Applying GARCH-EVT-Copula Models for Portfolio Value-at-Risk on G7 Currency Markets

Shian-Chang Huang

Department of Business Administration
National Changhua University of Education, Taiwan
E-mail: shhuang@cc.ncue.edu.tw
Tel: +886-4-7232105-7420

Yi-Hsin Chien

Department of Money and Banking
National Chengchi University, Taipei, Taiwan

Ruei-Ci Wang

Department of Mathematics
National Changhua University of Education, Taiwan

Abstract

This research estimates portfolio VaR (Value-at-Risk) on G7 exchange rates using a GJR-GARCH-EVT (extreme value theory)-Copula based approach. We first extracts the filtered residuals from each return series via an asymmetric GJR-GARCH model, then constructs the semi-parametric empirical marginal cumulative distribution function (CDF) of each asset using a Gaussian kernel estimate for the interior and a generalized Pareto distribution (GPD) estimate for the upper and lower tails (our approach focuses on the entire distribution rather than the tail distribution only). A Student's t copula is then fit to the data and used to induce correlation between the simulated residuals of each asset. In order to test the effectiveness of this model we backtest the estimated VaRs over a time window of 200 days. Empirical results demonstrate that our GJR-GARCH-EVT-Copula based approach outperforms traditional methods such as historical simulation or conditional Gaussian model.

Keywords: Value-at-Risk (VaR), Conditional Extreme Value Theory (CEVT), Generalized Pareto Distribution (GPD), Copula Function, Currency Risk

1. Intruduction

In globalization and economic liberalization, the foreign exchange market has become the market with largest transaction volume and involving most capital. Dynamics of exchange rate processes are the most complex in financial markets. In addition to trading volume, another feature of concern is that the foreign exchange market involves characteristics of long memory. Due to a large number of international goods and currencies transactions, exchange rate fluctuations or risk have become very important topics for multinational companies, individuals, or even domestic countries. Time varying dynamics of exchange rate fluctuations are the main concerns of individual investments, firm

production and distribution, and even business strategy. G7 currency indices play an important role in global markets. Therefore, this study takes G7 currency indices as the research object.

Value-at-Risk (VaR, Jorion, 2000) is a popular approach to quantifying market risk. It yields an estimate of the likely losses which could rise from price changes over a horizon at a given confidence level. VaR makes risk measure an intuitive criterion for asset management, and hence it very appeals to financial decision makers (Fischer, 2003; Miller, 2003; Rosengarten and Zangari, 2003). Inaccurate portfolio VaR estimates may lead firms to maintain insufficient risk capital reserves so that they have an inadequate capital cushion to absorb large financial shocks. For example, several major financial institutions crashed not long after the breakout of recent financial crises (e.g., East Asian financial crisis of 1997), and some of these failures have been associated with substantial portfolio VaR estimation errors.

Currently, most of the current research on VaR estimation focuses on the univariate case making it undesirable for portfolio risk management. Moreover, most of the significant research contributions to the literature on portfolio VaR are limited to estimators of marginal VaR, component VaR, and incremental VaR instead of portfolio VaR itself (Hallerbach, 2002). This study employs new framework for portfolio VaR estimations, which integrates asymmetric GJR-GARCH models for time-varying return distribution of individual assets, extreme value theory (EVT, Embrechts et al., 1997) for tail distributions, and copula functions (Nelsen, 1999) for the dependency structure on all assets of a portfolio.

Traditional VaR models assume the return series follow i.i.d (independent and identically distributed) Gaussian distributions. However, the general financial time series are leptokurtic with heavy-tailed which make VaR being underestimated for i.i.d. Gaussian distribution. Recent researchers (Ho et al., 2000; McNeil and Frey, 2000; Gencay et al., 2003) tend to adopt the extreme value theory (EVT) to solve the problem. EVT not only gets rid of the underestimation usually encountered in the Gaussian assumption but also possesses enough flexibility to model various tail distributions. Besides, researchers usually adopted MLE to estimate the parameters of EVT, but under limited samples MLE causes estimation bias easily.

On the time series characteristics, integrating EVT with time series model evolves into conditional version of EVT (CEVT, or dynamic EVT). Some literatures (McNeil and Frey, 2000; Nystrom and Skoglund, 2002) indicate that CEVT employing time series model filters the autocorrelations and heteroskedasticity in finance data. Consequently, the accuracy of VaR estimation is significantly enhanced.

On the dependence structure, it is extremely complex to fit the multivariate joint probability density function in investment portfolios. Hence, traditional research assumes the return series obey a simple multivariate normal distribution, but it usually underestimate the portfolio VaR. Recently, the concept of copula functions (Nelsen, 1999) is injected into financial field, offering a more simple and flexible method to model the multivariate dependence (Embrechts et al., 2000; Embrechts et al., 2001).

This study selects the G7 exchange rates to form a portfolio. We first transform the individual standardized residuals of GJR-GARCH (Glosten et al., 1993) models to uniform variates by the semi-parametric empirical CDF (cumulative distribution function), and then fit the t copula to the transformed data. Given the estimated parameters of a t copula, we can simulate jointly dependent equity index returns by first simulating the corresponding dependent uniform variates. Then, by extrapolating into the generalized Pareto distribution tails and interpolating into the smoothed interior, transform the uniform variates to standardized residuals via the inversion of the semi-parametric marginal CDF of each index (our approach focuses on the entire distribution rather than the tail distribution only (Byström, 2004)). This produces simulated standardized residuals consistent with those obtained from the GJR-GARCH filtering process. Longitudinally, each of the simulated standardized residuals represents an i.i.d. univariate stochastic process when viewed in isolation, whereas each cross section shares the rank correlation induced by the copula. Then the portfolio VaR could be simulated and back tested.

The contribution of this study lies in the combination of the GJR-GARCH model to condition the individual time series, the generalized Pareto distribution function to model the tail distribution for each asset, and the usage of copula functions to model the joint distribution of the correlated returns for all assets in the portfolio. Finally, the back testing is employed to compare the validity and performance of the proposed method relative to other popular methods.

The remainder of the paper is organized as follows. Section 2 introduces the extreme value theory. Section 3 describes the copula theory. Section 4 describes the data used in the study, and discusses the empirical findings. Finally, conclusions are given in Section 5.

2. Extreme Value Theories

There are two principal kinds of model for extreme values (Embrechts et al., 1997). The block maximum models are the oldest group of models. They are models for the largest observations collected from large samples of identically distributed observations. The peaks-over-threshold (POT) models are modern methods for EVT. They directly model all large observations which exceed a high threshold.

Within the POT class of models one may further distinguish two styles of analysis. One is the semi-parametric models built around the Hill estimator (Hill, 1975) and its relatives and the other is the fully parametric models based on the generalized Pareto distribution or GPD (Embrechts et al., 1997). This study applies to the latter style of analysis.

2.1. Generalized Pareto Distribution (GPD)

For the marginal return distributions, separate GP models are fit to both the lower and upper distribution tails. Under the parametrization of the GP tail model, the tail distribution is represented by the complement of the GP cumulative distribution function (CDF):

$$G_{\xi,\beta}(x) = \begin{cases} 1-(1+\xi x/\beta)^{-1/\xi} & \xi \neq 0 \\ 1-\exp(-x/\beta) & \xi = 0 \end{cases},$$

where ξ is the shape parameter, and β is the scale parameter. When $\xi > 0$ the GPD is heavy-tailed. When we consider the excess distribution over a threshold u,

$$F_u(y) = P\{X - u \leq y | X > u\}.$$

It is very easy to derive that in terms of the CDF of X (denoting it by F), we have

$$F_u(y) = \frac{F(y+u) - F(u)}{1 - F(u)}. \tag{1}$$

By the threoems of Pickands(1975), for a large class of underlying distributions, F_u will converge to $G_{\xi,\beta}$, namely

$$\lim_{u \to x_0} \sup_{0 \leq y \leq x_0} |F_u(y) - G_{\xi,\beta}(y)| = 0. \tag{2}$$

It means that for a large class of underlying distributions F, as the threshold u is progressively raised, the excess distribution F_u will converge to a generalized Pareto distribution. The resultant parameter estimations are functions of the selected threshold u. The choice of the threshold value u is crucial in order to obtain a good estimation in MLE. In fact, if u is too high, we have only a few exceedances data and the variance of the estimators is high. If u is too low, the estimators are biased because the relation (2) does not hold.

Setting $x = y + u$ and combining results of equations (1) and (2), our model can be written as

$$F(x) = (1 - F(u))G_{\xi,\beta}(x - u) + F(u) \qquad for \qquad x > u. \tag{3}$$

Using equation (3) to construct a tail estimator, the only additional element required is an estimate of $F(u)$. The empirical estimator $(N - N_u)/N$ is a good choice, where N_u is the number of exceedances beyond the high threshold u of $x_1, x_2, ..., x_N$. Putting the empirical estimator of $F(u)$ and our estimated parameters ($\hat{\xi}, \hat{\beta}$) of the GPD together, we arrive at the tail estimator:

$$\hat{F}(x) = 1 - \frac{N_u}{N}\left(1 + \hat{\xi}\frac{x - u}{\hat{\beta}}\right)^{-1/\hat{\xi}} .$$

(4)

3. Copula Theory

An n-dimensional copula is a multivariate cumulative distribution function, C, with uniform distributed margins in $[0,1]$ ($U(0,1)$) and the following properties: (Nelsen, 1999)

$$C : [0,1]^n \to [0,1] ;$$

C is grounded and n-increasing;

C has margins C_i which satisfy $C_i(u) = (1,...,1,u,1,...1) = u$ for all $u \in [0,1]$.

By Sklar theorem (Sklar, 1959), let F be an n-dimensional CDF with continuous margins $F_1,...,F_n$. Then it has the following unique copula representation:

$$F(x_1,...,x_n) = C(F_1(x_1),...,F_n(x_n))$$

(5)

It is obvious, from the above definition, that if $F_1,...,F_n$ are univariate distribution functions, $u_i = F_i(x_i), i = 1,...,n$, are uniform random variables, and $C(F_1(x_1),...,F_n(x_n))$ is the unique multivariate CDF with margins $F_1,...,F_n$. Eequation (5) is equivalent to the following representation (in variable u_i):

$$C(u_{1,...,}u_n) = F(F_1^{-1}(u_1),...,F_n^{-1}(u_n)) .$$

(6)

Sklar theorem implies that for multivariate distribution functions the univariate margins and the dependence structure can be separated. The dependence structure can be represented by an adequate copula function.

Copula functions are a useful tool to construct and simulate multivariate distributions. We introduce some popular copula functions below:

1. The bi-variant normal (or Gaussian) copula

$$C_N(u_1 u_2) = \Phi_\rho(\phi^{-1}(u_1), \phi^{-1}(u_2))$$

$$= \int_{-\infty}^{\phi^{-1}(u_1)} \int_{-\infty}^{\phi^{-1}(u_2)} \frac{1}{2\pi(1-\rho^2)^{1/2}} \exp\left(\frac{(x^2 - 2\rho xy + y^2)}{2(1-\rho^2)}\right) dx dy,$$

where Φ_ρ is the standard multivariate normal CDF, ϕ^{-1} is the inverse of the standard univariate normal CDF, and ρ is the linear correlation between X and Y.

2. The bi-variant t copula

$$C_t(u_1 u_2) = T_{v,\rho}\left(t_v^{-1}(u_1), t_v^{-1}(u_2)\right)$$

$$= \int_{-\infty}^{t_v^{-1}(u_1)} \int_{-\infty}^{t_v^{-1}(u_2)} \frac{1}{2\pi(1-\rho^2)^{1/2}} \exp\left(1 + \frac{x^2 - 2\rho xy + y^2}{v(1-\rho^2)}\right)^{-(v+2)/2} dx dy,$$

where $T_{v,\rho}$ is the bivariate Student's t-distribution with v degrees of freedom, t_v^{-1} is an inverse Student's t-distribution function, and ρ is the correlation between X and Y for $v > 2$.

3. The Clayton copula

$$C_c(u_1, u_2) = \left(u_1^{-\theta} + u_2^{-\theta}\right)^{-1/\theta}.$$

4. The Gumbel copula

$$C_G(u_1, u_2) = \exp\left(-\left[(-\ln(u_1))^\delta + (-\ln(u_2))^\delta\right]^{1/\delta}\right).$$

4. Empirecal Research
4.1. Data Collections

This study use the data set comprising the major G7 exchange rate indices, including the following daily currency indices: Pound/dollar (GBP/USD) , Canadian Dollar/dollar (CAD/USD) , Mark/dollar (DEM/USD) , Franc/dollar (FRF/USD) , Lira/dollar (ITL/USD) , Yen/dollar (JPY/USD) , Ruble/dollar (RUB/USD). These data are extracted from Datastream provided by Morgan Stanley Capital International (MSCI). The whole data set covers the period from January 3, 2005 to December 31, 2007, a total of 781 observations. These exchange rate indices are then transformed into daily returns. Figure 1 shows the G7 daily index returns. It's obvious that they are highly correlated.
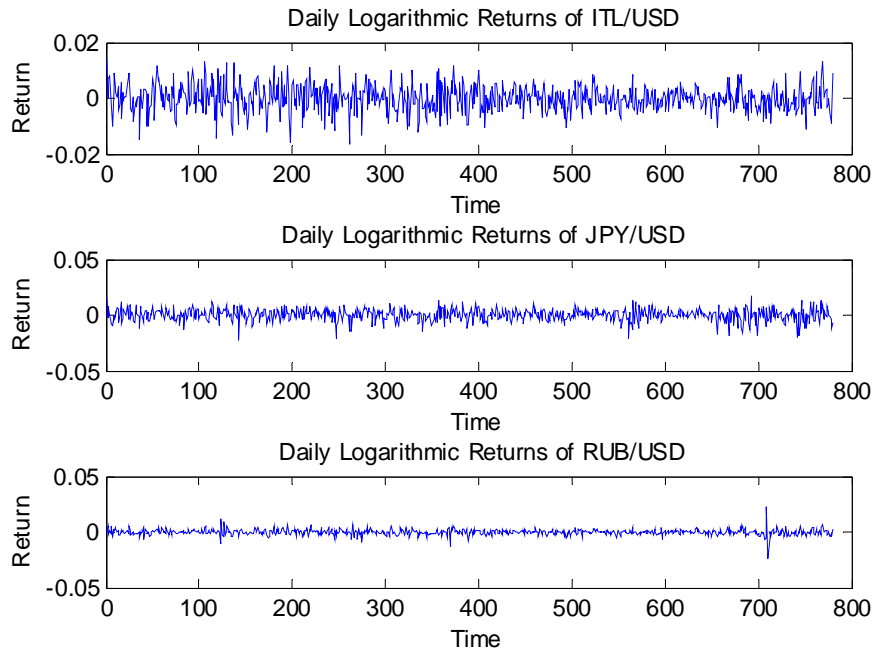
Figure 1: Daily returns of G7 exchange rate
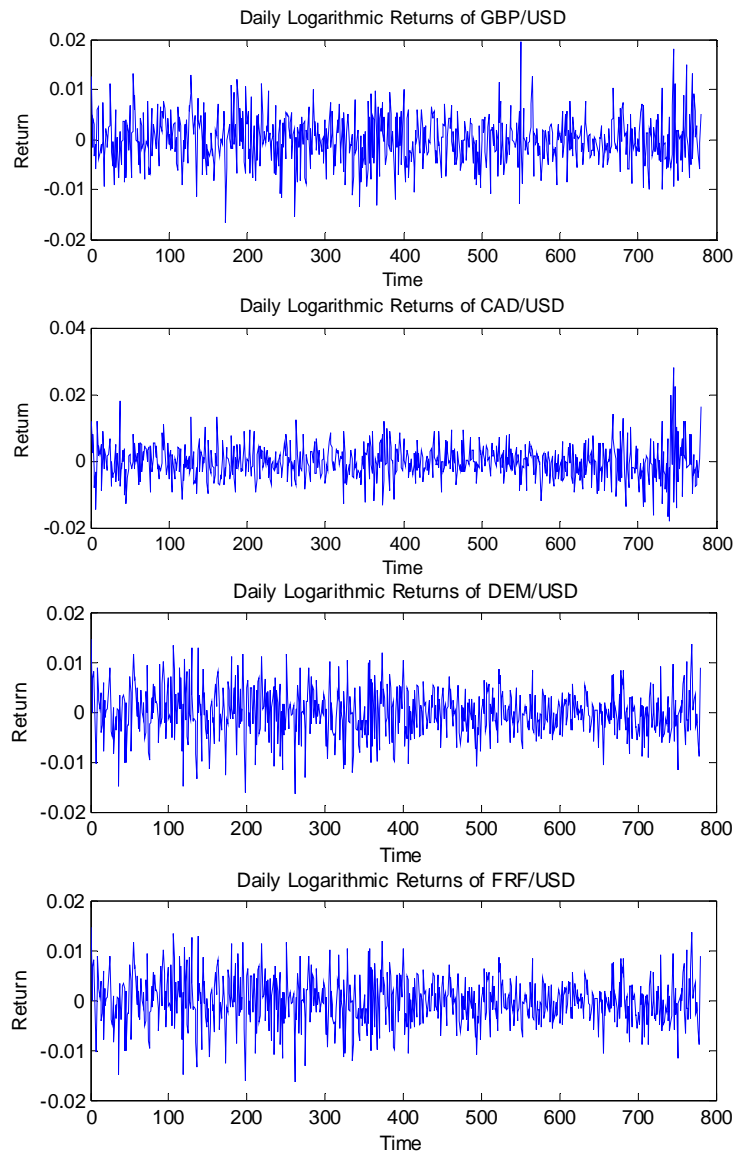
Figure 1: Daily returns of G7 exchange rate - continued

Modeling the tails of a distribution with a GPD requires the observations to be approximately independent and identically distributed (i.i.d.). However, most financial return series exhibit some degree of autocorrelation and, more importantly, heteroskedasticity. Figure 2 shows sample ACF (autocorrelation function) of returns and sample ACF of squared returns for the seven countries. The ACF of returns reveals some mild serial correlation. However, the sample ACF of the squared returns illustrates significant degree of persistence in variance, which implies that we need a GARCH model to condition the data for the subsequent tail estimation process.

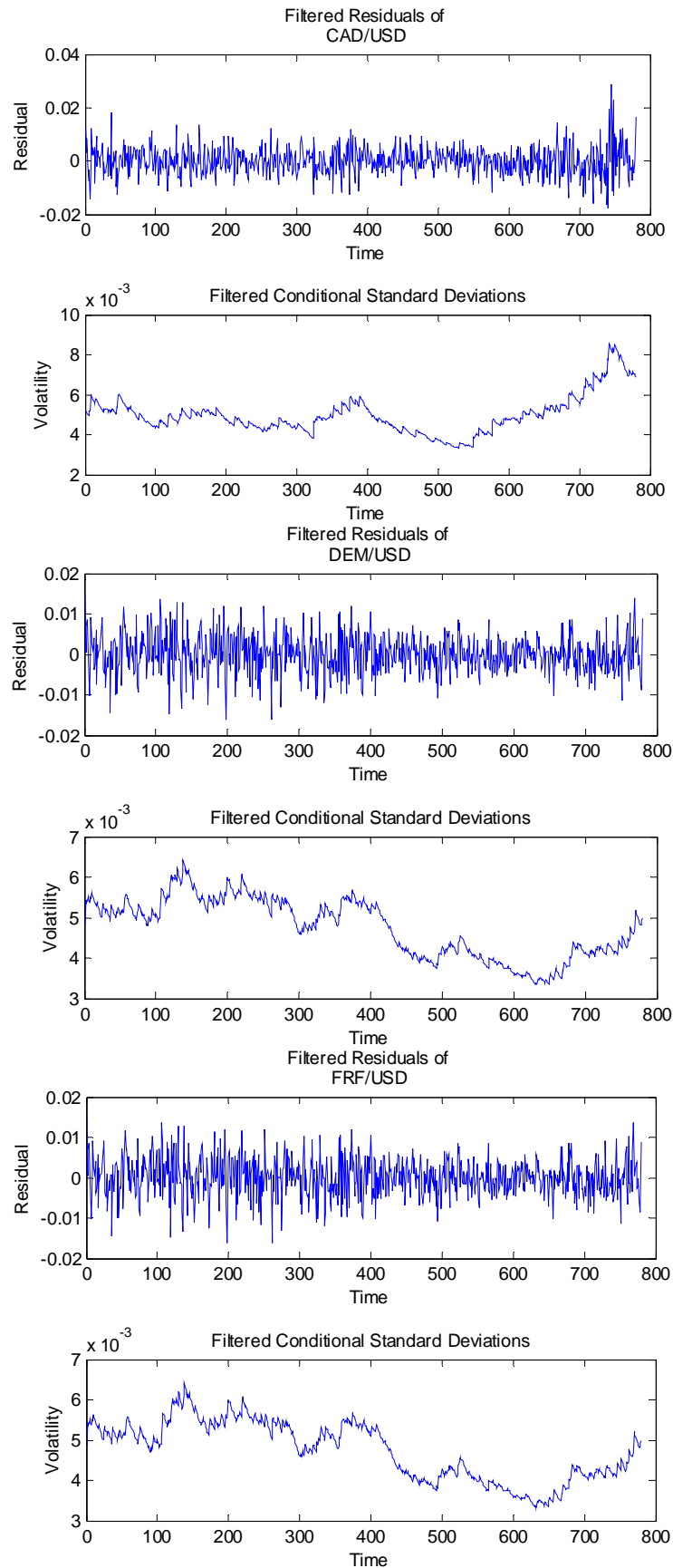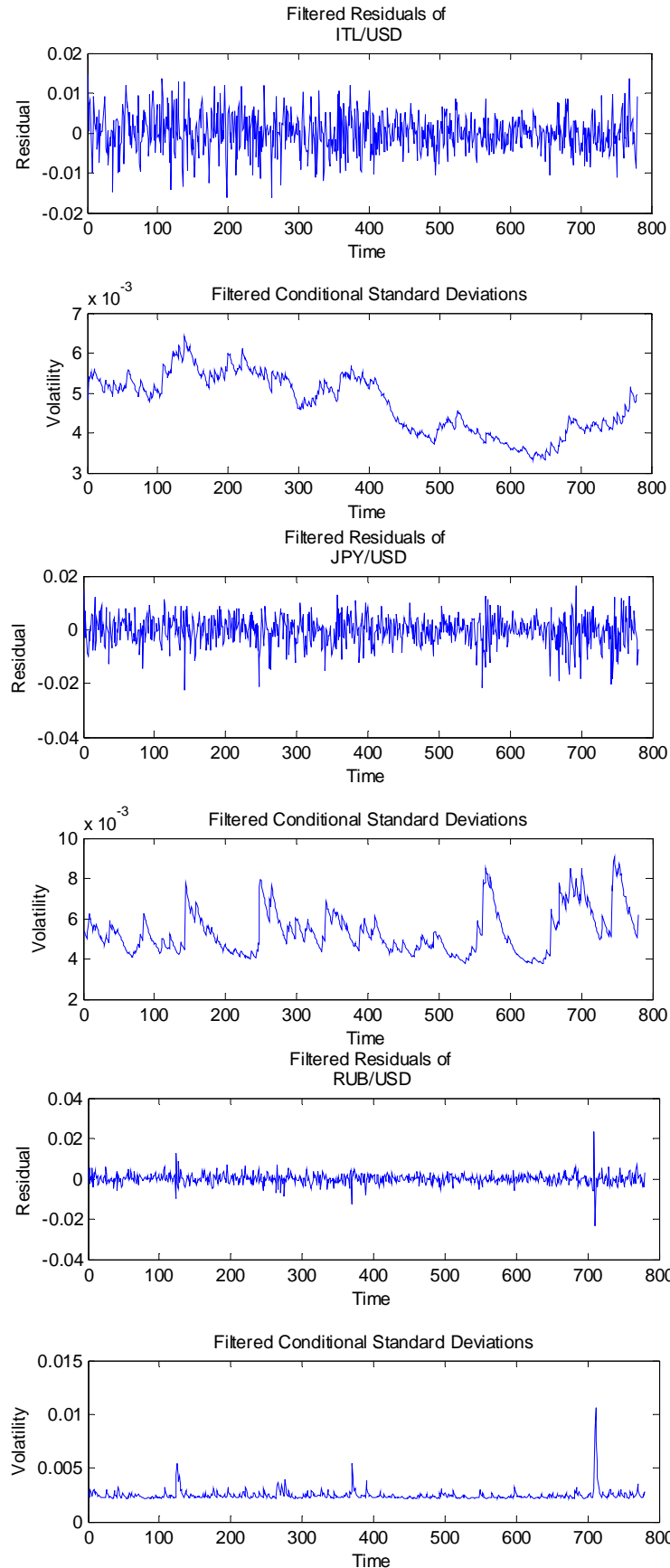**Figure 2:** Filtered residuals and volatility of seven markets
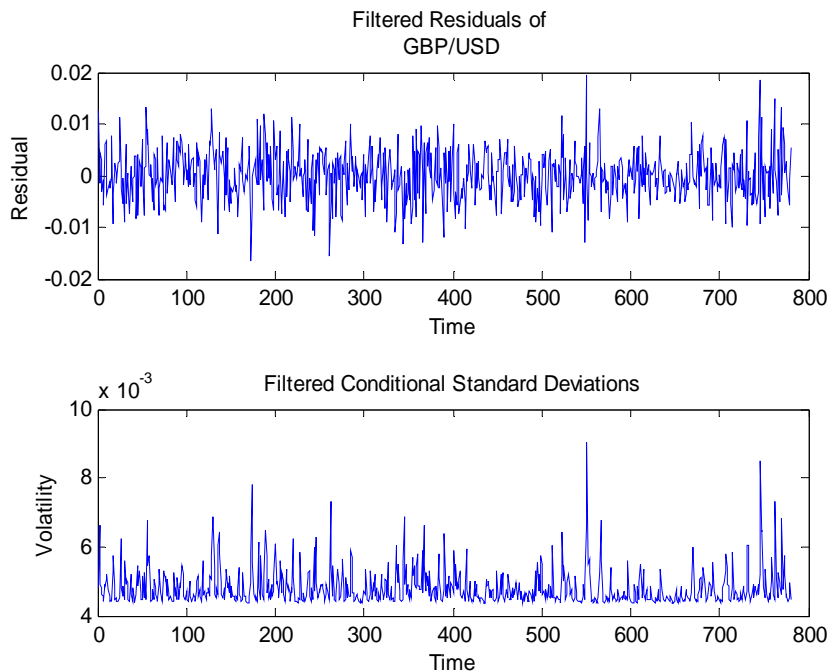
Figure 2: Filtered residuals and volatility of seven markets - continued

Figure 2: Filtered residuals and volatility of seven markets - continued

4.2. Model Estimations

To produce a series of i.i.d. observations, we fit a AR(1)-GJR-GARCH(1,1) model as follows to each index,

$$R_t = a_0 + a_1 R_{t-1} + \epsilon_t, \quad \epsilon_t \sim N(0, \sigma_t)$$

$$\sigma_t^2 = K + \delta \sigma_{t-1}^2 + \alpha \epsilon_{t-1}^2 + \varphi \epsilon_{t-1}^2 I_{t-1}$$

where

$$I_{t-1} = 0 \quad if \quad \epsilon_{t-1} \geq 0, \quad and \quad I_{t-1} = 1 \quad if \quad \epsilon_{t-1} < 0.$$

In the model, R_t is the index return, and σ_t the volatility. The GJR-GARCH model could incorporate asymmetric leverage effects for volatility clustering. Figure 3 are filtered model residuals from each index. Each lower graph of Figure 3 clearly illustrates the variation in volatility (heteroskedasticity) present in the filtered residuals. Subsequently, we standardize the residuals by the corresponding conditional standard deviation. These standardized residuals represent the underlying zero-mean, unit-variance, i.i.d. series upon which the EVT estimation of the sample CDF tails is based.

Given the standardized, i.i.d. residuals from the previous step, we estimate the empirical CDF of each index with a Gaussian kernel in interior and EVT in each tail, because the interior of a CDF is usually smooth, and non-parametric kernel estimates are well suited, but kernel smooth tends to perform poorly when applied to the upper and lower tails. To better estimate the tails of the distribution, we apply EVT to those residuals that fall in each tail.
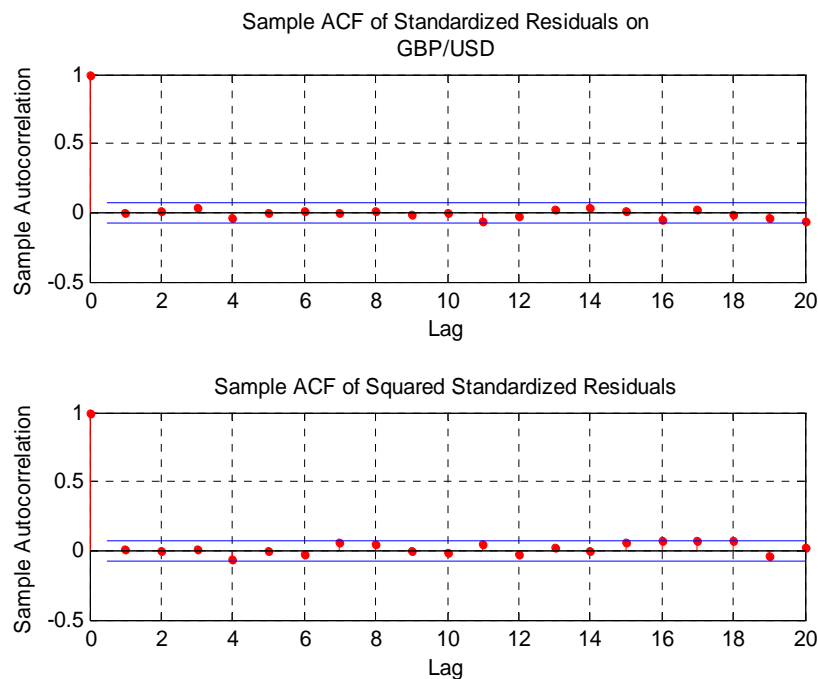
Figure 3: ACF plots of seven markets

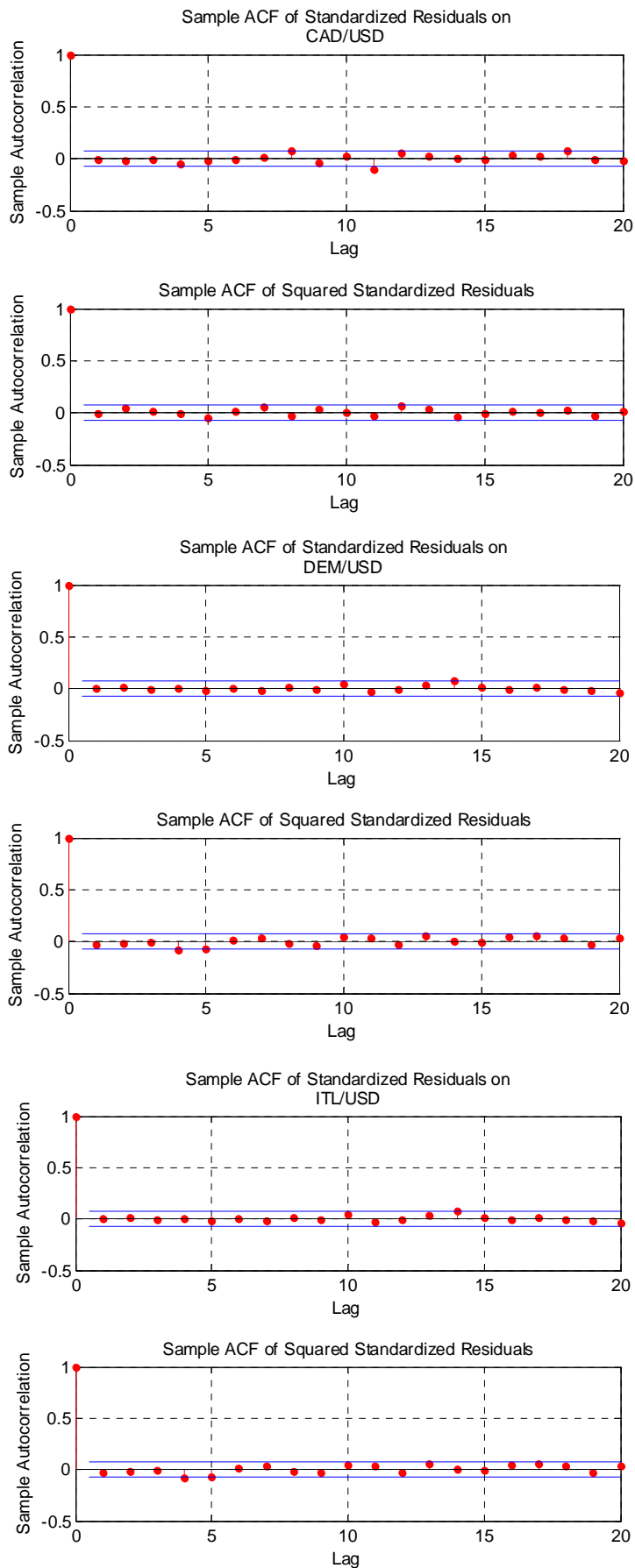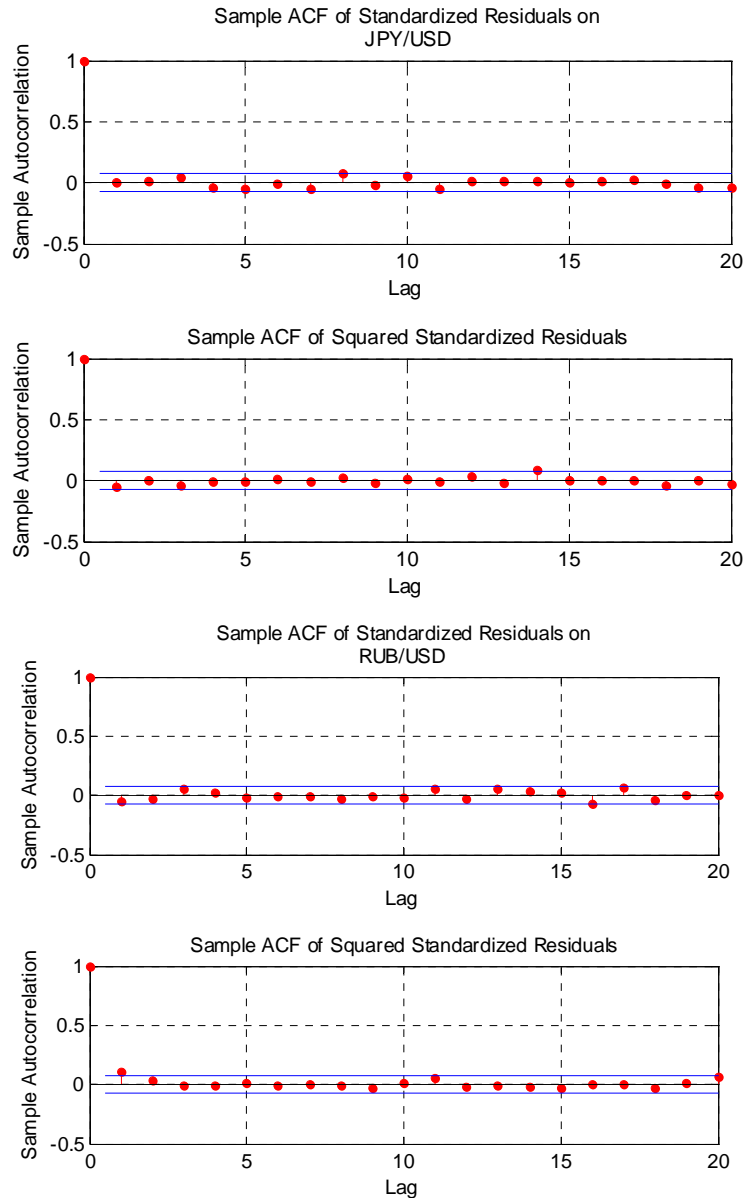**Figure 3:** ACF plots of seven markets - continued

Figure 3: ACF plots of seven markets - continued

4.3. VaR Calculations

We then transform the individual standardized residuals of AR(1)-GJR-GARCH(1,1) models to uniform variates by the semi-parametric empirical CDF, and then fit the t copula to the transformed data. The estimated optimal degree of freedom (v) of the t copula is 7.772. This study also adopts t copulas with v=10,15,20 for comparison. Subsequently, this study simulates jointly dependent currency index returns by reversing the above steps. We simulate 2000 independent random trials of dependent standardized index residuals over a one month horizon of 22 trading days. Then, using the simulated standardized residuals as the i.i.d. input noise process, reintroduce the autocorrelation and heteroskedasticity of GJR-GARCH model observed in the original index returns. Finally, given the simulated returns of each index, we form a 1/7 equally weighted index portfolio composed of the individual indices, and calculate the VaR at 99% confidence levels, over the one month risk horizon. The estimated 90 %, 95%, and 99% VaRs for t (7.772), t (10), t (15), t (20) and other models (historical simulation and GJR-GARCH +Gaussian distribution models) are listed in Table 1 for reference.

Finally, we backtest the 99% VaR estimations over a time window of 200 days, and compare the results with traditional models. We count the number of VaR exceedances for each model that is

the number of times in which the effective loss is greater than the 99% VaR estimation. The principal results of this backtesting procedure are displayed in Table 2. As shown in Tables 1 and 2, the failure rates of our model with t(7.772) is the nearest to 1%, 5%, 10%, respectively. Namely our model outperforms traditional VaR models. Empirical results clearly demonstrate that the CEVT-Copula based approach performs best. The historical simulation and GJR-GARCH-Gaussian overestimate the portfolio VaR. Figures 4. 5. 6 and 7 plot the profit and loss distributions of our CEVT-Copula models. Figures 8 plot the profit and loss distributions of the GJR-GARCH-Gaussian copula model and figure 9 plot the profit and loss distributions of our historical simulation model.

Table 1: VaRs of different models

	CEVT +t(7.772) copula	CEVT +t(10) copula	CEVT +t(15) copula	CEVT +t(20) copula	Historical simulation	GARCH +Gaussian
90% VaR	2.6448%	2.6408%	2.6220%	2.6475%	2.6011%	2.6329%
95% VaR	3.5189%	3.4333%	3.4549%	3.4176%	3.2794%	3.5056%
99% VaR	4.9892%	4.9796%	4.8469%	4.6833%	4.8276%	4.9581%
Max Loss	7.0218%	7.3190%	6.5106%	6.7471%	8.9168%	8.2375%
Max Gain	6.4193%	5.9001%	6.3755%	6.5223%	8.9168%	6.3911%

Table 2: Failure rate for each model

	Optimal DoF=7.772	DoF=10	DoF=15	DoF=20	Historical Simulation	GARCH +Gaussian
Failure Rate α=0.1	0.08	0.065	0.065	0.06	0	0.06
Failure Rate α=0.05	0.03	0.025	0.025	0.025	0	0.025
Failure Rate α=0.01	0.005	0	0	0	0	0

Figure 4: Portfolio profit and loss distribution (CEVT + t (7.772) copula)

Simulated One-Month Global Portfolio Returns PDF

Figure 5: Portfolio profit and loss distribution (CEVT + t (10) copula)

Simulated One-Month Global Portfolio Returns PDF

Figure 6: Portfolio profit and loss distribution (CEVT + t (15) copula)
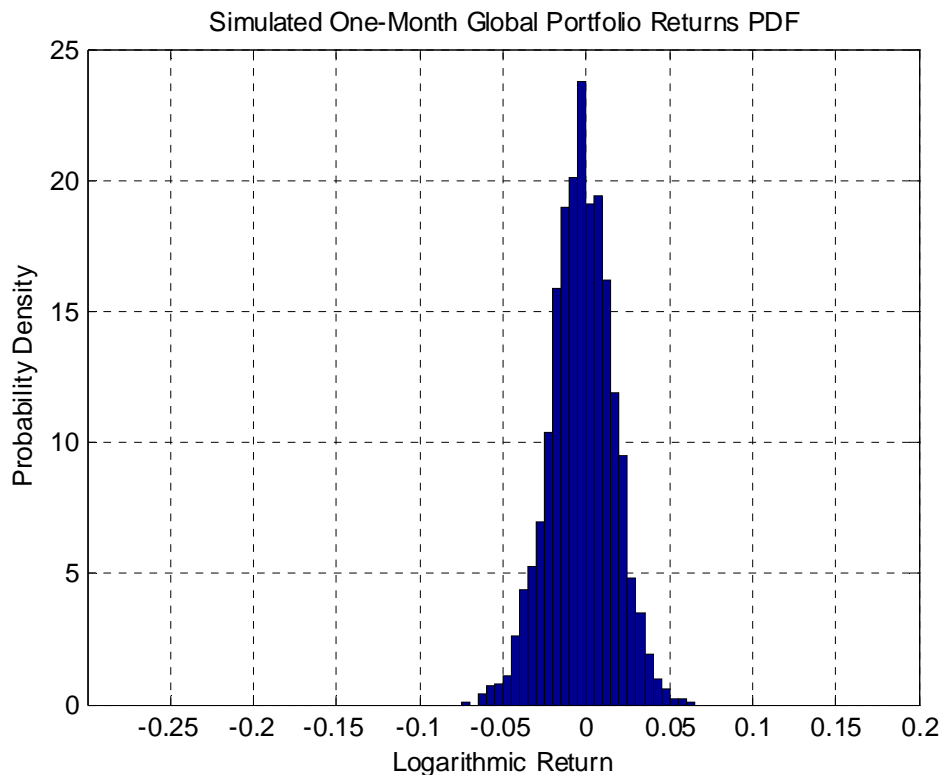

Simulated One-Month Global Portfolio Returns PDF

Figure 7: Portfolio profit and loss distribution (CEVT + t (20) copula)
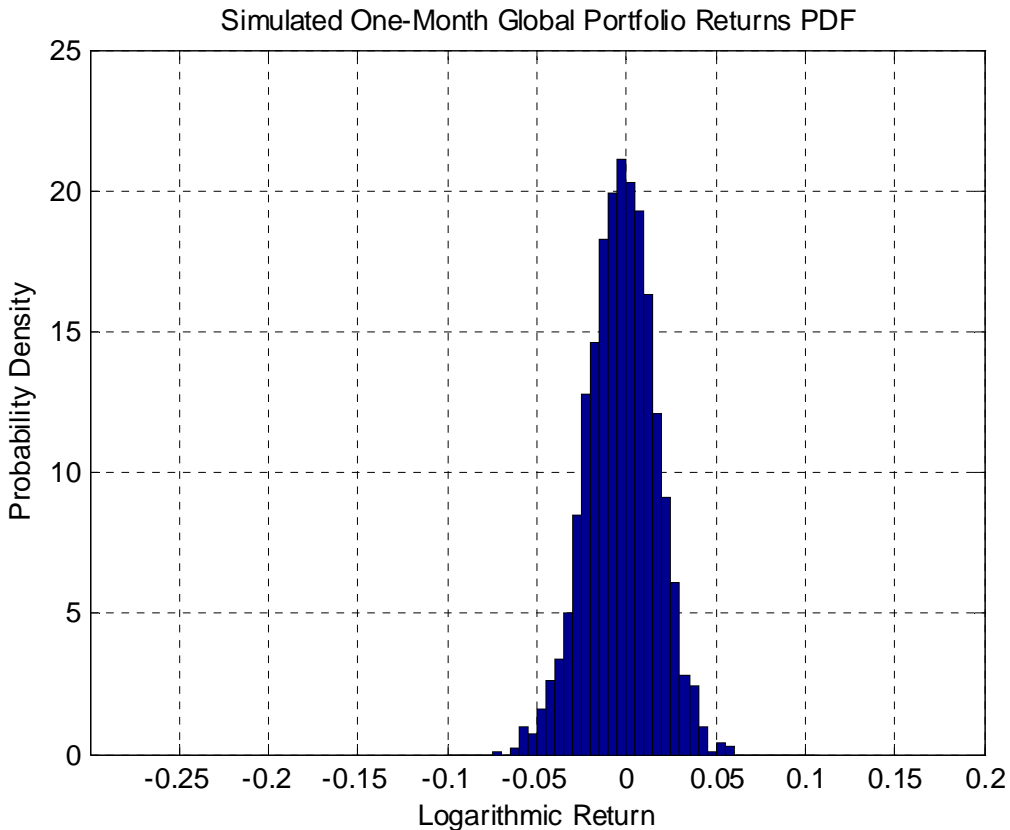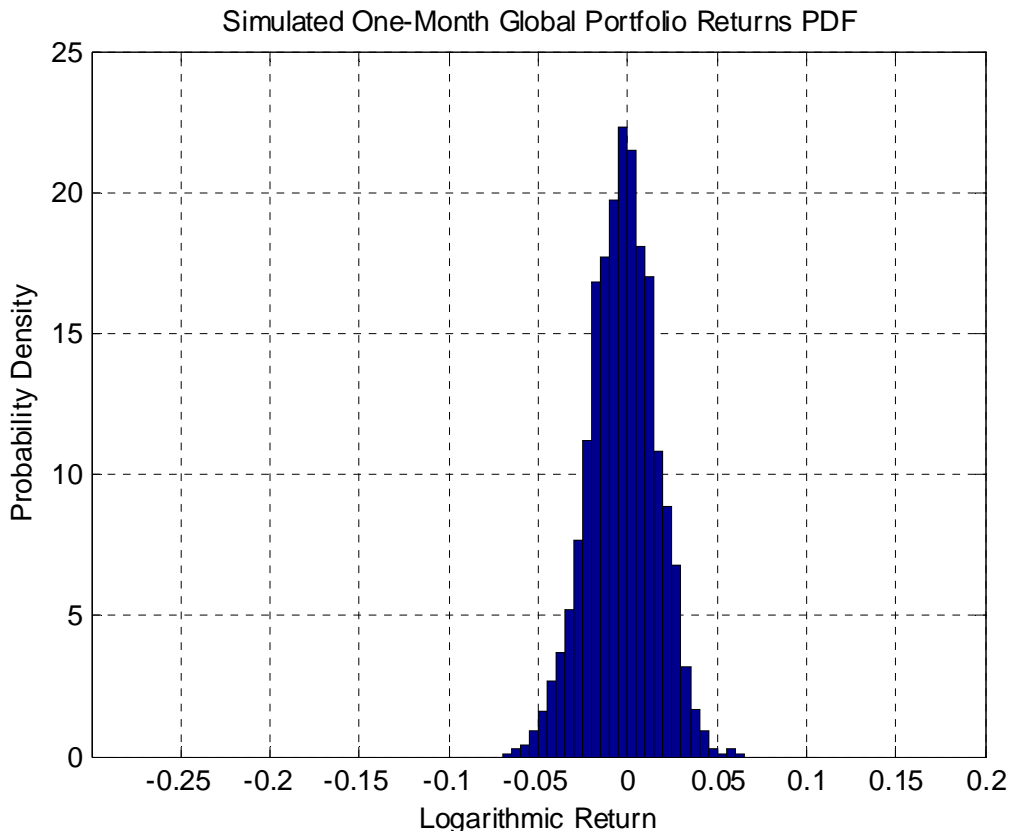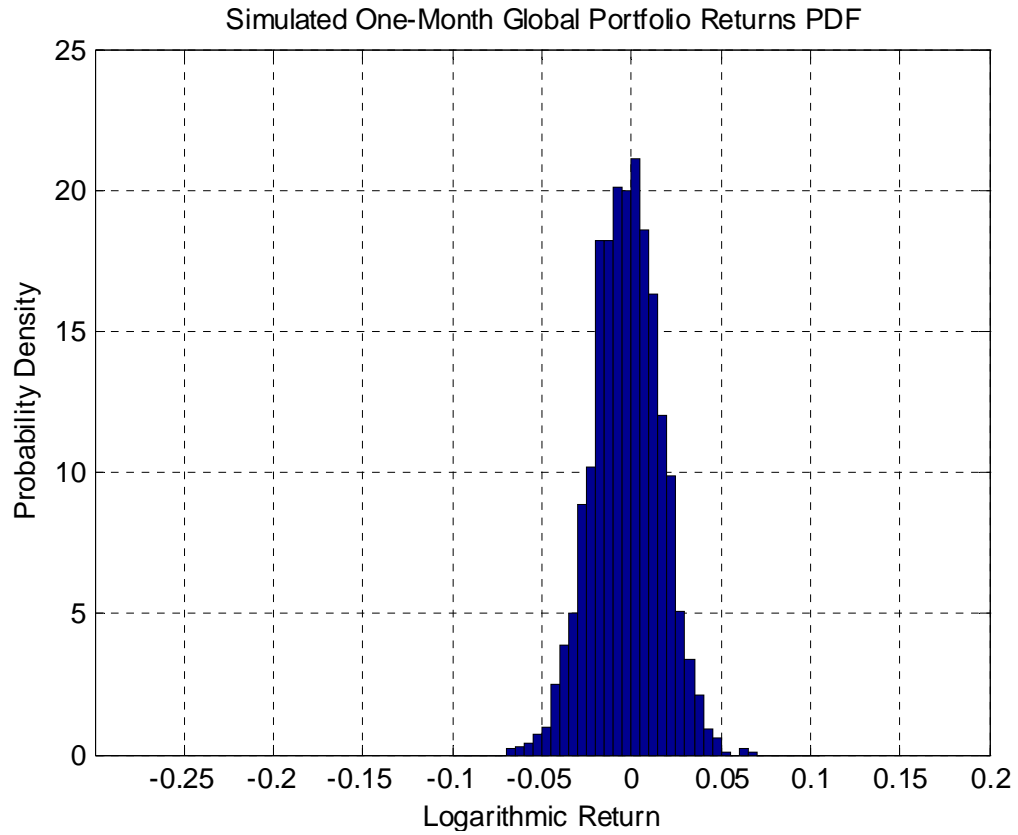

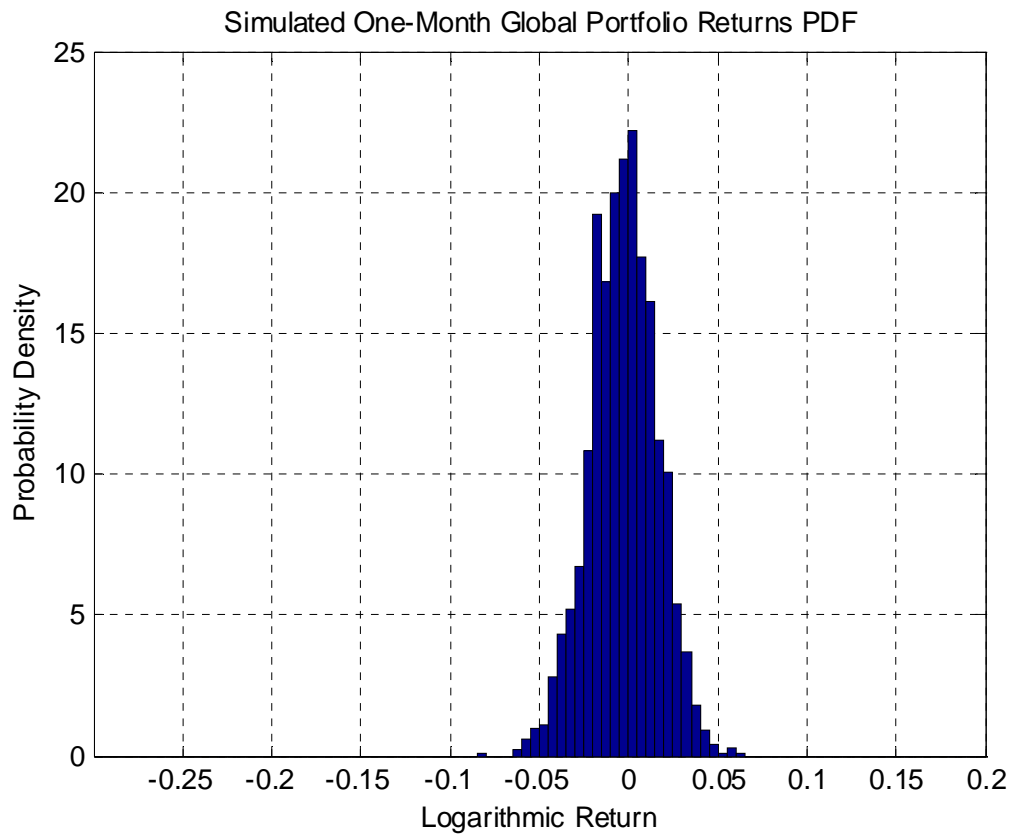**Figure 8:** Portfolio profit and loss distribution (GJR-GARCH+Gaussian copula)
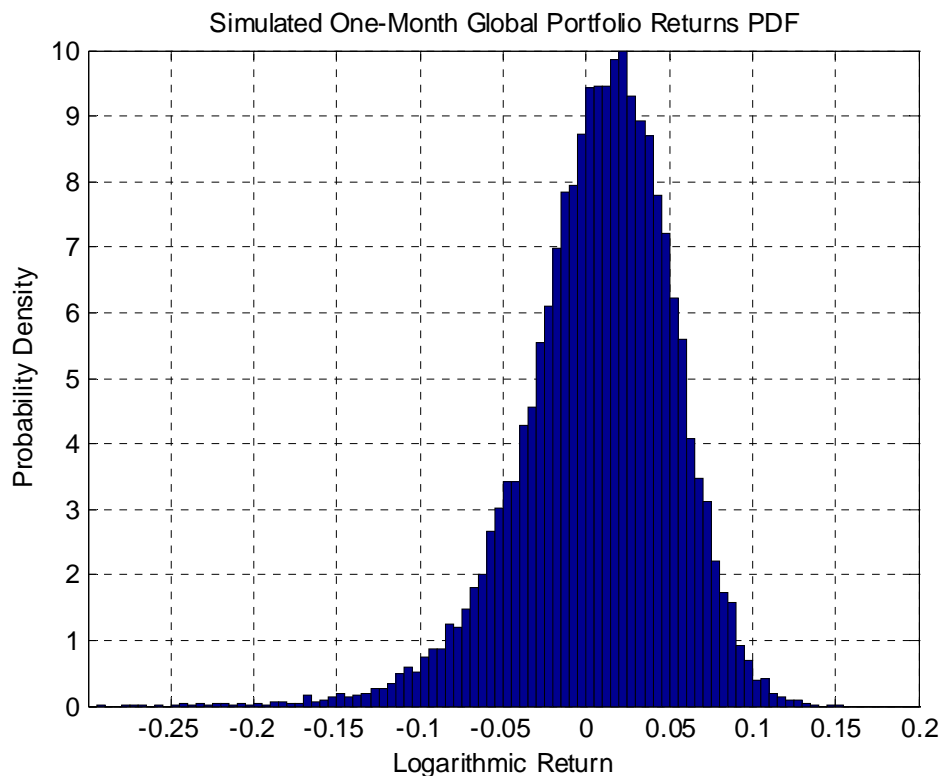
Figure 9: Portfolio profit and loss distribution (Historical Simulation)

Simulated One-Month Global Portfolio Returns PDF

5. Conclusions

The study incorporated a GJR-GARCH model with the copula-EVT to model the time-varying return distribution. This approach focuses on the entire distribution rather than the tail distribution only (Byström, 2004) and estimates portfolio VaR more accurately than traditional models.

Our procedure starts with the GJR-GARCH model to estimate the conditional mean and volatility of the each asset. Then, in the second stage, the POT method of EVT is used to model the tail distribution of the residual. Finally, a seven-dimensional t copula is fitted to the data and used to induce correlation between the simulated residuals of each asset.

In sum, the highly effective framework of this study can also be applied to other portfolio VaR problems. Results of this study can be used to perform a good risk management on global investments. Future research may consider dynamic copula in the dependence structure.

References

[1]   Byström, H. N. E. (2004). Managing extreme risks in tranquil and volatile markets using conditional extreme value theory. International Review of Financial Analysis, 13, 133–152.
[2]   Embrechts, P., C. Klüppelberg, and T. Mikosch (1997). Modelling extremal events for insurance and finance. Berlin: Spring Verlag.
[3]   Embrechts, P., A.J. McNeil, and D. Straumann (2000), Correlation and dependency in risk management: properties and pitfalls, in M. Dempster and H.K. Moffatt (Eds.), Risk Management: Value at Risk and Beyond, Cambridge University Press
[4]   Embrechts, P., F. Lindskog, and A. J. McNeil (2001), Modelling dependence with copulas and applications to risk management, ETH Zurich, preprint.
[5]   Fischer, T. (2003). Risk capital allocation by coherent risk measures based on one-sided moments. Insurance: Mathematics and Economics, 32, 135–146.

[6] Gencay R., F. Selcuk and A. Uluguelyagci (2003), High volatility, thick tail and extreme value theory in value-at-risk estimation, Mathematics and Economics, pp. 337-356.

[7] Glosten, L.R., Jagannathan, R., Runkle, D.E. (1993), On the Relation between Expected Value and the Volatility of the Nominal Excess Return on Stocks, The Journal of Finance, vol. 48, pp. 1779-1801.

[8] Hallerbach, W. G. (2002). Decomposing portfolio value-at-risk: A general analysis, Journal of Risk, 5.

[9] Hill, B. M. (1975), A Simple General Approach to Inference about the Tail of Distribution, Annals of Statistics, 3, pp.1163-1174.

[10] Ho, L.C., P. Burridge, J. Cadle, and M. Theobald (2000), Value-at-Risk: Applying the extreme value approach to Asian markets in recent financial turmoil, Pacific-Basin Finance Journal, 8, pp.249-275

[11] Jorion, P. (2000), Value at Risk: The Benchmark for Controlling Market Risk, second edition, McGraw-Hill, N.Y.

[12] Miller, D. E. (2003). The fundamentals of risk measurement. Financial Analysts Journal, 59, 108–109.

[13] McNeil, A.J. and R. Frey (2000), Estimation of Tail-Related Risk Measures for Heteroscedastic Financial Time Series: an Extreme Value Approach, Journal of Empirical Finance 7, Issues 3-4, November 2000, Pages 271-300

[14] Nelsen, R.B. (1999), An Introduction to Copulas, Lectures Notes in Statistics, 139, Springer Verlag, New York

[15] Nystrom, K., Skoglund, J. (2002), Univariate Extreme Value Theory, GARCH and Measures of Risk, Preprint, Swedbank.

[16] Pickands, J. (1975). Statistical inference using extreme order statistics, Annals of Statistics, **3**, 119-131.

[17] Rosengarten, J., & Zangari, P. (2003). Risk monitoring and performance measurement. In B. Litterman (Ed.), Modern investment management—An equilibrium approach, G.S.A. M. Quantitative Resources Group. New York: Wiley

[18] Sklar, A. (1959), "Fonctions de Répartition à n Dimentional et Leurs Marges," Publ.Inst. Statist. Paris, 8, pp. 229-231.