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ON THE EMDEN-FOWLER EQUATION $u''(t)u(t) = c_1 + c_2u'(t)^2$ WITH $c_1 \geq 0,\ c_2 \geq 0$*

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Abstract In this article, we study the following initial value problem for the nonlinear equation

$$\begin{cases} u''u(t) = c_1 + c_2u'(t)^2,\ c_1 \geq 0, c_2 \geq 0, \\ u(0) = u_0,\ u'(0) = u_1. \end{cases}$$

We are interested in properties of solutions of the above problem. We find the life-span, blow-up rate, blow-up constant and the regularity, null point, critical point, and asymptotic behavior at infinity of the solutions.

Key words Blow-up; Life-span; Blow-up constant; asymptotic behavior; null

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0 Introduction

In our articles, we studied the semi-linear wave equation $\Box u + f(u) = 0$ under some conditions, and found some results concerning blow-up, blow-up rate, and the estimates for the life-span of solutions. Here, we consider the following equation

$$u'' = u^p(c_1 + c_2(u'(t))^q), u(0) = u_0, u'(0) = u_1, c_1 > 0, c_2 > 0, \tag{0.1}$$

it is a particular form of the generalized Emden-Fowler equation

$$y''_{xx} = x^n y^m (A + B(y'_x)^l).$$

To study the behavior of the solutions for the above equation, we separate q into five parts, $q < 0$, $0 < q < 1$, $1 \leq q < 2$, $q = 2$ and $q > 2$. We considered the case that $p > 1$ and $q > 1$ in [10], and obtained some results on life-span, blow-up rates of solutions; this method in [10] cannot be applied to the case of $p = -1, q = 2$; here we focus on the study on such a particular case with $c_1 \geq 0$ and $c_2 \geq 0$. We also find the life-span, blow-up rate and blow-up constant and other properties of u. For further informations on such equation we refer the reader to [1][1].

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[1]For results on the blow-up character of solution of the equation $(| u' |^{m-2} u')' = u^p$, see [12]

We say that a function $g : \mathbb{R} \to \mathbb{R}$ having blow-up time T^* and a blow-up rate α means that there is a finite number T^* such that $g(t)$ exists for $t < T^*$ and $\lim\limits_{t\to T^*} g(t)^{-1} = 0$ and there exists a nonzero $\beta \in \mathbb{R}$ with $\lim\limits_{t\to T^*} (T^* - t)^\alpha g(t) = \beta$, in this case, β is called the blow-up constant of g. By the standard arguments of existence of solutions to ordinary differential equations, one can prove the local existence of solutions to the nonlinear equation

$$\begin{cases} u''(t)u(t) = c_1 + c_2 u'(t)^2, u(0) = u_0, u'(0) = u_1 \ c_1 \geq 0, c_2 \geq 0, \\ u(0) = u_0, u'(0) = u_1, \end{cases} \tag{0.2}$$

and for $u_0 \neq 0$, we have found the following result:

For (i) $c_2 > 1$, the blow-up rate and blow-up constant of solutions are obtained;

(ii) $c_2 \in (0, 1)$, the solution u can be characterized as the property of the function $t^{\frac{1}{1-c_2}}$, and we have got the results concerning critical point and asymptotic behavior at infinity of the solutions.

We will often use the following lemma:

Lemma 0.1 Suppose that $f \in C^1[t_0, \infty) \cap C^2(t_0, \infty)$, $f(t_0) > 0$, $f'(t) < 0$, and $f''(t) \leq 0$, for $t > t_0$. Then, there exists a finite positive number $T > t_0$, such that $f(T) = 0$.

Lemma 0.2 Suppose that u is the solution of (0.2). If $u_0 > 0$ and $u_1 > 0$, then, $u(t), u'(t)$, and $u''(t) > 0$, for $t \in [0, T^*)$, where T^* is the life-span of u.

Proof After some computations, one can obtain Lemma 0.1. We only prove Lemma 0.2. Suppose that there exists a positive number t_0, such that $u'(t_0) \leq 0$. Because $u \in C^2$ and $u_1 > 0$, there exists a positive number t_1, defined by $t_1 = \inf\{t \in (0, t_0] : u'(t) \leq 0\}$, then, $u'(t_1) = 0$, and $u'(t) > 0, u(t) > 0$ for $t \in [0, t_1)$ and $u''(t) > 0$, for $t \in [0, t_1)$; therefore, $u'(t_1) \geq u_1 > 0$. This result contradicts with $u'(t_1) = 0$; thus, we conclude that $u'(t) > 0$ for $t \in [0, T^*)$.

Together the equation (0.2) and the continuities of u, u', and u'', the lemma follows.

1 Blow-up Phenomena for $c_2 > 1$

For $u_0 \neq 0$, we obtain

Theorem 1 If T^* is the life-span of u and u is the solution of problem (0.2), then, T^* is finite, that is, u is only a local solution of (0.2), if one of the following is valid

(i) $u_0 u_1 \geq 0$ or (ii) $u_0 u_1 < 0$.

Further, in the case of (i), we have the estimate

$$T^* \leq T_1^*(u_0, u_1, c_2) = \frac{1}{c_2 - 1}\sqrt{\frac{c_2}{c_1}}(u_0^2)^{\frac{1}{2}}\Big(1 + \frac{c_2}{c_1}u_1^2\Big)^{\frac{-1}{2c_2}} \int_0^\alpha \frac{\mathrm{d}r}{\sqrt{1 - r^{\frac{2c_2}{c_2-1}}}}, \tag{1.1}$$

where $\alpha = (1 + \frac{c_2}{c_1}u_1^2)^{\frac{c_2-1}{-2c_2}}$. In the case of (ii), we have

$$T \leq T_2^*(u_0, u_1, c_2) = \frac{1}{c_2 - 1}\sqrt{\frac{c_2}{c_1}}(u_0^2)^{\frac{1}{2}}\Big(1 + \frac{c_2}{c_1}u_1^2\Big)^{\frac{-1}{2c_2}} \Big(\int_0^1 + \int_\alpha^1\Big) \frac{\mathrm{d}r}{\sqrt{1 - r^{\frac{2c_2}{c_2-1}}}}; \tag{1.2}$$

we have further

$$z_1(u_0, u_1, c_2) = \frac{1}{c_2 - 1}\sqrt{\frac{c_2}{c_1}}(u_0^2)^{\frac{1}{2}}\Big(1 + \frac{c_2}{c_1}u_1^2\Big)^{\frac{-1}{2c_2}} \int_\alpha^1 \frac{\mathrm{d}r}{\sqrt{1 - r^{\frac{2c_2}{c_2-1}}}} \tag{1.3}$$

is a critical point of u^2.

Remark According to this theorem, one can obtain the asymptotic behavior of life-span $T^*_{1,\varepsilon}(u_0, u_1, c_2)$ by $T^*_1(u_0, u_1, c_2)$: If T^* is the life-span of u and u is the solution of the problem

$$u''(t)u(t) = c_1 + c_2u'(t)^2,\ u(0) = \varepsilon u_0, u'(0) = \varepsilon u_1, \quad u_0u_1 > 0$$

then, T^* is given by

$$T^* \leq T^*_{1,\varepsilon}(u_0, u_1, c_2) = \frac{\varepsilon}{c_2 - 1}\sqrt{\frac{c_2}{c_1}}(u_0^2)^{\frac{1}{2}}(1 + \frac{c_2}{c_1}\varepsilon^2u_1^2)^{\frac{-1}{2c_2}} \int_0^{(1+\frac{c_2}{c_1}\varepsilon^2u_1^2)^{\frac{c_2-1}{-2c_2}}} \frac{\mathrm{d}r}{\sqrt{1 - r^{\frac{2c_2}{c_2-1}}}}$$

and

$$\begin{aligned} T^* &\sim \frac{\varepsilon}{c_2 - 1}\sqrt{\frac{c_2}{c_1}}(u_0^2)^{\frac{1}{2}}\left(1 + \frac{c_2}{c_1}\varepsilon^2u_1^2\right)^{\frac{-1}{2c_2}} \int_0^1 \frac{\mathrm{d}r}{\sqrt{1 - r^{\frac{2c_2}{c_2-1}}}} \\ &= \frac{\varepsilon c}{2}\sqrt{\frac{1}{c_1c_2}}(u_0^2)^{\frac{1}{2}}\left(1 + \frac{c_2}{c_1}\varepsilon^2u_1^2\right)^{\frac{-1}{2c_2}} \quad \text{as} \ \ \varepsilon\tilde{\ }0^+, \end{aligned} \tag{1.4}$$

where $c = \beta(\frac{1}{2}, \frac{2c_2}{c_2-1})$, that means there exists no classic solution to the equation (0.2) if the initial values are zero.

Proof of Theorem 1 Consider the function $J_u(t) = (u(t)^2)^{-\frac{c_2-1}{2}}$, we have

$$\begin{aligned} J'_u(t) &= (1 - c_2)(u(t)^2)^{-\frac{c_2-1}{2}-1}(u(t)u'(t)), \\ J''_u(t) &= (1 - c_2)(u(t)^2)^{-\frac{c_2-1}{2}-1}(u(t)u''(t) + u'(t)^2) \\ &\quad -(1 - c_2)(c_2 + 1)(u(t)u'(t))^2(u(t)^2)^{-\frac{c_2-1}{2}-2} \\ &= (1 - c_2)(u(t)^2)^{-\frac{c_2-1}{2}-1}(u(t)u''(t) - c_2u'(t)^2) \\ &= c_1(1 - c_2)(u(t)^2)^{-\frac{c_2+1}{2}} = c_1(1 - c_2)(J_u(t))^{\frac{c_2+1}{c_2-1}}. \end{aligned} \tag{1.5}$$

Apply the energy method to the equation for J_u, we obtain

$$J'_u(t)^2 + \frac{c_1}{c_2}(1 - c_2)^2J_u(t)^{\frac{2c_2}{c_2-1}} = E_J(0) = c_2^{-1}(1 - c_2)^2(u_0^2)^{-c_2}(c_1 + c_2u_1^2) > 0. \tag{1.6}$$

According to the fact that $c_1 > 0, c_2 > 1$, and (1.5), $J''_u(t) < 0$, for all $t \geq 0$. Thus, for

(i) $u_0u_1 > 0$, then, $J'_u(0) < 0$ and for $t \geq \frac{-J_u(0)}{J_u{}'(0)} = \frac{-u_0}{(1-c_2)u_1}$, $J''_u(t) < 0,\ J'_u(t) \leq J'_u(0)$, and $J_u(t) \leq J_u(0) + J'_u(0)t \leq 0$, which means that there exists a real finite number $T^*_1 \leq \frac{-u_0}{(1-c_2)u_1}$ with

$$J_u(T^*_1) = 0 = \lim_{t \to T^*_1}(u(t)^2)^{-\frac{c_2-1}{2}};$$

that is, u blows up at finite time T^*_1. By (1.6), we obtain

$$\begin{aligned} J'_u(t) &= -\sqrt{E_J(0) - \frac{c_1}{c_2}(1 - c_2)^2J_u(t)^{\frac{2c_2}{c_2-1}}} \\ &= -\frac{c_2 - 1}{\sqrt{c_2}}\sqrt{c_1}\sqrt{(u_0^2)^{-c_2}(1 + \frac{c_2}{c_1}u_1^2) - J_u(t)^{\frac{2c_2}{c_2-1}}}, \end{aligned} \tag{1.7}$$

$$t = -\frac{1}{c_2 - 1}\sqrt{\frac{c_2}{c_1}}\int_{J_u(0)}^{J_u(t)} \frac{\mathrm{d}r}{\sqrt{(u_0^2)^{-c_2}(1 + \frac{c_2}{c_1}u_1^2) - r^{\frac{2c_2}{c_2-1}}}},$$

$$T_1^*(u_0,u_1,c_2)=\frac{1}{c_2-1}\sqrt{\frac{c_2}{c_1}}\int_0^{J_u(0)}\frac{\mathrm{d}r}{\sqrt{(u_0^2)^{-c_2}(1+\frac{c_2}{c_1}u_1^2)-r^{\frac{2c_2}{c_2-1}}}}. \tag{1.8}$$

The estimates (1.1) and (1.8) are equivalent.

(ii) $u_0u_1<0$, then, by equation (0.2), we have

$$(u(t)u'(t))'=c_1+(1+c_2)u'(t)^2\geq c_1>0,\ u(t)u'(t)\geq u_0u_1+c_1t>0 \text{ for } t>\frac{-u_0u_1}{c_1};$$

therefore, there exists a critical point $z_1(u_0,u_1,c_2):=z_1$ of J_u, that is $J_u'(z_1)=0=u'(z_1)$. By (1.6), we have

$$\begin{aligned}
J_u'(t)&=\frac{c_2-1}{\sqrt{c_2}}\sqrt{c_1}\sqrt{(u_0^2)^{-c_2}(1+\frac{c_2}{c_1}u_1^2)-J_u(t)^{\frac{2c_2}{c_2-1}}},\\
J_u(z_1)&=\Big((u_0^2)^{-c_2}\Big(1+\frac{c_2}{c_1}u_1^2\Big)\Big)^{\frac{c_2-1}{2c_2}}=J_u(0)\Big(1+\frac{c_2}{c_1}u_1^2\Big)^{\frac{c_2-1}{2c_2}},\\
z_1&=\frac{1}{c_2-1}\sqrt{\frac{c_2}{c_1}}\int_{J_u(0)}^{J_u(z_1)}\frac{\mathrm{d}r}{\sqrt{(u_0^2)^{-c_2}(1+\frac{c_2}{c_1}u_1^2)-r^{\frac{2c_2}{c_2-1}}}}\\
&=\frac{1}{c_2-1}\sqrt{\frac{c_2}{c_1}}\int_{J_u(0)}^{J_u(0)(1+\frac{c_2}{c_1}u_1^2)^{\frac{c_2-1}{2c_2}}}\frac{\mathrm{d}r}{\sqrt{(u_0^2)^{-c_2}(1+\frac{c_2}{c_1}u_1^2)-r^{\frac{2c_2}{c_2-1}}}}\\
&=\frac{1}{c_2-1}\sqrt{\frac{c_2}{c_1}}\frac{J_u(0)(1+\frac{c_2}{c_1}u_1^2)^{\frac{c_2-1}{2c_2}}}{\sqrt{(u_0^2)^{-c_2}(1+\frac{c_2}{c_1}u_1^2)}}\int_{(1+\frac{c_2}{c_1}u_1^2)^{\frac{c_2-1}{-2c_2}}}^{1}\frac{\mathrm{d}r}{\sqrt{1-r^{\frac{2c_2}{c_2-1}}}}.
\end{aligned}$$

We obtain (1.3). Using (1.6), we obtain $J_u(z_1)=((u_0^2)^{-c_2}(1+\frac{c_2}{c_1}u_1^2))^{\frac{c_2-1}{2c_2}}$ and for $t\geq z_1$,

$$\begin{aligned}
J_u'(t)&=-\frac{c_2-1}{\sqrt{c_2}}\sqrt{c_1}\sqrt{(u_0^2)^{-c_2}(1+\frac{c_2}{c_1}u_1^2)-J_u(t)^{\frac{2c_2}{c_2-1}}},\\
t-z_1&=-\frac{1}{c_2-1}\sqrt{\frac{c_2}{c_1}}\int_{J_u(z_1)}^{J_u(t)}\frac{\mathrm{d}r}{\sqrt{(u_0^2)^{-c_2}(1+\frac{c_2}{c_1}u_1^2)-r^{\frac{2c_2}{c_2-1}}}},\\
T_2^*&=z_1+\frac{1}{c_2-1}\sqrt{\frac{c_2}{c_1}}\int_0^{J_u(z_1)}\frac{\mathrm{d}r}{\sqrt{(u_0^2)^{-c_2}(1+\frac{c_2}{c_1}u_1^2)-r^{\frac{2c_2}{c_2-1}}}}.
\end{aligned}\tag{1.9}$$

From (1.9), we get estimate (1.2).

2 Blow-up Rate and Blow-up Constant for $c_2>1$

In this section, we study the blow-up rate and blow-up constant for u^2, $(u^2)'$, and $(u^2)''$ under the conditions in Section 1. We have the following results.

Theorem 2 If u satisfies one of the conditions in Theorem 1. Then, the blow-up rate of u^2 is $2/(c_2-1)$, and the blow-up constant of u^2 is $(\frac{1}{c_2-1})^{\frac{2}{c_2-1}}(\frac{c_2}{c_1+c_2u_1^2})^{\frac{1}{c_2-1}}|u_0|^{\frac{2c_2}{c_2-1}}$, that is, for $m\in\{1,2\}$

$$\lim_{t\to T_m^*(u_0,u_1,c_2)^-}(T_m^*(u_0,u_1,c_2)-t)^{\frac{2}{c_2-1}}u(t)^2=\Big(\frac{1}{c_2-1}\sqrt{\frac{c_2}{c_1+c_2u_1^2}}\,|u_0|^{c_2}\Big)^{\frac{2}{c_2-1}}. \tag{2.1}$$

The blow-up rate of $(u^2)'$ is $(c_2+1)/(c_2-1)$, and the blow-up constant of $(u^2)'$ is $2c_2^{\frac{1}{c_2-1}}\ |u_0|^{\frac{2c_2}{c_2-1}}(c_1+c_2u_1^2)^{-\frac{1}{c_2-1}}(c_2-1)^{-\frac{c_2+1}{c_2-1}}$, that is,

$$\begin{aligned}&\lim_{t\to T_m^*(u_0,u_1,c_2)^-}(T_m^*(u_0,u_1,c_2)-t)^{\frac{c_2+1}{c_2-1}}(u^2)'(t)\\&=2c_2^{\frac{1}{c_2-1}}\ |u_0|^{\frac{2c_2}{c_2-1}}\ (c_1+c_2u_1^2)^{-\frac{1}{c_2-1}}(c_2-1)^{-\frac{c_2+1}{c_2-1}}.\end{aligned}\tag{2.2}$$

The blow-up rate of $(u^2)''$ is $2c_2/(c_2-1)$, and the blow-up constant of $(u^2)''$ is $2(c_2+1)c_2^{\frac{1}{c_2-1}}\ |u_0|^{\frac{2c_2}{c_2-1}}(c_1+c_2u_1^2)^{\frac{-c_2-1}{c_2-1}}(\frac{1}{c_2-1})^{\frac{2c_2}{c_2-1}}$, that is,

$$\begin{aligned}&\lim_{t\to T_m^*(u_0,u_1,c_2)^-}(T_m^*(u_0,u_1,c_2)-t)^{\frac{2c_2}{c_2-1}}(u^2)''(t)\\&=2(c_2+1)c_2^{\frac{1}{c_2-1}}\ |u_0|^{\frac{2c_2}{c_2-1}}\ (c_1+c_2u_1^2)^{\frac{-c_2-1}{c_2-1}}\Big(\frac{1}{c_2-1}\Big)^{\frac{2c_2}{c_2-1}}.\end{aligned}\tag{2.3}$$

Proof For (i) $u_0u_1\geq 0$, by (1.8), we obtain

$$T_1^*(u_0,u_1,c_2)-t=\frac{1}{c_2-1}\sqrt{\frac{c_2}{c_1}}\int_0^{J_u(t)}\frac{\mathrm{d}r}{\sqrt{(u_0^2)^{-c_2}(1+\frac{c_2}{c_1}u_1^2)-r^{\frac{2c_2}{c_2-1}}}},$$

$$\lim_{t\to T_1^*(u_0,u_1,c_2)^-}(T_1^*(u_0,u_1,c_2)-t)J_u^{-1}(t)=\frac{1}{c_2-1}\sqrt{\frac{c_2}{c_1+c_2u_1^2}}\,|u_0|^{c_2},$$

$$\lim_{t\to T_1^*(u_0,u_1,c_2)^-}(T_1^*(u_0,u_1,c_2)-t)(u(t)^2)^{\frac{c_2-1}{2}}=\frac{1}{c_2-1}\sqrt{\frac{c_2}{c_1+c_2u_1^2}}\,|u_0|^{c_2},$$

$$\lim_{t\to T_1^*(u_0,u_1,c_2)^-}(T_1^*(u_0,u_1,c_2)-t)^{\frac{2}{c_2-1}}u(t)^2=\Big(\frac{1}{c_2-1}\sqrt{\frac{c_2}{c_1+c_2u_1^2}}\,|u_0|^{c_2}\Big)^{\frac{2}{c_2-1}}.$$

Thus, the assertion (2.1) is completely proved for $u_0u_1\geq 0$. By (1.7) and (2.1), we have

$$\begin{aligned}&\lim_{t\to T_1^*(u_0,u_1,c_2)^-}(1-c_2)(u(t)^2(T_1^*(u_0,u_1,c_2)-t)^{\frac{2}{c_2-1}})^{-\frac{c_2-1}{2}-1}\\&\times\lim_{t\to T_1^*(u_0,u_1,c_2)^-}(u(t)u'(t))((T_1^*(u_0,u_1,c_2)-t)^{\frac{c_2+1}{c_2-1}})\\&=-\frac{c_2-1}{\sqrt{c_2}}\sqrt{c_1}\sqrt{(u_0^2)^{-c_2}(1+\frac{c_2}{c_1}u_1^2)}=-\frac{c_2-1}{\sqrt{c_2}}\,|u_0|^{-c_2}\ \sqrt{c_1+c_2u_1^2},\\&\lim_{t\to T_1^*(u_0,u_1,c_2)^-}(u(t)u'(t))((T_1^*(u_0,u_1,c_2)-t)^{\frac{c_2+1}{c_2-1}})\\&=\frac{1}{\sqrt{c_2}}\,|u_0|^{-c_2}\ \sqrt{c_1+c_2u_1^2}\Big(\frac{1}{c_2-1}\sqrt{\frac{c_2}{c_1+c_2u_1^2}}\,|u_0|^{c_2}\Big)^{\frac{c_2+1}{c_2-1}}.\end{aligned}$$

Thus, (2.2) is obtained for $u_0u_1\geq 0$. By (1.5), (2.1), and (2.2), we conclude that

$$\begin{aligned}&\lim_{t\to T_1^*(u_0,u_1,c_2)^-}(u^2(t))''(T_1^*(u_0,u_1,c_2)-t)^{\frac{2c_2}{c_2-1}}\\&=2(c_2+1)\lim_{t\to T_1^*(u_0,u_1,c_2)^-}(u(t)u'(t)(T_1^*(u_0,u_1,c_2)-t)^{\frac{c_2+1}{c_2-1}})^2\\&\quad\times\lim_{t\to T_1^*(u_0,u_1,c_2)^-}(u(t)^2(T_1^*(u_0,u_1,c_2)-t)^{\frac{2}{c_2-1}})^{-1}\\&\quad+2c_1\lim_{t\to T_1^*(u_0,u_1,c_2)^-}(T_1^*(u_0,u_1,c_2)-t)^{\frac{2c_2}{c_2-1}}\\&=2(c_2+1)\frac{1}{c_2}\,|u_0|^{-2c_2}\ (c_1+c_2u_1^2)\Big(\frac{1}{c_2-1}\sqrt{\frac{c_2}{c_1+c_2u_1^2}}\,|u_0|^{c_2}\Big)^{\frac{2c_2}{c_2-1}}.\end{aligned}$$

Therefore, (2.3) is obtained for $u_0u_1 \geq 0$.

For (ii) $u_0u_1 < 0$, one can get the conclusions through the same argument as above using (1.9). We do not repeat the steps.

3 Solution Property for $c_2 \in (0.5, 1)$

For $c_2 = 1/2$, then $2u''u(t) = 2c_1 + u'(t)^2$. After some computation, we find that $u(t) = u_0 + u_1t + \frac{1}{2}(c_1 + \frac{u_1^2}{2})u_0^{-1}t^2$ is the solution of equation (0.2). Thus, we have the fallowing result.

Suppose that u is a solution of equation (0.2) for $c_2 = 1/2$. Then,

$$\lim_{t\to\infty} u(t)t^{-\frac{1}{c_2}} = \frac{1}{2}u_0^{-2c_2}(c_1 + c_2u_1^2), \quad \lim_{t\to\infty} u'(t)t^{-\frac{1+c_2}{c_2}} = u_0^{-2c_2}(c_1 + c_2u_1^2).$$

Here, we discuss the case $u_0 \neq 0$. We have the following result on critical point and asymptotic behavior at infinity of the solutions for equation (0.2):

Theorem 3 Suppose that u is a solution of problem (0.2) with $u_0 \neq 0$. Then, for

(i) $u_0 > 0$, $u_1 > 0$,

$$\lim_{t\to\infty} u(t)t^{-\frac{1}{1-c_2}} = \Big(\frac{1-c_2}{\sqrt{c_2}}\Big)^{\frac{1}{1-c_2}} |u_0|^{\frac{-c_2}{1-c_2}} (c_1 + c_2u_1^2)^{\frac{-1}{2c_2-2}};$$

(ii) $u_0 > 0,\ u_1 < 0$, there exists a constant $Z_2(u_0, u_1, c_1, c_2) := Z_2$, such that $\lim\limits_{t\to Z_2} u'(t) = 0$ and

$$Z_2 = \frac{1}{2}c_2^{\frac{-1}{2}}u_0(c_1 + c_2u_1^2)^{\frac{-1}{2c_2}} \int_{\frac{c_1}{c_1+c_2u_1^2}}^1 s^{\frac{-c_2-1}{2c_2}}(1-s)^{-\frac{1}{2}}\mathrm{d}s;$$

(iii) $u_0 < 0,\ u_1 < 0$,

$$\lim_{t\to\infty} u(t)t^{-\frac{1}{1-c_2}} = -\Big(\frac{1-c_2}{\sqrt{c_2}}\Big)^{\frac{1}{1-c_2}} |u_0|^{\frac{-c_2}{1-c_2}} (c_1 + c_2u_1^2)^{\frac{-1}{2c_2-2}};$$

(iv) $u_0 < 0,\ u_1 > 0$, there exists a constant $Z_3(u_0, u_1, c_1, c_2) := Z_3$, such that $\lim\limits_{t\to Z_3} u'(t) = 0$ and

$$Z_3 = \frac{1}{2}c_2^{\frac{-1}{2}}u_0(c_1 + c_2u_1^2)^{\frac{-1}{2c_2}} \int_{\frac{c_1}{c_1+c_2u_1^2}}^1 s^{\frac{-c_2-1}{2c_2}}(1-s)^{-\frac{1}{2}}\mathrm{d}s.$$

Remark After some verification, the following argumentations are also valid for the case of $c_2 \in (0, 0.5)$.

Proof (1) For $u_0 > 0$ and $u_1 > 0$, by (1.5), (1.6), we have

$$J_u'(t) = (1-c_2)(u(t)^2)^{\frac{1-c_2}{2}-1}(u(t)u'(t)),$$

$$J_u''(t) = c_1(1-c_2)J_u(t)^{-\frac{c_2+1}{1-c_2}} > 0,$$

$$J_u'(t) \geq (1-c_2)u_0^{-c_2}u_1 > 0,$$

$$J_u'(t)^2 \leq E_J(0) = c_2^{-1}(1-c_2)^2(u_0^2)^{-c_2}(c_1 + c_2u_1^2),$$

$$J_u(t) \leq J_u(0) + \sqrt{E_J(0)}t.$$

In contrast, we can see that

$$J_u'(t) \geq \sqrt{E_J(0)} - \sqrt{\frac{c_1}{c_2}}(1-c_2)J_u(t)^{\frac{-c_2}{1-c_2}} > 0,$$

$$J_u(t) \geq J_u(0) + \sqrt{E_J(0)}t - \sqrt{\frac{c_1}{c_2}}(1-c_2)\int_0^t J_u(r)^{\frac{-c_2}{1-c_2}}\mathrm{d}r$$

and

$$\begin{aligned}
\int_0^t J_u(r)^{\frac{-c_2}{1-c_2}}\mathrm{d}r &= \int_{J_u(t)^{\frac{-c_2}{1-c_2}}}^{J_u(0)^{\frac{-c_2}{1-c_2}}} \frac{1-c_2}{c_2}s^{1-\frac{1}{c_2}}\Big(E_J(0)-\frac{c_1}{c_2}(1-c_2)^2s^2\Big)^{-\frac{1}{2}}\mathrm{d}s \\
&= \frac{1}{2\sqrt{c_1c_2}}\Big(\sqrt{\frac{c_2E_J(0)}{c_1}}\frac{1}{1-c_2}\Big)^{1-\frac{1}{c_2}}\int_{\frac{c_1}{c_2}(1-c_2)^2E_J(0)^{-1}J_u(t)^{\frac{-2c_2}{1-c_2}}}^{\frac{c_1}{c_2}(1-c_2)^2E_J(0)^{-1}J_u(0)^{\frac{-2c_2}{1-c_2}}}(1-s)^{\frac{1}{2}-1}s^{\frac{2c_2-1}{2c_2}-1}\mathrm{d}s \\
&\leq \frac{1}{2\sqrt{c_1c_2}}\Big(\sqrt{\frac{c_2E_J(0)}{c_1}}\frac{1}{1-c_2}\Big)^{1-\frac{1}{c_2}}\beta\Big(\frac{1}{2},\frac{2c_2-1}{2c_2}\Big);
\end{aligned}$$

therefore,

$$J_u(t) \geq J_u(0) + \sqrt{E_J(0)}t - \frac{1-c_2}{2c_2}\Big(\sqrt{\frac{c_2E_J(0)}{c_1}}\frac{1}{1-c_2}\Big)^{1-\frac{1}{c_2}}\beta\Big(\frac{1}{2},\frac{2c_2-1}{2c_2}\Big)$$

and then, we conclude that $\lim\limits_{t\to\infty} J_u(t)t^{-1} = \sqrt{E_J(0)}$, and obtain the conclusion under (i).

(2) For $u_0 > 0,\ u_1 < 0$, using (1.5) and (1.6), we have

$$J_u'(t) = -\sqrt{E_J(0) - \frac{c_1}{c_2}(1-c_2)^2J_u(t)^{\frac{2c_2}{c_2-1}}}.$$

Suppose that $J_u'(t) < 0$ for all $t \geq 0$. Then,

$$J_u(t) \leq \Big(\frac{c_2}{c_1(1-c_2)^2}\Big)^{\frac{c_2-1}{2c_2}}E_J(0)^{\frac{c_2-1}{2c_2}},$$

$$\begin{aligned}
t &= \int_{J_u(t)}^{J_u(0)}\frac{\mathrm{d}r}{\sqrt{E_J(0)-\frac{c_1}{c_2}(1-c_2)^2r^{\frac{2c_2}{c_2-1}}}} \\
&= \frac{1}{2}(1-c_2)^{\frac{1}{c_2}}c_2^{\frac{-c_2-1}{2c_2}}E_J(0)^{\frac{-1}{2c_2}}\int_{\frac{c_1}{c_2E_J(0)}(1-c_2)^2J_u(0)^{\frac{-2c_2}{1-c_2}}}^{\frac{c_1}{c_2E_J(0)}(1-c_2)^2J_u(t)^{\frac{-2c_2}{1-c_2}}} s^{\frac{-c_2-1}{2c_2}}(1-s)^{\frac{1}{2}-1}\mathrm{d}s \\
&\leq \frac{1}{2(1-c_2)}E_J(0)^{\frac{1}{2}}c_1^{\frac{-c_2-1}{2c_2}}J_u(0)^{\frac{c_2+1}{1-c_2}}\int_{\frac{c_1}{c_2E_J(0)}(1-c_2)^2J_u(0)^{\frac{-2c_2}{1-c_2}}}^{1}(1-s)^{-\frac{1}{2}}\mathrm{d}s \\
&= \frac{1}{1-c_2}E_J(0)^{\frac{1}{2}}c_1^{\frac{-c_2-1}{2c_2}}J_u(0)^{\frac{c_2+1}{1-c_2}}\Big(1-\frac{c_1}{c_2E_J(0)}(1-c_2)^2J_u(0)^{\frac{-2c_2}{1-c_2}}\Big)^{\frac{1}{2}}.
\end{aligned}$$

It creates a contradiction; thus, there exists a constant $Z_2(u_0,u_1,c_1,c_2) := Z_2$, such that $\lim\limits_{t\to Z_2} u'(t) = 0 = \lim\limits_{t\to Z_2} J_u'(t)$ and $J_u(Z_2) = (\frac{c_2}{c_1(1-c_2)^2})^{\frac{c_2-1}{2c_2}}E_J(0)^{\frac{c_2-1}{2c_2}}$, also,

$$Z_2 = \int_{(\frac{c_2}{c_1(1-c_2)^2})^{\frac{c_2-1}{2c_2}}E_J(0)^{\frac{c_2-1}{2c_2}}}^{J_u(0)}\frac{\mathrm{d}r}{\sqrt{E_J(0)-\frac{c_1}{c_2}(1-c_2)^2r^{\frac{2c_2}{c_2-1}}}}$$

$$
\begin{aligned}
&= \frac{1}{2}(1-c_2)^{\frac{1}{c_2}} c_2^{\frac{-c_2-1}{2c_2}} E_J(0)^{\frac{-1}{2c_2}} \int_{\frac{c_1}{c_2 E_J(0)}(1-c_2)^2 J_u(0)^{\frac{-2c_2}{1-c_2}}}^{1} s^{\frac{-c_2-1}{2c_2}} (1-s)^{\frac{-1}{2}} \mathrm{d}s \\
&= \frac{1}{2} c_2^{\frac{-1}{2}} u_0 (c_1 + c_2 u_1^2)^{\frac{-1}{2c_2}} \int_{\frac{c_1}{c_1+c_2 u_1^2}}^{1} s^{\frac{-c_2-1}{2c_2}} (1-s)^{\frac{-1}{2}} \mathrm{d}s.
\end{aligned}
$$

The estimates under (ii) are completely proved.

(3) Similar to the above arguments, it results in the estimates under (iii) and (iv).

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