.[1] This is reasonable because calls and conversions usually occur far from maturity. The present structural model considers not only tax benefits and bankruptcy costs of the bond issuer, but also refunding costs and a call notice period of the redemption. Moreover, the debt size and debt structure of the issuer are considered in particular.

The numerical results with realistic parameters reveal the predicted optimal call policies of the convertible bond are generally consistent with the results of the recent empirical findings, such as Sarkar (2003) and Altintig and Butler (2005). Specifically, the observed late (early) calls, that is, the underestimation (overestimation) of the optimal call trigger, can be rational. The present model also provides additional insights as well as complements earlier studies about the valuation and optimal strategies of the callable convertible bond.

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[1]Huang, Ju, and Ou-Yang (2003) extend Leland and Toft (1996) and Longstaff and Schwartz (1995) to value a finite-maturity corporate bond under stochastic interest rates, but the rolled-over issuances of a corporate bond could not apply to the case of a callable convertible bond.

The remainder of this article is organized as follows. The valuation framework is provided. The valuation and optimal strategies of a noncallable convertible bond are given, followed by those of a call-forcing convertible bond. Next the valuation and optimal strategies of a callable convertible bond are examined, with a call notice period taken into consideration as well. Some numerical analyses are provided, followed by concluding remarks.

VALUATION FRAMEWORK

Consider a bond issuer (or an objective firm) where the callable convertible bond is a senior issue, which continuously pays constant coupon flows, C_1, with infinite maturity and the par value, P_1. According to Altintig and Butler (2005), the convertible bond is usually not the only debt issue, and thus the debt size and structure of the issuer will significantly affect the valuation and optimal call policy of callable convertible bonds. For this reason, a perpetual straight bond with par value P_2 and continuous coupon payments C_2 is further added into the debt structure of the issuer. These two bonds are assumed to have equal priority, and the remaining claim on the firm would be the common stocks. Let $V(t)$ designate the unlevered asset value of the bond issuer at time t. The dynamics of $V(t)$ on the risk-neutral filtered probability space are given by

$$dV(t) = V(t)[(r - q)dt + \sigma \, dW^Q(t)] \tag{1}$$

where r denotes the constant risk-free interest rate,[2] q is the constant payout ratio of the issuer,[3] σ is the constant return volatility, and W^Q is a Wiener process. Owing to Harrison and Kreps (1979) and Harrison and Pliska (1981), risk neutrality is utilized to bypass the original rate of return of the unlevered asset. Here the unlevered asset is assumed to be a continuously tradable asset in a frictionless market; it is additionally assumed there is a risk-free asset earning the instantaneous risk-free interest rate without any restriction. The unlevered asset value is set to be independent of the capital structure of the firm, which is a standard assumption in structural models. This also implies the validity of the Modigliani-Miller theorem. Furthermore, there is no incentive conflict

[2]Although the assumption of constant interest rate seems to be a limitation, Brennan and Schwartz (1980) have indicated the pricing errors are likely to be small.
[3]Because the optimal bankruptcy strategy V_B^* is usually far away from zero, the unlevered asset value V would never be really small. Therefore, both of the coupon payments would be fulfilled and the constant payout ratio q would not be a problem of the model. The reviewer is thanked for this valuable suggestion.

between the management and shareholders of the objective firm in this framework. In other words, the intention of the management is assumed to agree with that of the shareholders always, that is, to maximize the common shareholders' wealth subject to the constraints placed upon the firm.

Next, the characteristics of the callable convertible bond in the present framework are analyzed. As the title indicates, only the call and conversion provisions are involved in these convertible bonds, with no other exotic provisions prevailing, such as the reset of the call price. Most convertible bonds are also puttable bonds, and bondholders own the right to sell their bonds back to the issuer. However, this flexibility plays only a minor role in the valuation, and can be ignored for simplicity.

One advantage of the structural model for valuing convertible bonds is that it captures the dilution effect of the conversion, which is represented by the ratio between the total converted shares and the total outstanding shares after conversion, as in Schönbuchner (2003, p. 266) where the state variable is the total firm value. It should be emphasized that convertible bonds are converted into *equities,* and then could be expressed as a fraction of the total firm value by the accounting identity that the total firm value always equals the sum of values of convertibles and equities. This is still true when the unlevered asset value is used as the state variable and the convertible bond is the only debt in the issuer's capital structure, because the unlevered asset value would equal the total firm value after all the outstanding convertible bonds are converted. In the case that there are straight bonds and convertible bonds in the issuer's debt structure, the advantage of showing dilution effect could not be entirely reserved.[4] Nevertheless, it would be assumed that the bondholder will receive a conversion fraction γ of the unlevered asset value of the issuer when convertible bonds are converted. This simplified assumption not only keeps the model tractable, but also makes the model comparable with previous structural models. Moreover, the effects of tax benefits and bankruptcy costs on conversion value could be partly captured through the optimal conversion strategy, which would be explained in a later section.

The conversion here is assumed to be a "block conversion"; that is, all the bondholders will convert the convertible bonds into common shares once and for all. Constantinides (1984) shows that there is at least one Nash equilibrium in the various set of conversion strategies; moreover, the highest value of the convertible bond in these Nash

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[4]The authors are indebted to the reviewer for mentioning this issue.

equilibria coincides with the bond value in the case of block conversion. If the issuer calls back all of the outstanding callable convertible bonds, all the bondholders must immediately choose either to convert the bonds into common shares or to receive the prespecified call price (the redemption value), K. The transaction costs associated with the conversion are ignored, whereas the proportional refunding costs, β multiplied by the par value of the convertible bond, are incorporated when the call is redeemed for cash, where $0 < \beta < 1$. In addition, the call and conversion protected periods are neglected in this article.

At the initial time (assumed for simplicity to be time zero), the optimal call barrier, V^*_{Call}, and the optimal voluntary conversion barrier, V^*_{Con}, which will be determined in a later section, are both greater than the initial unlevered asset value, $V(0) \equiv V$. As soon as the unlevered asset value goes up and touches V^*_{Call} or V^*_{Con}, then either the call of the bond issuer or the voluntary conversion of the bondholders is triggered. In addition to the results of being called or being voluntarily converted, there are still two other possible outcomes for the callable convertible bond. One is that the bond issuer declares bankruptcy prior to the call and the voluntary conversion; the other one is that the callable convertible bond remains alive with nothing happening. In the same manner, the optimal bankruptcy barrier, V^*_B, which is less than V, can be defined as well. As soon as the unlevered asset value of the issuer goes down and touches V^*_B, the bankruptcy is triggered by the issuer. Once the issuer declares bankruptcy, the convertible bondholders receive the recovery value, $(P_1/P_1 + P_2)(1 - \alpha)V^*_B$, at the time of default, where α is the ratio of bankruptcy costs between 0 and 1, and the recovery rate is weighted by the par values of the two bonds showing that both bondholders share equal priority. In the later sections, the optimal strategies for call, voluntary conversion, and bankruptcy will be endogenously determined by taking the objectives of the issuer and bondholders into consideration.

Theorem VII of Ingersoll (1977a) shows that whenever it is optimal to voluntarily convert a noncallable convertible bond, it will also be optimal to convert a callable convertible bond that is otherwise identical. This theorem relies solely on the fact the callable convertible bond is no more valuable than the noncallable convertible bond, which is otherwise the same, and therefore would be shown to hold in the present model. Furthermore, in a later section it will be shown that whenever it is optimal to call back a call-forcing convertible bond, it will also be optimal to call back an otherwise-identical callable convertible bond. Under the assumption that the above information is common knowledge for both

the bond issuer and bondholder, a callable convertible bond would turn out to be either the otherwise-identical call-forcing convertible bond when the optimal call trigger is less than the optimal voluntary conversion trigger, or the otherwise-identical noncallable convertible bond when the optimal call trigger is greater than or equal to the optimal voluntary conversion trigger. In sum, the value of a callable convertible bond as well as the corresponding optimal strategies can be deduced from those of the call-forcing convertible bond and the noncallable convertible bond, which are otherwise the same, which is also in line with Sîrbu et al. (2004). For this reason, first the pricing formulas are derived, followed by the optimal strategies of a noncallable convertible bond and a call-forcing convertible bond in the next two sections.

VALUATION AND OPTIMAL STRATEGIES OF A NONCALLABLE CONVERTIBLE BOND

For a noncallable convertible bond, the issuer can decide when to go bankrupt, and the bondholder can determine when to voluntarily convert the bonds into common shares. The present article utilizes a double-barrier options valuation approach to pricing a risky noncallable convertible bond, where the initial lower barrier V_{B1} represents the bankruptcy trigger of the issuer, and the initial upper barrier V_{Con} denotes the voluntary conversion trigger of the bondholders. These two barriers are now treated as exogenously given constants and will be later endogenously determined through the Nash-equilibrium argument.

Under the present risk-neutral framework, the initial value of a risky straight bond, $B(V; V_{B1})$, can be priced as

$$B(V; V_{B1}) = E^Q \left[\int_0^{\tau_{B1}} C_2 e^{-rt} dt + \frac{P_2}{P_1 + P_2}(1 - \alpha)V_{B1}e^{-r\tau_{B1}} \right]$$

$$= \frac{C_2}{r}(1 - F_{\tau_{B1}}) + \frac{P_2}{P_1 + P_2}(1 - \alpha)V_{B1}F_{\tau_{B1}} \qquad (2)$$

where $\tau_{B1} \equiv \inf(t > 0 : V(t) \leq V_{B1})$ is a stopping time representing the time of bankruptcy, and $F_{\tau_{B1}} \equiv E^Q[e^{-r\tau_{B1}}1_{\{\tau_{B1} < \infty\}}]$ is the state price of bankruptcy, showing the present value of one dollar upon the bankruptcy filing, where $1_{\{A\}}$ denotes the indicator function with value 1 if event A occurs, and with value 0 otherwise. The first term of Equation (2) demonstrates the present value of the coupon payments conditioned on the bankruptcy not occurring, and the second term shows the present value of the bond's recovery value if the issuer goes under. Note that the

recovery rate is weighted by the par values of the two bonds due to the equal priority, and the expectation is taken under the risk-neutral probability measure.

Next, the initial value of a noncallable convertible bond, $NCCB(V; V_{B1}, V_{Con})$, can be written as

$$NCCB(V; V_{B1}, V_{Con}) = E^Q\left[e^{-r\tau_{B1}}1_{\{\tau_{B1}<\tau_{Con}, \tau_{B1}<\infty\}}\frac{P_1}{P_1 + P_2}(1-\alpha)V_{B1}\right]$$

$$+ E^Q[e^{-r\tau_{Con}}1_{\{\tau_{Con}<\tau_{B1},\tau_{Con}<\infty\}}\gamma V_{Con}] + E^Q\left[\int_0^{\tau_{B1}\wedge\tau_{Con}} C_1 e^{-rt}\,dt\right]$$

$$(3)$$

where $t \wedge s \equiv \min(t, s)$ and $\tau_{Con} \equiv \inf[t > 0 : V(t) \geq V_{Con}]$ is a stopping time standing for the time of voluntary conversion. On the right-hand side of Equation (3), the first term denotes the discounted recovery value of the noncallable convertible bond when bankruptcy occurs prior to voluntary conversion. The second term represents the discounted voluntary conversion value when the block conversion happens before bankruptcy, and the last term designates the discounted value of the cumulative coupon payments, which may be truncated either by voluntary conversion or by bankruptcy.

Equation (3) can be rewritten as

$$NCCB(V; V_{B1}, V_{Con})$$

$$= \frac{C_1}{r} + E^Q\left[e^{-r\tau_{B1}}1_{\{\tau_{B1}<\tau_{Con},\tau_{B1}<\infty\}}\left(\frac{P_1}{P_1 + P_2}(1-\alpha)V_{B1} - \frac{C_1}{r}\right)\right]$$

$$+ E^Q\left[e^{-r\tau_{Con}}1_{\{\tau_{Con}<\tau_{B1},\tau_{Con}<\infty\}}\left(\gamma V_{Con} - \frac{C_1}{r}\right)\right] \qquad (4)$$

Because V_{B1} and V_{Con} are two constants, Equation (4) can be further simplified as follows:

$$NCCB(V; V_{B1}, V_{Con})$$

$$= \frac{C_1}{r} + \left(\frac{P_1}{P_1 + P_2}(1-\alpha)V_{B1} - \frac{C_1}{r}\right)G_{\tau_{B1}} + \left(\gamma V_{Con} - \frac{C_1}{r}\right)G_{\tau_{Con}} \qquad (5)$$

where $G_{\tau_{B1}} \equiv E^Q[e^{-r\tau_{B1}}1_{\{\tau_{B1}<\tau_{Con},\tau_{B1}<\infty\}}]$ is the risk-neutral state price of bankruptcy, which happens before voluntary conversion, and $G_{\tau_{Con}} \equiv E^Q[e^{-r\tau_{Con}}1_{\{\tau_{Con}<\tau_{B1},\tau_{Con}<\infty\}}]$ is the risk-neutral state price of voluntary conversion occurring prior to bankruptcy. When V_{Con} approaches infinity, meaning that the bondholder will never voluntarily convert, Equation (5)

would simply reduce to Equation (2), the pricing formula of a perpetual risky straight bond with the identical recovery rate. This further justifies the validity of Equation (5).

Following Leland (1994), the initial tax benefits of the future coupon payments, $\mathrm{TB}(V; V_{B1}, V_{\mathrm{Con}})$, and the initial value of the potential bankruptcy costs, $\mathrm{BC}(V; V_{B1}, V_{\mathrm{Con}})$, are viewed as two contingent claims upon the unlevered asset value. By the risk-neutral pricing method, the cumulative discounted tax benefits at the initial time can be represented by

$$
\begin{aligned}
\mathrm{TB}(V; V_{B1}, V_{\mathrm{Con}}) &= E^Q\left[\int_0^{\tau_{B1}\wedge\tau_{\mathrm{Con}}} \tau C_1 e^{-rt}\, dt + \int_0^{\tau_{B1}} \tau C_2 e^{-rt}\, dt\right] \\
&= \frac{\tau C_1}{r}(1 - G_{\tau_{B1}} - G_{\tau_{\mathrm{Con}}}) + \frac{\tau C_2}{r}(1 - F_{\tau_{B1}}) \quad (6)
\end{aligned}
$$

where τ is the constant effective tax rate for the bond issuer. In view of Equation (6), the first term is the tax benefits of the noncallable convertible bond, which are accumulated from the initial time to infinity and may be truncated either by bankruptcy or by voluntary conversion. The second term represents the tax benefits of the risky straight bond, which may be truncated only by bankruptcy. Similarly, the discounted bankruptcy costs at the initial time can be written as

$$
\mathrm{BC}(V; V_{B1}, V_{\mathrm{Con}}) = E^Q[e^{-r\tau_{B1}}\alpha V(\tau_{B1}) 1_{\{\tau_{B1}<\infty\}}] = \alpha V_{B1} F_{\tau_{B1}} \quad (7)
$$

Although the bankruptcy costs seem to be irrelevant to the voluntary conversion strategy, it will be shown later that the optimal bankruptcy strategy and optimal voluntary conversion strategy mutually interact.

The initial total firm value, $F_{\mathrm{NCCB}}(V; V_{B1}, V_{\mathrm{Con}})$, is therefore equal to the initial unlevered asset value plus the initial tax benefits less the initial bankruptcy costs; that is,

$$
F_{\mathrm{NCCB}}(V; V_{B1}, V_{\mathrm{Con}}) = V + \mathrm{TB}(V; V_{B1}, V_{\mathrm{Con}}) - \mathrm{BC}(V; V_{B1}, V_{\mathrm{Con}})
$$

Because of the accounting identity of the balance sheet, the initial equity value of the bond issuer, $E_{\mathrm{NCCB}}(V; V_{B1}, V_{\mathrm{Con}})$, must be equal to the initial total firm value minus the initial values of the noncallable convertible bond and straight bond; that is,

$$
\begin{aligned}
&E_{\mathrm{NCCB}}(V; V_{B1}, V_{\mathrm{Con}}) \\
&\quad = F_{\mathrm{NCCB}}(V; V_{B1}, V_{\mathrm{Con}}) - \mathrm{NCCB}(V; V_{B1}, V_{\mathrm{Con}}) - B(V; V_{B1})
\end{aligned}
$$

To complete the pricing formulas for $\text{NCCB}(V; V_{B1}, V_{\text{Con}})$, $F_{\text{NCCB}}(V; V_{B1}, V_{\text{Con}})$, and $E_{\text{NCCB}}(V; V_{B1}, V_{\text{Con}})$, the analytical expressions for $G_{\tau_{B1}}$, $G_{\tau_{\text{Con}}}$, and $F_{\tau_{B1}}$ are provided in the Appendix.

As for the optimal voluntary conversion policy, V_{Con}^*, and the optimal bankruptcy strategy, V_{B1}^*, the following smooth-pasting conditions are applied:

$$\frac{\partial E_{\text{NCCB}}(V; V_{B1}, V_{\text{Con}})}{\partial V}\Bigg|_{V=V_{B1}=V_{B1}^*} = \frac{\partial E_{\text{NCCB}}(V; V_{B1}, V_{\text{Con}})|_{V=V_{B1}=V_{B1}^*}}{\partial V_{B1}^*} = 0 \quad (8)$$

$$\frac{\partial \text{NCCB}(V; V_{B1}, V_{\text{Con}})}{\partial V}\Bigg|_{V=V_{\text{Con}}=V_{\text{Con}}^*} = \frac{\partial \text{NCCB}(V; V_{B1}, V_{\text{Con}})|_{V=V_{\text{Con}}=V_{\text{Con}}^*}}{\partial V_{\text{Con}}^*} = \gamma \quad (9)$$

These two conditions represent that at the initial time, the shareholder (the issuer) chooses V_{B1}^* to maximize the equity value, and the bondholder determines V_{Con}^* to maximize the value of the noncallable convertible bond, respectively. Furthermore, the Nash-equilibrium argument can be employed to explain the optimal strategies for voluntary conversion and bankruptcy. Given any V_{Con}^*, the shareholder determines the optimal bankruptcy strategy as a function of V_{Con}^*, denoted as $V_{B1}^*(V_{\text{Con}}^*)$; on the other hand, given any V_{B1}^*, the bondholder also decides the optimal conversion strategy as a function of V_{B1}^*, denoted as $V_{\text{Con}}^*(V_{B1}^*)$. Under the assumption that both the shareholder and the bondholder are fully informed, the optimal (Nash equilibrium) strategies for voluntary conversion and bankruptcy can be obtained by jointly solving Equations (8) and (9) numerically. Note that because the optimal strategies are independent of the unlevered asset value, they do not involve any uncertainty and will not vary with the time to maturity. This time-independent characteristic is consistent with a previous assumption. Finally, putting $V_{\text{Con}} = V_{\text{Con}}^*$ and $V_{B1} = V_{B1}^*$ back into Equation (5) finishes the derivation of the analytical valuation of a noncallable convertible bond subject to the issuer's default risk. Subsequently, a similar framework is applied to the call-forcing convertible bond.

VALUATION AND OPTIMAL STRATEGIES OF A CALL-FORCING CONVERTIBLE BOND

Consider a call-forcing convertible bond in which the issuer can decide when to go bankrupt and when to call the bonds back, but the bondholders cannot convert voluntarily. When the unlevered asset value goes up and touches the upper call trigger, V_{Call}, the issuer will announce to

call back the bonds. Meanwhile, the bondholder can then choose to either accept and receive the call price, K, or be forced to convert the bond into common shares and obtain the conversion value, γV_{Call}. On the other hand, when the unlevered asset value goes down and touches the lower default trigger, V_{B2}, the issuer will declare bankruptcy. Similarly, the risk-neutral pricing method implies the initial value of a call-forcing convertible bond, CFCB($V; V_{B2}, V_{Call}$), can be written as

$$\text{CFCB}(V; V_{B2}, V_{Call}) = E^Q\left[e^{-r\tau_{B2}}1_{\{\tau_{B2}<\tau_{Call}, \tau_{B2}<\infty\}}\frac{P_1}{P_1+P_2}(1-\alpha)V_{B2}\right]$$

$$+ E^Q[e^{-r\tau_{Call}}1_{\{\tau_{Call}<\tau_{B2}, \tau_{Call}<\infty\}}\max(\gamma V_{Call}, K)]$$

$$+ E^Q\left[\int_0^{\tau_{B2}\wedge\tau_{Call}} C_1 e^{-rt}\,dt\right] \tag{10}$$

where $\tau_{B2} \equiv \inf(t > 0 : V(t) \le V_{B2})$ and $\tau_{Call} \equiv \inf(t > 0 : V(t) \ge V_{Call})$ are also two stopping times that stand for time of bankruptcy and time of call, respectively. On the right-hand side of Equation (10), the first term denotes the discounted recovery value of a call-forcing convertible bond as the bankruptcy occurs prior to the call. The second term is the expected present value of the payoff at the time of call, where the payoff is equal to the maximum of the forced conversion value and redemption value. Here the issuer is assumed to stay solvent when the bond is redeemed for cash. The last term is the discounted value of the cumulative coupon payments, which may be truncated either by the call or by the bankruptcy of the issuer.

Similar to the previous section, Equation (10) can be simplified as follows:

$$\text{CFCB}(V; V_{B2}, V_{Call}) = \frac{C_1}{r} + \left(\frac{P_1}{P_1+P_2}(1-\alpha)V_{B2} - \frac{C_1}{r}\right)H_{\tau_{B2}}$$

$$+ \left(\max(\gamma V_{Call}, K) - \frac{C_1}{r}\right)H_{\tau_{Call}} \tag{11}$$

where $H_{\tau_{B2}} \equiv E^Q[e^{-r\tau_{B2}}1_{\{\tau_{B2}<\tau_{Call}, \tau_{B2}<\infty\}}]$ and $H_{\tau_{Call}} \equiv E^Q[e^{-r\tau_{Call}}1_{\{\tau_{Call}<\tau_{B2}, \tau_{Call}<\infty\}}]$ are the risk-neutral state price of bankruptcy before the call and that of call before the bankruptcy, respectively. The straight bond value and the total firm value in the call-forcing convertible bond case are, respectively, expressed as

$$B(V; V_{B2}) = \frac{C_2}{r}(1 - F_{\tau_{B2}}) + \frac{P_2}{P_1+P_2}(1-\alpha)V_{B2}F_{\tau_{B2}} \tag{12}$$

$$F_{\mathrm{CFCB}}(V; V_{B2}, V_{\mathrm{Call}})$$

$$= V + \frac{\tau C_1}{r}(1 - H_{\tau_{B2}} - H_{\tau_{\mathrm{Call}}}) + \frac{\tau C_2}{r}(1 - F_{\tau_{B2}})$$

$$+ (\tau \max(K - P_1, 0) - \beta P_1) 1_{\{\gamma V_{\mathrm{Call}} < K\}} F_{\tau_{\mathrm{Call}}} - \alpha V_{B2} F_{\tau_{B2}} \qquad (13)$$

where $F_{\tau_{B2}} \equiv E^Q[e^{-r\tau_{B2}}1_{\{\tau_{B2}<\infty\}}]$ and $F_{\tau_{\mathrm{Call}}} \equiv E^Q[e^{-r\tau_{\mathrm{Call}}}1_{\{\tau_{\mathrm{Call}}<\infty\}}]$ are two risk-neutral state prices as well. Equation (12) has similar meanings as Equation (2). The first term of Equation (13) is the initial unlevered asset value, and the second and third terms, respectively, represent the cumulative discounted tax benefits of the coupon payments for the call-forcing convertible bond and the straight bond, which may be truncated by the call or by the bankruptcy. Next, the fourth term stands for the additional discounted tax benefits of the call price over the par value minus the additional discounted refunding costs when the bond is redeemed for cash. The last term expresses the corresponding discounted bankruptcy costs. Again by using the accounting identity of the balance sheet, the initial equity value in this case, $E_{\mathrm{CFCB}}(V; V_{B2}, V_{\mathrm{Call}})$, can be calculated by the total firm value minus the values of the call-forcing convertible bond and straight bond. The analytical expressions for $H_{\tau_{B2}}$, $H_{\tau_{\mathrm{Call}}}$, $F_{\tau_{\mathrm{Call}}}$, and $F_{\tau_{B2}}$ are also given in the Appendix.

The optimal call and bankruptcy policies for the bond issuer can be determined by the corresponding smooth-pasting conditions:

$$\frac{\partial E_{\mathrm{CFCB}}(V; V_{B2}, V_{\mathrm{Call}})}{\partial V}\bigg|_{V=V_{B2}=V_{B2}^*} = \frac{\partial E_{\mathrm{CFCB}}(V; V_{B2}, V_{\mathrm{Call}})|_{V=V_{B2}=V_{B2}^*}}{\partial V_{B2}^*} = 0 \quad (14)$$

$$\frac{\partial E_{\mathrm{CFCB}}(V; V_{B2}, V_{\mathrm{Call}})}{\partial V}\bigg|_{V=V_{\mathrm{Call}}=V_{\mathrm{Call}}^*} = \frac{\partial E_{\mathrm{CFCB}}(V; V_{B2}, V_{\mathrm{Call}})|_{V=V_{\mathrm{Call}}=V_{\mathrm{Call}}^*}}{\partial V_{\mathrm{Call}}^*}$$

$$= \begin{cases} 1 - \gamma, & \text{if } \gamma V_{\mathrm{Call}}^* \geq K \\ 1, & \text{if } \gamma V_{\mathrm{Call}}^* < K \end{cases} \qquad (15)$$

Note that Equation (14) is similar to Equation (8), and Equation (15) is completely different from Equation (9). Equation (15) is equal to either $(1 - \gamma)$ when the forced conversion occurs, or 1 when the call redemption happens.[5] Equations (14) and (15) represent that the issuer makes decisions on the optimal call and bankruptcy strategies to maximize the equity value. As noted in Sarkar (2003), the shareholders of the firm must choose the optimal call policy to maximize the equity value rather than to minimize the convertible bond value (such as Ingersoll, 1977a,

[5]The equity value evaluated at V_{Call}^* is equal to $V_{\mathrm{Call}}^* - (\max(\gamma V_{\mathrm{Call}}^*, K) - C_1/r)$.

and Brennan and Schwartz, 1977). These two objectives are equivalent in a perfect capital market, but in a market with frictions (such as tax benefits, bankruptcy costs, and refunding costs), minimizing the convertible bond value does not imply maximizing the equity value.

By jointly solving V_{Call}^* and V_{B2}^* from Equations (14) and (15) and then substituting them back into Equation (11), this section completes the analytical valuation of the call-forcing convertible bond with consideration of the issuer's default risk. The next section is devoted to a discussion of the callable convertible bond to which the optimal strategies of the noncallable and call-forcing convertible bonds can be employed.

VALUATION AND OPTIMAL STRATEGIES OF A CALLABLE CONVERTIBLE BOND

In regard to a callable convertible bond, the optimal strategies for call, voluntary conversion, and bankruptcy all have to be determined. This is because the bond issuer can make decisions when to call the bond back and whether to declare bankruptcy, and the bondholders also have the flexibility to choose when to voluntarily convert the bond. For this purpose, Theorem VII of Ingersoll (1977a) is stated and proved here as Proposition 1, with our notations, and Proposition 2 is provided to show how the callable convertible bond can be deduced either from the noncallable convertible bond or from the call-forcing convertible bond, which are otherwise identical.

Proposition 1:   *Whenever it is optimal to voluntarily convert a noncallable convertible bond, it will also be optimal to convert a callable convertible bond that is otherwise identical.*

Proof.   First, recall that the optimal voluntary conversion strategy, V_{Con}^*, is chosen to maximize the value of the non-callable convertible bond by the bondholder. In addition, observe the fact the value of the noncallable convertible bond is no less than that of the callable convertible bond, which is otherwise the same, because the call feature will reduce the bond value. Accordingly, here $\text{NCCB}(V_{Con}^*) = \gamma V_{Con}^* \geq \text{NCCB}(V) \geq \text{CCB}(V)$. Therefore, *ceteris paribus*, V_{Con}^* would also be optimal for the holder of the otherwise-identical callable convertible bond to obtain the maximal bond value, γV_{Con}^*.   ❑

Proposition 2:   *Whenever it is optimal to call back a call-forcing convertible bond, it will also be optimal to call back a callable convertible bond that is otherwise identical.*

Proof. First, the optimal call policy, V_{Call}^*, is determined to maximize the bond issuer's equity value, which is not equivalent to but implies minimizing the bond value. In addition, because of the voluntary conversion option, the value of the callable convertible bond is no less than that of the call-forcing convertible bond, which is otherwise the same. As a result, $\text{CCB}(V) \geq \text{CFCB}(V) \geq \text{CFCB}(V_{\text{Call}}^*) = \max(\gamma V_{\text{Call}}^*, K)$. Finally, *ceteris paribus*, V_{Call}^* would also be optimal for the bond issuer of the otherwise-identical callable convertible bond to obtain the minimal bond value, $\max(\gamma V_{\text{Call}}^*, K)$. ❑

Assuming the above propositions are common knowledge for both the bond issuer and the bondholder, it can be concluded that if the optimal call trigger is greater than or equal to the optimal voluntary conversion trigger, the voluntary conversion may happen, whereas the call will not. That is to say, the valuation of the callable convertible bond is exactly that of the noncallable convertible bond, which is otherwise identical. On the other hand, if the optimal call trigger is less than the optimal voluntary conversion trigger, the valuation of the callable convertible bond is just that of the call-forcing convertible bond, which is otherwise the same. In summary, the analytical valuation of a callable convertible bond subject to the issuer's default risk can be expressed as follows:

$$\text{CCB}(V; V_B^*, V_{\text{Con}}^*, V_{\text{Call}}^*) = \begin{cases} \text{NCCB}(V; V_{B1}^*, V_{\text{Con}}^*), \text{ if } V_{B1}^* < V < V_{\text{Con}}^* \leq V_{\text{Call}}^* \\ \text{CFCB}(V; V_{B2}^*, V_{\text{Call}}^*), \text{ if } V_{B2}^* < V < V_{\text{Call}}^* < V_{\text{Con}}^* \end{cases} \quad (16)$$

In what follows, a call notice period is incorporated into this structural model. The call notice period (usually 30 days) can be used to partially explain the observed call policies in the market, particularly late calls.

When an in-the-money call is announced by the bond issuer, the bondholder is given an implicit put, which entitles him or her to sell the convertibles back to the bond issuer at the call price plus the accrued coupons. Because the payment will not be rendered until the call date regardless of when the bondholder decides not to convert, the put is therefore a European-style option. Butler (2002) and Lau and Kwok (2004) both provide models to show longer call notice periods will result in greater optimal call triggers. As for the empirical studies, such as Asquith (1995) and Altintig and Butler (2005), the call notice period will give a notable impact on the valuation and the late calls of the callable convertible bond.

Taking a call notice period into consideration will modify the original valuation and optimal strategies of the call-forcing convertible bond.

Following the concept of Ingersoll (1977b), first the value of the call-forcing convertible bond at the call announcement date is denoted by $\text{CFCB}(V_{\text{Call}}; T, K_T)$, where T is the length of the call notice period, and K_T is the effective call price including the after-tax accrued coupons paid at the end of the period; that is, $K_T = K + T \times C_1$. Therefore, the price of the call-forcing convertible bond at the end of the period would be $\max[\gamma V(\tau_{\text{Call}} + T), KT]$, and the value evaluated at the time of call can be priced via the risk-neutral pricing method as below:

$$\begin{aligned}
\text{CFCB}&(V_{\text{Call}}; T, K_T) \\
&= e^{-rT} E^Q[\max(\gamma V(\tau_{\text{Call}} + T), K_T)|V_{\text{Call}}] \\
&= \gamma V_{\text{Call}} e^{-qT} + e^{-rT} E^Q[\max(K_T - \gamma V(\tau_{\text{Call}} + T), 0)|V_{\text{Call}}] \\
&= \gamma V_{\text{Call}} e^{-qT} + (-\gamma V_{\text{Call}} e^{-qT} N(-d_1) + K_T e^{-rT} N(-d_2)) \qquad (17)
\end{aligned}$$

where

$$d_1 = (\ln(\gamma V_{\text{Call}}/K_T) + (r - q + 0.5\sigma^2)T)/\sigma\sqrt{T}$$

$$d_2 = (\ln(\gamma V_{\text{Call}}/K_T) + (r - q - 0.5\sigma^2)T)/\sigma\sqrt{T}$$

and $N(\cdot)$ denotes the cumulative standard normal distribution. In view of Equation (17), the value of the call-forcing convertible bond, evaluated at the call announcement date, approximates the conversion value plus a European put option on the conversion value with strike price K_T, the effective call price. As a result, the valuation of the call-forcing convertible bond with a call notice period must be modified as

$$\begin{aligned}
\text{CFCB}(V; V_{B2}, V_{\text{Call}}, T, K_T) &= \frac{C_1}{r} + \left(\frac{P_1}{P_1 + P_2}(1 - \alpha)V_{B2} - \frac{C_1}{r}\right)H_{\tau_{B2}} \\
&\quad + \left(\text{CFCB}(V_{\text{Call}}; T, K_T) - \frac{C_1}{r}\right)H_{\tau_{\text{Call}}} \qquad (11')
\end{aligned}$$

and the corresponding total firm value is

$$\begin{aligned}
F_{\text{CFCB}}&(V; V_{B2}, V_{\text{Call}}, T, K_T) \\
&= V + \frac{\tau C_1}{r}(1 - H_{\tau_{B2}} - H_{\tau_{\text{Call}}}) + \frac{\tau C_2}{r}(1 - F_{\tau_{B2}}) - \alpha V_{B2} F_{\tau_{B2}} \\
&\quad + (\tau \max(K_T - P_1, 0) - \beta P_1)e^{-rT} F_{\tau_{\text{Call}}} N(-d_2) \qquad (13')
\end{aligned}$$

The last term of the right-hand side of Equation (13′) shows the additional value upon the call announcement, including not only the potential tax benefits but also the proportional refunding costs from the cash redemption. Note that the additional tax benefits and refunding costs

would be relevant only when the call is announced, and in turn the bond is redeemed for cash at the end of the call notice period.[6] By the same token, the modified value of the equity, $E_{\text{CFCB}}(V; V_{B2}, V_{\text{Call}}, T, K_T)$, is equal to the modified total firm value minus the sum of the straight bond value and the modified value of the call-forcing convertible bond. Moreover, the optimal call and bankruptcy strategies for the bond issuer with a call notice period are given by the solutions of the corresponding smooth-pasting conditions:

$$
\left. \frac{\partial E_{\text{CFCB}}(V; V_{B2}, V_{\text{Call}}, T, K_T)}{\partial V} \right|_{V = V_{B2} = V_{B2}^*}
$$

$$
= \frac{\partial E_{\text{CFCB}}(V; V_{B2}, V_{\text{Call}}, T, K_T)|_{V = V_{B2} = V_{B2}^*}}{\partial V_{B2}^*} = 0 \qquad (14')
$$

$$
\left. \frac{\partial E_{\text{CFCB}}(V; V_{B2}, V_{\text{Call}}, T, K_T)}{\partial V} \right|_{V = V_{\text{Call}} = V_{\text{Call}}^*}
$$

$$
= \frac{\partial E_{\text{CFCB}}(V; V_{B2}, V_{\text{Call}}, T, K_T)|_{V = V_{\text{Call}} = V_{\text{Call}}^*}}{\partial V_{\text{Call}}^*}
$$

$$
= 1 + (\tau \max(K_T - P_{1,}) - \beta P_1) e^{-rT} \frac{\partial N(-d_2|_{V = V_{\text{Call}} = V_{\text{Call}}^*})}{\partial V_{\text{Call}}^*}
$$

$$
- \frac{\partial \text{CFCB}(V_{\text{Call}}^*; T, K_T)}{\partial V_{\text{Call}}^*} - \frac{\partial B(V_{\text{Call}}^*; V_{B2})}{\partial V_{\text{Call}}^*} \qquad (15')
$$

Note that Equation (14′) is similar to Equation (14), whereas Equation (15′) is significantly different from Equation (15) because of the call notice period requirement.

It is worth noticing that the optimal voluntary conversion is not affected by the call notice period, which therefore does not alter the valuation of the noncallable convertible bond. As for the callable convertible bond with a call notice period, the valuation and the optimal strategies must be further modified. If the optimal voluntary conversion trigger is less than or equal to the optimal call trigger, the issuer may decide to call back the bond at the same time as when the bondholder voluntarily converts the bond into common stocks. In contrast, this will not occur when there is no call notice period requirement. To see how this could happen, first assume the case that the optimal call trigger is no less than

[6]The conditional probability, $F_{\tau_{\text{Call}}} N(-d_2)$, showing the probability the call not only is announced, but also will be redeemed for cash, is simply equal to the product of the probabilities of the two events via the Markov property of a Wiener process.

the optimal voluntary conversion trigger and that the bondholder would convert the bond at the end of the call notice period. In this case, the issuer will choose the suboptimal call policy, that is, call back the bond at the last instant before the bondholder voluntarily converts the bond, and therefore he can save the payouts during the call notice period.[7] The expected present value of the savings can therefore be calculated as

$$\int_{\tau_{\text{Call}}^{**}}^{\tau_{\text{Call}}^{**}+T} e^{-rt}qE^{Q}[\gamma V(t)|V_{\text{Call}}^{**} = V_{\text{Con}}^{*}]\, dt = \gamma V_{\text{Call}}^{**}(1 - e^{-qT})$$

which is greater than 0 whenever the payout ratio q is positive, where V_{Call}^{**} is the suboptimal call policy of the issuer (identical to the optimal voluntary conversion strategy of the bondholder), and $\tau_{\text{Call}}^{**} \equiv \inf\{t > 0 : V(t) \geq V_{\text{Call}}^{**} = V_{\text{Con}}^{*}\}$ is the corresponding time of call. As a result, the callable convertible bond with a call notice period will turn into the call-forcing convertible bond with the additional suboptimal call strategy, which is otherwise the same.

According to the above analysis, the value of the callable convertible bond with a call notice period can be modified by

$$\text{CCB}(V; V_B^*, V_{\text{Con}}^* = V_{\text{Call}}^{**}, V_{\text{Call}}^*, T, K_T)$$

$$= \begin{cases} \text{CFCB}(V; V_{B2}^{**}, V_{\text{Call}}^{**} = V_{\text{Con}}^{*}, T, K_T), & \text{if } V_{B2}^{**} < V < V_{\text{Call}}^{**} \leq V_{\text{Call}}^{*} \\ \text{CFCB}(V; V_{B2}^{*}, V_{\text{Call}}^{*}, T, K_T), & \text{if } V_{B2}^{*} < V < V_{\text{Call}}^{*} < V_{\text{Call}}^{**} \end{cases} \quad (16')$$

Notice that V_{B2}^{**} is the suboptimal bankruptcy strategy, which can be numerically solved from Equation (14′), given that $V_{\text{Call}} = V_{\text{Call}}^{**}$. In addition, Equation (16′) implicitly clarifies the interaction of the optimal strategies of the callable convertible bond. For example, the optimal voluntary conversion strategy, derived from the noncallable convertible bond, is the suboptimal call strategy, and hence has a significant impact on the optimal call and bankruptcy policies as well as on the valuation of the callable convertible bond.

NUMERICAL ANALYSIS

In this section, the optimal strategies of the model are first characterized, and then some numerical examples are implemented to analyze the optimal strategies for the callable convertible bond. In particular,

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[7]The present article focuses on the case mentioned here, but other possibilities, such as the occurrence of voluntary conversion, could also be addressed in the present framework.

comparative static analyses are conducted with respect to the parameters that are rarely discussed in the previous theoretical literature. Once the optimal strategies are obtained, the corresponding value of the callable convertible bond can be easily calculated by the present pricing formula, and is therefore omitted from the numerical analysis.

The optimal strategies for call, voluntary conversion, and bankruptcy, which are numerically solved from Equations (14′) and (15′),[8] may result in multiple solutions due to the nonlinearity of the equations. In these numerical cases with reasonable parameters, there are at most three possible optimal call policies and one optimal voluntary conversion strategy, together with the corresponding optimal bankruptcy strategy, for the callable convertible bond. It is worth noting that only one optimal call strategy takes effect according to the initial unlevered asset values, which would be further explained later. The relationship between the optimal call and voluntary conversion strategies can be ranked as $V_{\text{Call}}^{*,1} > V_{\text{Con}}^{*} > V_{\text{Call}}^{*,2} > K/\gamma > V_{\text{Call}}^{*,3}$, where K/γ is the optimal call trigger provided by Ingersoll (1977a). In the base case of the present model, the parameters are as follows: $P_1 = 50$, $C_1 = 1.5$ (3%), $P_2 = 250$, $C_2 = 20$ (8%), $\tau = 0.35$, $\alpha = 0.5$, $\beta = 0.2$, $r = 5\%$, $q = 3\%$, $\sigma = 0.2$, $K = 55$, $T = 30$ days (1/12 years), and $\gamma = 0.1$.[9] All of the parameters in this article are the same as the base case, unless otherwise stated. Besides, in the numerical calculation of this article, the desired pricing formulas involving some infinite series (from zero to infinity) have been replaced with the finite series (assumed from 0 to 30).[10] In the base case, the three optimal call policies (the corresponding optimal bankruptcy strategies) ranked from high to low are: $V_{\text{Call}}^{*,1} = 939.32$ (177.84), $V_{\text{Call}}^{*,2} = 582.20$ (179.08), and $V_{\text{Call}}^{*,3} = 477.78$ (182.55). According to the previous section, the suboptimal call policy (the corresponding bankruptcy trigger), that is, the optimal voluntary conversion strategy, is $V_{\text{Call}}^{**} = V_{\text{Con}}^{*} = 686.95$ (178.56). Here the optimal call trigger of Ingersoll (1977a), K/γ, is 550, and the relationship mentioned above, as $V_{\text{Call}}^{*,1} > V_{\text{Call}}^{**} > V_{\text{Call}}^{*,2} > K/\gamma > V_{\text{Call}}^{*,3}$, is confirmed. Although there may be three possible optimal call policies, only one would be realized. By the previous section, the

[8]The smooth-pasting conditions are checked for satisfaction with the precision at the 1×10^{-4} level.

[9]All of the parameters are selected to be generally consistent with the 292 data of Altintig and Butler (2005).

[10]The similar finite sums from 0 to 6 in Kolkiewicz (2002) achieve the precision at the 1×10^{-4} level. For this numerical analysis, it takes an extremely short time to calculate the finite sums from 0 to 30 with stable values.

value of the callable convertible bond is one of the following four call-forcing convertible bond values, expressed as follows:

$$\text{CCB}(V; V_B^*, V_{\text{Con}}^*, V_{\text{Call}}^*, T, K_T)$$

$$= \begin{cases} \text{CFCB}(V; V_{B2}^{*,3}, V_{\text{Call}}^{*,3}, T, K_T), & \text{if } V_{B2}^{*,3} < V \leq V_{\text{Call}}^{*,3}.\,(\text{early calls}) \\ \text{CFCB}(V; V_{B2}^{*,2}, V_{\text{Call}}^{*,2}, T, K_T), & \text{if } V_{\text{Call}}^{*,3} < V \leq V_{\text{Call}}^{*,2}.\,(\text{late calls}) \\ \text{CFCB}(V; V_{B2}^{**}, V_{\text{Call}}^{**}, T, K_T), & \text{if } V_{\text{Call}}^{*,2} < V \leq V_{\text{Call}}^{**}.\,(\text{late calls}) \\ \text{CFCB}(V; V_{B2}^{*,1}, V_{\text{Call}}^{*,1}, T, K_T), & \text{if } V_{\text{Call}}^{**} < V \leq V_{\text{Call}}^{*,1}.\,(\text{late calls}) \end{cases}$$

There are four possibilities (one early call and three late calls) in the base case. For example, other things being equal, if the issuer's initial unlevered asset value is relatively low, say 450, then $V_{B2}^{*,3} < V \leq V_{\text{Call}}^{*,3}$. Therefore, the optimal call and bankruptcy policies of the issuer are $V_{\text{Call}}^{*,3}$ and $V_{B2}^{*,3}$, and the value of the callable convertible bond is equal to that of the call-forcing convertible bond with $V_{\text{Call}}^{*,3}$ and $V_{B2}^{*,3}$. This is the case of early call due to $V_{\text{Call}}^{*,3} < K/\gamma$. Similarly, if the unlevered asset value is relatively high, say 950, then $V_{\text{Call}}^{**} < V < V_{\text{Call}}^{*,1}$. Accordingly, the optimal call and bankruptcy policies are $V_{\text{Call}}^{*,1}$ and $V_{B2}^{*,1}$, which would be the case of late calls because of $K/\gamma < V_{\text{Call}}^{*,1}$.

In empirical studies, most callable convertible bonds are called too late with respect to the optimal call policy provided by Ingersoll (1977a). For example, Ingersoll (1977b) first reports the evidence from the data of 179 convertibles that firms would not announce the call until the conversion value is much higher than the call price. The excess call premium, defined as the premium of the conversion value over the call price, has a median of 44%. Recently, Altintig and Butler (2005), collecting data from 1986 to 2000, indicate the 25th, 50th, and 75th percentiles of the excess call premium are 22%, 35%, and 54%, respectively. In the above studies, the excess call premium is calculated only from the cases of late calls, and in the present base case, the three possible cases of late calls have the average excess call premium of 34%, which generally coincides with the empirical findings. On the other hand, the early call discount, defined as the premium of the call price over the conversion value, is about 13% in the base case. This agrees with Cowan, Nayar, and Singh (1993), who gather the out-of-the-money calls from 1963 to 1987 and show the median of the early call discount is about 14%.

To compare with Ingersoll (1977a) in a perfect setting, the tax benefits, bankruptcy costs, refunding costs, call notice period, and debt size of the straight bond are all ignored; that is, $\tau = \alpha = \beta = T = P_2 = C_2 = 0$. The optimal call and voluntary conversion strategies (the corresponding

optimal bankruptcy strategies) then have the following relationship: $V_{\text{Call}}^{*,1} = V_{\text{Con}}^* = 814.64\ (18.54) > K/\gamma = 550 = V_{\text{Call}}^{*,2} = 550.00\ (18.52)$. In other words, if the initial unlevered asset value is between 18.52 and 550 in this setting of the perfect market, the optimal call policy of the present model is equivalent to that of Ingersoll (1977a). On the other hand, if the unlevered asset value is greater than 550, the optimal call policy is $V_{\text{Call}}^{*,1} = V_{\text{Con}}^*$, resulting in the case of late call. When $P_2 = 250$, the relationship will change to $V_{\text{Call}}^{*,1} = V_{\text{Con}}^* = 874.74\ (274.46) > K/\gamma = 550 = V_{\text{Call}}^{*,2} = 550.00\ (272.42)$, showing the debt size and debt structure of the issuer have no impact on the second optimal call policy because the tax benefits, bankruptcy costs, and refunding costs are all neglected. However, the first optimal call policy is affected, and there is a significant impact on the corresponding optimal bankruptcy policy. The effects of these parameters will be further investigated later.

Comparative Static Analysis

Table I presents the effects of the tax benefits, bankruptcy costs, and refunding costs on the optimal strategies, which are all normalized by the optimal call trigger of Ingersoll (1977a). The effect of tax rates on the second optimal call policy is consistent with the empirical finding of Ederington, Gaton, and Campbell (1997), showing greater tax rates result in higher optimal call triggers. This can be explained via the cash flow advantage hypothesis; that is, the net cash flow resulting from increasing tax rates is positive. On the other hand, the first and third optimal call policies as well as all the optimal bankruptcy strategies are decreasing as the tax rate rises, whereas the suboptimal call policy is generally positively correlated with the tax rate due to the voluntary conversion of the bondholder. Next, the increasing proportional bankruptcy cost has a positive impact on the first and third optimal call policies, but a negative impact on the second and suboptimal call policies. The effect of bankruptcy costs on the optimal bankruptcy policy is unclear yet small. Finally, the refunding costs have a positive (negative) impact on the second optimal call policy (the third optimal call policy), but have a negative (positive) effect on the corresponding bankruptcy strategies. In addition, the first optimal and the suboptimal call policies and the corresponding bankruptcy strategies are uncorrelated with the refunding costs.

Table II demonstrates the effects of the coupon payments of the convertible bond and the par value and coupon payments of the straight bond on the optimal strategies, which are all normalized by the optimal

TABLE I

Effects of the Tax Rates, Proportional Bankruptcy Costs, and Proportional Refunding Costs on the Optimal Strategies for the Callable Convertible Bond[a]

	$\alpha = 0.2$		$\alpha = 0.5$		$\alpha = 0.8$	
	$\beta = 0.1$	$\beta = 0.2$	$\beta = 0.1$	$\beta = 0.2$	$\beta = 0.1$	$\beta = 0.2$
$\tau = 0.3$	$V^{*,1}_{Call} = 1.6792(0.3485)$[b] $V^{**}_{Call} = 1.3735(0.3492)$ $V^{*,2}_{Call} = 1.0211(0.3504)$ $V^{*,3}_{Call} = 0.8588(0.3555)$	$V^{*,1}_{Call} = 1.6792(0.3485)$ $V^{**}_{Call} = 1.3735(0.3492)$ $V^{*,2}_{Call} = 1.0682(0.3501)$ $V^{*,3}_{Call} = 0.8025(0.3600)$	$V^{*,1}_{Call} = 1.7719(0.3484)$ $V^{**}_{Call} = 1.2144(0.3500)$ $V^{*,3}_{Call} = 0.9442(0.3533)$	$V^{*,1}_{Call} = 1.7719(0.3484)$ $V^{**}_{Call} = 1.2144(0.3500)$ $V^{*,2}_{Call} = 1.0540(0.3510)$ $V^{*,3}_{Call} = 0.8862(0.3572)$	$V^{*,1}_{Call} = 1.8551(0.3483)$	$V^{*,1}_{Call} = 1.8551(0.3483)$ $V^{*,2}_{Call} = 1.0396(0.3520)$ $V^{*,3}_{Call} = 0.9196(0.3568)$
$\tau = 0.35$	$V^{*,1}_{Call} = 1.6291(0.3234)$ $V^{**}_{Call} = 1.3732(0.3241)$ $V^{*,2}_{Call} = 1.0236(0.3252)$ $V^{*,3}_{Call} = 0.8403(0.3306)$	$V^{*,1}_{Call} = 1.6291(0.3234)$ $V^{**}_{Call} = 1.3732(0.3241)$ $V^{*,2}_{Call} = 1.0708(0.3249)$ $V^{*,3}_{Call} = 0.7855(0.3349)$	$V^{*,1}_{Call} = 1.7079(0.3233)$ $V^{**}_{Call} = 1.2490(0.3247)$ $V^{*,3}_{Call} = 0.9186(0.3286)$	$V^{*,1}_{Call} = 1.7079(0.3233)$ $V^{**}_{Call} = 1.2490(0.3247)$ $V^{*,2}_{Call} = 1.0585(0.3256)$ $V^{*,3}_{Call} = 0.8687(0.3319)$	$V^{*,1}_{Call} = 1.7795(0.3233)$ $V^{**}_{Call} = 0.9690(0.3262)$	$V^{*,1}_{Call} = 1.7795(0.3233)$ $V^{*,2}_{Call} = 1.0465(0.3263)$ $V^{**}_{Call} = 0.9690(0.3262)$ $V^{*,3}_{Call} = 0.9052(0.3313)$
$\tau = 0.4$	$V^{*,1}_{Call} = 1.5721(0.2984)$ $V^{**}_{Call} = 1.3755(0.2989)$ $V^{*,2}_{Call} = 1.0282(0.3000)$ $V^{*,3}_{Call} = 0.8115(0.3061)$	$V^{*,1}_{Call} = 1.5721(0.2984)$ $V^{**}_{Call} = 1.3755(0.2989)$ $V^{*,2}_{Call} = 1.0749(0.2998)$ $V^{*,3}_{Call} = 0.7600(0.3100)$	$V^{*,1}_{Call} = 1.6386(0.2983)$ $V^{**}_{Call} = 1.2796(0.2993)$ $V^{*,2}_{Call} = 1.0082(0.3007)$ $V^{*,3}_{Call} = 0.8875(0.3039)$	$V^{*,1}_{Call} = 1.6386(0.2983)$ $V^{**}_{Call} = 1.2796(0.2993)$ $V^{*,2}_{Call} = 1.0643(0.3002)$ $V^{*,3}_{Call} = 0.8380(0.3071)$	$V^{*,1}_{Call} = 1.6998(0.2982)$ $V^{**}_{Call} = 1.1358(0.3000)$ $V^{*,2}_{Call} = 0.9520(0.3025)$	$V^{*,1}_{Call} = 1.6998(0.2982)$ $V^{**}_{Call} = 1.1358(0.3000)$ $V^{*,2}_{Call} = 1.0542(0.3007)$ $V^{*,3}_{Call} = 0.8844(0.3059)$

[a] The optimal triggers are all normalized by the optimal call trigger of Ingersoll (1977a), which is equal to $K/\gamma = 550$.

[b] The values within the parentheses represent the corresponding optimal bankruptcy triggers normalized by the optimal call trigger of Ingersoll (1977a).

TABLE II

Effects of the Coupon Payments of the Convertible Bond, and the Par Value and Coupon Payments of the Straight Bond on the Optimal Strategies for the Callable Convertible Bond[a]

	$P_2 = 230$		$P_2 = 250$		$P_2 = 270$	
	$C_2 = 18.4(8\%)$	$C_2 = 20.7(9\%)$	$C_2 = 20(8\%)$	$C_2 = 22.5(9\%)$	$C_2 = 21.6(8\%)$	$C_2 = 24.3(9\%)$
$C_1 = 1.5$ (3%)	$V_{Call}^{*,1} = 1.6200(0.2989)$[b] $V_{Call}^{**} = 1.2921(0.2997)$ $V_{Call}^{*,2} = 1.0721(0.3005)$ $V_{Call}^{*,3} = 0.7807(0.3102)$	$V_{Call}^{*,1} = 1.7531(0.3340)$ $V_{Call}^{**} = 1.2548(0.3355)$ $V_{Call}^{*,2} = 1.0512(0.3365)$ $V_{Call}^{*,3} = 0.8932(0.3419)$	$V_{Call}^{*,1} = 1.7079(0.3233)$ $V_{Call}^{**} = 1.2490(0.3247)$ $V_{Call}^{*,2} = 1.0585(0.3256)$ $V_{Call}^{*,3} = 0.8687(0.3319)$	$V_{Call}^{*,1} = 1.8541(0.3615)$ $V_{Call}^{**} = 1.1935(0.3637)$ $V_{Call}^{*,2} = 1.0327(0.3652)$ $V_{Call}^{*,3} = 0.9297(0.3692)$	$V_{Call}^{*,1} = 1.7969(0.3478)$ $V_{Call}^{**} = 1.1921(0.3497)$ $V_{Call}^{*,2} = 1.0437(0.3509)$ $V_{Call}^{*,3} = 0.9105(0.3559)$	$V_{Call}^{*,1} = 1.9560(0.3889)$ $V_{Call}^{**} = 1.0968(0.3924)$ $V_{Call}^{*,2} = 1.0012(0.3948)$
$C_1 = 2$ (4%)	$V_{Call}^{*,1} = 1.8206(0.3053)$ $V_{Call}^{**} = 1.8083(0.3053)$ $V_{Call}^{*,2} = 1.0597(0.3065)$ $V_{Call}^{*,3} = 0.8426(0.3124)$	$V_{Call}^{*,1} = 1.9446(0.3403)$ $V_{Call}^{**} = 1.7781(0.3406)$ $V_{Call}^{*,2} = 1.0397(0.3423)$ $V_{Call}^{*,3} = 0.9147(0.3461)$	$V_{Call}^{*,1} = 1.9020(0.3296)$ $V_{Call}^{**} = 1.7785(0.3299)$ $V_{Call}^{*,2} = 1.0470(0.3314)$ $V_{Call}^{*,3} = 0.8980(0.3358)$	$V_{Call}^{*,1} = 2.0397(0.3677)$ $V_{Call}^{**} = 1.7399(0.3683)$ $V_{Call}^{*,2} = 1.0191(0.3708)$	$V_{Call}^{*,1} = 1.9854(0.3540)$ $V_{Call}^{**} = 1.7440(0.3545)$ $V_{Call}^{*,2} = 1.0319(0.3566)$ $V_{Call}^{*,3} = 0.9287(0.3601)$	$V_{Call}^{*,1} = 2.1365(0.3951)$ $V_{Call}^{**} = 1.6942(0.3962)$
$C_1 = 2.5$ (5%)	$V_{Call}^{**} = 2.3157(0.3116)$ $V_{Call}^{*,1} = 2.0431(0.3119)$ $V_{Call}^{*,2} = 1.0484(0.3123)$ $V_{Call}^{*,3} = 0.8866(0.3161)$	$V_{Call}^{**} = 2.2888(0.3465)$ $V_{Call}^{*,1} = 2.1565(0.3467)$ $V_{Call}^{*,2} = 1.0276(0.3479)$	$V_{Call}^{**} = 2.2912(0.3359)$ $V_{Call}^{*,1} = 2.1170(0.3361)$ $V_{Call}^{*,2} = 1.0356(0.3371)$ $V_{Call}^{*,3} = 0.9193(0.3402)$	$V_{Call}^{**} = 2.2584(0.3740)$ $V_{Call}^{*,1} = 2.2445(0.3741)$	$V_{Call}^{**} = 2.2639(0.3603)$ $V_{Call}^{*,1} = 2.1938(0.3604)$ $V_{Call}^{*,2} = 1.0184(0.3622)$	$V_{Call}^{*,1} = 2.3351(0.4014)$ $V_{Call}^{**} = 2.2240(0.4016)$

[a]The optimal triggers are all normalized by the optimal call trigger of Ingersoll (1977a), which is equal to $K/\gamma = 550$.

[b]The values within the parentheses represent the corresponding optimal bankruptcy triggers normalized by the optimal call trigger of Ingersoll (1977a).

call trigger of Ingersoll (1977a). First of all, the increase of the convertible bond coupon payments exerts a negative effect on the second optimal call policy, which again agrees well with the empirical finding of Sarkar (2003), whereas it has a positive impact on the first and third optimal and suboptimal call policies. Next, the par value of the straight bond, which represents the leverage/debt ratio of the issuer, is positively correlated with the first and third optimal call policies, but is negatively correlated with the suboptimal and second optimal call policies. In particular, the effect of the debt ratio on the second optimal call policy is quite consistent to a certain extent with Altintig and Butler (2005), showing the higher the debt to asset ratio, the lower the optimal call trigger. The reason might be that highly levered firms delay calling to avoid the risk that the stock price might fall, precipitating redemptions from the convertible bondholders. In addition, the effect of the straight bond coupon payments on all the optimal call policies totally agrees with that of the straight bond par value. Finally, all the above payments of the bond issuer have a significant positive effect on the corresponding optimal bankruptcy strategies, in line with Leland (1994), which only considers risky straight bonds.

Remarks

In general, there is one perspective on the numerical analysis—the behavior of the second optimal call policy is generally consistent with the empirical findings of late calls. This is because the second optimal call policy of the present model is suitable for the issuer with the middle unlevered asset value. In other words, the observed behavior of late calls, on average, is in line with the second optimal call strategy. All the empirical studies do not recognize that the bond issuer may adopt entirely distinct optimal call policies in different situations. For example, in the present base case, the issuer with a higher initial unlevered asset value (lower leverage ratio), say 700, should choose the first optimal call policy with the excess call premium of 71%; on the other hand, the issuer with a lower initial unlevered asset value (higher leverage ratio), say 500, is suitable for the second optimal call policy with the excess call premium of 5.9%.

There is a particular observation that the first optimal call policy sometimes behaves like the suboptimal call policy (the voluntary conversion strategy of the bondholder), in the cases with parameters associated with the redemption, such as the call price and the call notice period. However, the behavior of first optimal call policy is not in line with that of the suboptimal call policy, in the cases with parameters concerning the issuer's capital structure such as the tax rates and proportional

bankruptcy costs. The reason might be that both the bond issuer and bondholder care about the redemption of the bond, whereas the firm's capital structure is not the major concern of the bondholder.

CONCLUSION

This article constructs a structural model to derive an analytical valuation and optimal strategies of a callable convertible bond by the pricing method of double-barrier options, where the call, voluntary conversion, and bankruptcy can occur at anytime of the bond's duration. Not only does the shareholder endogenously determine the optimal call and bankruptcy policies as the equity value is maximized, but also the convertible bondholder obtains the optimal voluntary conversion strategy as the convertible bond value is maximized. This model further takes account of the bankruptcy costs, tax benefits, and capital structure of the bond issuer as well as the refunding costs and a call notice period. It is shown that the callable convertible bond can be reduced to either the otherwise-identical noncallable convertible bond if the optimal voluntary conversion trigger is less than or equal to the optimal call trigger, or the otherwise-identical call-forcing convertible bond, when the optimal voluntary conversion trigger is greater than the optimal call trigger.

The numerical results show that the optimal call policy of the present model agrees well with Ingersoll (1977a) in a perfect environment. When the bankruptcy costs, tax benefits, refunding costs, debt structure, and a call notice period are incorporated, the model can be further used to explain the empirical findings of late calls (Ederington et al., 1997; Sarkar, 2003; Altintig and Butler, 2005) and early calls (Cowan et al., 1993). Therefore, calling convertible bonds too late or too early can be rational.

Undoubtedly, this model provides additional corporate financing insights and complements earlier studies about the valuation and optimal strategies of a callable convertible bond with the issuer's credit risk. It would be an important but challenging work to extend this model to take account of finite maturity convertibles and stochastic interest rates.

APPENDIX[11]

This Appendix provides the analytical expressions of $G_{\tau_{B1}}$, $G_{\tau_{Con}}$, $F_{\tau_{B1}}$, $H_{\tau_{B2}}$, $H_{\tau_{Call}}$, $F_{\tau_{Call}}$, and $F_{\tau_{B2}}$, which can be divided into two parts. One is related to the distribution of a stopping time of a standard Brownian motion with a

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[11]In the Appendix, all notations are defined in the text. The derivation of the formulas is rather lengthy; it is available upon request.

constant drift and diffusion. The other is associated with the joint distribution of two stopping times of a standard Brownian motion with a constant drift and diffusion.

For the first part, many books that discuss Brownian motions would provide the desired distribution, such as Chapter 1 of Harrison (1985). By using the Girsanov theorem, the technique of completing the squares, and the Fubini theorem, the following formulas can be derived

$$F_{\tau_{B1}} = \left(\frac{V}{V_{B1}}\right)^{-\lambda^*-\lambda/\sigma^2}, \quad F_{\tau_{Call}} = \left(\frac{V_{Call}}{V}\right)^{\lambda-\lambda^*/\sigma^2}, \quad \text{and} \quad F_{\tau_{B2}} = \left(\frac{V}{V_{B2}}\right)^{-\lambda^*-\lambda/\sigma^2}$$

where $\lambda = r - q - 0.5\sigma^2$, and $\lambda^* = \sqrt{\lambda^2 + 2r\sigma^2}$.

As for the second part, refer to Kolkiewicz (2002), where the desired distributions are provided in a systematic way. Note the reasonable assumption that the upper barrier (the call or voluntary conversion triggers) and the lower barrier (the bankruptcy trigger) will never intersect. Again, using the Girsanov theorem and completing the squares with the Fubini Theorem will result in the following formulas:

$$G_{\tau_{B1}} = \left(\frac{V_{B1}}{V}\right)^{\lambda/\sigma^2} \sum_{n=0}^{\infty} \left\{\left(\frac{(V_{B1})^{2n+1}}{(V_{Con})^{2n}V}\right)^{\lambda^*/\sigma^2} - \left(\frac{(V_{B1})^{2n+1}V}{(V_{Con})^{2n+2}}\right)^{\lambda^*/\sigma^2}\right\}$$

$$G_{\tau_{Con}} = \left(\frac{V_{Con}}{V}\right)^{\lambda/\sigma^2} \sum_{n=0}^{\infty} \left\{\left(\frac{(V_{B1})^{2n}V}{(V_{Con})^{2n+1}}\right)^{\lambda^*/\sigma^2} - \left(\frac{(V_{B1})^{2n+2}}{(V_{Con})^{2n+1}V}\right)^{\lambda^*/\sigma^2}\right\}$$

$$H_{\tau_{B2}} = \left(\frac{V_{B2}}{V}\right)^{\lambda/\sigma^2} \sum_{n=0}^{\infty} \left\{\left(\frac{(V_{B2})^{2n+1}}{(V_{Con})^{2n}V}\right)^{\lambda^*/\sigma^2} - \left(\frac{(V_{B2})^{2n+1}V}{(V_{Con})^{2n+2}}\right)^{\lambda^*/\sigma^2}\right\}$$

and

$$H_{\tau_{Call}} = \left(\frac{V_{Call}}{V}\right)^{\lambda/\sigma^2} \sum_{n=0}^{\infty} \left\{\left(\frac{(V_{B2})^{2n}V}{(V_{Call})^{2n+1}}\right)^{\lambda^*/\sigma^2} - \left(\frac{(V_{B2})^{2n+2}}{(V_{Call})^{2n+1}V}\right)^{\lambda^*/\sigma^2}\right\}$$

BIBLIOGRAPHY

Altintig, Z. A., & Butler, A. W. (2005). Are they still called late? The effect of notice period on calls of convertible bonds. Journal of Corporate Finance, 11, 337–350.

Anderson, R., & Sundaresan, S. (1996). Design and valuation of debt contracts. Review of Financial Studies, 9, 37–68.

Asquith, P. (1995). Convertible bonds are not called late. Journal of Finance, 50, 1275–1289.

Asquith, P., & Mullins, D. W. (1991). Convertible debt: Corporate call policy and voluntary conversion. Journal of Finance, 46, 1273–1289.

Ayache, E., Forsyth, P. A., & Vetzal, K. R. (2003). The valuation of convertible bonds with credit risk. Journal of Derivatives, 11(Fall), 9–29.

Brennan, M., & Schwartz, E. (1977). Convertible bonds: valuation and optimal strategies for call and conversion. Journal of Finance, 32, 1699–1715.

Brennan, M., & Schwartz, E. (1980). Analyzing convertible bonds. Journal of Financial and Quantitative Analysis, 15, 907–929.

Butler, A. W. (2002). Revisiting optimal call policy for convertibles. Financial Analysts Journal, 58, 50–55.

Campbell, C. J., Ederington, L. H., & Vankudre, P. (1991). Tax shields, sample-selection bias, and the information content of conversion-forcing bond calls. Journal of Finance, 46, 1291–1324.

Constantinides, G. M. (1984). Warrant exercise and bond conversion in competitive markets. Journal of Financial Economics, 13, 371–397.

Cowan, A. R., Nayar, N., & Singh, A. K. (1993, Winter). Calls of out-of-the-money convertible bonds. Financial Management, 22, 106–116.

Ederington, L., Gaton, G., & Campbell, C. (1997, Spring). To call or not call convertible debt. Financial Management, 26, 22–31.

Harrison, J. M. (1985). Brownian motion and stochastic flow systems. New York: Wiley.

Harrison, J. M., & Kreps, D. M. (1979). Martingales and arbitrage in multiperiod securities markets. Journal of Economic Theory, 20, 381–408.

Harrison, J. M., & Pliska, S. R. (1981). Martingales and stochastic integrals in the theory of continuous trading. Stochastic Processes and Their Applications, 11, 215–260.

Huang, J. J., Ju, N., & Ou-Yang, H. (2003). A model of optimal capital structure with stochastic interest rates. Stern School of Business Working Paper No. FIN-03- 014. New York: New York University.

Ingersoll, J. E. (1977a). A contingent-claims valuation of convertible securities. Journal of Financial Economics, 4, 289–322.

Ingersoll, J. E. (1977b). An examination of corporate call policies on convertible securities. Journal of Finance, 32, 463–478.

Kolkiewicz, A. W. (2002). Pricing and hedging more general double-barrier options. Journal of Computational Finance, 5, 1–26.

Lau, K. W., & Kwok, Y. K. (2004). Anatomy of option features in convertible bonds. Journal of Futures Markets, 24, 513–532.

Leland, H. E. (1994). Corporate debt value, bond covenants, and optimal capital structure. Journal of Finance, 49, 1213–1252.

Leland, H. E., & Toft, K. B. (1996). Optimal capital structure, endogenous bankruptcy, and the term structure of credit spreads. Journal of Finance, 51, 987–1019.

Longstaff, F. A., & Schwartz, E. S. (1995). A simple approach to valuing risky fixed and floating rate debt. Journal of Finance, 50, 789–819.

Mella-Barral, P., & Perraudin, W. (1997). Strategic debt service. Journal of Finance, 52, 531–556.

Merton, R. C. (1974). On the pricing of corporate debt: The risk structure of interest rates. Journal of Finance, 29, 449–470.

Nyborg, K. G. (1996). The use and pricing of convertible bonds. Applied Mathematical Finance, 3, 167–190.

Rogers, L. C. G. (1999). Modelling credit risk. Working paper, University of Bath.

Sarkar, S. (2003). Early and late calls of convertible bonds: Theory and evidence. Journal of Banking and Finance, 27, 1349–1374.

Schönbucher, P. J. (2003). Credit derivatives pricing models: Models, pricing, and implementation. New York: Wiley.

Sîrbu, M., Pikovsky, I., & Shreve, S. E. (2004). Perpetual convertible bonds. SIAM Journal of Control and Optimization, 43, 58–85.