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Almost marginal conditional stochastic dominance

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ABSTRACT

Marginal Conditional Stochastic Dominance (MCSD) developed by Shalit and Yitzhaki (1994) gives the conditions under which all risk-averse individuals prefer to increase the share of one risky asset over another in a given portfolio. In this paper, we extend this concept to provide conditions under which most (and not all) risk-averse investors behave in this way. Instead of stochastic dominance rules, almost stochastic dominance is used to assess the superiority of one asset over another in a given portfolio. Switching from MCSD to Almost MCSD (AMCSD) helps to reconcile common practices in asset allocation and the decision rules supporting stochastic dominance relations. A financial application is further provided to demonstrate that using AMCSD can indeed improve investment efficiency.

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1. Introduction

The most common investment rule is certainly the mean–variance (MV) rule. It is easy to compute, and in some cases even to express analytically, which explains why the MV rule has become most widely accepted throughout the financial profession (see Lizyayev and Ruszczyński, 2012). On the other hand, Expected utility (EU) maximization lies at the heart of modern investment theory and practice. To be analytically consistent with EU maximization, the MV rule requires strong assumptions (such as quadratic utility functions or normally distributed returns), which seldom hold in practice. However, EU requires the specification of the investor's utility function which appears extremely difficult.

Stochastic dominance (SD) is an alternative approach which avoids all these shortcomings by considering the preferences shared by all the rational decision-makers. Therefore, it does not require a specific utility function nor a specific return distribution. Furthermore, it uses the whole probability distribution rather than the usual MV parameters of standard deviation and mean return. The second-degree stochastic dominance (SSD) rule is appropriate for the class of all risk-averse EU maximizers. It has the advantage that it requires no restrictions on probability distributions nor on investors' utility functions outside of the requirement that investors be risk-averse, EU maximizers.

Given a portfolio of assets, marginal conditional stochastic dominance (MCSD) has been introduced by Yitzhaki and Olkin (1991) and Shalit and Yitzhaki (1994) as a condition under which all risk-averse EU maximizer individuals prefer to increase the share of one risky asset over that of another. Specifically, these authors consider risk-averse investors holding a given portfolio of risky assets and derive criteria expressed in terms of the joint probability distribution of the assets and of the underlying portfolio to ensure that the share of an asset is increased at the expense of another in the portfolio. This helps to detect inefficiency and to improve inefficient portfolios. MCSD has been successfully applied to solve asset allocation problems by several authors, including Clark et al. (2011), Clark and Kassimatis (2012, 2013), Shalit and Yitzhaki (2010). MCSD expresses the conditions under which all risk-averse investors holding a specific portfolio prefer one asset to another. Furthermore, MCSD has been shown to involve more than pairwise comparisons as developed by Shalit and Yitzhaki (2003). It is a less demanding concept and more adapted to empirical analysis than SSD because it considers only marginal changes of holding risky assets in a given portfolio.

Despite their theoretical attractiveness, MV and SSD rules may create paradoxes in the sense that they fail to distinguish between some risky prospects, whereas it is obvious that the vast majority of investors would prefer one over the other. This is why Bali et al. (2009) considered almost stochastic dominance (ASD) as a viable

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alternative. ASD corresponds to all utility functions after eliminating pathological preferences, keeping only the economically relevant utility functions. Bali et al. (2009) demonstrated that the ASD rule unambiguously supports some common practice, like advising a higher stock to bond ratio for long investment horizons. Switching from SD to ASD thus allows for the provision of a theoretical support for the practitioners' view within the EU paradigm. The study conducted by these authors suggests that modifying MCSD into almost MCSD (AMCSD) may also help in the analysis of economic behavior under risk. This is the subject of the present work.

In this paper, MCSD is weakened to ensure that most (but not all) risk-averse decision-makers increase the share of one risky asset over another. This extension of MCSD to AMCSD is inspired by almost stochastic dominance rules introduced by Leshno and Levy (2002), and suitably corrected by Tzeng et al. (2013).[2] Specifically, restrictions are imposed on the marginal utility function and on its derivative to exclude extreme forms of preferences that are not shared by real-world investors. Then, the condition leading to MCSD is adapted to correspond to utilities defining almost second-degree stochastic dominance. As pointed out by Levy et al. (2010), investment rules based on stochastic dominance may cover "theoretical preferences that are not encountered in practice": there are situations where stochastic dominance is unable to rank two portfolios, whereas experimentally 100% of the subjects reveal a clear-cut ranking. The switch from MCSD to AMCSD can be expected to avoid such paradoxical results.

The remainder of this paper is organized as follows. The next section extends MCSD to AMCSD. Section 3 discusses a numerical example comparing the two concepts. Section 4 allows for changes in multiple assets in MCSD rules. Section 5 provides empirical illustrations. Section 6 briefly concludes the paper and discusses how to extend AMCSD rules to higher orders.

2. Almost marginal conditional stochastic dominance

2.1. Marginal conditional stochastic dominance

Assume that a risk-averse investor with a utility function u holds a portfolio with n risky assets. Let w_0 be the initial wealth, X_i denote the rate of return on risky asset i and α_i be the investment proportion on asset $i, i = 1, 2, \ldots, n$. A portfolio $\boldsymbol{\alpha}$ is defined by the shares α_i such that $\sum_{i=1}^n \alpha_i = 1$. The final wealth of the investor is given by $W = w_0(1 + \sum_{i=1}^n \alpha_i X_i)$. Henceforth, we normalize the initial wealth w_0 to unity so that $W = 1 + \sum_{i=1}^n \alpha_i X_i$.

The goal of the investor is to select the weights to maximize $E[u(W)]$. Given a portfolio $\boldsymbol{\alpha}$, Shalit and Yitzhaki (1994) have established that it is optimal to increase the weight α_k of asset k at the expense of asset j if, and only if,

$$E\big[u'(W)(X_k - X_j)\big] \geqslant 0. \tag{1}$$

Asset k dominates asset j according to MCSD if condition (1) is fulfilled for all risk-averse investors, that is, for all concave utility u.

Let R denote the portfolio return, i.e.,

$$R = \sum_{i=1}^n \alpha_i X_i.$$

Shalit and Yitzhaki (1994) proved that for a given portfolio $\boldsymbol{\alpha}$, asset k dominates asset j according to MCSD if, and only if, the inequality

$$E[X_k | R \leqslant r] \geqslant E[X_j | R \leqslant r]$$

holds for all the return levels r. This is easily deduced from (1) by taking the kinked utilities $u(x) = \min\{x, r\}$. In words, MCSD favors assets performing better in adverse situations (i.e., when the portfolio underperforms $\iff R \leqslant r$).

The next section shows how to define AMCSD as distinct from MCSD, avoiding extreme forms of preferences.

2.2. From MCSD to AMCSD

MCSD is based on all the non-decreasing and concave utility functions, that is, on the utility functions in

$$U_2 = \{\text{utility functions } u | u' \geqslant 0 \text{ and } u'' \leqslant 0\}.$$

As explained in Leshno and Levy (2002), U_2 contains some extreme utility functions which presumably rarely represent real-world investors' preferences. The prototype is $u(x) = \min\{x, r\}$ for some constant r. Note that such utilities form the representative set of non-decreasing and concave utility functions used by Hadar and Seo (1988).

To reveal a preference for most investors, but not for all of them, we restrict U_2 to a subset of it. Specifically, following Leshno and Levy (2002), let us further impose restrictions on the utility function and define

$$U_2^*(\varepsilon) = \left\{ u \in U_2 | -u''(x) \leqslant \inf\{-u''(x)\}\left(\frac{1}{\varepsilon} - 1\right) \text{ for all } x \right\}, \tag{2}$$

where $\varepsilon \in \left(0, \frac{1}{2}\right)$. The range of the parameter ε which controls the area of violation has been discussed empirically by Levy et al. (2010).

The following result characterizes the situations where asset j is dominated by asset k for all investors with $u \in U_2^*(\varepsilon)$. Before stating it formally, we need to introduce some additional notation. Let $\mu_i(r)$ denote the conditional expected return of asset i when the portfolio return is r, i.e.,

$$\mu_i(r) = E[X_i | R = r].$$

Henceforth, we assume without real loss of generality that the return is bounded and valued over some interval $[a, b]$ of the real line. Furthermore, define

$$\begin{aligned} B(t) = & \int_a^t \Big(\mu_k(r) - \mu_j(r)\Big) dF_R(r) \\ = & \Big(E[X_k | R \leqslant t] - E[X_j | R \leqslant t]\Big) F_R(t) \\ \Omega = & \{t \in [a, b] | B(t) < 0\} \end{aligned}$$

and let Ω^c denote the complement of Ω in $[a, b]$. MCSD requires $B(t) \geqslant 0$ for all t, that is, $\Omega = \emptyset$. If this is not the case, Ω represents the set of violations for MCSD.

Proposition 1. *Given portfolio $\boldsymbol{\alpha}$, asset k dominates asset j for all individuals with preferences represented by the utility function $u \in U_2^*(\varepsilon)$ if, and only if,*

$$\int_\Omega (-B(t) dt) \leqslant \varepsilon \int_a^b | B(t) | \, dt \tag{3}$$

and $E[X_k] \geqslant E[X_j]$.

The proof of this result can be found in the appendix. Together with the comparison of expected returns, condition (3) provides the operational way to check for AMCSD in a given portfolio.

Tzeng et al. (2013) have shown that a distribution is preferred to another one by all decision makers with utility function $u \in U_2^*(\varepsilon)$ if and only if the distribution dominates the other one in terms of almost second-degree stochastic dominance, which contains two conditions. The first one is that the mean of the distribution is greater than that of the other one, which corresponds

[2] Lizyayev and Ruszczyński (2012) provided an alternative definition of almost stochastic dominance called tractable almost stochastic dominance due to its benefits in regard to tractability in computation.

to $E[X_k] \geqslant E[X_j]$ in Proposition 1. The second condition is that the area which violates second-degree stochastic dominance cannot be too large, which indeed is expressed by the inequality $\int_\Omega (-B(t)dt) \leqslant \varepsilon \int_a^b |B(t)|dt$ in Proposition 1. In other words, AMCSD is an extension of ASSD.

Note that Shalit and Yitzhaki (1994, Theorem 3) further provided necessary conditions for MCSD. Specifically, if asset k dominates asset j according to MCSD then

$$E[X_k] \geqslant E[X_j], \tag{4}$$

and

$$E[X_k] - 2Cov[X_k, F_R(R)] \geqslant E[X_j] - 2Cov[X_j, F_R(R)], \tag{5}$$

These inequalities can be further expressed by means of the Gini index of the portfolio. It turns out that these conditions are also necessary for AMCSD. The first necessary condition is obviously true for AMCSD as it explicitly appears in Proposition 1. Shalit and Yitzhaki (1994) indicated that the second necessary condition can be rewritten as

$$\int_a^b (B(t))dt \geqslant 0.$$

For the AMCSD rule, if condition (3) holds, then

$$\int_\Omega (-B(t)dt) \leqslant \frac{1}{2} \int_a^b \mid B(t) \mid dt \tag{6}$$

so that condition (5) is also true.

3. Numerical illustrations

An example is introduced to demonstrate the application of Proposition 1. Assume that there exist three independently distributed risky assets. The distributions of the rates of return for these three assets are, respectively,

$$X_1 = \begin{cases} -10\% \text{ with probability } \frac{1}{2} \\ +15\% \text{ with probability } \frac{1}{2} \end{cases}$$

$$X_2 = \begin{cases} -11\% \text{ with probability } \frac{1}{2} \\ +50\% \text{ with probability } \frac{1}{2} \end{cases}$$

$$X_3 = \begin{cases} -15\% \text{ with probability } \frac{1}{2} \\ +25\% \text{ with probability } \frac{1}{2} \end{cases}$$

We further assume that the weights in the current portfolio are $\alpha_1 = 25\%, \alpha_2 = 50\%$ and $\alpha_3 = 25\%$. Table 1 shows the distribution of the portfolio returns and the assets' conditional expected returns. Column 1 in Table 1 denotes the portfolio returns ranked from the lowest to the highest. Column 2 represents the probabilities for the corresponding portfolio returns in Column 1 and Column 3 lists the cumulative distribution of the portfolio. Columns 4–6 provide the expected returns conditional on the portfolio return given by Column 1. For example, the portfolio return -1.75% $(= 25\% \times (-10\%) + 50\% \times (-11\%) + 25\% \times 25\%)$ has a 12.5% probability of occurring, the cumulative probability is 37.5% and the expected returns on assets 1, 2 and 3 are -10%, -11% and 25%, respectively.

Shalit and Yitzhaki (1994) related MCSD to Absolute Concentration Curves (ACCs) defined as follows. The ACC for asset i with respect to the portfolio $\boldsymbol{\alpha}$ is

$$ACC_i(p) = \int_{-\infty}^{F_R^{-1}(p)} \mu_i(r) dF_R(r) = E[X_i | R \leqslant F_R^{-1}(p)] \times p \tag{7}$$

where $F_R^{-1}(p) = \inf\{r \in \mathbb{R} | F_R(r) \geqslant p\}$. Table 1 further shows the ACCs. When the cumulative probability is 37.5%,

$$ACC_1(0.375) = -10\% \times 12.5\% + 15\% \times 12.5\% + (-10\%) \times 12.5\% \\ = -0.625\%,$$

$$ACC_2(0.375) = -11\% \times 12.5\% + (-11\%) \times 12.5\% + (-11\%) \\ \times 12.5\% = -4.125\%,$$

$$ACC_3(0.375) = -15\% \times 12.5\% + (-15\%) \times 12.5\% + 25\% \times 12.5\% \\ = -0.625\%.$$

From Table 1, the ACCs for the three assets reveal that they do not MCSD dominate each other since the violation set $\Omega \neq \emptyset$. On basis of the criteria of MCSD, no suggestion can be made to improve the efficiency of the portfolio, but AMCSD can further suggest how to improve the investment.

Among the two criteria of AMCSD in Proposition 1, the criterion that $E[X_k] \geqslant E[X_j]$ can help us to reduce the comparison among assets. Note that $E[X_k] = ACC_k(1)$. Table 1 indicates that $E[X_2] = 19.5 > E[X_3] = 5 > E[X_1] = 2.5$. Thus, we only need the results for $B(\cdot)$ in Proposition 1 of 2 vs. 1, 2 vs. 3 and 3 vs. 1 to make further suggestions. Table 2 displays the corresponding $B(\cdot)$.

In the case of 2 vs. 1, Table 2 shows that $B(t) = ACC_2(0.375) - ACC_1(0.375) = -4.125\% - (-0.625\%) = -3.5\%$ when the cumulative probability is 37.5%. Since Ω denotes the set that $B(t) < 0$, in this case, Ω includes the portfolio returns which are less than $F_R^{-1}(50\%) = 4.5\%$. Thus, $\int_\Omega(-B(t))dt = 13.75\%$ and $\int_a^b \mid B(t) \mid dt = 49.25\%$. Similarly, Table 2 further shows that

$$2 \text{ vs. } 3: \int_\Omega (-B(t))dt = 11.5\%, \int_a^b |B(t)|dt = 47.25\%;$$

$$3 \text{ vs. } 1: \int_\Omega (-B(t))dt = 8.125\%, \int_a^b |B(t)|dt = 13.75\%.$$

When $\int_a^b \mid B(t) \mid dt \neq 0$, condition (3) in Proposition 1 can be rewritten as

$$\frac{\int_\Omega(-B(t)dt)}{\int_a^b \mid B(t) \mid dt} \leqslant \varepsilon. \tag{8}$$

Table 1
Portfolio returns, asset conditional expected returns and ACCs (in %).

r	$\Pr[R = r]$	$F_R(r)$	$\mu_1(r)$	$\mu_2(r)$	$\mu_3(r)$	ACC_1	ACC_2	ACC_3
−11.750	12.500	12.500	−10.000	−11.000	−15.000	−1.250	−1.375	−1.875
−5.500	12.500	25.000	15.000	−11.000	−15.000	0.625	−2.750	−3.750
−1.750	12.500	37.500	−10.000	−11.000	25.000	−0.625	−4.125	−0.625
4.500	12.500	50.000	15.000	−11.000	25.000	1.250	−5.500	2.500
18.750	12.500	62.500	−10.000	50.000	−15.000	0.000	0.750	0.625
25.000	12.500	75.000	15.000	50.000	−15.000	1.875	7.000	−1.250
28.750	12.500	87.500	−10.000	50.000	25.000	0.625	13.250	1.875
35.000	12.500	100.000	15.000	50.000	25.000	2.500	19.500	5.000

Table 2
$B(\cdot)$ and he criteria of AMCSD (in %).

F_R	2 vs. 1			2 vs. 3			3 vs. 1		
	$B(t)$	$\lvert B(t)\rvert$ in Ω	$\lvert B(t)\rvert$	$B(t)$	$\lvert B(t)\rvert$ in Ω	$\lvert B(t)\rvert$	$B(t)$	$\lvert B(t)\rvert$ in Ω	$\lvert B(t)\rvert$
12.500	−0.125	0.125	0.125	0.500		0.500	−0.625	0.625	0.625
25.000	−3.375	3.375	3.375	1.000		1.000	−4.375	4.375	4.375
37.500	−3.500	3.500	3.500	−3.500	3.500	3.500	0.000		0.000
50.000	−6.750	6.750	6.750	−8.000	8.000	8.000	1.250		1.250
62.500	0.750		0.750	0.125		0.125	0.625		0.625
75.000	5.125		5.125	8.250		8.250	−3.125	3.125	3.125
87.500	12.625		12.625	11.375		11.375	1.250		1.250
100.000	17.000		17.000	14.500		14.500	2.500		2.500
Summation		13.750	49.250		11.500	47.250		8.125	13.750

Table 3
The criteria of AMCSD (in %).

	2 vs. 1	2 vs. 3	3 vs. 1
Differences in expectations	17.000	14.500	2.500
$\int_\Omega [-B(t)]dt / \int_a^b \lvert B(t)\rvert dt$	0.279	0.243	0.591

Table 3 shows the two criteria of AMCSD suggested: the differences in expectations and $\int_\Omega(-B(t)dt)/\int_a^b \mid B(t)\mid dt$. In the case of 2 vs. 1, $E[X_2]-E[X_1]=19.5\%-2.5\%=17\%$, and $\int_\Omega(-B(t)dt)/\int_a^b \mid B(t)\mid dt = 13.75\%/49.25\% = 0.2792$.

Now, let us assume that $\varepsilon = 0.3$. From Table 3, asset 2 AMCSD dominates both assets 1 and 3 for all investors in $U_2^*(\varepsilon=0.3)$. Thus, the expected utility of all investors in $U_2^*(\varepsilon=0.3)$ can be further improved by increasing the weight on asset 2. On the basis of AMCSD, the current portfolio is not efficient. Asset 2 does not MCSD dominate either assets 1 or 3 because MCSD seeks the condition for all risk-averse investors. In this example, it is obvious that asset 2 could be an attractive alternative for most investors. Therefore, with respect to the current portfolio, most investors, e.g., those with utility function $u\in U_2^*(\varepsilon=0.3)$, are inclined to invest more in asset 2.

4. AMCSD allowing changes in multiple assets

Using AMCSD rules, a dominating asset can be found by comparing two assets in a given portfolio. However, building a portfolio based on pairwise comparisons may be not effective in practical use. In this section, we extend the AMCSD analysis to allow for simultaneous changes in several assets of a portfolio and using the concept of Lorenz dominance, which is suggested by Shalit and Yitzhaki (2003).

As in Shalit and Yitzhaki (2003), define the Lorenz curve for portfolio $\boldsymbol{\alpha}$ as

$$L\left[F_R^{-1}(p)\right] = \sum_{i=1}^n \alpha_i ACC_i(p).$$

Suppose that there exists an alternative portfolio $\boldsymbol{\alpha}+\sum_{i=1}^n d\alpha_i$ where $\sum_{i=1}^n d\alpha_i = 0$. Therefore,

$$dL\left[F_R^{-1}(p)\right] = \sum_{i=1}^n ACC_i(p)d\alpha_i$$

The following result provides AMCSD rules allowing for changes in multiple assets.

Proposition 2. *Given a portfolio $\boldsymbol{\alpha}$. An alternative portfolio, $\boldsymbol{\alpha}+\sum_{i=1}^n d\alpha_i$, is preferred by all individuals with preferences represented by the utility function $u\in U_2^*(\varepsilon)$ if, and only if,*

$$-\int_\Omega \sum_{i=1}^n ACC_i(p)d\alpha_i dp \leqslant \varepsilon \int_a^b \left|\sum_{i=1}^n ACC_i(p)d\alpha_i\right| dp \tag{9}$$

and $\sum_{i=1}^n E[X_i]d\alpha_i \geqslant 0$, where $\Omega = \{p\in[0,1] | \sum_{i=1}^n ACC_i(p)d\alpha_i < 0\}$.

The proof of Proposition 2 is similar to that of Proposition 1 and is omitted. By Proposition 2, we can test whether a portfolio is efficient by checking all possible variations in the assets' weights conditional on $\sum_{i=1}^n d\alpha_i = 0$ and $\sum_{i=1}^n E[X_i]d\alpha_i \geqslant 0$. Given ε, if there are no $d\alpha_i$ such that Eq. (9) is satisfied, the initial portfolio $\boldsymbol{\alpha}$ is efficient.

We further demonstrate how to implement Proposition 2 to check whether a portfolio is efficient. On the basis of Proposition 2, the following programming can help us to identify efficient portfolios:

$$\min_{d\alpha_i} -\int_\Omega \sum_{i=1}^n ACC_i(p)d\alpha_i dp - \varepsilon \int_a^b \left|\sum_{i=1}^n ACC_i(p)d\alpha_i\right| dp \tag{10}$$

$$s.t. \sum_{i=1}^n E[X_i]d\alpha_i \geqslant \quad 0 \tag{11}$$

$$\sum_{i=1}^n d\alpha_i = \quad 0 \tag{12}$$

$$-\alpha_i \leqslant \quad d\alpha_i \leqslant 1-\alpha_i, \quad i=1,\ldots,n \tag{13}$$

where Ω is as defined in Proposition 2. The objective function and the first constraint in the above problem directly come from Proposition 2. The second constraint makes sure that the sum of the portfolio weights is equal to one. The third constraint is added to avoid short selling. Thus, given a portfolio $\boldsymbol{\alpha}$, if the optimal value of the objective function is negative, then the portfolio $\boldsymbol{\alpha}$ is inefficient in terms of AMCSD.

In practice, the empirical distribution of the portfolio return R is discrete rather than continuous. Let $r_1 \leqslant r_2 \leqslant \ldots \leqslant r_T$ represent all the observations of R ranked in ascending order. Therefore, the objective function (10) can be rewritten as

$$\min_{d\alpha_i} -\sum_{j\in\Omega}\sum_{i=1}^n ACC_i\left(\frac{j}{T}\right)d\alpha_i - \varepsilon\sum_{j=1}^T\left|\sum_{i=1}^n ACC_i\left(\frac{j}{T}\right)d\alpha_i\right| \tag{14}$$

To establish an algorithm which can be solved by linear programming, some notations are defined as follows. Let

$$Z_j^- = \max\left\{-\sum_{i=1}^n ACC_i\left(\frac{j}{T}\right)d\alpha_i, 0\right\}$$

and

$$Z_j^+ = \max\left\{\sum_{i=1}^n ACC_i\left(\frac{j}{T}\right)d\alpha_i, 0\right\}.$$

Thus, the programming model can be rewritten as

$$\min_{Z_j^-, Z_j^+, d\alpha_i} (1-\varepsilon)\sum_{j=1}^{T} Z_j^- - \varepsilon \sum_{j=1}^{T} Z_j^+ \tag{15}$$

$$\text{s.t.} \quad Z_j^+ - Z_j^- = \sum_{i=1}^{n} ACC_i\left(\frac{j}{T}\right) d\alpha_i$$
$$\sum_{i=1}^{n} E[X_i] d\alpha_i \geqslant 0$$
$$\sum_{i=1}^{n} d\alpha_i = 0$$
$$-\alpha_i \leqslant d\alpha_i \leqslant 1-\alpha_i, \quad i = 1, \ldots, n.$$

Note that the objective function and all constraints are linear functions of Z_j^-, Z_j^+ and $d\alpha_i$. Thus, the program can be easily implemented by standard linear programming.

5. Empirical works

This section provides a financial application to demonstrate that using AMCSD can indeed improve investment efficiency. In a recent work, Bali et al. (2013) apply the almost stochastic dominance (ASD) approach to test whether hedge funds outperform stocks and bonds. They use historical and simulated return distributions and find that the US equity market is dominated in terms of ASD by the Long/Short Equity Hedge and Emerging Markets hedge fund strategies, and that the US Treasury market is dominated by the Long/Short Equity Hedge, Multi-strategy, Managed Futures, and Global Macro hedge fund strategies. Since both ASD and AMCSD correspond to all utility functions after eliminating pathological preferences, we can rely on their findings and further ask how to improve investment efficiency by the AMCSD approach.

Furthermore, although some hedge fund strategies dominate the US equity market and/or the US Treasury market, it does not mean that a 100% holding in the hedge funds is the only efficient portfolio for investors. Due to the correlations among the return distributions of hedge funds, stocks and bonds, we might find that some portfolios including positive weights on these assets are efficient. The purpose of this section is to show that using AMCSD can improve the efficiency of existing portfolios by constructing portfolios including hedge funds, stocks and bonds.

5.1. Data

The hedge fund data are obtained from the Hedge Fund Research database. Our data period is from January 1994 to December 2011, which is the same as in Bali et al. (2013). During that period, the database contains 11,867 defunct funds and 6853 live funds. As in Bali et al. (2013), the size of a fund is measured by the average monthly assets under management over the life of the fund. Our data show that the mean and the median hedge fund size are \$149.5 million and \$28.2 million, respectively.

We further follow the steps in Bali et al. (2013) to screen our data. We include both live (6853 funds) and dead funds (11,867) to avoid survivorship bias. The first 12-month return histories of all individual hedge funds in our sample are deleted to eliminate back-fill bias. All hedge funds in our final sample need to have at least 24 months of return history to mitigate the impact of multi-period sampling bias and to obtain sensible measures of risk for funds. After the above screening processes, we have 12,816 hedge funds in our sample including 7443 dead funds and 5373 live funds.

In the Hedge Fund Research database, there are 7 investment styles reported: Emerging Markets, Equity Market Neutral, Event Driven, Fund of Funds, Macro, Relative Value and Equity Hedge. Following Bali et al. (2013), we calculate the equally-weighted average returns of funds for each of the 7 investment strategies. Furthermore, the S&P500 index returns and the 1-year Treasury Bond returns are used to proxy the performance of the US equity market and the performance of the short-term US Treasury securities.

Table 4 shows the basic statistics for the performance of the above 7 hedge fund strategies, the S&P500 index, and the 1-year Treasury Bond. The performance in our study is similar to that in Bali et al. (2013). The monthly average return of the S&P 500 is 0.56% and the standard deviation is 4.53% in our sample. The average return of Emerging Market is 1.08%, almost twice that of the S&P 500, and the standard deviation is 4.82%. Only the return distributions of Macro and the 1-year T-Bond are skewed to the right. The other 6 hedge fund strategies and the S&P 500 are characterized by negative skewness. According to the mean–variance criterion, 5 hedge fund strategies dominate the US equity market, i.e., Macro, Equity Hedge, Event Driven, Relative Value and Equity Market Neutral. However, the mean–variance criterion may not be appropriate for hedge funds since the Jarque–Bera (JB) statistics reject the hypothesis that the return distribution of the hedge funds follows a normal distribution, unless the utility function is quadratic.

5.2. Applying AMCSD

5.2.1. Pairwise comparison

To demonstrate how to apply the AMCSD criterion to improve investment efficiency, we assume that the current investment weights on hedge funds, the S&P 500 and T-Bonds are 5%, 85% and 10%, respectively. We examine whether investors should increase the proportion on one hedge fund strategy by decreasing that on the S&P 500.

Table 4
Descriptive statistics.

Asset	Mean (%)	Median (%)	Std. Dev. (%)	Skewness	Kurtosis	Min (%)	Max (%)	JB	p-Value
Emerging Market	1.08	1.74	4.82	−0.9066	4.6121	−25.28	17.79	209.68	<0.001
Macro	0.84	0.74	2.07	0.3655	0.0977	−4.06	7.24	4.78	0.0718
Equity Hedge	0.89	1.12	2.76	−0.4416	2.3616	−10.89	11.06	53.69	<0.001
Event Driven	0.83	1.10	1.86	−1.6426	5.9061	−8.70	4.64	392.59	<0.001
Relative Value	0.69	0.84	1.31	−2.9249	18.0144	−9.06	4.01	3082.83	<0.001
Fund of Fund	0.50	0.62	1.66	−0.6570	3.2184	−6.43	5.95	102.75	<0.001
Equity Neutral	0.59	0.58	0.87	−0.1481	2.8234	−3.45	3.68	67.90	<0.001
S&P500	0.56	1.12	4.53	−0.6416	0.9372	−16.94	10.77	21.71	0.0019
1-year T-Bond	0.32	0.32	0.29	0.4620	0.3400	−0.33	1.31	8.41	0.0218

This table presents the descriptive statistics of the monthly returns on the hedge fund portfolios, S&P500 index, and 1-year Treasury Bond for the sample period from January 1994 to December 2011. We compute the equal-weighted average monthly returns of funds for each of the 7 investment styles. The Jarque–Bera statistic, $JB = n[S^2/6 + (K-3)^2/24]$, is a formal statistic for testing whether the returns are normally distributed, where n denotes the number of observations, S skewness and K kurtosis. JB follows a Chi-square distribution with two degrees of freedom. The last column reports the corresponding p-value.

Similar to the process in the numerical example, the criteria $E[X_k] \geqslant E[X_j]$ of AMCSD in Proposition 1 can help us to reduce the comparisons among assets. Table 4 shows that the expected return of all hedge funds is greater than the return of the S&P 500 except for the Fund of Fund strategy. Thus, we only provide the comparisons between the other 6 hedge fund strategies and the S&P 500.

The empirical ε is calculated as follows:

$$\varepsilon = \frac{\int_{\Omega_{hs}} (-B_{hs}(t)dt)}{\int_a^b |B_{hs}(t)|dt}, \tag{16}$$

where

$$B_{hs}(t) = \left(E[X_{hedge}|R \leqslant t] - E[X_{stock}|R \leqslant t]\right)F_R(t),$$
$$\Omega_{hs} = \{t \in [a,b]|B_{hs}(t) < 0\},$$

X_{hedge} is the random return of hedge funds and X_{stock} is that of the S&P 500.

The same ε can be obtained by using ACCs introduced in Section 3. Let us use Fig. 1 to show how to obtain ε. The x-axis in Fig. 1 is the cumulative probability of the given portfolio and the y-axis is the value of ACC as in Eq. (7). As shown in Fig. 1, ACC_{stock} may cross ACC_{hedge}. The area Q denotes the "violation area" which violates the rule of MCSD and the areas P and T are the areas consistent with the rule of MCSD. Thus, the empirical ε can be obtained by

$$\varepsilon = \frac{Q}{P+Q+T}. \tag{17}$$

Fig. 2 shows the ACCs for the portfolios with a weight of 5% for the Emerging Market hedge fund, 85% for the S&P 500 and 10% for the T-Bond. The dashed, solid, and dotted lines represent the ACCs for the Emerging Market hedge fund, S&P 500 and T-Bond, respectively. Fig. 2 indicates that ACC_{hedge} exceeds ACC_{stock} for most level of the cumulative probability except when the portfolio return is smaller than the 0.46% quantile. Thus, according to MCSD, the investment cannot be improved. However, the violation area ε is only 0.0003 while comparing the Emerging Market hedge fund and S&P 500. According to the experiments proposed by Levy et al. (2010), the critical value of ε is equal to 3.2%. In other words, most investors will accept the cases with a violation area smaller than 3.2%. Thus, the current portfolio is inefficient for most investors since the empirical $\varepsilon = 0.0003 \leqslant 3.2\%$. For most investors, the current portfolio can be improved by increasing the weight on the Emerging Market hedge fund.

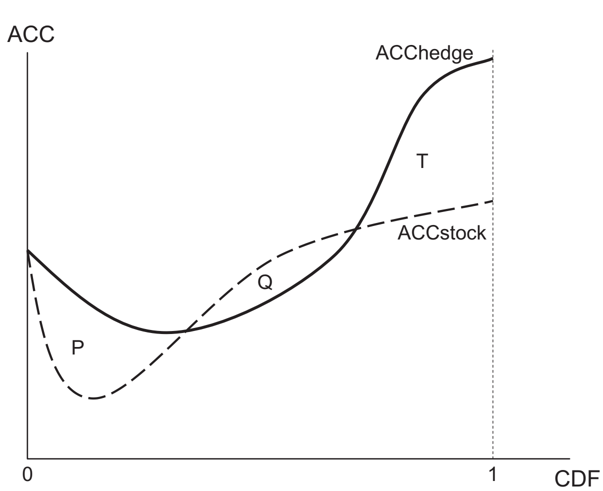

Fig. 1. The ACCs.

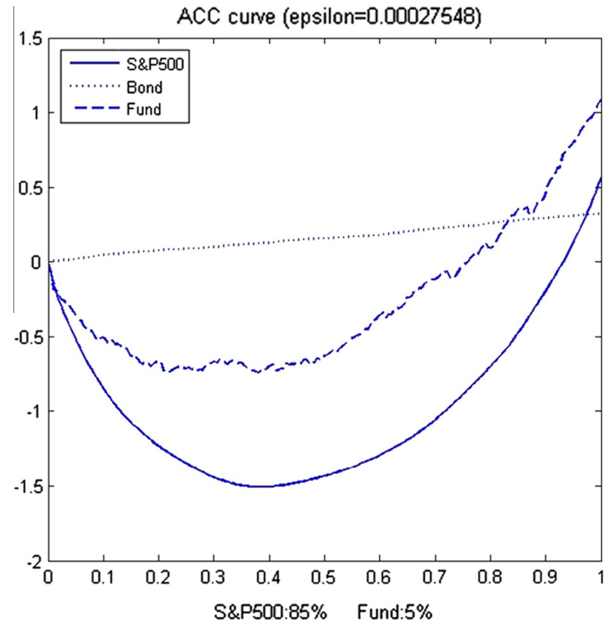

Fig. 2. The ACCs and the corresponding εs for the portfolios with weight 5% on the Emerging Market hedge fund, 85% on the S&P 500 and 10% on the T-Bond.

We further fix the weight on the T-Bond (10%) and vary the weights on the Emerging Market hedge fund and S&P 500 with a 5% increment in the hedge fund. The ACCs are shown in Fig. 3. Similar to Fig. 2, the dashed, solid, and dotted lines represent the ACCs for the Emerging Market hedge fund, S&P 500 and T-Bond, respectively. The weights on the S&P 500 and Emerging Market hedge fund are shown at the bottom of each sub-figure, whereas the corresponding ε is shown at the top of each sub-figure. For example, for the current portfolio with weights of 30% on the Emerging Market hedge fund, 60% on the S&P 500 and 10% on the T-Bond, $\varepsilon = 0.0016276$. Fig. 3 demonstrates that we cannot

Table 5
Efficient portfolios.

Hedge fund	Efficiency test	Inefficient	Efficient	Difference
Event Driven	**MCSD** ($\varepsilon = 0$)	983	17	
	AMCSD ($\varepsilon = 0.01$)	984	16	1
	AMCSD ($\varepsilon = 0.05$)	984	16	1
Equity Hedge	**MCSD** ($\varepsilon = 0$)	982	18	
	AMCSD ($\varepsilon = 0.01$)	984	16	2
	AMCSD ($\varepsilon = 0.05$)	984	16	2
Macro	**MCSD** ($\varepsilon = 0$)	933	67	
	AMCSD ($\varepsilon = 0.01$)	972	28	39
	AMCSD ($\varepsilon = 0.05$)	977	23	44
Relative Value	**MCSD** ($\varepsilon = 0$)	989	11	
	AMCSD ($\varepsilon = 0.01$)	989	11	0
	AMCSD ($\varepsilon = 0.05$)	989	11	0
FOF	**MCSD** ($\varepsilon = 0$)	878	122	
	AMCSD ($\varepsilon = 0.01$)	954	46	76
	AMCSD ($\varepsilon = 0.05$)	973	26	96
Emerging market	**MCSD** ($\varepsilon = 0$)	987	13	
	AMCSD ($\varepsilon = 0.01$)	987	13	0
	AMCSD ($\varepsilon = 0.05$)	987	13	0
Market Neutral	**MCSD** ($\varepsilon = 0$)	988	12	
	AMCSD ($\varepsilon = 0.01$)	989	11	1
	AMCSD ($\varepsilon = 0.05$)	990	10	2

The simulation results are generated from 1000 random initial weights for seven cases with alternative hedge funds. The table shows the numbers of portfolios that are recognized as "Efficient" and "Inefficient" under three different efficiency criteria: (i) MCSD, (ii) AMCSD under $\varepsilon = 0.01$ and (iii) AMCSD under $\varepsilon = 0.05$. The numbers of portfolios, which are efficient under MCSD but inefficient under AMCSD, are shown in the last column.

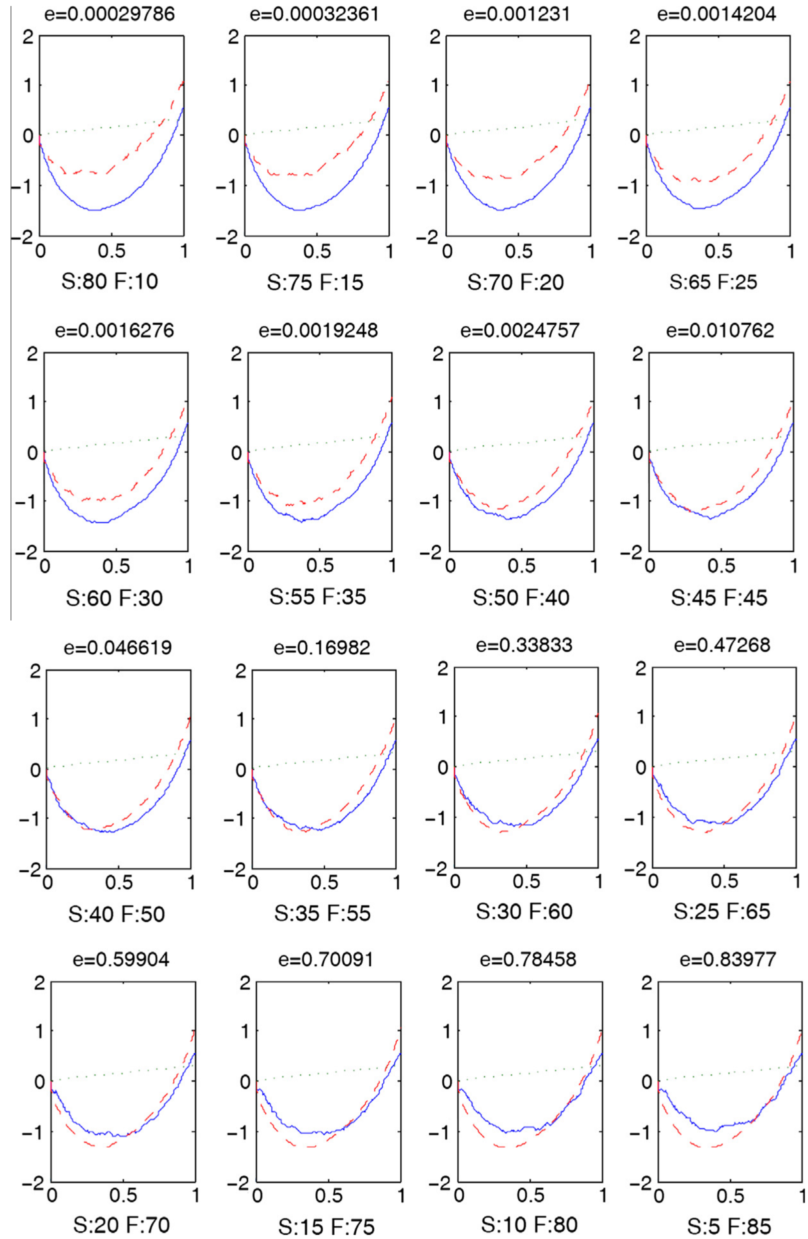

Fig. 3. The ACCs and the corresponding εs for the portfolios with different weights on the Emerging Market hedge fund and S&P 500 given a 10% weight on the T-Bond.

find any ACC of one asset located above the ACC of another one for all levels of cumulative probability for the given portfolio. In other words, MCSD cannot tell investors whether they should change the weights on the Emerging Market hedge fund, S&P 500 and T-Bond, but AMCSD can. According to AMCSD, the portfolios with $\varepsilon \leqslant \varepsilon^*$ can be further improved and should be considered to be inefficient. By using $\varepsilon^* = 3.2\%$, Fig. 3 indicates that the portfolios with investment weights on the Emerging Market hedge fund lower than 45% are dominated portfolios in terms of AMCSD when the investment weight on the T-Bond is fixed at 10%.

5.2.2. Changes in multiple assets

In this part, we employ programming (15) to implement AMCSD rules allowing changes in multiple assets. In other words, the weights of the hedge funds, S&P 500 and T-Bond are allowed to change simultaneously. As an illustration, we generate 1000 sets of portfolios with random weights and process the algorithm with $\varepsilon = 0, \varepsilon = 0.01$ and $\varepsilon = 0.05$. Under the MCSD rule ($\varepsilon = 0$), if the minimum value of the objective function is equal to zero, this means that we can change the proportion of assets and obtain an alternative combination preferred by all risk-averse investors. However, if the minimum value of the objective function is greater than zero for all possible $d\alpha_i$, the initial weight of the portfolio will be regarded as efficient. On the other hand, under AMCSD rules ($\varepsilon = 0.01$ and $\varepsilon = 0.05$), if the optimal value of the objective function is positive, then the initial portfolio is efficient. Otherwise, it is inefficient.

Table 5 shows the simulation results generated from 1,000 random initial weights for seven cases with alternative hedge funds. The simulation in Table 5 demonstrates that AMCSD rules could substantially reduce the set of efficient portfolios in cases with the Macro or Fund of Funds (FOF) hedge funds.

The findings in Table 5 are kind of on the basis of the point estimator of ε where

$$\varepsilon = \frac{-\int_\Omega \sum_{i=1}^n ACC_i(p)\, d\alpha_i\, dp}{\int_a^b \left|\sum_{i=1}^n ACC_i(p)\, d\alpha_i\right| dp}. \tag{18}$$

Table 6
Efficiency test.

Hedge fund	S&P500	Bond	Hedge fund	S&P500	Bond	Confidence interval	Hedge fund	S&P500	Bond	Confidence interval
Panel A: Efficient portfolio under MCSD			*Panel B: Dominating portfolio under AMCSD with* $\varepsilon = 0.01$				*Panel C: Dominating portfolio under AMCSD with* $\varepsilon = 0.05$			
0.0000	0.0200	0.9800	1.0000	0.0000	0.0000	(0.0007,0.0068)*	1.0000	0.0000	0.0000	(0.0009,0.007)*
0.0100	0.0200	0.9700	1.0000	0.0000	0.0000	(0.0018,0.0192)	1.0000	0.0000	0.0000	(0.0016,0.0198)*
0.0600	0.0300	0.9100	0.0748	0.0000	0.9252	(0,0.0896)	0.1165	0.0000	0.8835	(0,0.0122)*
0.0700	0.0200	0.9100	0.9077	0.0000	0.0923	(0.0001,0.511)	0.9070	0.0000	0.0930	(0.0002,0.4852)
0.1600	0.0200	0.8200								
0.2000	0.0300	0.7700	0.4786	0.5214	0.0000	(0,0.4175)	0.4786	0.5214	0.0000	(0,0.4046)
0.2200	0.0000	0.7800	0.0000	0.5829	0.4171	(0,0.1163)	0.0000	0.8558	0.1442	(0.0002,0.2485)
0.2400	0.0600	0.7000	0.2678	0.0000	0.7322	(0,0.0436)	0.2678	0.0000	0.7322	(0,0.0446)*
0.2400	0.0800	0.6800	0.2770	0.0000	0.7230	(0,0.0255)	0.2770	0.0000	0.7230	(0,0.0244)*
0.2600	0.0900	0.6500	0.3017	0.0000	0.6983	(0,0.0222)	0.3017	0.0000	0.6983	(0,0.0222)*
0.2600	0.0900	0.6500	0.3017	0.0000	0.6983	(0,0.0239)	0.3017	0.0000	0.6983	(0,0.0246)*
0.2900	0.0000	0.7100	0.0000	0.6264	0.3736	(0,0.15)	0.0000	0.6429	0.3571	(0,0.1706)
0.2900	0.0500	0.6600	0.3132	0.0000	0.6868	(0,0.1253)	0.3131	0.0000	0.6869	(0,0.1492)
0.2900	0.0900	0.6200	0.3317	0.0000	0.6683	(0,0.0277)	0.3317	0.0000	0.6683	(0,0.0254)*
0.3100	0.0300	0.6600								
0.3100	0.0300	0.6600								
0.3300	0.0300	0.6400								
0.3300	0.0600	0.6100	0.3579	0.0000	0.6421	(0,0.0688)	0.3578	0.0000	0.6422	(0,0.0667)
0.3400	0.0200	0.6400								
0.3600	0.0200	0.6200								
0.3600	0.0600	0.5800	0.3879	0.0000	0.6121	(0,0.1056)	0.3878	0.0000	0.6122	(0,0.0902)
0.3600	0.0600	0.5800	0.3879	0.0000	0.6121	(0,0.0797)	0.3878	0.0000	0.6122	(0,0.1072)
0.3600	0.0700	0.5700	0.3925	0.0000	0.6075	(0,0.0605)	0.3924	0.0000	0.6076	(0,0.0579)
0.3600	0.0800	0.5600	0.3972	0.0000	0.6028	(0,0.0448)	0.3970	0.0000	0.6030	(0,0.0431)*
0.3700	0.0000	0.6300	0.0000	0.8217	0.1783	(0,0.2141)	0.0000	0.8271	0.1729	(0,0.2701)
0.3700	0.0300	0.6000								
0.4200	0.0300	0.5500								
0.4500	0.0800	0.4700	0.4872	0.0000	0.5128	(0,0.0694)	0.4870	0.0000	0.5130	(0,0.0718)
0.4600	0.0000	0.5400	0.0000	0.9936	0.0064	(0,0.2938)	0.0000	0.9936	0.0064	(0,0.2495)
0.4600	0.0000	0.5400	0.0000	0.9936	0.0064	(0,0.1883)	0.0000	0.9936	0.0064	(0,0.2586)
0.4600	0.0600	0.4800	0.4879	0.0000	0.5121	(0,0.215)	0.4878	0.0000	0.5122	(0,0.2395)
0.5100	0.0400	0.4500								
0.5200	0.0300	0.4500								
0.5200	0.0400	0.4400								
0.5200	0.0800	0.4000	0.5572	0.0000	0.4428	(0,0.0949)	0.5570	0.0000	0.4430	(0,0.0853)
0.5300	0.0500	0.4200					0.5531	0.0000	0.4469	(0,0.6131)
0.5400	0.0000	0.4600	0.1435	0.8565	0.0000	(0,0.2948)	0.1435	0.8565	0.0000	(0,0.3606)
0.5500	0.0400	0.4100								
0.5500	0.0500	0.4000					0.5731	0.0000	0.4269	(0.0002,0.5861)
0.5500	0.0600	0.3900	0.5783	0.0000	0.4217	(0,0.3795)	0.5778	0.0000	0.4222	(0,0.3227)
0.5700	0.0500	0.3800					0.5931	0.0000	0.4069	(0.0003,0.6557)
0.6400	0.0300	0.3300								
0.6400	0.0700	0.2900	0.6725	0.0000	0.3275	(0,0.3212)	0.6724	0.0000	0.3276	(0,0.3092)
0.6600	0.1100	0.2300	0.7111	0.0000	0.2889	(0,0.0549)	0.7109	0.0000	0.2891	(0,0.063)
0.7100	0.0600	0.2300								
0.7200	0.0000	0.2800	0.7021	0.2979	0.0000	(0,0.3642)	0.7021	0.2979	0.0000	(0,0.4063)
0.7200	0.0500	0.2300								
0.7300	0.0400	0.2300								
0.7300	0.1100	0.1600	0.7826	0.0000	0.2174	(0,0.0907)	0.7809	0.0000	0.2191	(0,0.0847)
0.7300	0.1100	0.1600	0.7826	0.0000	0.2174	(0,0.0815)	0.7809	0.0000	0.2191	(0,0.0874)
0.7500	0.0300	0.2200								
0.7500	0.0500	0.2000								
0.7700	0.0600	0.1700								
0.7700	0.0700	0.1600					0.2139	0.0000	0.7861	(0.0001,0.389)
0.7800	0.1400	0.0800	0.8448	0.0000	0.1552	(0,0.0405)	0.8448	0.0000	0.1552	(0,0.042)*
0.7900	0.1500	0.0600	0.8594	0.0000	0.1406	(0,0.0425)	0.8594	0.0000	0.1406	(0,0.0396)*
0.8100	0.1400	0.0500	0.8748	0.0000	0.1252	(0,0.0523)	0.8748	0.0000	0.1252	(0,0.0487)*
0.8300	0.0600	0.1100								
0.8300	0.0600	0.1100								
0.8400	0.0000	0.1600	0.7021	0.2979	0.0000	(0,0.4349)	0.7021	0.2979	0.0000	(0,0.4428)
0.8400	0.0000	0.1600	0.8024	0.0000	0.1976	(0.0003,0.5253)	0.8024	0.0000	0.1976	(0.0002,0.4811)
0.8500	0.1500	0.0000	0.9194	0.0000	0.0806	(0,0.0465)	0.9194	0.0000	0.0806	(0,0.0446)*
0.8500	0.1500	0.0000	0.9194	0.0000	0.0806	(0,0.049)	0.9194	0.0000	0.0806	(0,0.0472)*
0.8700	0.0800	0.0500					0.0878	0.0000	0.9122	(0.0001,0.4687)
0.9100	0.0900	0.0000	0.9524	0.0000	0.0476	(0.0002,0.4509)	0.9517	0.0000	0.0483	(0.0002,0.3559)
0.9300	0.0600	0.0100								
1.0000	0.0000	0.0000								

This table presents the efficient portfolio weights generated from 1000 random initial weights for Macro hedge funds. Panel A shows the efficient portfolio weights on the hedge funds, S&P 500 and Bond under the MCSD rule, respectively. Panel B shows the efficient portfolio weights on the hedge funds, S&P 500 and Bond under the AMCSD rule with $\varepsilon = 0.01$ and the corresponding 95% confidence interval for ε. Panel C shows the efficient portfolio weights on the hedge funds, S&P 500 and Bond under the AMCSD rule with $\varepsilon = 0.05$ and the corresponding 95% confidence interval for ε. If the initial portfolio is efficient under the MCSD rule as well as under the AMCSD rule, then the dominating portfolio weights are not shown in Panels B and C. The asterisk denotes the case where the estimated ε is smaller than the critical value at the 5% significance level.

Following Bali et al. (2013), we further estimate confidence bands for ε by using the bootstrap method.[3] Since 216 monthly observations on asset returns can only estimate one ε at a time, the bootstrap resampling method is used to simulate actual data and generates a confidence interval of the estimator. First, each ε is simulated by random sampling with replacement from actual data repeated 2000 times, and then these series returns will be used to compute Eq. (18). This process is repeated 1000 times. Finally, the 95% confidence intervals are obtained by the percentile bootstrap method. Specifically, 1000 ε estimators are sorted. Then, the 25th smallest and the 25th largest ε estimators are used to form the lower bound and upper bound of the confidence interval.

In Table 6, we only focus on the case of the Macro hedge fund in our illustration. We show that, when $\varepsilon = 0.01$, 28 portfolios are inefficient under point estimation but only one portfolio is significant under the 95% confidence interval, $(0.0007, 0.0068)$. The portfolio happens to be the one allocating all money in the Macro hedge fund. Similarly, the last column of Table 6 shows that, among 23 inefficient portfolios, there are 14 portfolios that are significant at the 5% level.

The benefits and costs of AMCSD vs. MCSD can be identified according to the above empirical findings. On the one hand, AMCSD indeed improves investment efficiency by constructing a smaller set of efficient portfolios than MCSD (e.g., in the cases where the portfolio contains a Macro hedge fund or Fund of Funds strategy). The transaction costs could be substantially reduced by applying AMCSD. On the other hand, the benefit of AMCSD is sustained by the cost of an increase in ε. That is, the decision rule can be applied for a smaller set of decision makers.

6. Discussion

This paper develops a new methodology for improving existing portfolios, based on MCSD but replacing stochastic dominance with almost stochastic dominance. This helps to exclude extreme forms of preferences, not shared by real-world investors. Given the increasing importance of MCSD in constructing efficient portfolios, we believe that the extension proposed in the present paper will be useful for active asset management. In this respect, switching from MCSD to AMCSD helps to reduce the inconsistency between common practice in asset allocation and the elegant decision rules in modern portfolio theory inspired from stochastic dominance relations. This is done by considering only economically relevant utility functions, i.e. by excluding pathological preferences. In our simple numerical example, we discovered that a portfolio without possible MCSD improvement may appear to be dominated based on AMCSD. In this case, AMCSD tends to favor investing less in assets with lower means which is in line with common practice in asset management.

To conclude, let us mention that AMCSD can also be extended to higher orders that account for attitudes towards risk beyond risk aversion. Recall that the Nth order stochastic dominance is based on the common preferences shared by all the decision-makers with the utility function

$$U_N = \left\{\text{utility function } u|(-1)^{n+1}u^{(n)} \geqslant 0, \quad n = 1, 2, \ldots, N\right\},$$

where $u^{(n)}$ denotes the nth derivative of the utility function u, and $N > 2$. Besides risk aversion, U_N entails behavioral traits such as prudence ($N = 3$), temperance ($N = 4$), and more generally risk apportionment of order N in the terminology of Eeckhoudt and Schlesinger (2006).

Almost Nth order stochastic dominance introduced by Tzeng et al. (2013) excludes extreme forms of preferences by being restricted to

$$U_N^*(\varepsilon_N) = \left\{u \in U_N|(-1)^{N+1}u^{(N)}(x) \leqslant \inf\left\{(-1)^{N+1}u^{(N)}(x)\right\}\left(\frac{1}{\varepsilon_N} - 1\right) \text{ for all } x\right\}.$$

Now, starting from $B^{(1)}(t) = B(t)$, let us define iteratively for $n = 2, 3, \ldots N$

$$B^{(n)}(t) = \int_a^t B^{(n-1)}(s)ds,$$

$\Omega_n = \left\{t \in [a, b] : B^{(n)}(t) < 0\right\}$, and Ω_n^c as the complement of Ω_n in $[a, b]$. Following the reasoning that leads to Proposition 1, we can show that given portfolio $\boldsymbol{\alpha}$, asset k dominates asset j for all individuals with preferences $u \in U_N^*(\varepsilon_N), N > 2$, if and only if

$$\int_{\Omega_N}\left(-B^{(N)}(t)\right)dt \leqslant \varepsilon_N \int_a^b \left|B^{(N)}(t)\right|dt$$

and $B^{(n)}(b) \geqslant 0, n = 2, 3, \ldots N$. This provides the necessary and sufficient condition that asset j is dominated by asset k for all investors with $u \in U_N^*(\varepsilon_N)$.

Acknowledgement

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Appendix A. Proof of Proposition 1

A.1. "If" part

Let us first prove that (3) together with $E[X_k] \geqslant E[X_j]$ ensure that inequality (1) holds true. To this end,

$$E\left[u'(W)(X_k - X_j)\right] = E\left[u'(W)\left(E[X_k|W] - E[X_j|W]\right)\right] = \int_a^b u'(t)(\mu_k(t) - \mu_j(t))dF_R(t) = \int_a^b u'(t)dB(t).$$

Integrating the above equation by parts yields

$$E\left[u'(W)(X_k - X_j)\right] = u'(b)B(b) + \int_a^b (-u''(t))B(t)dt.$$

Since

$$B(b) = \int_a^b \left(\mu_k(t) - \mu_j(t)\right)dF_R(t) = E[X_k] - E[X_j],$$

the conditions $u' > 0$ and $E[X_k] \geqslant E[X_j]$ ensure that $u'(b)B(b) \geqslant 0$. Furthermore,

$$\begin{aligned}
\int_a^b (-u''(t))B(t)dt &= \int_\Omega (-u''(t))B(t)dt + \int_{\Omega^c} (-u''(t))B(t)dt \\
&\geqslant \int_\Omega \sup\{-u''(t)\}B(t)dt + \int_{\Omega^c} \inf\{-u''(t)\}B(t)dt \\
&= \sup\{-u''(t)\}\int_\Omega B(t)dt + \inf\{-u''(t)\}\int_{\Omega^c} B(t)dt \\
&= (\sup\{-u''(t)\} + \inf\{-u''(t)\})\int_\Omega B(t)dt \\
&\quad + \inf\{-u''(t)\}\left(\int_{\Omega^c} B(t)dt - \int_\Omega B(t)dt\right) \\
&= -(\sup\{-u''(t)\} + \inf\{-u''(t)\})\int_\Omega (-B(t))dt \\
&\quad + \inf\{-u''(t)\}\int_a^b |B(t)|dt.
\end{aligned}$$

[3] MCSD benefits from the statistical test developed by Chow (2001) and Schechtman et al. (2008). The tests have been implemented in many papers, e.g., Clark et al. (2011), Clark and Kassimatis (2012). To the best of our knowledge, AMCSD involves set estimation and lacks proper statistics with well behaved asymptotic distributions in the literature. Thus, we follow Bali et al. (2013) to estimate ε.

Since $u \in U_2^*(\varepsilon)$, based on the definition of $U_2^*(\varepsilon)$, we have

$$\sup\{-u''(t)\} \leqslant \inf\{-u''(t)\}\left(\frac{1}{\varepsilon}-1\right)$$

$$\Longleftrightarrow \varepsilon \leqslant \frac{\inf\{-u''(t)\}}{\sup\{-u''(t)\}+\inf\{-u''(t)\}}.$$

Thus, condition (3) ensures that

$$\int_\Omega(-B(t))dt \leqslant \frac{\inf\{-u''(t)\}}{\sup\{-u''(t)\}+\inf\{-u''(t)\}}\int_a^b|B(t)|dt.$$

Rearranging the above equation yields

$$-(\sup\{-u''(t)\}+\inf\{-u''(t)\})\int_\Omega(-B(t))dt + \inf\{-u''(t)\}\int_a^b|B(t)|dt \geqslant 0.$$

Thus, we have $\int_a^b(-u''(t))B(t)dt \geqslant 0$ and hence $E\left[u'(W)(X_k-X_j)\right] \geqslant 0$, which ends the proof of the "if" part.

A.2. "Only if" part

We would like to prove that if

$$\int_\Omega(-B(t))dt > \varepsilon\int_a^b|B(t)|dt \text{ or } E[X_k] < E[X_j],$$

then there exists one individual with preferences $u \in U_2^*(\varepsilon)$ such that inequality (1) is violated. Let us first show the following statement:

$$\int_\Omega(-B(t))dt > \varepsilon\int_a^b|B(t)|dt \Rightarrow \exists u \in U_2^*(\varepsilon) \text{ such that (1) is violated.}$$

Let $\underline{\theta}$ and $\bar{\theta}$ be two positive real numbers such that $\varepsilon = \frac{\underline{\theta}}{\bar{\theta}+\underline{\theta}}$. Assume that $\Omega = [c,d]$, where $a \leqslant c \leqslant d \leqslant b$. Define a marginal utility function as follows:

$$u'(x) = \begin{cases} \underline{\theta}(b-d)+\bar{\theta}(d-c)+\underline{\theta}(c-x) & \text{if } a \leqslant x \leqslant c \\ \underline{\theta}(b-d)+\bar{\theta}(d-x) & \text{if } c \leqslant x \leqslant d\,, \\ \underline{\theta}(b-x) & \text{if } d \leqslant x \leqslant b \end{cases}$$

which belongs to $U_2^*(\varepsilon)$. Since $u'(b)=0$, we have

$$E\left[u'(W)(X_k-X_j)\right] = \int_a^b(-u''(t))B(t)dt = \bar{\theta}\int_c^d B(t)dt + \underline{\theta}\int_{\Omega^C}B(t)dt$$
$$= -(\bar{\theta}+\underline{\theta})\int_a^b(-B(t))dt + \underline{\theta}\int_a^b|B(t)|dt.$$

Since $\varepsilon = \frac{\underline{\theta}}{\bar{\theta}+\underline{\theta}}$, and

$$\int_\Omega(-B(t))dt > \frac{\underline{\theta}}{\bar{\theta}+\underline{\theta}}\int_a^b|B(t)|dt,$$

we have $E\left[u'(W)(X_k-X_j)\right] < 0$, which is the desired result.

Now, let us prove that

$$E[X_k] < E[X_j] \;\Rightarrow\; \exists u \in U_2^*(\varepsilon) \text{ such that (1) is violated.}$$

Define

$$u'(x) = c - \delta x,$$

where δ is a positive constant small enough to ensure $c > \delta b$. Thus, $u \in U_2^*(\varepsilon)$. Furthermore,

$$E\left[u'(W)(X_k-X_j)\right] = u'(b)B(b) + \int_a^b(-u''(t))B(t)dt$$
$$= (c-\delta b)B(b) + \delta\int_a^b B(t)dt.$$

Now, if c is such that

$$(c-\delta b)\left(E[X_k]-E[X_j]\right) + \delta\int_a^b B(t)dt < 0 \iff c > \delta b - \frac{\delta\int_a^b B(t)dt}{E[X_k]-E[X_j]},$$

we have $E\left[u'(W)(X_k-X_j)\right] < 0$ which contradicts (1) and completes the proof.

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