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# NESTING HORIZONTAL AND VERTICAL DIFFERENTIATION WITH LOCATION CHOICES

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*Abstract.* This study solves a location-then-price game in which horizontal and vertical differentiation are combined using an asymmetric distribution of consumers' taste. Boundary locations are robust when the taste disparity of the population is not large and out-of-market locations are not allowed. Firms may have incentives to move either inside or outside the market in other situations, so the equilibrium prices are never differentiated. The restrictions of vertical differentiation under this framework are further examined. A model with the entrance of a vertically differentiated product is also discussed.

## 1. INTRODUCTION

Horizontal and vertical differentiation are important issues for industrial organizations. Hotelling (1929) and d'Aspremont *et al.* (1979) provide the standard framework of horizontal differentiation, while vertical differentiation originates in the work of Mussa and Rosen (1978) and Shaked and Sutton (1982).[1] The difference between horizontal and vertical differentiation is defined by whether consumers are unanimous or not in their ranking of products in the market when prices are identical. If they are, then the differentiation is vertical, whereas if they are not, the differentiation is horizontal.

Several studies discuss both horizontal and vertical differentiation. Neven and Thisse (1990) apply a two-dimensional framework to discuss vertical and horizontal differentiation simultaneously. Gabszewicz and Thisse (1986) use two kinds of models to discuss vertical and horizontal differentiations: the former involves a restriction that the locations of firms should be outside of the unit length interval, while the latter does not involve such a restriction. Dos Santos Ferreira and Thisse (1996) adopt different transport rates to represent vertical differentiation and, thus, horizontal differentiation emerges only when both firms have the same transport rate.

Gabszewicz and Wauthy (2012) present a creative framework to demonstrate a combination of horizontal and vertical differentiation for an asymmetric distribution of consumers' tastes. They define a natural market for one firm as the market segment of this firm when both firms quote the same price. Obviously, horizontal differentiation emerges only when the natural markets of both

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[1] Vertical differentiation is intensively applied to trade models in Flam and Helpman (1987), Huang *et al.* (2010) and Wang and Wang (2011).

firms are identical. Gabszewicz and Wauthy (2012) show that when the natural market of one firm is sufficiently small, the equilibrium price for the large natural market is higher than that of the smaller natural market. However, the locations of firms in their model are presumed to be at the end points of the linear market.

The present study solves a location-then-price game in which horizontal and vertical differentiation are combined using an asymmetric distribution of consumers' taste as described by Gabszewicz and Wauthy (2012). In the present study, a quadratic transport cost is used to analyse the location-then-price game to eliminate the possibility of the nonexistence of a price equilibrium as shown in d'Aspremont *et al.* (1979). When the locations of firms are endogenous and restricted to be inside the market, the presumed boundary locations in Gabszewicz and Wauthy (2012) are robust when the disparity of the population's tastes is small. However, when the disparity is large, the firm with a smaller natural market will move closer to the market centre. The robustness of the triopoly case in Gabszewicz and Wauthy (2012) is also discussed in Section III.

The present study also finds several restrictions on the framework of Gabszewicz and Wauthy (2012) when locations are endogenous. First, in the case that firms are allowed to locate outside of the market, the equilibrium prices and profits for firms are identical, no matter whether the disparity of populations' tastes is large or small. The intuition is that firms will exert their location strategies to balance the differences in their natural markets. Second, the framework of vertical differentiation in Gabszewicz and Wauthy (2012) may not be well defined. The definition of vertical differentiation requires that all consumers prefer one brand to another if both firms quote the same prices. In their framework, both firms have their natural markets when firms' prices are identical, which violates the definition of vertical differentiation. The present study provides a modification to the Gabszewicz and Wauthy's (2012) framework such that locations of firms are restricted inside some ranges to satisfy the definition of vertical differentiation. With this modification, the present study finds that the firm with a larger natural market will charge a higher price than other firms.

The rest of this paper is organized as follows. Section II includes the model and equilibrium analysis. Section III discusses the entry conditions for a potential entrant with a product of higher or lower quality. Section IV concludes. All proofs are listed in the Appendix.

## 2. THE MODEL AND EQUILIBRIUM ANALYSIS

Consider a location-then-price game with duopolistic firms. Suppose there is a linear market with different types of consumer distributed in [0, 1] and two firms (1 and 2) whose natural markets are defined by $\mu/2$ and $(1 - \mu)/2$, respectively (see Fig. 1), where $\mu$ $(1 - \mu)$ is the density of type 1 (2) consumers who prefer to buy a product from firm 1 (2) if both firms set an identical price, with $\mu \in [0, 1/2]$.

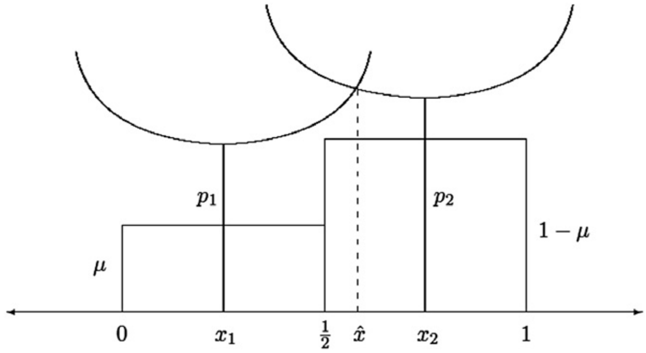


*Figure 1. Configuration of the market*

The indifferent consumer is located at the right half of the market ($\hat{x} \geq 1/2$).[2] Solving $S - p_1 - k(x - x_1)^2 = S - p_2 - k(x_2 - x)^2$ yields

$$\hat{x} = \frac{p_1 - p_2 + k(x_1^2 - x_2^2)}{2k(x_1 - x_2)},$$

where $S$ is the reservation price, $k$ is the unit transport rate, and $p_i$ and $x_i$ are the price and the location of firm $i$. Without loss of generality, assume $k = 1$, as done in Gabszewicz and Wauthy (2012). The profit functions are

$$\pi_1 = p_1 \cdot \left( \frac{\mu}{2} + \left( \hat{x} - \frac{1}{2} \right)(1-\mu) \right),$$
$$\pi_2 = p_2 \cdot (1-\mu)(1-\hat{x}).$$

Solving $\partial \pi_1 / \partial p_1 = 0$ and $\partial \pi_2 / \partial p_2 = 0$ yields

$$p_1 = \frac{(x_1 - x_2)(\mu(x_1 + x_2 - 2) - (x_1 + x_2))}{3(1-\mu)}, \tag{1}$$

$$p_2 = \frac{(x_1 - x_2)(x_1 + x_2 - 3 - \mu(x_1 + x_2 - 2))}{3(1-\mu)}. \tag{2}$$

Plugging equations (1) and (2) into the profit functions and solving $\partial \pi_1 / \partial x_1 = 0$ and $\partial \pi_2 / \partial x_2 = 0$ yields the following proposition and corollary.

**Proposition 1.** *When the locations of firms are allowed to exceed the boundaries, (1)* $x_1^* = \dfrac{3-8\mu}{8(1-\mu)}$, $x_2^* = \dfrac{9-8\mu}{8(1-\mu)}$, $p_1^* = p_2^* = \dfrac{3k}{8(1-\mu)^2}$,

[2] In fact, no equilibrium exists when $\hat{x} < 1/2$. Detailed proof is available upon request to the authors.

$\pi_1^* = \pi_2^* = \dfrac{3k}{32(1-\mu)^2}$, *and the market shares are the same; (2) if* $\mu < 3/8$, *then* $x_1 > 0$, $x_2 > 1$*; (3) if* $3/8 < \mu \le 1/2$, *then* $x_1 < \mathbf{0}$, $x_2 > 1$.

**COROLLARY 1.** *The equilibrium location differential increases with* $\mu$*. The equilibrium prices and profits are identical. In addition, the equilibrium prices and profits increase with* $\mu$*.*

The intuition of Proposition 1 and Corollary 1 is as follows. Although the natural markets of these two firms differ, the firms can alter their product varieties (locations) to yield identical price and profits. This phenomenon arises commonly in Hotelling-like models, such as those of d'Aspremont *et al*. (1979), Tabuchi and Thisse (1995) and Tsai and Lai (2005). In the current model, vertical differentiation does not lead to price (profit) differentiation. This finding differs from that of Gabszewicz and Wauthy (2012), in which the firm with a larger natural market can charge a higher price and, thus, earn a higher profit. Moreover, the location choices are affected by the taste disparity of the population. As $\mu$ decreases, the asymmetry in the tastes of the population increases, and the firms select a smaller differentiation in the product variety (location), because firms need not move farther apart to mitigate the price competition, owing to the large exogenous difference between their natural markets. When the locations are not allowed to fall outside the boundaries, that is, $x_1 \in [0, 1/2]$, $x_2 \in [1/2, 1]$, the results can be summarized as the following proposition.

**PROPOSITION 2.** *When locations are not allowed to exceed the boundaries, (1) when* $0 < \mu < 1/3$, *the equilibrium locations* $(x_1^*, x_2^*) = \left(\dfrac{1-3\mu}{3(1-\mu)}, 1\right)$. *As* $\mu$ *increases, the location differential, the equilibrium price differential and the absolute price levels all increase. (2) When* $1/3 \le \boldsymbol{\mu} \le 1/2$, *the equilibrium locations* $(x_1^*, x_2^*) = (0, 1)$. *As* $\mu$ *increases, differential and absolute price levels all increase.*

The second part of Proposition 2 suggests that when $\mu$ is large ($1/3 \le \mu \le 1/2$), the location choices are the boundary points, and the equilibrium prices are the same as in Gabszewicz and Wauthy (2012). However, when $\mu$ is small, such that $0 < \mu < 1/3$, the equilibrium location of firm 1 is interior ($x_1^* > 0$), while firm 2 is still at the boundary ($x_2^* = 1$). In this case, a larger asymmetry in the population's taste induces firm 1 to locate closer to the centre of the market, reducing the location differential. More importantly, the equilibrium price differential increases with $\mu$, owing to this location effect. Notably, if $1/3 \le \mu \le 1/2$, then $x_1^* = 0$ and $[p_1^*, p_2^*, \pi_1^*, \pi_2^*] = \left[\dfrac{1+\mu}{3(1-\mu)}, \dfrac{2-\mu}{3(1-\mu)}, \dfrac{(1+\mu)^2}{18(1-\mu)}, \dfrac{(2-\mu)^2}{18(1-\mu)}\right]$. In this case, the equilibrium prices are the same as those obtained by Gabszewicz and Wauthy (2012).

Consider the definition of vertical differentiation in which all consumers prefer one brand to another one if both firms charge the same prices, as in Gabszewicz and Wauthy (2012), where the locations are exogenous at $x_1 = 0$ and $x_2 = 1$. One restriction of their framework is that firms have their non-zero natural markets when $\mu \neq 0$, even if prices are the same, which violates the definition of vertical differentiation.[3] In fact, if the locations of firms are endogenous, to fulfil the definition of vertical differentiation, one more condition is required such that $x_1 \leq 1 - x_2$, to ensure the indifferent consumer located in [0, 1/2] (and, thus, firm 2 takes the whole market) when $p_1 = p_2$. In our model, if we assume a zoning constraint on firms' locations such that $x_1 \leq \beta \in [0, 1/2]$ and $x_2 \in [1/2, 1 - \beta]$, then our model satisfies the definition of vertical differentiation. This result is summarized in the following proposition.

**Proposition 3.** *When locations of firms are restricted by* $x_1 \in [0, \beta]$ *and* $x_2 \in [1/2, 1-\beta]$, $\beta \in (0, 1/2)$ *to satisfy the definition of vertical differentiation, there are three cases of equilibrium locations: (1) When* $\frac{4\beta-1}{4\beta-3} \leq \mu \leq \frac{1-\beta}{3\beta}$, $(x_1^{**}, x_2^{**}) = \left(\frac{1-\beta+\mu(\beta-3)}{3(1-\mu)}, 1-\beta\right)$. *(2) When* $0 \leq \mu \leq \frac{4\beta-1}{4\beta-3}$, $(x_1^{**}, x_2^{**}) = (0, 1-\beta)$. *(3). When* $\frac{1-\beta}{3\beta} < \mu \leq \frac{1}{2}$, $(x_1^{**}, x_2^{**}) = (\beta, 1-\beta)$. *In all the above cases, firm 2 charges a higher price than firm 1* ($p_2^{**} > p_1^{**}$). *However,* $p_2^{**} - p_1^{**}$ *may increase or decrease when* $\beta$ *or* $\mu$ *increases.*

With restrictions on firms' locations to satisfy the definition of vertical differentiation, Proposition 3 shows that the equilibrium locations depend on the ranges of $\mu$ and $\beta$, and the firm with a larger natural market charges higher prices ($p_2^{**} > p_1^{**}$), which coincides with the traditional literature.

## 3. ENTRY

Following Gabszewicz and Wauthy (2012), suppose a potential entrant, firm 3, is located at the end of a linear segment of length $L$. No consumer is present along this segment. Note that this linear segment connects to the main street at $x = 1/2$. First, consider firm 3 as a potential entrant with a product of inferior quality.

First, if firms are allowed to locate outside the market boundaries, the conditions that firm 3 is excluded from the market are

$$S - \left(\frac{1}{2} - x_1^*\right)^2 - p_1^* \geq S - L^2, \tag{3}$$

and

$$S - \left(x_2^* - \frac{1}{2}\right)^2 - p_2^* \geq S - L^2, \tag{4}$$

[3] We are grateful to one of the anonymous referees for pointing out this view.

where equation (3) means that the closest consumer to firm 3 (at 1/2) will buy from firm 1, instead of firm 3, even when $p_3 = 0$, and similarly for firms 2 and 3 in equation (4). After some calculations, we have the following proposition.

**PROPOSITION 4.** *(1) If the locations of firms are allowed to exceed the market boundaries, then when* $L > L_2 = \dfrac{16\mu^2 - 40\mu + 49}{8(1-\mu)}$, *firm 3 is excluded from the market when the price equilibrium is* $(p_1^*, p_2^*, 0)$. *(2) If firms are not allowed to locate outside the market, then when* $L > \max\{L_3, L_4\}$, *firm 3 is excluded from the market under equilibrium prices* $(p_1^*, p_2^*, 0)$.

Intuitively, when incumbent firms are located farther apart, the potential entrant is more likely to enter the market at its centre. Notice that two critical boundaries of $L_2$ and $\max\{L_3, L_4\}$ in Proposition 3 both increase in $\mu$.[4] Intuitively, as $\mu$ increases, the market moves toward horizontal differentiation, incumbent firms are more likely to locate farther apart, and preventing the entrance of firm 3 becomes more difficult, so the critical boundaries increase.

Next, consider firm 3 as a potential entrant with a product of higher quality. Assume that the location of firm 3 is exogenously determined. Without loss of generality, follow Gabszewicz and Wauthy (2012) and assume $L = 1/2$. Let $\hat{S} > S$ be the reservation price for this high quality product. The results can be summarized as the following proposition.[5]

**PROPOSITION 5.** *When the third firm selling a higher quality product of intrinsic quality* $\hat{S}$ *satisfies* $S + 1 < \hat{S} < S + 2$, *there exist equilibrium locations at the boundaries* $x_1^* = 0$ *and* $x_2^* = 1$, *and equilibrium prices* $p_1^* = p_2^* = \dfrac{2 + S - \hat{S}}{3}$ *and* $p_3^* = \dfrac{\hat{S} - S + 1}{3}$.

That these equilibrium prices are double those obtained by Gabszewicz and Wauthy (2012) is of particular interest. This is because the price competition among firms is reduced by quadratic transportation costs. Notably, $S + 1 < \hat{S} < S + 2$ is required to guarantee positive equilibrium prices in both the current framework and that of Gabszewicz and Wauthy (2012).

## 4. CONCLUSIONS

Gabszewicz and Wauthy (2012) provide a creative framework that combines horizontal and vertical product differentiation, with an asymmetric distribution of consumers' tastes. Using quadratic transport costs, this study provides a foundation for their model by further solving the location and price game. The

[4] Note that $L_2 \in [\sqrt{119}/8, \sqrt{33}/4] \approx [0.8750, 1.4361]$ and $\max\{L_3, L_4\} \in [\sqrt{201}/18, \sqrt{609}/18] \approx [0.7876, 1.3710]$ because $\mu \in [0, 1/2]$.

[5] We have checked two possible sets of locations, ($x_1 = 0$, $x_2 \in (0, 1/2, 1)$) and ($x_1 \in (0, 1/2)$, $x_2 = 1$), and find that no equilibrium exists. Other interior locations ($x_1 \in (0, 1/2)$, $x_2 \in (1/2, 0)$) are very complicated to solve. However, we believe that such interior solutions are unlikely to exist.

boundary locations, presumed by Gabszewicz and Wauthy (2012), are shown to be the equilibrium choices when taste disparity of the population is not large and locations are not allowed to be outside the market. However, the firm with a smaller natural market will choose an interior location when the population's disparity is large. Furthermore, firms always have an incentive to choose outside points when they are allowed to do so. In this situation, the equilibrium prices are not differentiated, because the location choices of the firms are non-restricted. The restrictions of vertical differentiation under their framework are further examined. An extension of this model that allows a potential entrant with a vertically differentiated product is also considered.

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## APPENDIX

### *Proof of Proposition 1*

From $\partial\pi_1 / \partial x_1 = 0$ and $\partial\pi_2 / \partial x_2 = 0$, the equilibrium locations are $x_1^* = \dfrac{3-8\mu}{8(1-\mu)}$ and $x_2^* = \dfrac{9-8\mu}{8(1-\mu)}$. Other solutions are $\left\{x_1 = \dfrac{3-4\mu}{4(1-\mu)}, x_2 = \dfrac{9-4\mu}{4(1-\mu)}\right\}$ and $\left\{x_1 = \dfrac{4\mu+3}{4(\mu-1)}, x_2 = \dfrac{4\mu-3}{4(\mu-1)}\right\}$. However, they violate the second-order conditions. Notably, $x_2$ is always greater than one. If $\mu = 1/2$, then $(x_1^*, x_2^*) = (-1/4, 5/4)$, which is reduced to the traditional result (see Tabuchi and Thisse, 1995). If $\mu = 0$, then $(x_1^*, x_2^*) = (3/8, 9/8)$. When $\mu = 0$, the total market size is 1/2 and $(x_1^*, x_2^*) = (3/8, 9/8)$ is equivalent to the relative position of (–1/4, 5/4) when the market size is one, because 1/2 – (1/4 × 1/2) = 3/8 and 1 + 1/4 × 1/2 = 9/8. Substituting ($x_1^*$ and $x_2^*$) into equation (1) and (2) yields $p_1^* = p_2^* = \dfrac{3k}{8(1-\mu)^2}$. The equilibrium profits are $\pi_1^* = \pi_2^* = \dfrac{3k}{32(1-\mu)^2}$.

*Proof of Corollary 1*

For comparative statics:

$$\frac{\partial x_1^*}{\partial \mu} = \frac{-5}{8(1-\mu)^2} < 0,$$
$$\frac{\partial x_2^*}{\partial \mu} = \frac{1}{8(1-\mu)^2} > 0.$$

The above results reveal that the equilibrium locations become farther apart as $\mu$ increases. In addition,

$$\frac{\partial p_i^*}{\partial \mu} = \frac{3}{4(1-\mu)^3} > 0, \quad i = 1, 2,$$
$$\frac{\partial \pi_i^*}{\partial \mu} = \frac{3}{16(1-\mu)^3} > 0, \quad i = 1, 2.$$

*Proof of Proposition 2*

When $x_2$ is restricted in [1/2, 1], we first show that $x_2 = 1$ is the unique optimal solution for firm 2. Solving $\partial \pi_2 / \partial x_2 = 0$ yields two interior solutions, $\hat{x}_2 = \dfrac{x_1(1-\mu)+3-2\mu}{3(1-\mu)}$ and $\hat{\hat{x}}_2 = \dfrac{x_1(\mu-1)+3-2\mu}{(1-\mu)}$. Because $\hat{x}_2 - \hat{\hat{x}}_2 = \dfrac{-2(3-2x_1+2\mu x_1-2\mu)}{3(1-\mu)} < 0$, thus $\hat{x}_2 < \hat{\hat{x}}_2$. Moreover, $\hat{x}_2 - 1 = \dfrac{x_1+\mu-\mu x_1}{3(1-\mu)} > 0$, resulting in $\hat{x}_2 > 1$. Because $\partial^3 \pi_2 / \partial x_2^3 = 1/3(1-\mu) > 0$, $\pi_2$ is a third-order polynomial function and so $\pi_2$ is strictly increasing for $x_2 < \hat{x}_2$. Therefore, $x_2^* = 1$ is the unique optimal solution for firm 2. Substituting $x_2 = 1$ into $\left.\dfrac{\partial \pi_1}{\partial x_1}\right|_{x_2=1}$ and solving for $x_1$ yields $x_1^* = \dfrac{1-3\mu}{3(1-\mu)}$. The conditions for $x_2 = 1$ as a corner solution are satisfied, because $\partial \pi_2 / \partial x_2 > 0$ at $(x_1, x_2) = \left(\dfrac{1-3\mu}{3(1-\mu)}, 1\right)$ and $(x_1, x_2) = (0, 1)$. Solving $x_1^* = 0$ yields $\mu = 1/3$. Therefore, if $0 < \mu < 1/3$, then $0 < x_1^* < 1/2$ and $[p_1^*, p_2^*, \pi_1^*, \pi_2^*] = \left[\dfrac{8}{27(1-\mu)^2}, \dfrac{10}{27(1-\mu)^2}, \dfrac{16}{243(1-\mu)^2}, \dfrac{25}{243(1-\mu)^2}\right]$. In this case,

$p_2^* - p_1^* = \dfrac{2}{27(1-\mu)^2}$ and $\pi_2^* - \pi_1^* = \dfrac{9}{243(1-\mu)^2}$. Thus, $\partial x_1^*/\partial\mu = \dfrac{-2}{3(1-\mu)^2} < 0$, $\partial p_1^*/\partial\mu > 0$, $\partial p_2^*/\partial\mu > 0$ and $\partial(p_2^* - p_1^*)/\partial\mu = \dfrac{4}{27(1-\mu)^3} > 0$. In this bounded location case, the equilibrium price differential increases with $\mu$. This result is in contrast to Proposition 2 of Gabszewicz and Wauthy (2012). For $\mu = 0$, the equilibrium locations are reduced to $(x_1, x_2) = (1/2, 1)$, because the market now ranges from 1/2 to 1.

*Proof of Proposition 3*

When the location is not allowed to fall outside the boundary, similarly to the proof of Proposition 2, $x_2^* = 1 - \beta$ is the unique solution. Substituting $x_2 = 1 - \beta$ into $\left.\dfrac{\partial \pi_1}{\partial x_1}\right|_{x_2 = 1-\beta}$ and solving for $x_1$ yields

$$x_1^{**} = \frac{1-\beta+\mu(\beta-3)}{3(1-\mu)}.$$

There are three cases of optimal location for firm 1. The first case is $(x_1 = x_1^{**}, x_2 = 1-\beta)$, with the following two necessary conditions:

$$x_1^{**} \geq 0 \quad \Rightarrow \quad \mu \leq \frac{1-\beta}{3-\beta},$$

$$x_1^{**} \leq \beta \quad \Rightarrow \quad \mu \geq \frac{4\beta-1}{4\beta-3}.$$

Notably, $\dfrac{4\beta-1}{4\beta-3} < \dfrac{1-\beta}{3-\beta}$, resulting in a region of $\mu$ for these two conditions. Moreover, the equilibrium prices are:

$$p_1^{**} = \frac{8}{27}\frac{(\mu\beta+1-\beta)^2}{(1-\mu)^2}, \quad p_2^{**} = \frac{2(1-\beta+\mu\beta)(5+4\beta-4\mu\beta)}{27(1-\mu)^2},$$

and so

$$p_2^{**} - p_1^{**} = \frac{2(\mu\beta+1-\beta)(1+8\beta-8\mu\beta)}{27(1-\mu)^2} > 0.$$

Furthermore, $\dfrac{\partial(p_2^{**} - p_1^{**})}{\partial\mu} = \dfrac{2(2+7\beta-7\beta\mu)}{27(1-\mu)^3} > 0$, $\dfrac{\partial(p_2^{**} - p_1^{**})}{\partial\beta} = \dfrac{2(7+16\beta\mu-16\beta)}{27(1-\mu)} > 0$.

The second case is $(x_1 = 0, x_2 = 1 - \beta)$, if $\mu < \dfrac{4\beta-1}{4\beta-3}$. In this case, the equilibrium prices are:

$$p_1^{**} = \frac{(1-\beta)(1+\mu-\beta+\mu\beta)}{3(1-\mu)}, \quad p_2^{**} = \frac{(1-\beta)(2+\beta-\mu-\mu\beta)}{3(1-\mu)},$$
$$p_2^{**} - p_1^{**} = \frac{(1-\beta)(1+2\beta(1-\mu)-2\mu)}{3(1-\mu)} > 0.$$

Furthermore, $\frac{\partial(p_2^{**} - p_1^{**})}{\partial\mu} = \frac{\beta-1}{3(1-\mu)^2} < 0$, $\frac{\partial(p_2^{**} - p_1^{**})}{\partial\beta} = \frac{1+4\mu\beta-4\beta}{3(1-\mu)}$, which can be either positive or negative.

The last case is ($x_1 = \beta$, $x_2 = 1 - \beta$), if $\mu > \frac{1-\beta}{3-\beta}$. Then, the equilibrium prices become:

$$p_1^{**} = \frac{(1-2\beta)(1+\mu)}{3(1-\mu)}, \quad p_2^{**} = \frac{(1-2\beta)(2-\mu)}{3(1-\mu)},$$
$$p_2^{**} - p_1^{**} = \frac{(1-2\beta)(1-2\mu)}{3(1-\mu)} > 0.$$

Furthermore, $\frac{\partial(p_2^{**} - p_1^{**})}{\partial\mu} = \frac{2\beta-1}{3(1-\mu)^2} < 0$, $\frac{\partial(p_2^{**} - p_1^{**})}{\partial\beta} = \frac{2(1-2\mu)}{3(1-\mu)} < 0$.

*Proof of Proposition 4*

Given $p_3 = 0$, equation (3) states that the closest consumer to firm 3 (i.e. $x = 1/2$) cannot obtain a larger surplus by buying from firm 3 than by buying from firm 1. Equation (4) describes a similar condition for comparison with firm 2. Equation (3) implies

$$L \geq \frac{\sqrt{16\mu^2 + 8\mu + 25}}{8(1-\mu)} \equiv L_1,$$

and equation (4) implies

$$L \geq \frac{\sqrt{16\mu^2 - 40\mu + 49}}{8(1-\mu)} \equiv L_2 > L_1.$$

Therefore, when $L > L_2$, firm 3 is excluded from the market at the vector of equilibrium prices $(p_1^*, p_2^*, 0)$.

Second, when firms are not permitted to locate outside the market, since the results are identical to those of Gabszewicz and Wauthy (2012) when $1/3 \leq \mu \leq 1/2$, only $0 < \mu < 1/3$ needs to be discussed. Substituting ($p_1^*$, $p_2^*$, $x_1^* = \frac{1-3\mu}{3(1-\mu)}$) and $x_2^* = 1$ into equations (3) and (4) yields

$$L \geq \frac{81\mu^2 + 54\mu + 105}{18(1-\mu)} \equiv L_3,$$

$$L \geq \frac{81\mu^2 - 162\mu + 201}{18(1-\mu)} \equiv L_4 > L_3 \quad \text{iff } \mu < \frac{4}{9}.$$

Therefore, when $L > \max\{L_3, L_4\}$, firm 3 is excluded from the market at the vector of equilibrium prices $(p_1^*, p_2^*, 0)$. Moreover, if $L_2 - \max\{L_3, L_4\} > 0$, then firm 3 is more likely to be excluded when incumbent firms are allowed to locate outside the market.

*Proof of Proposition 5*

Solving $S - p_1 - k(x_1 - x)^2 = \hat{S} - k(L + 1/2 - x)^2 - p_3$ yields the indifferent consumer $\hat{x}_l = \dfrac{S - p_1 - x_2^2 - \hat{S} + 1 + p_3}{2(1-x_1)}$. Similarly, solving $S - p_2 - k(x_2 - x)^2 = \hat{S} - k(x - 1/2 + L)^2 - p_3$ yields the indifferent consumer $\hat{x}_r = \dfrac{\hat{S} - S + p_2 - p_3 + x_2^2}{2x_2}$. The profit functions become $\pi_1 = p_1 \cdot \hat{x}_l$, $\pi_2 = p_2 \cdot (1 - \hat{x}_r)$ and $\pi_3 = p_3 \cdot (\hat{x}_r - \hat{x}_l)$. Using backward induction and solving the price game $\partial \pi_i / \partial p_i = 0$, $i = 1, 2, 3$ yields equilibrium prices:

$$p_1 = \frac{1}{6}(2S - 3x_1^2 - 2\hat{S} + 3 - x_1 x_2 + x_2),$$

$$p_2 = \frac{1}{6}(7x_2 + 2S - 3x_2^2 - 2\hat{S} - x_1 x_2),$$

$$p_3 = \frac{1}{3}(\hat{S} + x_2 - x_1 x_2 - S).$$

The conditions of boundary locations are

$$\left.\frac{\partial \pi_1}{\partial x_1}\right|_{x_1=0, x_2=1} = \frac{\mu(2 + S - \hat{S})(1 + S - \hat{S})}{18} < 0 \quad \text{iff} \quad S + 1 < \hat{S} < S + 2,$$

$$\left.\frac{\partial \pi_2}{\partial x_2}\right|_{x_1=0, x_2=1} = \frac{(1-\mu)(\hat{S} - S - 2)(1 + S - \hat{S})}{18} > 0 \quad \text{iff} \quad S + 1 < \hat{S} < S + 2.$$

Plugging $x_1 = 0$ and $x_2 = 1$ into prices yields

$$p_1^* = p_2^* = \frac{2 + S - \hat{S}}{3}, \quad p_3^* = \frac{\hat{S} - S + 1}{3}.$$