國立政治大學金融系研究所

碩士學位論文

預測S&P500指數實現波動度與VIX－探討VIX、VIX選擇權與VVIX之資訊內涵

The S&P 500 Index Realized Volatility and VIX Forecasting - The Information Content of VIX, VIX Options and VVIX

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摘要

波動度對於金融市場影響甚多，同時為金融資產定價的重要參數以及市場穩定度的衡量指標，尤其在金融危機發生時，波動度指數的驟升反映資產價格震盪。本篇論文嘗試捕捉S&P500指數實現波動度與VIX變動率未來之動態，並將VIX、VIX 選擇權與 VVIX 納入預測模型中，探討其資訊內涵。透過研究 S&P500 指數實現波動度，能夠預測 S&P500 指數未來之波動度與報酬，除了能夠觀察市場變動，亦能使未來選擇權定價更為準確；而藉由模型預測 VIX，能夠藉由 VIX 選擇權或 VIX 期貨，提供避險或投資之依據。文章採用 2006 年至 2011 年之 S&P500 指數、VIX、VIX 選擇權與 VVIX 資料。

在 S&P500 指數之實現波動度預測當中，本篇論文的模型改良自先前文獻，結合實現波動度、隱含波動度與 S&P500 指數選擇權之風險中立偏態，所構成之異質自我回歸模型 (HAR-RV-IV-SK model)。論文額外加入 VIX 變動率以及 VIX 指數選擇權之風險中立偏態作為模型因子，預測未來 S&P500 指數實現波動度。研究結果表示，加入 VIX 變動率作為 S&P500 指數實現波動度預測模型變數後，可增加 S&P500 指數實現波動度預測模型之準確性。

在 VIX 變動率預測模型之中，論文採用動態轉換模型，作為高低波動度之下，區分預測模型的方法。以 VIX 過去的變動率、VIX 選擇權之風險中立動差以及 VIX 之波動度指數(VVIX)作為變數，預測未來 VIX 變動率。結果顯示動態轉換模型能夠提升 VIX 預測模型的解釋能力，並且在動態轉換模型下，VVIX 與 VIX 選擇權之風險中立動差，對於 VIX 預測具有相當之資訊隱涵於其中。

關鍵字：VIX 選擇權、VVIX、資訊內涵、S&P500 指數實現波動度、

動態轉換模型、風險中立動差

Abstract

This paper tries to capture the future dynamic of S&P 500 index realized volatility and VIX. We add the VIX change rate and the risk neutral skewness of VIX options into the Heterogeneous Autoregressive model of Realized Volatility, Implied Volatility and Skewness (HAR-RV-IV-SK) model to forecast the S&P 500 realized volatility. Also, this paper uses the regime switching model and joins the VIX, risk neutral moments of VIX options and VVIX variables to raise the explanatory ability in the VIX forecasting. The result shows that the VIX change rate has additional information on the S&P 500 realized volatility. By using the regime switching model, the VVIX and the risk neutral moments of VIX options variables have information contents in VIX forecasting. These models can be used for hedging or investment purposes.

Keywords: VIX Options, VVIX, Information Content, S&P 500 Realized Volatility, Regime Switching Model, Risk Neutral Moments

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Acronyms List

Acronym	Definition
ARCH	AutoRegressive Conditional Heteroskedasticity
CBOE	Chicago Board Options Exchange
CRV	Change Rate of VIX
EM	Expectation–Maximization
GARCH	Generalized AutoRegressive Conditional Heteroskedasticity
HAR	Heterogeneous Autoregressive
IV	Implied Volatility (= VIX)
KURV	Risk Neutral Kurtosis of VIX Options
RS	Regime Switching
RV	Realized Volatility of S&P 500 Index
S&P	Standard & Poor's
SK	Risk Neutral Skewness of S&P 500 Options
SKV	Risk Neutral Skewness of VIX Options
VIX	CBOE Volatility Index of S&P 500 Index
VOLV	Risk Neutral Volatility of VIX Options
VVIX	Volatility Index of VIX
VXO	CBOE S&P 100 Volatility Index

1. Introduction

In the recent times, the rapid growth of financial markets and technology generate more kinds of financial products and derivatives. People can construct their portfolio via various tools to pursue the higher return or manage risk. The volatilities of most assets are the key factors to determine the extent of profit and loss. The higher volatility may bring a higher return, but also the higher risk.

Thus, the management of volatility is an important issue in the current financial environment. Many researchers use different kinds of model to describe the volatility of assets. Some of them use models to forecast the volatilities in the future. It is meaningful and useful to capture the future dynamic of volatility by using the quantitative models. When we are able to foresee the future dynamics of volatility, we can use financial tools to hedge or speculate early.

This paper uses the risk neutral volatility, skewness and kurtosis (risk neutral moments) of VIX options to find the information contents on the future S&P 500 realized volatility and VIX change rate. We uses the formula in the research by Bakshi, Kapadia and Madan (2003) to calculate the risk neutral moments and take the risk neutral skewness of the S&P 500 options for the S&P 500 index realized volatility forecasting. Also, the risk neutral skewness of VIX options can be used in both the S&P 500 index realized volatility and the VIX change rate forecasting.

For S&P 500 realized volatility forecasting, this paper extends the HAR-RV-IV-SK model from Byun and Kim (2013), and take VIX change rate (CRV) and the risk neutral skewness of VIX options (SKV) as new variables in the S&P 500 realized volatility forecasting model. We hope to have a more precise model to forecast the S&P 500 realized volatility by including the informations of VIX and VIX options.

In VIX forecasting, this paper takes the VIX change rate, risk neutral volatility (VOLV), skewness (SKV), kurtosis (KURV) of VIX options and VVIX as variables to capture the 5-day and 21-day future dynamic of VIX change rate. Moreover, we join the regime switching model to predict the future change rate of VIX conditional in the different states. By joining the regime switching model into the Heterogeneous Autoregressive of VIX change rate (HAR-CRV) model, the variables become more informative and flexible in the VIX forecasting.

The rest of this study is organized as follows. Literature review is in Section 2. The description of index and options data is in Section 3. In Section 4, we set the Heterogeneous Autoregressive model of Realized Volatility, Implied Volatility, VIX Change Rate, Skewness of S&P 500 Index and Skewness of VIX (HAR-RV-IV-CRV-SK-SKV) model for the S&P 500 index realized volatility forecasting. Section 5 is the Regime Switching Heterogeneous Autoregressive model of VIX Change Rate, Skewness of VIX, and VVIX (HAR-CRV-SKV-VVIX) model for VIX forecasting. Finally, we have the conclusions in Section 6.

2. Literature Review

2.1 VIX, VIX Options and VVIX

In 1993, Chicago Board Options Exchange (CBOE) introduced the Volatility index (VIX) based on the S&P 100 options. After ten years, in 2003, CBOE created a new formula and replaced the old index, called VXO. The new formula considered the weighted average of the at-the-money and out-of–the-money call and put options of S&P 500 index. In 2014, CBOE enhanced the VIX index by joining S&P 500 weekly options into the formula. VIX measures the 30-day volatility of the S&P 500 options and weekly options.

Since VIX is useful in evaluating the volatility of the S&P 500 index. Also, the dynamic of VIX differs from normal stock indexes: mean-reverting, strong autocorrelation and volatility clustering properties. CBOE describes VIX as the powerful and flexible trading and risk management tool. Moreover, VIX has negative correlation with many assets, so we can use VIX derivatives to hedge or diversify portfolios.

VIX options started trading from February 2006, followed the introduction of VIX futures in 2004. The trade volume of VIX options and futures rises significantly in the current years. The VIX options and futures markets are still rapidly growing. Since the dynamic of VIX is quite special, VIX options have many features which are different from the S&P 500 options.

The Volatility of Volatility index (VVIX) is an indicator of 30-day volatility of VIX forward price. CBOE use the same algorithm as the volatility index of S&P 500 options to calculate VVIX from the VIX options. VVIX has mean-reverting property and downward slope of term structure and can be used as the realized volatility of VIX futures price.

2.2 Risk Neutral Skewness

The risk neutral volatility, skewness and kurtosis of options are the moments of the implied distribution from the options chain under the risk neutral measure. The risk neutral moments contain the information of how people expect the underlying value will be in the future.

To find the information content of risk neutral skewness, Bakshi, Kapadia and Madan (2003) use a model-free methodology to calculate the risk neutral density and to explain the presence and evolution of risk-neutral skewness over time and in the cross section of individual stocks. About the risk neutral volatility and kurtosis, Neumann and Skiadopoulosb (2012) investigate the patterns in the dynamics of higher order risk neutral moments extracted from the market prices of S&P 500 index options. Stilger et al. (2014) documents a relationship between the option- implied risk neutral skewness of individual stock returns distribution and future realized stock returns. Seo and Kim (2015) find the risk-neutral skewness has the explanatory power regarding future volatility during high sentiment periods.

2.3 HAR-RV-IV-SK Model

The Heterogeneous Autoregressive model of Realized Volatility (HAR-RV) model is proposed by Corsi (2009). The HAR-RV model successfully reproduces the main empirical features of financial returns. Fradkin (2009) add the jump component and realized absolute value to the HAR-RV model. He find the information content of implied volatility (IV) in the individual stocks and the market. Busch et al. (2011) use the Vector HAR model in the foreign exchange, stock, and bond markets realized volatility forecasting. Byun and Kim (2013) incorporate the risk neutral skewness of S&P 500 options (SK) into the HAR-RV-IV model. They investigate that risk neutral skewness of S&P 500 options has incremental explanatory power for future volatility in the S&P 500 index.

2.4 Volatility Forecasting

For the VIX forecasting, Fernandesa, Medeirosc, and Scharth (2014) perform a thorough statistical examination of the time-series properties of the daily market VIX, from the CBOE. They show that it is pretty hard to beat the pure HAR process because of the very persistent nature of the VIX index. Lin (2013) integrates CBOE VIX Term Structure and VIX futures to simplify VIX option pricing in multifactor models. Many researches use ARCH or GARCH model to capture the dynamic of volatility and use these models in the option pricing. Kanniainen, Lin, and Yang (2014) use information on VIX to improve the empirical performance of GARCH models for pricing the S&P 500 options.

2.5 Regime Switching Model

The regime switching (RS) model, also known as Markov switching model, is pronounced by Hamilton (1989). Regime switching model can separate the time series into two or more regimes according to the property of data and the research purpose. Also, it can give the distinct coefficients of mean, coefficient and volatility variables in the different regimes. The estimation of the regimes, transition probability, and the parameters can be easily computed by the expectation–maximization (EM) algorithm to maximize the likelihood function of the regression formula.

Regime switching model is widely used in the business cycle analysis and volatility forecasting. Khalifa et al. (2014) examine volatility transmission patterns for pairs of six stock markets of countries of the Gulf Cooperation Council (GCC) and pairs of these markets with the S&P 500 index, Oil-WTI prices and MSCI-world market by using the Multi-Chain Markov Switching (MCMS) model. Miao, Wu, and Su (2013) examine latent shifts in the conditional volatility and correlation for the U.S. stock and T-bond data using the two-state Markov-switching range-based volatility and correlation models.

3. Data

3.1 CBOE Data

The S&P 500 index, VIX, VIX options and VVIX data come from CBOE. The risk neutral skewness of S&P 500 options is from the SKEW index of CBOE. The calculation formula of VIX (and VVIX) is shown in Appendix I. The risk neutral volatility, skewness, and kurtosis of VIX options are computed from the closing price of VIX options in each day. We choose the 3-month treasury bills rate as the risk-free rate in the imply volatility calculation. By considering the liquidity and the price accuracy of VIX options, the data period is chosen from 15 May, 2006 to 16 December, 2011, total 1411 trade days. Figure 1 gives the S&P 500 index and VIX in the sample period.

Figure 1 The relationship between S&P 500 index and VIX in the sample period from 15 May 2006 to 16 Dec. 2011. The dynamics between S&P 500 index and VIX have negative relation.

3.2 Realized Volatility Data

In the realized volatility calculation, we calculate from the intraday return of S&P 500 index. The realized volatility of one day is the root sum square of all 5-minute return in one day, and the h-day realized volatility is the root mean square of daily realized volatility. In formulas:

$$RV_{t-1,t} = \sqrt{\sum_{i-1}^{n} r_{t,i}^{2}}\ ,\quad RV_{t-h,t} = \sqrt{\frac{1}{h}\sum_{i-1}^{h} RV_{t-i,t-i+1}^{2}}$$

The dynamic of S&P 500 realized volatility is similar to VIX, but the daily S&P 500 realized volatility is more volatile by comparing with the previous period. Since S&P 500 realized volatility reflects the intraday volatility, while VIX reflects the 30 days expectation, S&P 500 realized volatility has weaker autoregression property than VIX.[1]

3.3 VIX Options Data

In the data period, we have total 82,738 call and 44,188 put options of VIX come from every different date, expiration date, and exercise prices. Figure 2 shows the moneyness and daily frequency of VIX options in the sample period. The distribution of the VIX options moneyness is quietly different from the S&P 500 options - VIX options have wider range of moneyness.

3.4 Risk Neutral Moments of VIX Options

The near-month options data is chosen between 5- and 37-day time-to- maturity options and remove the options which have arbitrage opportunity. We have the 20,914 call and 13,611 put options of VIX to calculate the risk neutral moments. In the

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[1] See Long memory and nonlinearities in realized volatility: A Markov switching approach, Raggia and Bordignon (2012)

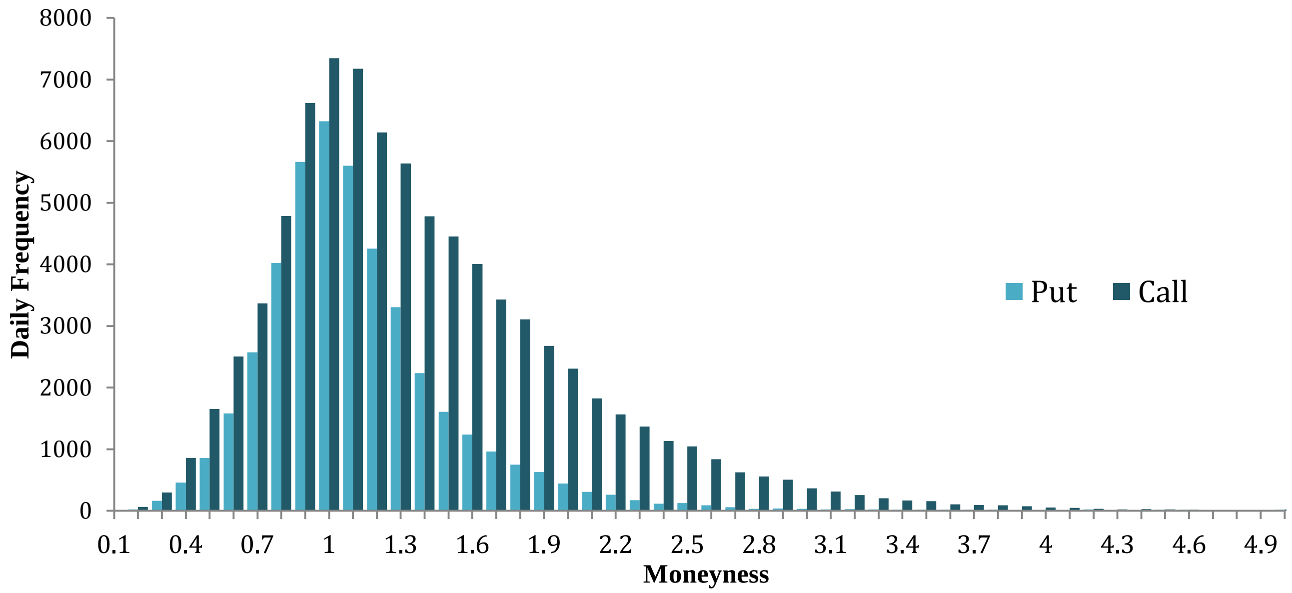


Figure 2 Moneyness of VIX options (exercise price (K)/ spot price (S)) verse the daily frequency. The call options have longer tail on the right hand side (moneyness>2), and have heavy body in the 1.5 to 2 times the spot price. The positive skewness means that there are more investors trade in out-of-the- money VIX call options.

calculation of the risk neutral momentum of VIX Options, we use the methodology from Bakshi, Kapadia and Madan (2003), and then get the daily risk neutral volatility, skewness, and kurtosis of VIX options. The formula of risk neutral moments is in Appendix II.

The risk neutral volatility of VIX is a kind of implied volatility of VIX, and the correlation between risk neutral volatility and VVIX is 0.84. The risk neutral skewness of VIX is usually positive. This fact reflects the property that the volatility smile of VIX options is a curve from lower left to the upper right side as the moneyness get higher.

If we compare the risk neutral skewness of VIX options to the risk neutral skewness of S&P 500 options, we can find that S&P 500 options have negative skewness. The reason is that people expect S&P 500 index is easy to fall down and hard to have a huge rise, while VIX usually increase shapely and return to normal slower.

When VIX get higher and higher, the risk neutral skewness of VIX will close to zero and the risk neutral kurtosis of VIX will be low (See Figure 3). Hence, the VIX options volatility smile curve will tend to be a U-shape when VIX is high. Since the range of exercise price goes wider and the volatility of VIX increase, the implied volatility of the deep out-of-the-money and deep in-the-money VIX options will be higher than the normal situation.[2]

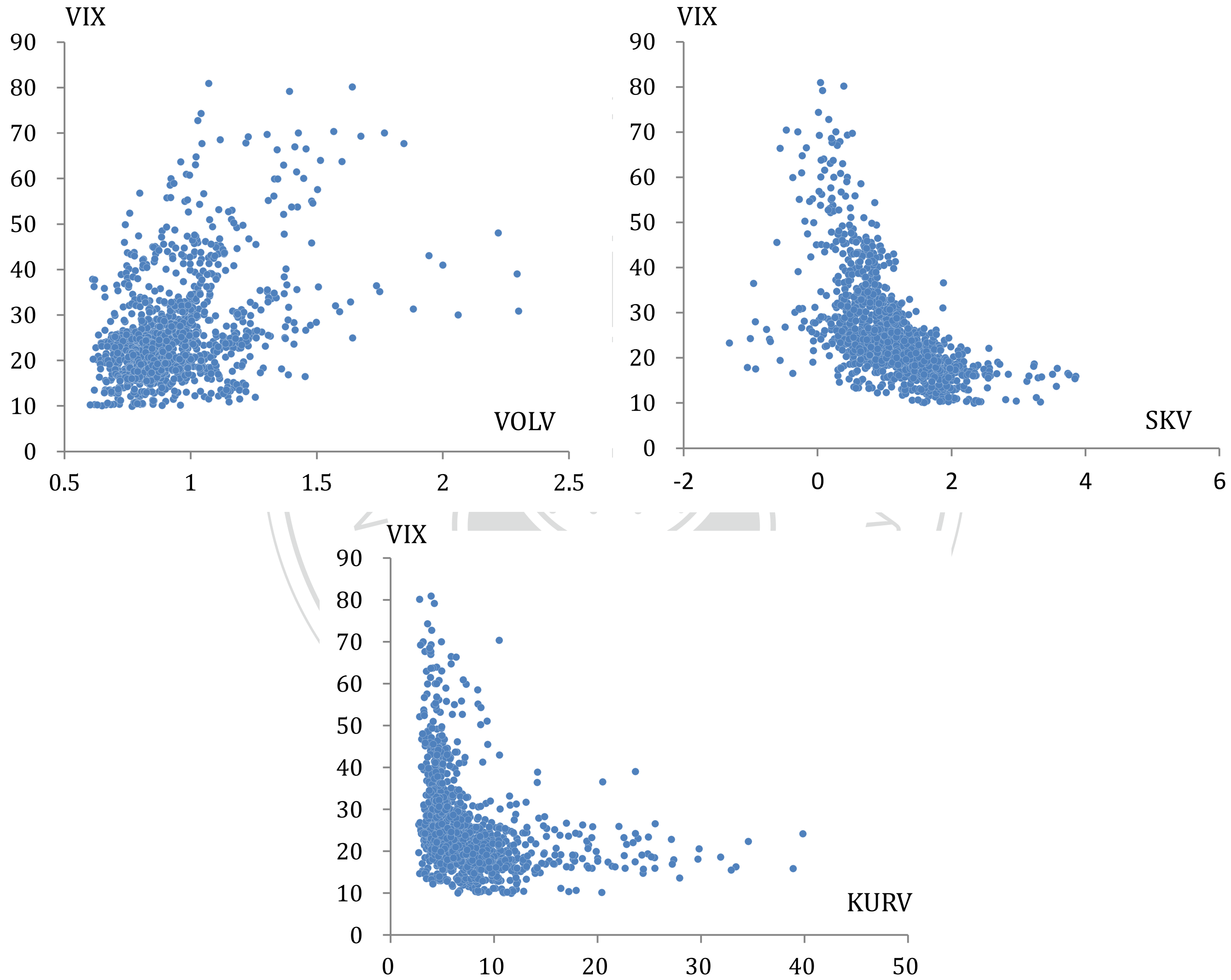


Figure 3 The scatter plots of risk neutral a) volatility, b) skewness, and c) kurtosis of VIX options to VIX. High risk neutral volatility occurs when VIX is higher than 30. As VIX increase, the risk neutral skewness tends to be zero and the risk neutral kurtosis also tends to be lower.

[2] See Consistent Modeling of SPX and VIX options, Gatheral (2008)

3.5 VVIX Data

We take the VVIX data from February 1, 2007 to 16 December, 2011 in the VIX forecasting model. Figure 4 presents the historical VIX and VVIX data in 2007-2011. From Figure 4, we can see that VVIX is more sensitive to the sharp decreasing of S&P 500 index. We use VVIX as a variable in the regime switching model to predict the VIX change rate.

Table 1 shows the descriptive statistics of our data. We choose the 1-day, 5-day and 21-day future S&P 500 realized volatility as the dependent variables in the model to predict the future S&P 500 realized volatility. The data period is from 15 May, 2006 to 16 December, 2011. For the VIX forecasting model, we choose the 5-day and 21-day VIX change rate as the dependent variables. Since the time range of the VVIX data started from 2007, we reduce the data period in the VIX forecasting model. The data period for VIX forecasting is from5 May, 2006 to 16 December, 2011.

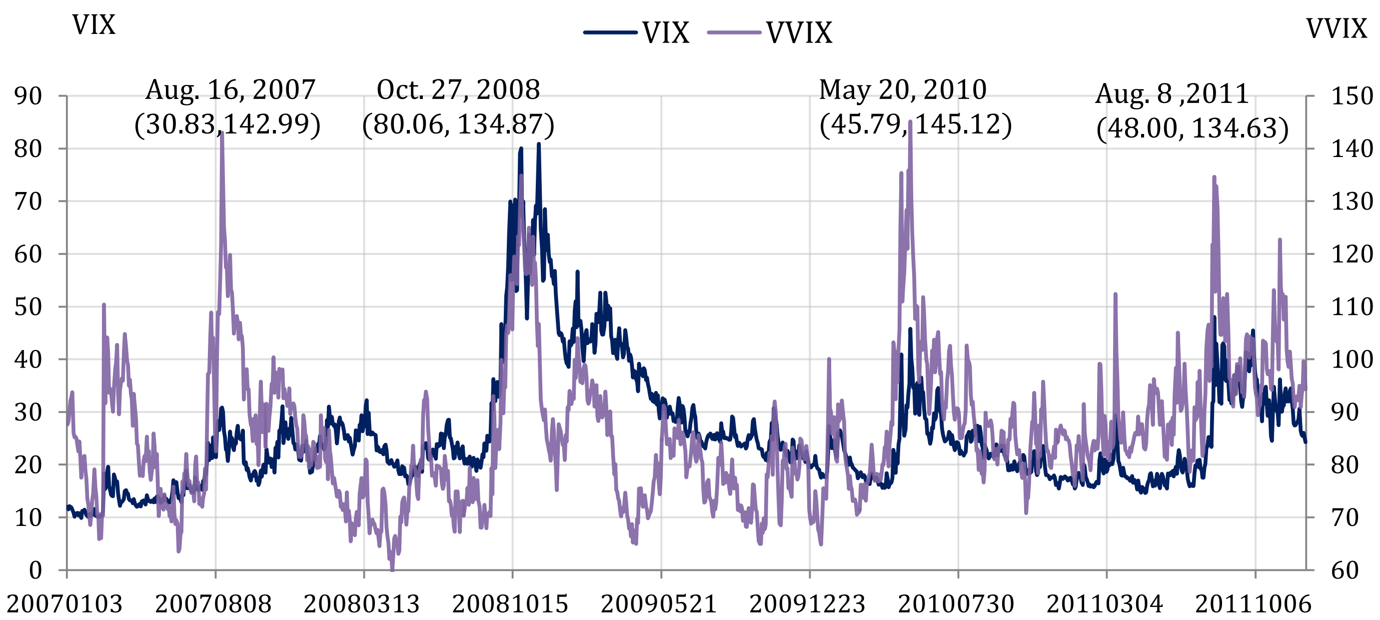


Figure 4 Volatility index (VIX) of S&P 500 index and volatility of volatility index (VVIX) form 2007 to 2011. The tags are the local maximums of VVIX.

Table 1 The Descriptive Statistics of Data

The Descriptive Statistics of 1-day, 5-day, 21-day S&P 500 realized volatilities (RVs), risk neutral skewness of S&P 500 options (SK), VIX, 1-day, 5-day, 21-day VIX change rates (CRVs) , risk neutral volatility, skewness, and kurtosis of VIX options (VOLV, SKV, KURV), and VVIX. The data period is from 15 May, 2006 to 16 December, 2011, except the data of VVIX, which is from February 1, 2007 to 16 December, 2011.

	$RV_{t-1,t}$	$RV_{t-5,t}$	$RV_{t-21,t}$	SK	VIX	$CRV_{t-1,t}$	$CRV_{t-5,t}$	$CRV_{t-21,t}$	VOLV	SKV	KURV	VVIX
Mean	1.1096	1.1386	1.1599	-1.9213	24.3123	0.0471	0.2649	1.3232	0.9233	1.1452	7.5535	86.1197
Std. Error	0.0216	0.0204	0.0193	0.0134	0.3033	0.1981	0.3661	0.6360	0.0055	0.0187	0.1253	0.3824
Std. Dev.	0.8110	0.7649	0.7258	0.5025	11.3900	7.4476	13.757	23.9006	0.2068	0.7031	4.7080	13.4044
Skewness	2.6851	7.5927	5.6174	-0.2033	1.7241	4.1438	2.0295	2.1598	1.8867	0.6181	2.7799	0.9895
Kurtosis	11.3956	2.3696	2.1404	-0.1086	3.7736	0.7284	0.5813	1.1234	6.5465	1.9327	10.4052	1.2769
Min	0.2283	0.2626	0.4202	-3.6930	9.8900	-35.0588	-43.3592	-61.0001	0.5618	-1.3093	2.7487	59.7400
Max	8.6813	5.7764	4.6438	-0.6430	80.8600	49.6007	70.7415	110.1742	2.3016	4.8172	39.8771	145.1200
Numbers	1411	1411	1411	1411	1411	1411	1411	1411	1411	1411	1411	1229

4. S&P 500 Realized Volatility Forecasting

4.1 Single Variable Regression Testing

Before the introduce of the S&P 500 realized volatility forecasting models, we use single-variable linear regression to analysis the simple relationship between each variable and the 1-, 5-, 21-day S&P 500 realized volatility. From Table 2, the high adjusted R-square of RV terms show the autocorrelation property of the S&P 500 realized volatility. Also, the VIX change rate terms can provide some information to the future realized volatility.

Table 2 Single Variable Regression for S&P 500 Realized Volatility Forecasting Model

Single Variable Regression of the future 1-, 5-, 21-day S&P 500 realized volatility and each independent variable. All of the coefficients are 1% significant (except the α term of the VIX variable in 21-day forecasting). The regression function is $RV_{t,t+h} = \alpha + \beta x_t + \varepsilon_{t,t+h}$.

$RV_{t,t+h}$		$RV_{t-1,t}$	$RV_{t-5,t}$	$RV_{t-21,t}$	VIX	$CRV_{t-1,t}$	$CRV_{t-5,t}$	$CRV_{t-21,t}$	SK_t	SKV_t
	α	0.1966	0.1025	0.0983	- 0.3325	1.1092	1.1057	1.0904	1.9791	1.7616
h = 1	β	0.8230	0.8831	0.8705	0.0593	0.1572	0.1584	0.0144	0.4525	- 0.5692
	Adj.R^2	0.6770	0.6950	0.6070	0.6940	0.0201	0.0715	0.18	0.078	0.243
	α	0.2715	0.1661	0.1470	- 0.2428	1.1416	1.1384	1.1227	2.0206	1.7681
h = 5	β	0.7850	0.8560	0.8573	0.0570	0.0114	0.1270	0.0137	0.4573	- 0.5473
	Adj.R^2	0.6920	0.7330	0.6610	0.7190	0.0116	0.0513	0.1810	0.0895	0.2520
	α	0.4123	0.3079	0.2814	- 0.0255	1.1673	1.1644	1.1520	1.9902	1.7049
h=21	β	0.6841	0.7578	0.7678	0.0493	0.0080	0.0103	0.0110	0.4280	- 0.4689
	Adj.R^2	0.5820	0.6390	0.5890	0.5980	0.0059	0.0369	0.1310	0.0869	0.2060

The risk neutral skewness of VIX options has higher adjusted R-square value to the S&P 500 realized volatility than the risk neutral skewness of S&P 500 options: The SK variable has adjusted R-square of approximately 8%, while SKV variable has 20% to 25%. The relationship between risk neutral skewness of VIX options and the future S&P 500 realized volatility is negative, and the S&P 500 realized volatility has strong positive correlation with VIX.

4.2 Methodology

Our model of S&P 500 realized volatility follows the HAR-RV-IV-SK model proposed by Byun and Kim (2013). The HAR-RV-IV-SK model is as Model 1.1.

For the next model, we add the 1-, 5- and 21-day change rate of VIX (CRV) into the HAR-RV-IV-SK model. We hope the new CRV variables can increase the explanation power of the future realized volatility of S&P 500 index. Then we have the second S&P 500 realized volatility forecasting model in model 1.2. The change rate of VIX comes from:

$$CRV_{t-h,t} = \ln\left(VIX_t / VIX_{t\text{-}h}\right), h \in \mathbb{N}$$

In model 1.3, we add the risk neutral skewness of VIX options (SKV) into the regression. We want to know if the additions of risk neutral skewness of VIX options (SKV) variables can provide more information on the original HAR-RV-IV-SK model.The calculation of SKV is the same formula of risk neutral skewness of S&P 500 options (see Appendix II).

In the final model for S&P 500 realized volatility forecasting, model 1.4 discusses the synergy of CRV and SKV variables in the HAR-RV-IV-SK model. The results are in Table 3.

Model 1.1 HAR-RV-IV-SK Model (for S&P 500 Realized Volatility Forecasting)

$$RV_{t,t+h} = \beta_0 + \begin{bmatrix}\beta_1 & \beta_2 & \beta_3\end{bmatrix}\begin{bmatrix}RV_{t-1,t} \\ RV_{t-5,t} \\ RV_{t-21,t}\end{bmatrix} + \beta_4 IV_t + \beta_5 SK_t + \varepsilon_{t,t+h},\ \varepsilon_{t,t+h} \sim N(0,\sigma^2),\ h = 1,\ 5,\ 21$$

Model 1.2 HAR-RV-IV-CRV-SK Model

$$RV_{t,t+h} = \beta_0 + \begin{bmatrix}\beta_1 & \beta_2 & \beta_3\end{bmatrix}\begin{bmatrix}RV_{t-1,t} \\ RV_{t-5,t} \\ RV_{t-21,t}\end{bmatrix} + \beta_4 IV_t + \begin{bmatrix}\beta_5 & \beta_6 & \beta_7\end{bmatrix}\begin{bmatrix}CRV_{t-1,t} \\ CRV_{t-5,t} \\ CRV_{t-21,t}\end{bmatrix} + \beta_8 SK_t + \varepsilon_{t,t+h},\ \varepsilon_{t,t+h} \sim N(0,\sigma^2),\ h = 1,\ 5,\ 21$$

Model 1.3 HAR-RV-IV-SK-SKV Model

$$RV_{t,t+h} = \beta_0 + \begin{bmatrix}\beta_1 & \beta_2 & \beta_3\end{bmatrix}\begin{bmatrix}RV_{t-1,t} \\ RV_{t-5,t} \\ RV_{t-21,t}\end{bmatrix} + \beta_4 IV_t + \beta_5 SK_t + \beta_6 SKV_t + \varepsilon_{t,t+h},\ \varepsilon_{t,t+h} \sim N(0,\sigma^2),\ h = 1,\ 5,\ 21$$

Model 1.4 HAR-RV-IV-CRV-SK-SKV Model

$$RV_{t,t+h} = \beta_0 + \begin{bmatrix}\beta_1 & \beta_2 & \beta_3\end{bmatrix}\begin{bmatrix}RV_{t-1,t} \\ RV_{t-5,t} \\ RV_{t-21,t}\end{bmatrix} + \beta_4 IV_t + \begin{bmatrix}\beta_5 & \beta_6 & \beta_7\end{bmatrix}\begin{bmatrix}CRV_{t-1,t} \\ CRV_{t-5,t} \\ CRV_{t-21,t}\end{bmatrix} + \beta_8 SK_t + \beta_9 SKV_t + \varepsilon_{t,t+h},\ \varepsilon_{t,t+h} \sim N(0,\sigma^2),\ h = 1,\ 5,\ 21$$

Table 3　Results of S&P 500 Realized Volatility Forecasting Model

A) 1-day Future Realized Volatility of S&P 500 Index ($RV_{t,t+1}$)

The model for 1-day S&P 500 realized volatility forecasting. Model (1.1) is the HAR-RV-IV-SK model, and (1.2) is HAR-RV-IV-CRV-SK model, which contains the change rate of VIX into the HAR-RV-IV-SK model. (1.3) adds risk neutral skewness of VIX options, i.e. the HAR-RV-IV-SK-SKV model. (1.4) is the HAR-RV-IV-CRV-SK-SKV model. The number in the parentheses is the t-statistics, and the stars behind the coefficients are the significant levels of 1%, 5% and 10%.

Model	Intercept	$RV_{t-1,t}$	$RV_{t-5,t}$	$RV_{t-21,t}$	VIX_t	$CRV_{t-1,t}$	$CRV_{t-5,t}$	$CRV_{t-21,t}$	SK_t	SKV_t	$Adj.R^2$
A1.1	-0.0502	0.2693***	0.3205***	-0.1914***	0.0342***				0.0588***		0.750
	(-0.8855)	(8.3764)	(7.1068)	(-4.0270)	(11.1077)				(2.6170)		
A1.2	0.0353	0.1955***	0.2515***	0.0093	0.0281***	0.0051***	0.0032***	0.0030***	0.0670***		0.763
	(0.6288)	(5.9936)	(5.1122)	(0.1760)	(8.5399)	(3.1258)	(3.0595)	(4.9185)	(3.0153)		
A1.3	-0.0608	0.2693***	0.3209***	-0.1915***	0.0344***				0.0611***	0.0079	0.749
	(-0.9741)	(8.3748)	(7.1118)	(-4.0275)	(10.9905)				(2.6359)	(0.4066)	
A1.4	-0.0148	0.1929***	0.2454***	0.0178	0.0293***	0.0050***	0.0034***	0.0032***	0.0803***	0.0409**	0.764
	(-0.2435)	(5.9161)	(4.9841)	(0.3342)	(8.7834)	(3.0295)	(3.1897)	(5.2179)	(3.4805)	(2.1114)	

Table 3 Results of S&P 500 Realized Volatility Forecasting Model (Continued)

B) 5-day Future Realized Volatility of S&P 500 Index ($RV_{t,t+5}$)

The model for 5-day S&P 500 realized volatility forecasting. Model (1.1) is the HAR-RV-IV-SK model, and (1.2) is HAR-RV-IV-CRV-SK model, which contains the change rate of VIX into the HAR-RV-IV-SK model. (1.3) adds risk neutral skewness of VIX options, i.e. the HAR-RV-IV-SK-SKV model. (1.4) is the HAR-RV-IV-CRV-SK-SKV model. The number in the parentheses is the t-statistics, and the stars behind the coefficients are the significant levels of 1%, 5% and 10%.

Model	Intercept	$RV_{t-1,t}$	$RV_{t-5,t}$	$RV_{t-21,t}$	VIX_t	$CRV_{t-1,t}$	$CRV_{t-5,t}$	$CRV_{t-21,t}$	SK_t	SKV_t	$Adj.R^2$
B1.1	0.1175**	0.2253***	0.3216***	-0.0219	0.0241***				0.0791***		0.776
	(2.3215)	(7.8389)	(7.9770)	(-0.5157)	(8.7537)				(3.9398)		
B1.2	0.2000***	0.1679***	0.2238***	0.1381***	0.0212***	0.0029**	0.0017*	0.0036***	0.0938***		0.789
	(3.9882)	(5.7560)	(5.0899)	(2.9132)	(7.2001)	(1.9723)	(1.8018)	(6.6281)	(4.7241)		
B1.3	0.0983*	0.2254***	0.3223***	-0.0220	0.0245***				0.0833***	0.0143	0.776
	(1.7612)	(7.8402)	(7.9916)	(-0.5171)	(8.7557)				(4.0198)	(0.8187)	
B1.4	0.1426***	0.1650***	0.2167***	0.1478***	0.0225***	0.0027*	0.0019**	0.0038***	0.1090***	0.0468***	0.790
	(2.6242)	(5.6638)	(4.9328)	(3.1168)	(7.5660)	(1.8518)	(1.9708)	(7.0110)	(5.2924)	(2.7052)	

Table 3 Results of S&P 500 Realized Volatility Forecasting Model (Continued)

C) 21-day Future Realized Volatility of S&P 500 Index ($RV_{t,t+21}$)

The model for 21-day S&P 500 realized volatility forecasting. Model (1.1) is the HAR-RV-IV-SK model, and (1.2) is HAR-RV-IV-CRV-SK model, which contains the change rate of VIX into the HAR-RV-IV-SK model. (1.3) adds risk neutral skewness of VIX options, i.e. the HAR-RV-IV-SK-SKV model. (1.4) is the HAR-RV-IV-CRV-SK-SKV model. The number in the parentheses is the t-statistics, and the stars behind the coefficients are the significant levels of 1%, 5% and 10%.

Model	Intercept	$RV_{t-1,t}$	$RV_{t-5,t}$	$RV_{t-21,t}$	VIX_t	$CRV_{t-1,t}$	$CRV_{t-5,t}$	$CRV_{t-21,t}$	SK_t	SKV_t	$Adj.R^2$
C1.1	0.4109***	0.1759***	0.3266***	0.1714***	0.0073*				0.0954***		0.664
	(6.9197)	(5.1404)	(6.8560)	(3.4457)	(2.2679)				(4.0464)		
C1.2	0.4691***	0.1139***	0.2921***	0.3359***	0.0016	0.0031*	0.0040***	0.0014**	0.0979***		0.673
	(7.8578)	(3.2300)	(5.5283)	(5.9656)	(0.4503)	(1.7420)	(3.5335)	(2.2266)	(4.1333)		
C1.3	0.3754***	0.1759***	0.3281***	0.1713***	0.0081**				0.1032***	0.0265	0.664
	(5.7455)	(5.1412)	(6.8864)	(3.4450)	(2.4664)				(4.2442)	(1.3043)	
C1.4	0.4083***	0.1105***	0.2847***	0.3464***	0.0030	0.0029	0.0042***	0.0017***	0.1142***	0.0499**	0.674
	(6.3171)	(3.1353)	(5.3886)	(6.1449)	(0.8501)	(1.6338)	(3.6835)	(2.6184)	(4.6482)	(2.4306)	

4.3 Results

In the S&P 500 1-day future realized volatility forecasting model (Table 3A), all the coefficients of model A1.1, HAR-RV-IV-SK model, are significant except the interception term. Comparing to the original HAR-RV-IV-SK model by Byun and Kim (2013), our research has higher significance. The differences may come from the different data period.

In model A1.2, we add the past 1-day, 5-day, 21-day VIX change rate terms in the model, and the model become a HAR-RV-IV-CRV-SK model. The adjusted R-square of model A1.2 is 0.763 that increases 1.3% from model A1.1. We attribute the increase of adjusted R-square to the information content of VIX change rate. However, the coefficient of past 21-day S&P 500 realized volatility in A1.2 increases from -0.1914 to 0.0093 and then become both positive and insignificant.

The adding of VIX change rate variables reduces the coefficient values of 1-day, 5-day RV and VIX terms, but increase the coefficient values of 21-day and S&P 500 skewness term. VIX change rate also makes a structural change of the model. This situation can also be observed in the model A1.4 and the 5-day and 21-day S&P 500 realized volatility forecasting model.

Model A1.3 shows that the VIX skewness cannot promote the explanatory power, and the VIX skewness term is insignificant. However, in Model A1.4, the VIX change rate terms increase the power of the S&P 500 realized volatility forecasting model and make the coefficient of VIX skewness becomes significant but has little impact on the explanatory power of the model. From Model 1.3 to Model 1.4, the t-statistic of both S&P500 skewness and VIX skewness term increase that means the VIX change rate can increase the significance of both variables.

For the 5-day S&P500 realized volatility forecasting, the 21-day RV term is negative and insignificant. While we add 1-day, 5-day, and 21-day VIX change rates as new variables the 21-day RV term is positive and significant. The 1-day and 5-day VIX change rate terms in Model B1.2 and B1.4 provide less information to the 5-day future S&P500 realized volatility. Also, the adding of VIX change rates increases the significance of S&P500 skewness and VIX skewness and raises the explanatory power of the model.

In the model of 21-day S&P500 realized volatility forecasting, the VIX and 1-day VIX terms become less informative in the models. Still, the adding of VIX change rate makes the forecasting model becomes more powerful and redistributes the weight of each variable in the model. The changing of the model structure increases the importance of the past 21-day realized volatility, VIX change rates, S&P500 skewness and VIX skewness terms.

Note that the 5-day S&P500 realized volatility forecasting model has the highest adjusted R-square of the models. In the data period, the HAR-RV-IV-CRV- SK-SKV model provides more accuracy to the 5-day S&P500 realized volatility, comparing to the 1-day and 21-day models. The reasons may be that 1-day S&P500 realized volatility has too much noise to make the prediction less accurate and the 21-day time horizon contains more uncertainty in the future. We conclude that the 5 trading days is a more appropriate time horizon to forecast the S&P500 realized volatility.

4.4 Residual Analysis

To know the behaviors of the new-adding variables, we do a simple residual analysis on the VIX, VIX change rate, the risk neutral skewness of S&P 500 Options and VIX options variables. Figure 5 is the scatter chart of VIX to the regression on when the VIX change rate residual in 1-day, 5-day, and 21-day in HAR-RV-IV-CRV-SK-SKV model. The residual become wider distributed as VIX increase.

Figure 6 is the scatter chart of the past 5-day VIX change rate to the error. Most of the high residuals occur in the situation of high VIX and close-to-zero VIX change rate. On the other hand, we can have a more precise prediction when the VIX change rates have previous patterns. The residuals distribution of the 1-day and 21-day historical VIX change rate are similar to the 5-day VIX change rate.

In Figure 7, we show the relationship of both the risk neutral skewness of S&P 500 options and risk neutral skewness of VIX options to the error term of the regression model. In the normal situation S&P 500 options have negative skewness, and VIX options have positive skewness.

While in the extreme events, the volatility of S&P 500 index become higher and VIX will increase. The degrees of volatility smile in the both S&P 500 options and VIX options tend to U-shapes. Therefore, the risk neutral skewness of VIX options will decrease toward zero and the risk neutral skewness of S&P 500 options increase toward zero when VIX is high. So the S&P500 realized volatility become unpredictable in the high volatile situation.

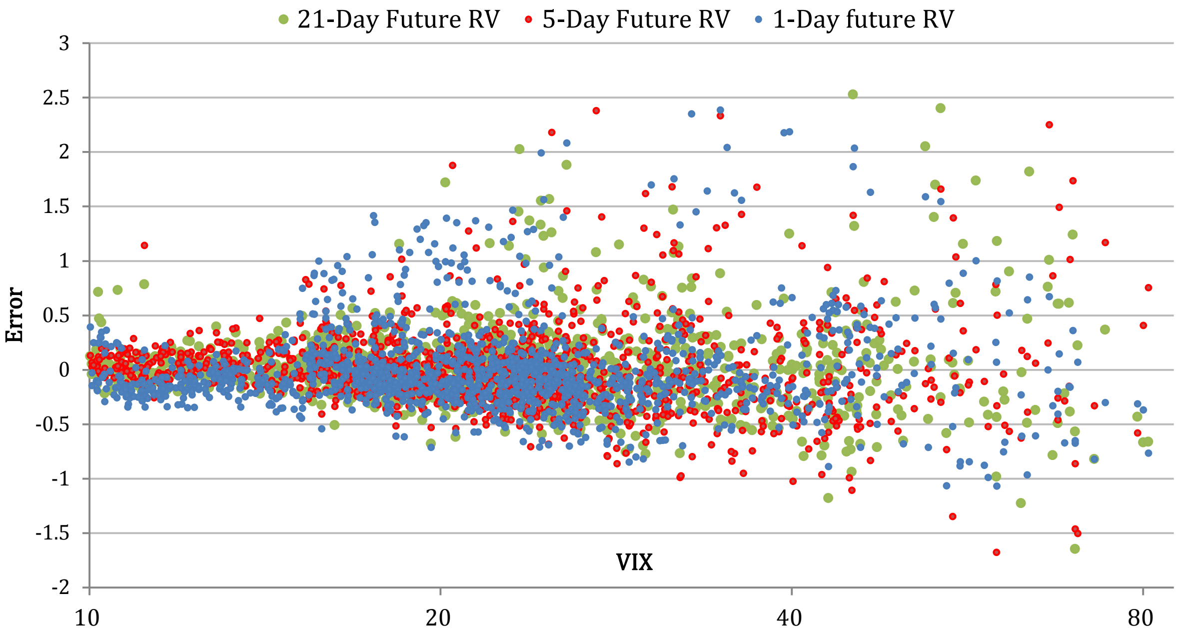


Figure 5 VIX verses the residual of model 1.4 for the 1-day, 5 -day, and 21-day S&P 500 future realized volatility. The scale of VIX (x-axis) is the exponential type. The error is small if VIX is low, and become higher as VIX increase. The errors separate widely in the range of 40 to 80.

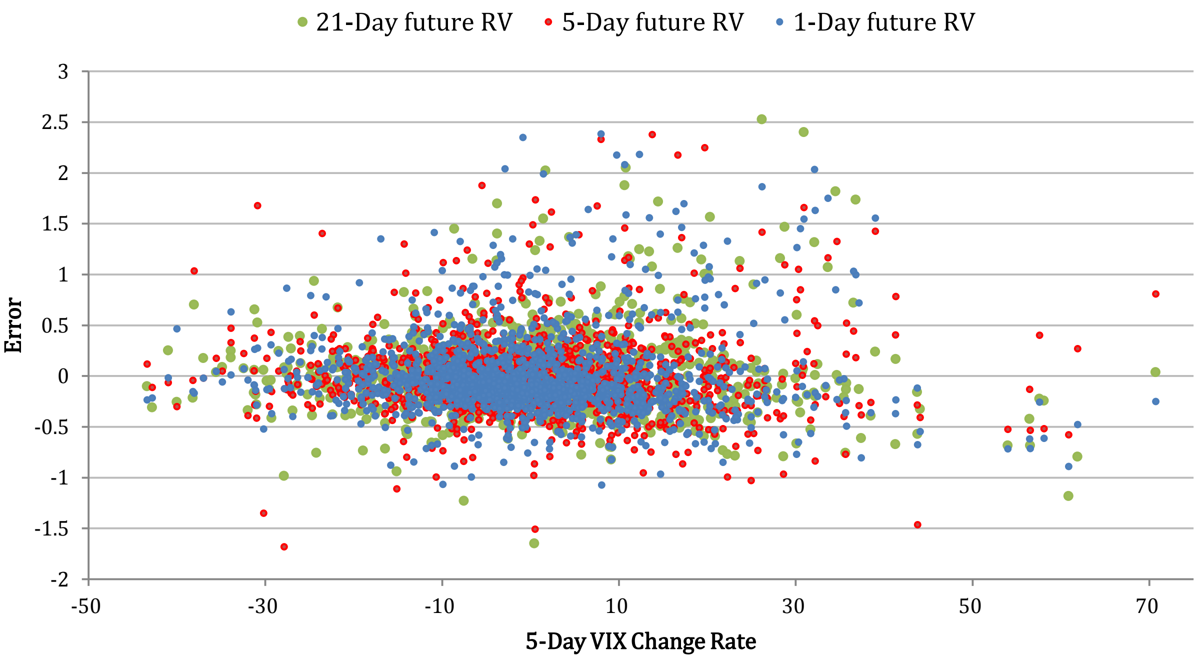
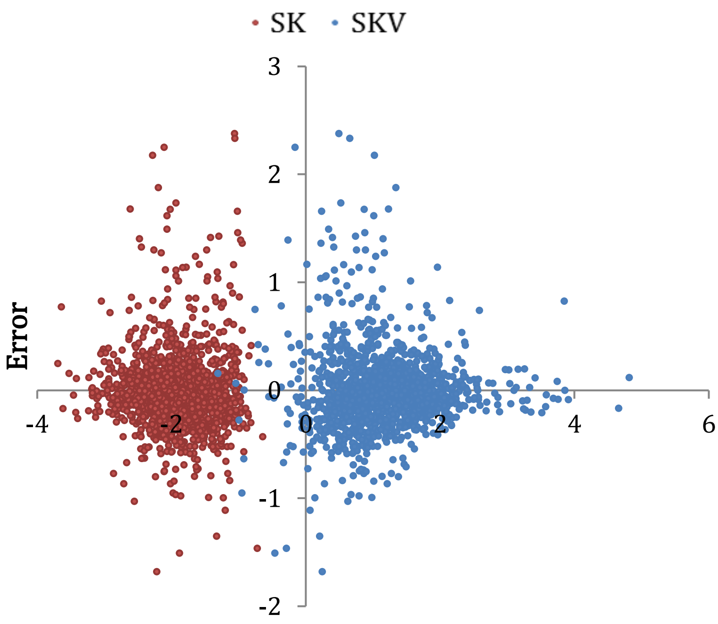
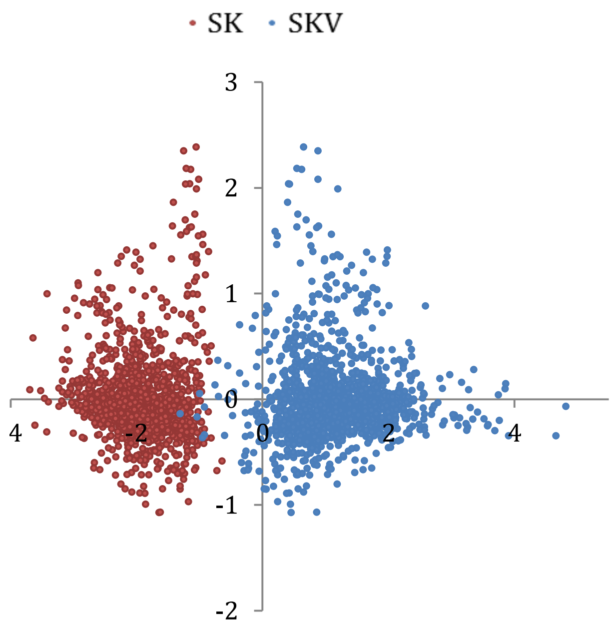


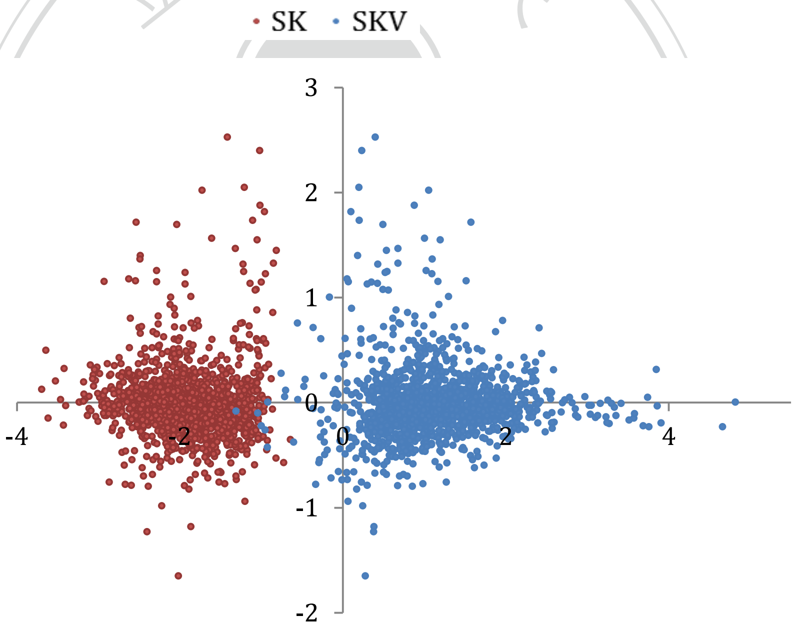
Figure 6 5-day VIX change rate verses the error of model 1.4 for the 1-, 5 -, and 21-day S&P 500 future realized volatility. Most of the residuals lie on the range of -0.5 to 0.5. When 5-day CRV is closed to zero, the variables become less informative. It is more difficult to capture the future dynamic of S&P 500 realized volatility from the VIX change rate.



a) 1-day RV forecasting model



b) 5-day RV forecasting model



c) 21-day RV forecasting model

Figure 7 The SK and SKV variables verse the regression error in the a) 1-day, b) 5-day, and c) 21-day S&P 500 realized volatility forecasting. S&P 500 options have positive skewness, and VIX options are usually positive. As the risk neutral skewness of both S&P 500 options and VIX options close to zero, the errors tent to be widely distributed.

5. VIX Forecasting

In this section, we focus on the VIX forecasting by using regime switching model in the VIX change rate prediction. Figure 7 shows the historical VIX and the 1-day, 5-day, and 21-day VIX change rate (CRVs). From the 1-day VIX change rate, we observe the volatility clustering of VIX. However, we cannot affirm any information but its amplitude in the graph of 1-day VIX change. In the 5-day and 21-day change rate of VIX, we can clearly see the change direction and the accumulated amplitude. Hence, we use the 5-day and 21- day VIX change rate to be our dependent variables in the regression.

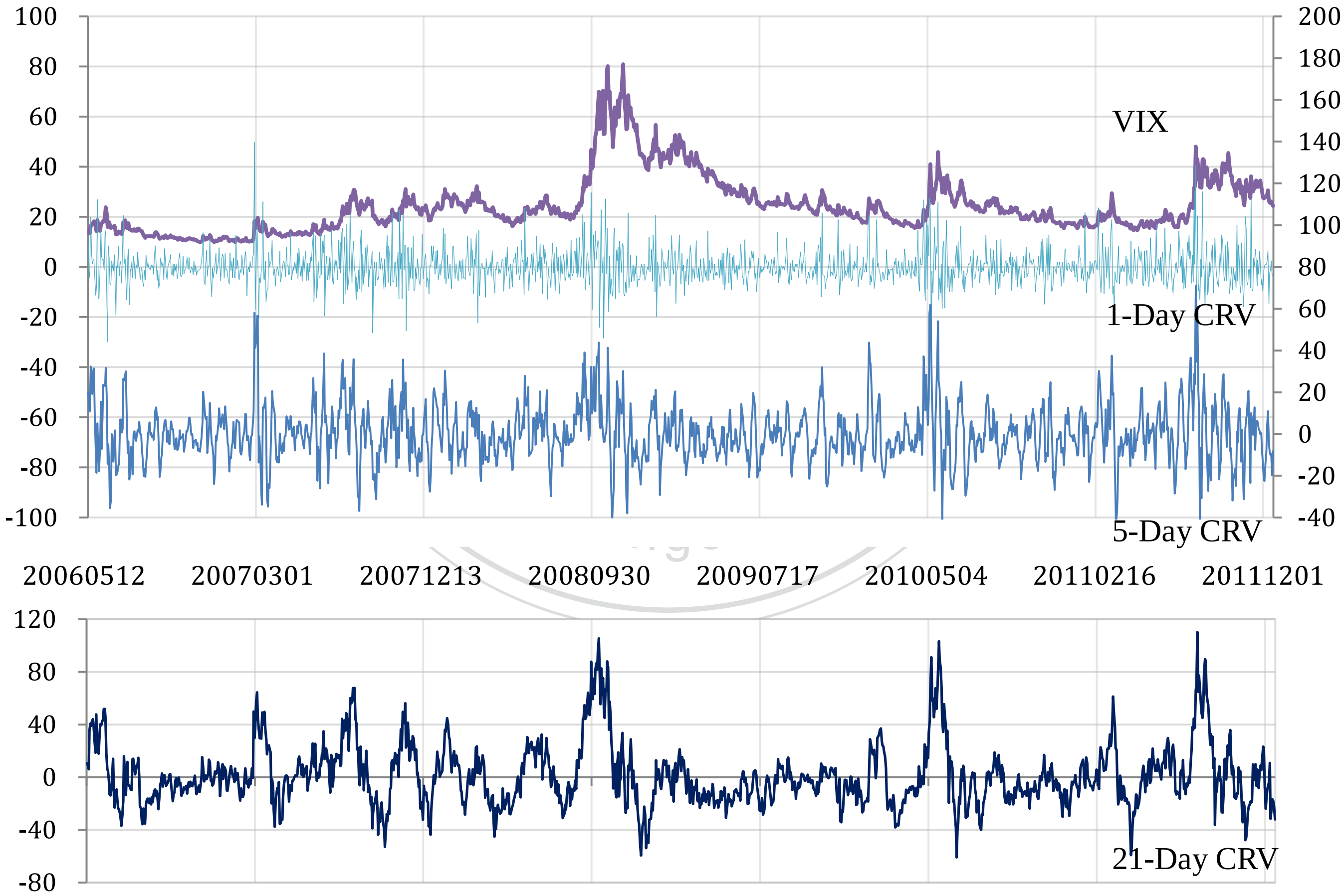


Figure 8 The relationship within VIX, 1-day, 5-day, and 21-day historical VIX change rate. The formula of VIX change rate is $CRV_{t-h,t} = \ln\left(VIX_t / VIX_{t\text{-}h}\right)$. As the period of the change rate variable gets longer, the pattern of accumulative change of VIX become more apparent.

5.1 Single Variable Regression Testing

For the independent variables, we test the past 1-day, 5-day, 21-day VIX change rate (CRV), risk neutral momentum (including volatility, skewness, and kurtosis) of VIX options (VOLV, SKV, KURV), and VIX of VIX (VVIX) in the VIX forecasting model. Table 4 is the result of single-variable linear regression of the VIX change rate forecasting model.

In the single-variable regression, all the independent variables have low explanatory power (adjusted $R^2 < 8\%$). However, if we use the two-state regime switching model to forecast the 5-day or 21-day futures VIX change rate, we can increase the power of these variables. We have different coefficients in the different states. This property will help us to capture the dynamic of VIX.

Table 4 Single Variable Regression for VIX Change Rate Forecasting Model

The single-variable regression of the future 5-day and 21-day VIX change rate with each independent variable. The regression function is $CRV_{t,t+h} = \alpha + \beta x_t + \varepsilon_{t,t+h}$.

$CRV_{t,t+h}$		$CRV_{t-1,t}$	$CRV_{t-5,t}$	$CRV_{t-21,t}$	$VOLV_t$	SKV_t	$KURV_t$	$VVIX_t$
	α	0.3953	0.4646	0.4613	9.9189***	-3.4180***	-1.4849**	12.2485***
h = 5	β	-0.2891***	-0.2404***	-0.0514	-10.2763***	3.5099***	0.2494***	-0.1380***
	Adj.R^2	0.0237	0.0569	0.00738	0.0226	0.0273	0.00583	0.0166
	α	1.7107	1.7541	1.8880***	24.8897***	-9.3314***	-3.7705***	37.1807***
h = 21	β	-0.2984***	-0.1609***	-0.1131***	-25.0186***	10.1982***	0.7304***	-0.4136***
	Adj.R^2	0.00758	0.00758	0.0121	0.0442	0.0769	0.0178	0.0491

5.2 Regime Switching Model[3]

The regime switching model is given by a structure of different regimes, we have:

$$y_t = \beta x_t + \varepsilon_t,\ \varepsilon_t \sim N(0, \sigma_{St}^2)$$

where under regime 1, parameters are given by β_1 and σ_1^2, and under regime 2, parameters are given by β_2 and σ_2^2. In this case, the likelihood function is given by:

$$\ln(L) = \sum_{t=1}^{T} \ln(f(y_t \mid S_t))$$

$$\text{where}\quad f(y_t \mid S_t) = \frac{1}{\sqrt{2\pi\sigma_{St}^2}} \exp\left(-\frac{(y_t - \beta_t x_t)^2}{2\sigma_{St}^2}\right)$$

We use the following two steps to determine the log likelihood function:

First, consider the joint density of y_t and the unobserved S_t variable, which is the product of the conditional and marginal densities:

$$f(y_t, S_t \mid F_{t-1}) = f(y_t \mid S_t, F_{t-1}) f(S_t \mid F_{t-1})$$

where F_{t-1} refers to information up to time $t-1$.

Second, to obtain the marginal density of y_t, integrate the S_t variable out of the above joint density by summing over all possible values of S_t:

$$f(y_t \mid F_{t-1}) = \sum_{S_t=0}^{1} f(y_t, S_t \mid F_{t-1}) = \sum_{S_t=0}^{1} f(y_t \mid S_t, F_{t-1}) f(S_t \mid F_{t-1})$$

$$= \frac{1}{\sqrt{2\pi\sigma_1^2}} \exp\left(-\frac{(y_t - \beta_1 x_1)^2}{2\sigma_1^2}\right) \times P(S_t = 0 \mid F_{t-1})$$

$$+ \frac{1}{\sqrt{2\pi\sigma_2^2}} \exp\left(-\frac{(y_t - \beta_2 x_t)^2}{2\sigma_2^2}\right) \times P(S_t = 1 \mid F_{t-1})$$

The log likelihood function is given by

$$\ln(L) = \sum_{t=1}^{T} \ln\left(\sum_{St=0}^{1} f(y_t \mid S_t, F_{t-1}) \times P(S_t \mid F_{t-1})\right)$$

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[3] The theorem comes from the book by Kim and Nelson, 1999. State-Space Models with Regime Switching: Classical and Gibbs-Sampling Approaches with Applications. Page 59-95.

5.3 VIX Forecasting Models

In the 5-day and 21- day VIX forecasting model, we separate the time series into regime 1 and regime 2. By adopting the regime switching model, all of the variables in the regression will have different coefficient in each state. Note that distinguish of two regimes differs from the different input and output variables.

First, in model 2.1, we do a previous observation that if we can separate the time line into two regimes: the normal volatility regime (Regime 1) and the extreme volatility regime (Regime 2).Then we add the past 1-day, 5-day, and 21-day VIX change rates (CRVs) in model 2.2. The model becomes a Regime Switching Heterogeneous Autoregressive model of VIX Change Rate (RS-HAR-CRV model).

The informations of the VIX derivatives are important to VIX forecasting. Under the regime switching model, we use the risk neutral volatility, skewness, of VIX options (VOLV, SKV, KURV) variables in the model 2.3. Also, in model 2.4, we use the volatility of volatility index (VVIX) to predict the future VIX change rate.

Next we show the synergy of the all variables in model 2.5 by adding all the variables above into the regime switching model. Finally, we reduce the independent variables to VIX change rate (CRV), risk neutral skewness of VIX options (SKV) and VVIX in model 2.6.

The regression formulas of the models are as below. Each model has two groups of the beta coefficients and the different coefficient of the standard deviation in the residual term in the different regime. The results of 5-day VIX forecasting model is in Table 5 and the results of 21-day VIX forecasting model is in Table 6.

Model 2.1 Simple Regime Switching Model (on VIX Change Rate)

$$CRV_{t,t+h} = \beta_{0,St} + \varepsilon_{t,t+h},\ \varepsilon_{t,t+h} \sim N(0,\sigma_{St}^2),\ \mathrm{S}_t = 1,\ 2\ ;\ h = 5,\ 21$$

Model 2.2 RS-HAR-CRV Model

$$CRV_{t,t+h} = \beta_{0,St} + \begin{bmatrix} \beta_{1,St} & \beta_{2,St} & \beta_{3,St} \end{bmatrix} \begin{bmatrix} CRV_{t-1,t} \\ CRV_{t-5,t} \\ CRV_{t-21,t} \end{bmatrix} + \varepsilon_{t,t+h},\ \varepsilon_{t,t+h} \sim N(0,\sigma_{St}^2),\ \mathrm{S}_t = 1,\ 2\ ;\ h = 5,\ 21$$

Model 2.3 RS-VOLV-SKV-KURV Model

$$CRV_{t,t+h} = \beta_{0,St} + \begin{bmatrix} \beta_{1,St} & \beta_{2,St} & \beta_{3,St} \end{bmatrix} \begin{bmatrix} VOLV_t \\ SKV_t \\ KURV_t \end{bmatrix} + \varepsilon_{t,t+h},\ \varepsilon_{t,t+h} \sim N(0,\sigma_{St}^2),\ \mathrm{S}_t = 1,\ 2\ ;\ h = 5,\ 21$$

Model 2.4 RS-VVIX Model

$$CRV_{t,t+h} = \beta_{0,St} + \beta_{1,St} VVIX_t + \varepsilon_{t,t+h},\ \varepsilon_{t,t+h} \sim N(0,\sigma_{St}^2),\ \mathrm{S}_t = 1,\ 2\ ;\ h = 5,\ 21$$

Model 2.5 RS--HAR-CRV-VOLV-SKV-KURV-VVIX Model

$$CRV_{t,t+h} = \beta_{0,St} + \begin{bmatrix} \beta_{1,St} & \beta_{2,St} & \beta_{3,St} \end{bmatrix} \begin{bmatrix} CRV_{t-1,t} \\ CRV_{t-5,t} \\ CRV_{t-21,t} \end{bmatrix} + \begin{bmatrix} \beta_{4,St} & \beta_{5,St} & \beta_{6,St} \end{bmatrix} \begin{bmatrix} VOLV_t \\ SKV_t \\ KURV_t \end{bmatrix} + \beta_{7,St} VVIX_t + \varepsilon_{t,t+h},\ \varepsilon_{t,t+h} \sim N(0, \sigma_{St}^2),\ S_t = 1,\ 2\ ;\ h = 5,\ 21$$

Model 2.6 RS--HAR-CRV -SKV -VVIX Model

$$CRV_{t,t+h} = \beta_{0,St} + \begin{bmatrix} \beta_{1,St} & \beta_{2,St} & \beta_{3,St} \end{bmatrix} \begin{bmatrix} CRV_{t-1,t} \\ CRV_{t-5,t} \\ CRV_{t-21,t} \end{bmatrix} + \beta_{4,St} SKV_t + \beta_{5,St} VVIX_t + \varepsilon_{t,t+h},\ \varepsilon_{t,t+h} \sim N(0, \sigma_{St}^2),\ S_t = 1,\ 2\ ;\ h = 5,\ 21$$

Table 5 Regime Switching Model for the 5-day VIX Change Rate Forecasting

The regime switching model for the 5-day VIX change rate forecasting. The number in the parentheses is the t-statistics, and the stars behind the coefficients are the significant levels of 1%, 5% and 10%. The transition probabilities are the probabilities that the regime change form i to j (i, j = 1, 2).

											Transition Probabilities		
Model		Intercept	$CRV_{t-1,t}$	$CRV_{t-5,t}$	$CRV_{t-21,t}$	$VOLV_t$	SKV_t	$KURV_t$	$VVIX_t$	Sigma	To Regime 1	Regime 2	Adj.R^2
A2.1	Regime 1	-2.9463***								7.7520***	0.95***	0.05	0.0422
		(-6.6090)								(3.4572)			
	Regime 2	5.0022***								18.7264***	0.07***	0.93	
		(5.0794)								(3.5509)			
A2.2	Regime 1	-2.0773***	-0.3006***	-0.3543***	0.101***					7.8059***	0.94***	0.06	0.1531
		(-2.8815)	(-5.2461)	(-8.9244)	(2.8134)					(3.1133)			
	Regime 2	4.1257**	0.3138**	-0.1025	-0.1404***					19.4833***	0.10***	0.90	
		(2.3844)	(2.1361)	(-1.1478)	(-3.2500)					(3.0879)			
A2.3	Regime 1	10.5673***				-23.0932***	1.8873***	0.2482***		8.1147***	0.93***	0.07	0.3520
		(5.5474)				(-13.1003)	(3.3981)	(3.6023)		(4.1466)			
	Regime 2	-1.6675				2.1469	7.5887***	0.7976***		11.7683***	0.13***	0.87	
		(-0.4491)				(0.7564)	(5.2318)	(3.2122)		(3.5860)			
A2.4	Regime 1	29.0157***							-0.4273***	7.9822***	0.92***	0.08	0.3410
		(12.5046)							(-15.2607)	(3.3316)			
	Regime 2	30.0116***							-0.2014***	12.2240***	0.13***	0.87	
		(5.2566)							(-3.5709)	(3.7066)			
A2.5	Regime 1	11.1525***	-0.1436***	-0.2418***	-0.0414**	-14.4967***	1.6968***	0.0832	-0.0928*	7.1229***	0.94***	0.06	0.4651
		(3.9680)	(-2.8893)	(-6.9683)	(-2.0909)	(-4.3678)	(3.1847)	(1.2308)	(-1.8232)	(3.8905)			
	Regime 2	9.0898*	-0.2827***	-0.2239***	0.0645**	1.8799	6.0646***	0.1275	-0.0748	12.0345***	0.09***	0.91	
		(1.7296)	(-3.9986)	(-4.2811)	(2.3455)	(0.2160)	(3.9742)	(0.5038)	(-0.6027)	(3.7336)			
A2.6	Regime 1	3.2268	-0.1502***	-0.2661***	-0.0793***		2.4308***		-0.1548***	7.2467***	0.94***	0.06	0.4673
		(1.1473)	(-2.9980)	(-7.2114)	(-3.8683)		(5.3225)		(-4.7052)	(3.7675)			
	Regime 2	24.2091***	-0.2119***	-0.2498***	0.0831***		4.3016***		-0.1907***	11.9934***	0.10***	0.90	
		(4.9420)	(-3.0057)	(-4.8038)	(3.0440)		(3.5161)		(-3.8448)	(3.7521)			

Table 6   Regime Switching Model for the 21-day VIX Change Rate Forecasting

The regime switching model for the 21-day VIX change rate forecasting. The number in the parentheses is the t-statistics, and the stars behind the coefficients are the significant levels of 1%, 5% and 10%. The transition probabilities are the probabilities that the regime change form i to j (i, j = 1, 2).

											Transition Probabilities		
Model		Intercept	$CRV_{t-1,t}$	$CRV_{t-5,t}$	$CRV_{t-21,t}$	$VOLV_t$	SKV_t	$KURV_t$	$VVIX_t$	Sigma	To Regime 1	Regime 2	Adj.R^2
B2.1	Regime 1	-8.2142***								14.1073***	0.99***	0.01	0.5237
		(-13.7799)								(4.3609)			
	Regime 2	34.8365***								22.9001***	0.05***	0.95	
		(17.7402)								(3.3837)			
B2.2	Regime 1	-9.8859***	-0.1112	-0.3172***	-0.2175***					11.6646***	0.98***	0.02	0.4436
		(-20.2621)	(1.6210)	(8.0508)	(9.7098)					(4.1347)			
	Regime 2	23.5203***	-0.4914**	0.8957***	-0.3233***					27.5125***	0.03***	0.97	
		(13.1186)	(2.6956)	(5.6906)	(5.0437)					(3.0284)			
B2.3	Regime 1	14.5879***				-30.6915***	7.5713***	-0.4276***		13.2329***	0.98***	0.02	0.5825
		(4.3970)				(-9.0758)	(8.3000)	(-3.4963)		(3.9221)			
	Regime 2	72.0552***				-46.8926***	2.7645	0.0632		21.2359***	0.05***	0.95	
		(8.7879)				(-8.0913)	(1.0446)	(0.2576)		(3.4901)			
B2.4	Regime 1	28.4374***							-0.4700***	12.2089***	0.98***	0.02	0.5682
		(6.9334)							(-9.1797)	(3.9465)			
	Regime 2	104.0067***							-0.9147***	21.0203***	0.04***	0.96	
		(14.1415)							(-11.0338)	(3.8271)			
B2.5	Regime 1	-6.5782	-0.1940***	-0.0535	-0.1172***	10.8242**	6.0724***	0.2030*	-0.2329***	12.5265***	0.98***	0.02	0.5577
		(-1.4416)	(-3.2826)	(1.2442)	(-6.1684)	(2.4963)	(5.7531)	(1.8174)	(-3.1774)	(5.2174)			
	Regime 2	-10.6672**	-0.8905***	0.0292	-0.1220	-54.7147***	5.9532	0.2157	0.9991***	24.3090***	0.05***	0.95	
		(-2.3895)	(-9.9720)	(0.3074)	(-1.4170)	(-3.9697)	(1.5867)	(0.4431)	(5.0562)	(3.6674)			
B2.6	Regime 1	-2.9040	-0.0958	-0.3389***	-0.1317***		5.0355***		-0.1423***	11.9434***	0.98***	0.02	0.5930
		(-0.7581)	(-1.3666)	(-8.1860)	(-5.2262)		(6.2701)		(3.2195)	(4.2267)			
	Regime 2	83.8596***	0.0280	-0.0775	-0.0872		0.7063		-0.6361***	22.2546***	0.04***	0.96	
		(6.6294)	(0.1649)	(-0.7092)	(-1.1489)		(0.3402)		(4.5763)	(6.3587)			

5.4 VIX Forecasting Results

In the 5-day VIX change rate forecasting, model A2.1 simply separate the data into regime 1 and regime 2. In the regime 1 of model A2.1, the 5-day VIX change rate follows a normal distribution with negative mean ($\mu_1 = -2.9463$) and relatively lower variance ($\sigma_1 = 7.7520$). This means that VIX tend to be stable and slowly decrease in the regime 1. In the regime 2, VIX increase ($\mu_2 = 5.0022$) and become more volatile ($\sigma_2 = 18.7264$), but the variance is too large to make sure the change direction of VIX.

Model A2.2 is the MS-HAR-CRV model. Although the CRV can increase the adjusted R-square, it does not have so many informations about the future dynamic of VIX if we only use the past VIX change rates as our variables. However, in model A2.3 we predict the 5-day VIX change rate by using the risk neutral moments of VIX options. The result shows that SKV and KURV terms are significant in both regimes.

Also, in Model A2.4, VVIX can provide more information than VIX change rate. Hence the risk neutral moments of VIX options and VVIX can improve the predict power of the VIX change rate forecasting under the two-state regime switching model. After we join the risk neutral moments of VIX options and VVIX, the variance of the regime 2 become smaller. So the definite separation of two regimes via adding effective variables can make higher accuracy in the forecasting of VIX change rate.

When we combine those variables above into one model, the explanatory power is increase. Model A2.5 ha adjusted R-square of 0.4651, which is about 10% higher than model A2.3 and A2.4, but some of the variables do not present well. The VOLV and VVIX variables are useful in the regime 1, while they are insignificant in the regime 2. KURV term is insignificant in the both regime. We stay the VVIX term in order to capture the volatility of VIX and VIX change rate.

In model A2.6, we use 1-, 5-, 21-day past VIX change rate (CRVs), risk neutral skewness of VIX options (SKV), and VVIX in the regime switching model to forecast the 5-day forward VIX change rate. All variables in the model are significant in two regimes, and we can see the obviously differences of the variables to the 5-day future VIX change rate.

Comparing with model A2.5, model A2.6 has more significant variables, and more simple. The coefficients of the same terms in the regime 1 and regime 2 imply the different impact of the variables to the 5-day future VIX change rate. The 21-day past VIX change rate term is a special case in model 2.6: In the regime 1, it has negative sign, which means the mean-reverting property of VIX. While in the regime 2, it become significantly positive. Just like the volatility clustering property of VIX, the higher VIX change rate will make more higher VIX in the future in the extreme situation.

For the 21-day future VIX change rate model, the differences between the two regimes are more obvious. Model B2.1 has 0.5237 adjusted R-square. So we can easily separate the two regimes by just give them the conditional mean and variance in each regime. Hence, most of the explanatory power of the 21-day VIX change rate forecasting model is contributed by the regime switching model.

In model B2.2, the 5-day and 21-day CRV terms are significant, but the addition of CRV decrease the distance between the means two regimes. So the effect of regime separation becomes weaker, and we have lower explanatory power. In model B2.3 and B2.4, the VOLV and VVIX variables have strong significance in the two regimes. However, most of the variables are only significant in regime 1, the normal situation. This situation presents an important fact that VVIX and the risk neutral volatility of VIX options play critical rules in the 21-day VIX forecasting.

Model B2.5 and B2.6 show the same result that VOLV and VVIX terms are the deterministic variables in regime 2. If we look back to the 21-day VIX change rate, risk neutral volatility of VIX options and VVIX, we can find that they have similar pattern. Since VOLV and VVIX are more sensitive to the change of VIX, these variables will sharply increase because a shock occurs. Therefore, the 21-day VIX change rate will be affected in the same time. After the shock, VOLV and VVIX become the important indicators of the recovering rate. The negative coefficients of VOLV and VVIX in the regime 2 in model B2.5 and B2.6 implies the strong mean reverting property of 21-day VIX change rate under the extreme regime.

5.5 Model Analysis

Figure 8 shows the results of the predicted VIX change rate, real VIX change rate, and the regimes of the 5-day and 21-day VIX change rate forecasting. Since the predicted value is constrained by the two regimes, the fitting of extreme changes of VIX are not so perfect. We cannot know how much percentage VIX will increase in the 5 or 21 future day. Nevertheless, we capture the shock when the regime goes from regime 1 to regime 2, and the change rate turn back to normal as the regime goes back to regime 1.

From the scatter graph of 5-day VIX change rate in Figure 9, the two regimes separate the whole data into the low change rate regime (regime1) and the high change rate regime (regime2). The regime separation of the 21-day model becomes much wider than the 5-day model. Regime switching model does not fit well if we look to the situation of high VIX change rate. Even if the prediction is not very precise, we still predict the overall direction of the VIX change rate in the longer term.

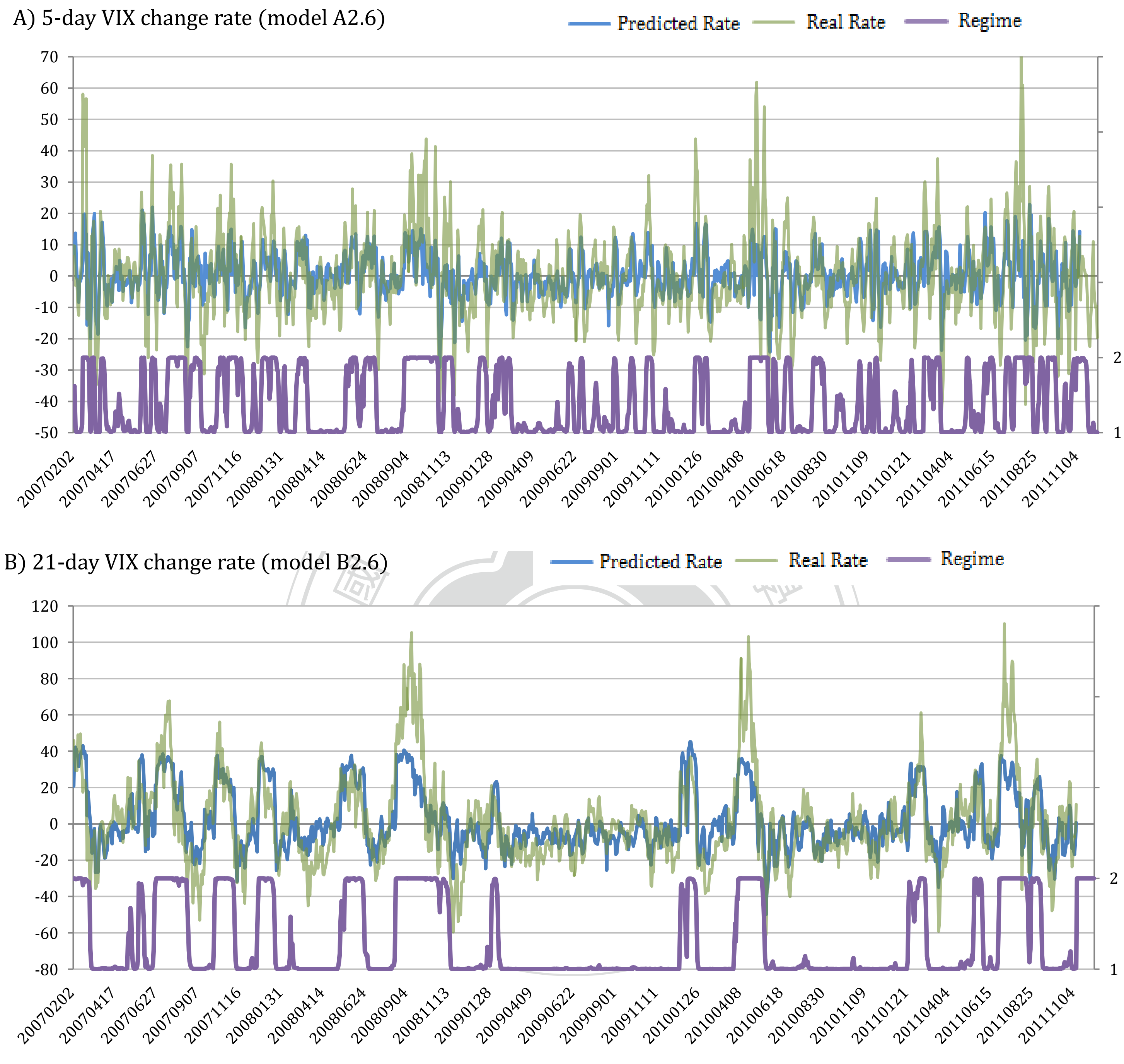
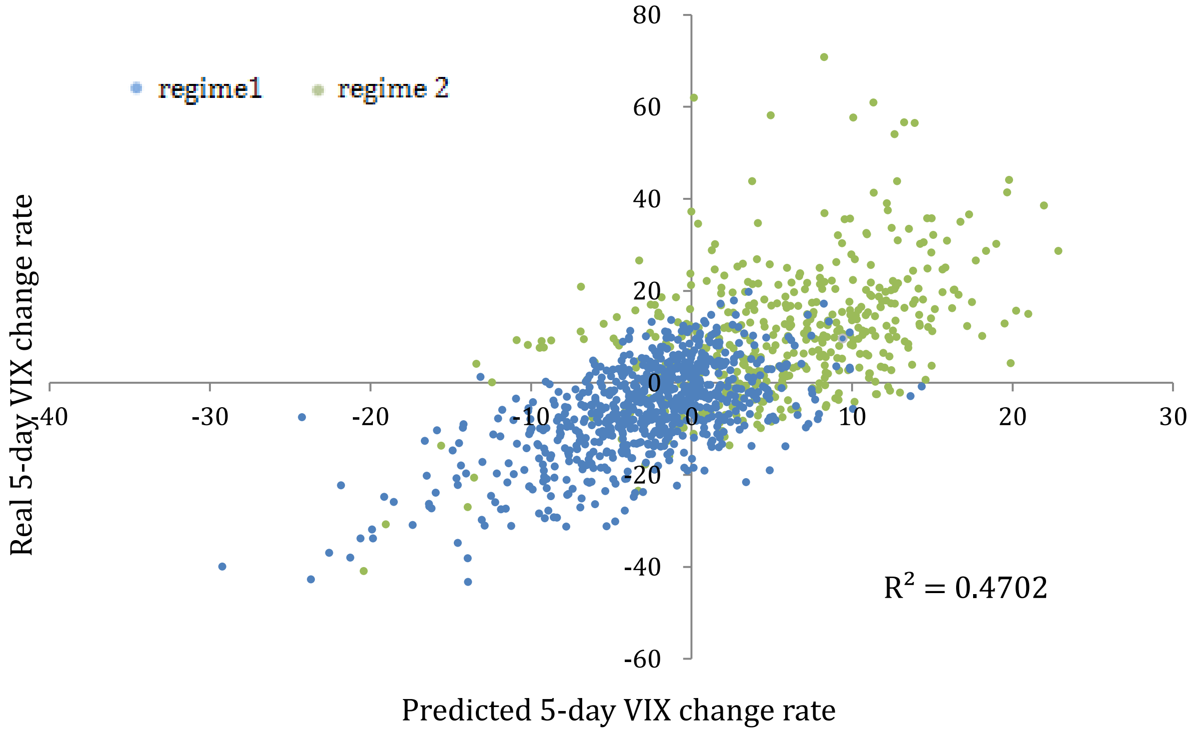


Figure 9 The regime switching model for the A) 5-day VIX change rate and B) 21-day VIX change rate in model 2.6 with the corresponding data time. The line below the real VIX change rate and predicted VIX change rate (from filtered probability) is the changing of regime. The data is from Feb. 2, 2007 to Dec. 8, 2011.



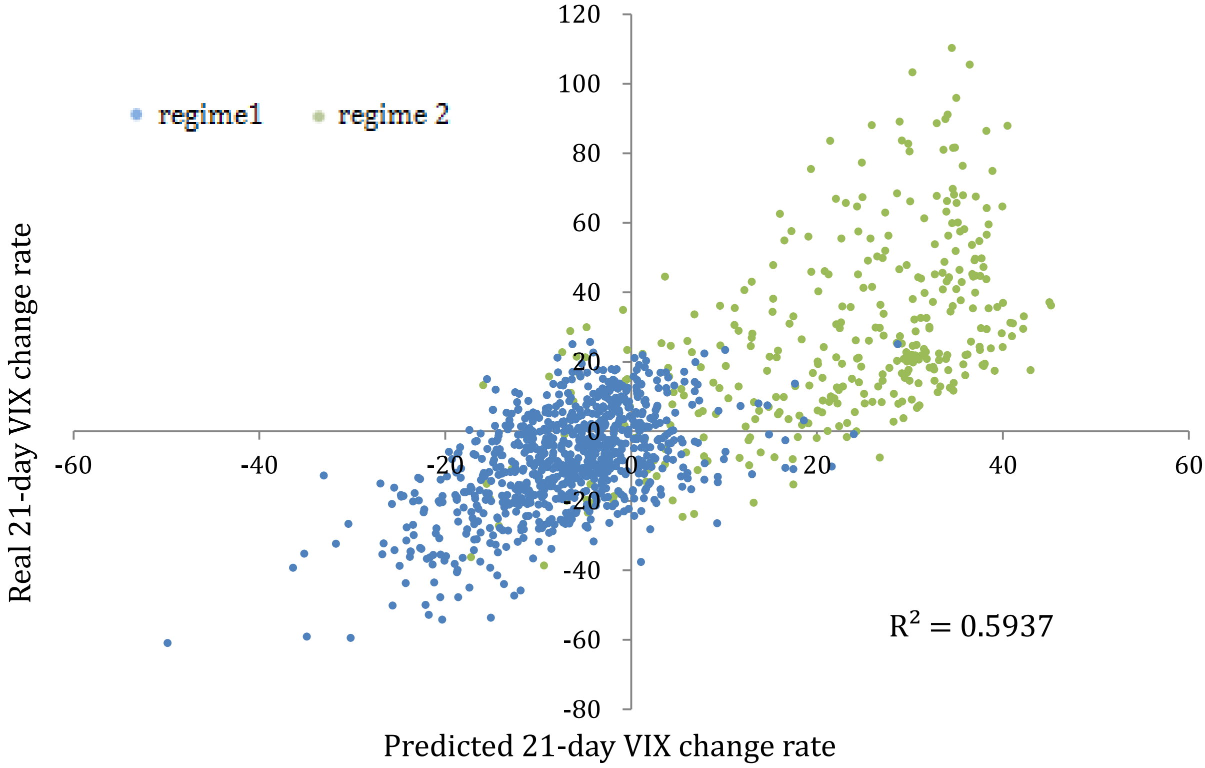


Figure 10 The scatter graph of the predicted value and real value of the A) 5-day VIX change rate and B) 21-day VIX change rate in model 2.6. The data near by the original point of each graph is in regime 1, and the data with high VIX change rate (real and predicted) is in regime 2

6. Conclusion

In the first part of this paper, we add the VIX change rate and risk neutral skewness of VIX options into the Heterogeneous Autoregressive model of Realized Volatility, Implied Volatility and Skewness (HAR-RV-IV-SK) model to capture the future dynamic of S&P 500 index realized volatility.

The result shows that the past 1-day, 5-day, and 21-day VIX change rate have information content to the future realized volatility of S&P index. The VIX change rate variables also increase the explanatory power of the model. In the same model, the risk neutral skewness of VIX options is significant when join with the VIX change rate, but provides less information about the future volatility of S&P 500 index.

From the residual analysis of VIX and 5-day VIX change rate, we find that the model has better predict power when VIX is low and the past VIX change rates are away from zero. The risk neutral skewness of S&P 500 options and VIX options are more useful when SK is strong negative and SKV is strong positive.

In the second model for VIX forecasting, we use the regime switching model to the capture dynamic of future 5-day and 21-day VIX. The regime switching model makes the forecasting model become more useful and flexible. Furthermore, we join the past VIX change rate, risk neutral volatility, skewness, kurtosis of VIX options, and VVIX variables in the VIX forecasting. The result indicates that risk neutral skewness of VIX options and VVIX is useful to forecast the 5-day future VIX change rate in the both regimes.

While in the 21-day VIX forecasting, the regime switching model can be more powerful, because the differences between the two regimes become wider. VVIX still contain strong information about the future 21-day VIX, while risk neutral skewness of VIX options contain less information relative to VVIX in the 21-day VIX forecasting model. In extreme situation, it is difficult to know how much VIX will increase. Nevertheless, our model provides a direct tool to forecast the future change rate of VIX.

This paper finds the properties of the VIX change rate, risk neutral moments of VIX options and VVIX variables. Also, we point out the information contents of these variables on the dynamics of S&P 500 realized volatility and VIX change rate. We can apply the S&P 500 realized volatility and VIX forecasting model in the detection of the abnormal change of S&P 500 index, VIX, or financial crisis.

Still, there are some details we need to think more in this paper. The first consideration is about the calculation of the risk neutral moments of VIX options. We take the 5-day to 21-day time-to-maturity VIX options to evaluate the risk neutral moments of VIX options variables in this paper. But the results may vary from the different expiration date of VIX options contracts.

Second, in the VIX forecasting model, the structure of regime switching can provide a preliminary observation to the dynamic of VIX. Nevertheless, the regime switching model is not so robust and reasonable enough. We need time to verify its accuracy and sustainability.

For the further research, we can apply the model into the different index, like the Dow Jones Industrial Average index (DJIA) and its volatility index (VXD). Also, we can use different kinds of model and take more useful variables to capture the dynamics of volatility index.

Appendix I – The formula of VIX

$$VIX = \sqrt{\frac{2}{T}\sum_i \frac{\Delta K_i}{K_i^2} e^{RT} Q(K_i) - \frac{1}{T}\left[\frac{F}{K_0} - 1\right]^2} \times 100$$

where

T: Time to expiration

F: Forward index level desired from index option prices

K_0: First strike below the forward index level F

K_i: Strike price of the ith out-of-the-money option; a call if $K_i > K_0$; and a put if $K_i < K_0$; both put and call if $K_i = K_0$

ΔK_i: Interval between strike prices - half the difference between the strike on either side of K_i

(*Note*: ΔK for the lowest strike is simply the difference between the lowest strike and the next higher strike. Likewise, ΔK for the highest strike is the difference between the highest strike and the next lower strike.)

R: Risk-free interest rate to expiration

$Q(K_i)$: The midpoint of the bid-ask spread for each option with strike K_i

Appendix II – The formula of risk neutral moments of VIX options

$$Skew(t,\tau) = \frac{e^{rt}W(t,\tau) - 3\mu(t,\tau)e^{rt}V(t,\tau) + 2\mu(t,\tau)^3}{[e^{rt}V(t,\tau) - \mu(t,\tau)^2]^{3/2}}$$

$$Kurt(t,\tau) = \frac{e^{rt}X(t,\tau) - 4\mu(t,\tau)e^{rt}W(t,\tau) + 6e^{rt}\mu(t,\tau)^2V(t,\tau) - 3\mu(t,\tau)^4}{[e^{rt}V(t,\tau) - \mu(t,\tau)^2]^2}$$

where

$$V(t,\tau) = \int_{S(t)}^{\infty} \frac{2\left(1 - ln\left[\frac{K}{S(t)}\right]\right)}{K^2} C(t,\tau;K)dK + \int_0^{S(t)} \frac{2\left(1 + ln\left[\frac{S(t)}{K}\right]\right)}{K^2} P(t,\tau;K)dK$$

$$W(t,\tau) = \int_{S(t)}^{\infty} \frac{6\,ln\left[\frac{K}{S(t)}\right] - 3\left(ln\left[\frac{K}{S(t)}\right]\right)^2}{K^2} C(t,\tau;K)dK$$

$$- \int_0^{S(t)} \frac{6\,ln\left[\frac{K}{S(t)}\right] + 3\left(ln\left[\frac{K}{S(t)}\right]\right)^2}{K^2} P(t,\tau;K)dK$$

$$X(t,\tau) = \int_{S(t)}^{\infty} \frac{12\left(ln\left[\frac{K}{S(t)}\right]\right)^2 - 4\left(ln\left[\frac{K}{S(t)}\right]\right)^3}{K^2} C(t,\tau;K)dK$$

$$+ \int_0^{S(t)} \frac{12\left(ln\left[\frac{K}{S(t)}\right]\right)^2 + 4\left(ln\left[\frac{K}{S(t)}\right]\right)^3}{K^2} P(t,\tau;K)dK$$

$$\mu(t,\tau) = e^{rt}[1 - e^{-rt} - \frac{1}{2}V(t,\tau) - \frac{1}{6}W(t,\tau) - \frac{1}{24}X(t,\tau)]$$

And we set the risk neutral volatility as:

$$Vol(t,\tau) = \frac{1}{[tV(t,\tau)]^{1/2}}$$

$C(t,\tau;K)$ is the market prices of call and $P(t,\tau;K)$ is the market prices of put options with strike price K, maturity τ from time t.

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