EQUITY SWAPS IN A LIBOR MARKET MODEL

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This study extends the BGM (A. Brace, D. Gatarek, & M. Musiela, 1997) interest rate model (the London Interbank Offered Rate [LIBOR] market model) by incorporating the stock price dynamics under the martingale measure. As compared with traditional interest rate models, the extended BGM model is both appropriate for pricing equity swaps and easy to calibrate. The general framework for pricing equity swaps is proposed and applied to the pricing of floating-for-equity swaps with either constant or variable notional principals. The calibration procedure and the practical implementation are also discussed. © 2007 Wiley Periodicals, Inc. Jrl Fut Mark 27:893–920, 2007

INTRODUCTION

An equity swap is an agreement that designates two counterparties to periodically exchange two payment streams over a prespecified period. One party promises to pay the return on an agreed stock market index on an agreed notional principal, while the other promises to pay an agreed fixed rate, a floating rate, or the return of another equity index on the

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same notional principal. The equity swap introduced by Bankers Trust in 1989, has continued to grow rapidly since its inception. Equity swaps have integrated money and equity markets in world financial markets. For example, equity swaps enable fund managers to transform their exposure from some stock to interest rate payments, without the need to buy or sell the stocks. Moreover, equity swaps can be used to engage in regulatory and tax arbitrage (such as reducing or avoiding transfer taxes, withholding taxes on dividends, capital gains taxes, etc.).

There are two classes of equity swaps; one with constant notional principal and the other with variable notional principal. Equity swaps with variable notional principal are used to create a self-financing stock index investment. For example, a portfolio manager who wants to obtain a portfolio's return that precisely matches a stock index's return may adopt this structure. Equity swaps with fixed notional principal are employed to simulate a stock market investment in which the investment principal is kept constant.

This study extends the BGM (Brace, Gatarek, & Musielan, 1997) model by incorporating stock price dynmics under the martingale measure, and then applies the resulting model to pricing floating-for-equity swaps with either constant or variable notional principal. Although the swaps with a constant notional principal can be priced in closed forms, those with a variable notional principal can be priced approximately. Unlike the Gaussian HJM model (Heath, Jarrow, & Morton, 1992) used in previous articles, the parameters in the swap pricing formula can be acquired from the market quantities. Because the LIBOR (London Interbank Offered Rate) rates are positive, no pricing error arises from the possible negative rates in the Gaussian HJM model.

Some earlier research has been conducted on the pricing of equity swaps. Marshall, Sorensen, and Tuncker (1992) provided a pricing model that is not an arbitrage-free valuation formula. Rich (1995) used a forward-start forward contract approach to value basic equity swaps. Jarrow and Turnbull (1996) provided a preference-free formula for equity-for-fixed swaps. In a deterministic interest rate environment, Chance and Rich (CR; 1998) contributed to the literature on equity swaps by providing the valuation formulas for several types of equity swaps within the framework of arbitrage-free replicating portfolios. Kijima and Muromachi (KM; 2001) provided the fixed-for-equity swap pricing models with constant and variable notional principals in a stochastic interest rate economy. Employing Amin and Jarrow's (1991) cross-country model setting, Wang and Liao (WL; 2003) adopted a risk-neutral valuation method for pricing several types of cross-currency two-way equity swaps.

Although many pricing models of the various types of equity swaps have been presented, few articles have been written on the floating-for-equity type. One reason for this is that the interest rate model used in KM (2001) and WL (2003) is the Gaussian HJM framework. The HJM models the term structure of interest rates by specifying the dynamics of instantaneous forward rates. The instantaneous forward rates are continuously compounded rather than simply compounded. They are not appropriate for pricing the equity-for-floating swaps with a paid-in-arrears feature. Furthermore, the instantaneous forward rates are not observable in the market, so recovering the parameters in the model from the market-observed data is a difficult task. In addition, the Gaussian HJM forward rates can become negative with a positive probability, which may cause some pricing error.

BGM have developed a continuous time model of simple forward LIBOR rates, which are market-observable quantities. Because the forward LIBOR rates in the BGM model (the LIBOR market model) are simply compounded, they are suitable for pricing the swaps with a paid-in-arrears feature. Because forward LIBOR rates are market-observable, the difficult task of transforming the market quantities into the model parameters is overcome. In addition, BGM assumed that forward LIBOR rates have a log-normal volatility structure that prevents the forward LIBOR rates from becoming negative with a positive probability.

The article is organized as follows. In the second section, an arbitrage-free extended HJM model is established.[1] Under the arbitrage-free relationship between the drift and the volatility terms in the extended HJM model, an arbitrage-free extended BGM model is derived. The pricing formulas for the floating-for-equity swaps are developed with constant and variable notional principals in section three. The calibration procedure and some numerical examples are presented in section four and conclusions are drawn in section five.

THE MODEL

In this section, we first derive an arbitrage-free extended HJM model, and then apply the arbitrage-free relationship between the drift and the volatility terms to developing an arbitrage-free extended BGM model. To do this, two steps must be taken as briefly described below.

First, we consider an extended HJM model under the natural measure. Then, we employ the insights of Harrison and Krep (1979) to

[1]We call the BGM model incorporating stock dynamics the *extended BGM model* and the HJM model including stock dynamics the *extended HJM model*.

characterize the conditions on the forward rate processes such that there exists a unique equivalent martingale probability measure. Under these conditions, the drift of the stock price process is the risk-free short rate of interest, whereas the drift of the forward rate process is of a special form determined by the diffusion term, called the *forward rate drift restriction*. HJM have shown that the restriction on the family of the drift processes is sufficient to guarantee the existence of a unique equivalent martingale probability measure.

Second we assume that the LIBOR rates and the stock price processes have log-normal volatility structures. Like the arbitrage-free mechanism in the BGM model, we use the arbitrage-free relationship between the drift and the volatility terms in the arbitrage-free extended HJM model to determine the drift terms of the LIBOR rates as well as the stock price processes. By this means, we obtain the arbitrage-free extended BGM model.

Step I: Arbitrage-Free Extended HJM Model

We assume that trading takes place continuously over a time interval $[0, \tau]$, $0 < \tau < \infty$. The uncertainty is described by the filtered probability space $(\Omega, F, P, \{F_t\}_{t\in[0,\tau]})$ where the filtration is generated by the independent standard Brownian motions: $\tilde{W}(t) = (\tilde{W}_1(t), \tilde{W}_2(t), \ldots, \tilde{W}_m(t))$. Note that P represents the actual probability measure. We list the notations as follows.

$f(t, T)$ = The forward interest rate contracted at time t for instantaneous borrowing and lending at time T with $0 \le t \le T \le \tau$.

$P(t, T)$ = The time t price of a zero-coupon bond paying one dollar at time T.

$S(t)$ = The time t price of a stock.

$r(t)$ = The risk-free short rate at time t.

$B(t) = \exp\left[\int_0^t r(u)\,du\right]$, the money market account at time t with initial value $B(0) = 1$.

$\overline{P}(t, T) = P(t, T) / B(t)$, the relative bond price for a T-maturity zero-coupon bond.

$\overline{S}(t) = S(t) / B(t)$, the relative stock price.

We follow the conditions as shown in the HJM model and add an additional condition to the stock price dynamics.

Condition 1. A family of forward rate processes.

For any given $T \in [0, \tau]$, $f(t, T)$ follows the following process:

$$df(t, T) = \mu(t, T)dt + \sigma(t, T) \cdot d\widetilde{W}(t), \qquad 0 \le t \le T \le \tau. \quad (1)$$

where $\{f(0, T) : T \in [0, \tau]\}$ is a nonrandom initial forward curve, $\mu(t, T)$ and $\sigma(t, T) = (\sigma_1(t, T), \ldots, \sigma_m(t, T))$ satisfy some regular conditions.[2]

Equation (1) is the HJM model. Through the different specifications for the volatility coefficients, the m random shocks generate significantly different qualitative characteristics of the forward rate processes.

The zero-coupon bond is defined as:

$$P(t,T) = \exp\left[-\int_t^T f(t,u)du\right]. \quad (2)$$

From Equations (1) and (2), HJM derived the bond price dynamics as given by

$$\begin{aligned} dP(t,T) &= P(t,T)[r(t) + b(t,T)]dt \\ &\quad - P(t,T)\sigma^*(t,T) \cdot d\widetilde{W}(t), \ \ 0 \le t \le T \le \tau. \end{aligned} \quad (3)$$

where $\sigma^*(t, T) = (\sigma_1^*(t, T), \ldots, \sigma_m^*(t, T))$ with

$$\sigma_i^*(t,T) = \int_t^T \sigma_i(t,u)du \quad \text{for} \quad i = 1,2, \ldots, m, \quad (4)$$

and

$$b(t,T) = -\int_t^T \mu(t,u)du + \frac{1}{2}\|\sigma^*(t,T)\|^2. \quad (5)$$

Condition 2. Equity price dynamics.

The dynamics of the equity price is provided as follows.

$$dS(t) = S(t)\eta(t)dt + S(t)\zeta(t) \cdot d\widetilde{W}(t). \quad (6)$$

[2] $\mu : \{(t, s)\} : 0 \le t \le s \le T\} \times \Omega \to R$ is jointly measurable, adapted and

$$\int_0^T |\mu(u, T)|du < +\infty \quad a.e. \quad P.$$

$\sigma_i : \{(t, s) : 0 \le t \le s \le T\} \times \Omega \to R$ are jointly measurable, adopted, and

$$\int_0^T |\sigma_i(u, T)|du < \infty \quad a.e. \quad P \text{ for } i = 1, 2, \ldots, m.$$

where $\eta(t)$ and $\zeta(t) = (\zeta_1(t), \zeta_2(t), \ldots, \zeta_m(t))$ satisfy some regular conditions.[3]

For greater flexibility, the number of random shocks, m, is not precisely designated, but rather depends on the simplicity and accuracy required by the user. For example, we may use four random shocks, i.e, $m = 4$, to capture all of the factors causing the shift of the entire forward rate curve and the movement of the stock price process. The first two random shocks can be interpreted, respectively, as the short-term and long-term factors causing the shift of different maturity ranges on the term structure. The third random shock can be regarded as the factor causing the unanticipated movement of the stock price. The correlation between the forward rates and the stock price is affected by the fourth random shock.

Condition 3. Existence of the market price of risk.

For any given $T_1, T_2, \ldots, T_{m-1} \in [0, \tau]$ with $T_1 \leq T_2 \leq \cdots \leq T_{m-1}$, assume that there exist maturity invariant solutions.

$\theta_i(\cdot, T_1, T_2, \ldots, T_{m-1}) = \theta_i(\cdot) : \Omega \times [0, T_1] \longrightarrow R,\ i = 1, 2, \ldots, m$ to the equations of the market price of risk:

$$\begin{bmatrix} b(t,T_1) \\ \vdots \\ b(t,T_{m-1}) \\ \eta(t) - r(t) \end{bmatrix} - \begin{bmatrix} \sigma_1^*(t,T_1) & \ldots & \sigma_m^*(t,T_1) \\ \vdots & \ddots & \vdots \\ \sigma_1^*(t,T_{m-1}) & \ldots & \sigma_m^*(t,T_{m-1}) \\ -\zeta_1(t) & \ldots & -\zeta_m(t) \end{bmatrix} \begin{bmatrix} \theta_1(t) \\ \vdots \\ \vdots \\ \theta_m(t) \end{bmatrix} = 0 \quad (7)$$

where $\{\theta_i(t)\}_{i=1,2,\ldots,m}$ satisty the following regular restrictions:

$$\int_0^{T_1} \theta_i^2(u)du < \infty^{\circ} \quad a.e.\ P \quad \text{for} \quad i = 1, 2, \ldots, m, \tag{8.1}$$

$$E^P\left[\exp\left(\sum_{i=1}^m \int_0^{T_1} \theta_i(u)d\widetilde{W}_i(u) - \frac{1}{2}\sum_{i=1}^m \int_0^{T_1} \theta_i^2(u)du\right)\right] = 1, \tag{8.2}$$

[3] $\zeta_i : [0, \tau] \rightarrow R$ is deterministic for $i = 1, 2, \ldots, m$. $\eta : [0, \tau] \rightarrow R$ is adapted, jointly measure, and satisfied

$$E\left[\int_0^T \left|\eta(u)\right|^2 du\right] < \infty.$$

$$E^P\left[\exp\left(\sum_{i=1}^{m}\int_0^{T_1}(\theta_i(u) - \sigma_i(u,T))d\widetilde{W}_i(u)\right.\right.$$
$$\left.\left. - \frac{1}{2}\sum_{i=1}^{m}\int_0^{T_1}(\theta_i^2(u) - \sigma_i(u,T))^2 du\right)\right] = 1 \quad for \quad T \in \{T_1, T_2, \ldots, T_{m-1}\}, \tag{8.3}$$

$$E^P\left[\exp\left(\sum_{i=1}^{m}\int_0^{T_1}(\theta_i(u) + \zeta_i(u))d\widetilde{W}_i(u)\right.\right.$$
$$\left.\left. - \frac{1}{2}\sum_{i=1}^{m}\int_0^{T_1}(\theta_i(u) + \zeta_i(u))^2 du\right)\right] = 1. \tag{8.4}$$

Equation (7) is the market price of risk equation and $\{\theta_i(\cdot)\}_{i=1,2,\ldots,m}$ are the market prices of risks associated with the random shocks $\{\widetilde{W}_i(\cdot)\}_{i=1,\,2,\ldots,m}$, respectively. Unlike HJM, we directly assume that $\{\theta_i(\cdot)\}_{i=1,\,2,\ldots,m}$ are not dependent on the maturities of the chosen bonds without an additional condition setting. By adding one more regular restriction in Condition 3, i.e., Equation (8.4), and following the proof as given by HJM,[4] we can show that Condition 3 holds if, and only if, there exists an equivalent martingale measure Q. According to the first fundamental theorem of asset pricing, the market admits no arbitrage opportunity.

Condition 4. Uniqueness of the equivalent Martingale measure.

For any given $T_1, T_2, \ldots, T_{m-1} \in [0, \tau]$ with $T_1 \le T_2 \le \cdots \le T_{m-1}$, assume that the diffusion terms matrix

$$\begin{bmatrix} \sigma_1^*(t, T_1) & \cdots & \sigma_m^*(t, T_1) \\ \vdots & \ddots & \vdots \\ \sigma_1^*(t, T_{m-1}) & \cdots & \sigma_m^*(t, T_{m-1}) \\ -\zeta_1(t) & \cdots & -\zeta_m(t) \end{bmatrix}$$

is nonsingular.

Condition 4 guarantees that the equations of the market price of risk have a unique solution, $\{\theta_i(\cdot)\}_{i=1,\,2,\ldots,m}$. HJM have shown that Condition 4 holds if, and only if, the martingale measure is unique. The second fundamental theorem of asset pricing tells us that if the martingale measure is unique, then the market pricing is complete.

[4]It is the proof of Proposition 1 as given by HJM. We may regard the volatility functions $\zeta_i(u)$ of the stock price process as the volatility functions $\sigma_i^*(u, \cdot)$ of the additional bond price process for $i = 1, 2, \ldots, m$ and then follow the same steps of the proof as given in HJM.

With the above conditions, we can define the Martingale measure Q with the following Radon-Nikodym derivatives:

$$\frac{dQ}{dP} = \exp\left[\sum_{i=1}^{m}\int_0^T \theta_i(t)d\tilde{W}_i(t) - \frac{1}{2}\sum_{i=1}^{m}\int_0^T \theta_i^2(t)dt\right]. \tag{9}$$

According to Girsanov's theorem, $\{W_1(t), W_2(t), \ldots, W_m(t)\}$ are m independent standard Brownian motions under the probability space (Ω, F, Q) and defined as follows:

$$W_i(t) = \tilde{W}_i(t) - \int_0^T \theta_i(u)du, \text{ for } i = 1, 2, \ldots, m. \tag{10}$$

Under the Conditions 3 and 4, we have the following equation

$$b(t, T) - \sum_{i=1}^{m} \sigma_i^*(t, T)\theta_i(t) = 0 \quad \forall T \in [0, \tau]. \tag{11}$$

Differentiating Equation (11) with respect to T, we obtain the forward rate drift restriction for no-arbitrage:

$$\mu(t, T) = -\sum_{i=1}^{m} \sigma_i(t, T)(\theta_i(t) - \sigma_i^*(t, T)). \tag{12}$$

Taking Equation (12) into Equation (1), we obtain the forward rate dynamics under the martingale measure.

We conclude the above results with Proposition 1:

Proposition 1. The dynamics under the Martingale measure.

Under the Martingale measure Q, the dynamics of forward rates, bond prices, and stock prices are as follows:

$$dP(t, T) = P(t, T)r(t)\,dt - P(t, T)\sigma^*(t, T) \cdot dW(t), \tag{13}$$

$$df(t, T) = \sigma(t, T) \cdot \sigma^*(t, T)dt + \sigma(t, T) \cdot dW(t), \tag{14}$$

$$dS(t) = S(t)r(t) + S(t)\zeta(t) \cdot dW(t), \tag{15}$$

where $\sigma(t, T) = (\sigma_1(t, T), \ldots, \sigma_m(t, T))$, $\sigma^*(t, T) = (\sigma_1^*(t, T), \ldots, \sigma_m^*(t, T))$ and $\sigma_i^*(t, T) = \int_t^T \sigma_i(t, u)du$ *for* $i = 1, 2, \ldots, m$ *for all* $T \in [0, \tau]$.

It is worth emphasizing that even if we incorporate the stock price dynamics into the HJM interest rate model, the forward rate drift restriction for no-arbitrage remains unchanged. Therefore, we can use this fact to derive the arbitrage-free extended BGM model and then apply it to pricing equity swaps.

Step II : The Arbitrage-Free Extended BGM Model

It is important to note that, hereafter, we model the term structure of interest rates by specifying the forward LIBOR rate dynamics rather than the instantaneous forward rate dynamics, although we still use the notations, economic environment and forward rate drift restriction for no-arbitrage in the above subsection to derive the extended BGM model under the Martingale measure.

Fix some $\delta > 0$ and $T \in [0, \tau]$, define the forward LIBOR rate process $\{L(t, T); 0 \leq t \leq T\}$ as

$$1 + \delta L(t, T) = \exp\left(\int_T^{T+\delta} f(t, u)\, du\right). \tag{16}$$

Assumption 1. A family of LIBOR rate processes.

We assume that $L(t, T)$ has a log-normal volatility structure and its stochastic process is given by

$$dL(t,T) = \mu_L(t,T)dt + L(t,T)\gamma(t,T) \cdot dW(t) \tag{17}$$

where $\gamma(\cdot, T)$: $[0, \tau] \to R^m$ is deterministic, bounded, and piecewise continuous volatility functions and $\mu_L(t, T) : [0, T] \to R$ is some drift function.

Assumption 2. The equity price dynamics.

The dynamics of the equity price is as follows:

$$dS(t) = S(t)\mu_S(t) + S(t)\zeta(t) \cdot dW(t). \tag{18}$$

where $\mu_S(t) : [0, \tau] \to R$ is some drift function and the volatility vector function $\zeta(t) : [0, \tau] \to R^m$ satisfies some regular conditions as given by Condition 2.

It is important to emphasize that the drift terms of the LIBOR rates and the stock price processes are not yet determined. The specific forms of their drift terms must make the economy arbitrage-free. We will use the arbitrage-free relationship between the drift and volatility terms in Proposition 1 to determine the drift terms in Equations (17) and (18).

First, we determine $\mu_L(t, T)$ in the LIBOR rate process. Assume that $Y(t) = \int_T^{T+\delta} f(t, u)\, du$ and $L(t, T) = (1/\delta)\, (\exp(Y(t)) - 1)$. Making use of Itô's lemma, we have

$$dL(t, T) = \frac{1}{\delta}\exp\left(\int_T^{T+\delta} f(t, u)\, du\right)\left\{dY(t) + \frac{1}{2} dY(t) \cdot dY(t)\right\} \tag{19}$$

and

$$dY(t) = \frac{1}{2}(\|\sigma^*(t, T+\delta)\|^2 - \|\sigma^*(t, T)\|^2)\,dt$$
$$+ (\sigma^*(t, T+\delta) - \sigma^*(t, T)) \cdot dW(t). \tag{20}$$

Combining Equations (19) and (20), we have

$$\begin{aligned}
dL(t, T) &= \frac{1}{\delta}\exp\left(\int_T^{T+\delta} f(t, u)du\right)\left\{\frac{1}{2}(\|\sigma^*(t, T+\delta)\|^2 - \|\sigma^*(t, T)\|^2\right. \\
&\quad \left. + \frac{1}{2}\|\sigma^*(t, T+\delta) - \sigma^*(t, T)\|^2\right\} dt \\
&\quad + \frac{1}{\delta}\exp\left(\int_T^{T+\delta} f(t, u)\,du\right)\{(\sigma^*(t, T+\delta) - \sigma^*(t, T)) \cdot dW(t)\} \\
&= \frac{1}{\delta}(1 + \delta L(t, T))(\|\sigma^*(t, T+\delta)\|^2 - \sigma^*(t, T) \cdot \sigma^*(t, T+\delta))\,dt \\
&\quad + \frac{1}{\delta}(1 + \delta L(t, T))(\sigma^*(t, T+\delta) - \sigma^*(t, T)) \cdot dW(t) \\
&= \frac{1}{\delta}(1 + \delta L(t, T))(\sigma^*(t, T+\delta) - \sigma^*(t, T)) \cdot \sigma^*(t, T+\delta)\,dt \\
&\quad + \frac{1}{\delta}(1 + \delta L(t, T))(\sigma^*(t, T+\delta) - \sigma^*(t, T)) \cdot dW(t).
\end{aligned} \tag{21}$$

Equation (21) indicates the relationship between the drift and the diffusion terms of the LIBOR rate process under the martingale measure. We can use this relationship to determine $\mu_L(t, T)$.

In Assumption 1, we have assumed that the LIBOR rate's volatility structure is log-normal, and thus

$$\frac{1}{\delta}(1 + \delta L(t, T))(\sigma^*(t, T+\delta) - \sigma^*(t, T)) = L(t, T)\gamma(t, T). \tag{22}$$

Substituting Equation (22) for the drift term in Equation (21), we have

$$dL(t, T) = L(t, T)\gamma(t, T) \cdot \sigma^*(t, T+\delta)\,dt$$
$$+ L(t, T)\gamma(t, T) \cdot dW(t). \tag{23}$$

The bond volatility vector, $\sigma^*(t, T)$, is not yet specified in the model. To make use of the arbitrage-free structure in HJM, we must specify $\sigma^*(t, T)$. By the recurrent relationship of Equation (22), the bond volatility process, $\sigma^*(t, T)$ for any given $T \in [0, \tau]$, can be represented as follows:

$$\sigma^*(t, T) = \begin{cases} \displaystyle\sum_{k=1}^{\lfloor \delta^{-1}(T-t) \rfloor} \frac{\delta L(t, T - k\delta)\gamma(t, T - k\delta)}{1 + \delta L(t, T - k\delta)} & t \in [0, T - \delta] \ \& \ T - \delta > 0, \\ 0 & \text{otherwise.} \end{cases} \tag{24}$$

where $\lfloor \delta^{-1}(T - t) \rfloor$ denotes the greatest integer that is less than $\delta^{-1}(T - t)$. Thus, under the Martingale measure, the LIBOR rate dynamics is

$$dL(t, T) = L(t, T)\gamma(t, T) \cdot \sigma^*(t, T + \delta)\, dt + L(t, T)\gamma(t, T) \cdot dW(t),$$

where $\sigma^*(t, T + \delta)$ is defined by Equation (24).

Additionally we take advantage of the drift-volatility relationship of the stock price process in Proposition 1 to determine the drift function $\mu_S(t)$ under the martingale measure. Due to the identical volatility structure in Proposition 1 and Assumption 2, we have

$$\mu_S(t) = r(t).$$

Thus, under the Martingale measure, the stock price process is

$$dS(t) = S(t)r(t) + S(t)\zeta(t) \cdot dW(t).$$

Proposition 2. The extended LIBOR market model under the Martingale measure.

Under the Martingale measure, the LIBOR rates and the stock price processes are as follows:

$$dS(t) = S(t)r(t) + S(t)\zeta(t) \cdot dW(t), \tag{25}$$

$$dL(t, T) = L(t, T)\gamma(t, T) \cdot \sigma^*(t, T + \delta)\, dt + L(t, T)\gamma(t, T) \cdot dW(t), \tag{26}$$

where $t \in [0, T]$, $T \in [0, \tau]$ and $\sigma^*(t, T + \delta)$ is defined in Equation (24).

Unlike the forward rates in the HJM model, the forward LIBOR rates are market-observable. Therefore, the volatility $\gamma(t, T)$, $T \in [0, \tau]$, can be obtained from the quoted prices of interest rate derivatives actively traded in the market and $\sigma^*(t, T)$, $T \in [0, \tau]$, can be calculated from Equation (24).

By changing the Martingale measure Q to the forward Martingale measure, denoted by $Q_{T+\delta}$[5], the above processes result as follows:

Proposition 3. The extended LIBOR market model under the forward Martingale measure.[6]

Under the forward measure $Q_{T+\delta}$,

$$dS(t) = S(t)[r(t) - \zeta(t) \cdot \sigma^*(t, T + \delta)]\,dt + S(t)\zeta(t) \cdot dW_{T+\delta}(t), \quad (27)$$

$$dL(t, T) = L(t, T)\gamma(t, T) \cdot dW_{T+\delta}(t), \quad (28)$$

where $t \in [0, T]$ $T \in [0, \tau]$ and $\sigma^*(t, T + \delta)$ is defined in Equation (24).

$L(t, T)$ can be expressed as $\frac{1}{\delta}(P(t, T) - P(t, T + \delta))/P(t, T + \delta)$ where $\frac{1}{\delta}(P(t, T) - P(t, T + \delta))$ represents the price of a tradable asset. When any tradable asset's price is expressed with respect to the numéraire $P(t, T + \delta)$, it has to be a Martingale under the measure $Q_{T+\delta}$. Therefore, $L(t, T)$ is a Martingale under the measure $Q_{T+\delta}$.

With Propositions 2 and 3, we can price equity swaps, especially those which are floating-for-equity swaps. The pricing formulas are to be developed in the next section.

PRICING EQUITY SWAPS

In this section, we use the model derived in the previous section to first price the floating-for-equity swap using the constant notional principal, and then the variable notional principal. The procedure for deriving the two instruments serves to illustrate the techniques for derivatives pricing under the extended LIBOR market model. Although former can actually be priced in a closed form, only an approximate formula can be derived for the latter. The accuracy of the approximate formula will be examined in Appendix C.

Pricing Floating-for-Equity Swaps With a Constant Notional Principal

A floating-for-equity swap with a constant notional principal (FESC) is defined as follows: The contract starts at time t_0 with the reset dates $t_0 \leq t_1 \leq \cdots \leq t_{n-1}$ and the payment dates $t_1 \leq t_2 \leq \cdots \leq t_n$. In accordance with practice, we define $\delta = t_{k+1} - t_k$, $k = 0, 1, \ldots, n - 1$. During the tenor of the swap, the notional principal of the contract is fixed, and is assumed to be \$1. At each payment data t_k, for $k = 1, \ldots, n$, one party pays the return on the underlying equity, $S(t_k)/S(t_{k-1}) - 1$, to the counterparty

[5] $Q_{T+\delta}$ is the forward Martingale measure with respect to the numéraire $P(t, T + \delta)$.

[6] See Shreve (2004) for details on the changing-numéraire mechanism.

and receives from the counterparty a floating payment, $\delta(L(t_{k-1}, t_{k-1}) + K)$, where $L(t, t_{k-1})$ is signified by the simply compounded forward LIBOR rate for the period $[t_{k-1}, t_k]$ observed at time t and $L(t_{k-1}, t_{k-1})$ is thus explained accordingly, whereas K is a spread in basis points. For the party who pays the floating rate and receives the equity return, the cash flow stream is given as follows:

$$\begin{aligned}
&\text{At time } t_1 &:\quad & [S(t_1)/S(t_0) - 1] - \delta[L(t_0, t_0) + K] \\
&\text{At time } t_2 &:\quad & [S(t_2)/S(t_1) - 1] - \delta[L(t_1, t_1)+K] \\
&\quad\vdots & & \qquad\qquad\vdots \\
&\text{At time } t_n &:\quad & [S(t_n)/S(t_{n-1}) - 1] - \delta[L(t_{n-1}, t_{n-1}) + K]
\end{aligned}$$

The pricing formula of an FESC is presented in the following theorem, and the proof is provided in Appendix A.

Theorem 1. The pricing formula of an FESC.
Under the extended BGM model, the price of an FESC at time s, $t_0 \leq s \leq t_1$, is given by

$$\begin{aligned}
FESC = \frac{S(s)}{S(t_0)} - \delta \sum_{k=2}^{n} L(s, t_{k-1})P(s, t_k) - \delta L(t_0, t_0)P(s, t_1) \\
- P(s, t_n) - \delta K \sum_{k=1}^{n} P(s, t_k). \qquad (29)
\end{aligned}$$

Equation (29) is the FESC pricing formula that should be implemented in the market because it provides the theoretical foundation under the BGM model. If the floating rates are replaced with a fixed rate, i.e., $L(t, t_{k-1}) = R\ \forall t \in [s, t_{k-1}]$ for $k = 1, 2, \ldots, n$, $K = 0$ and $\delta = 1$[7], then the pricing formula of the FESC degenerates to the fixed-for-equity swap pricing formula given by KM (2001). Equation (29) thus provides an FESC pricing model that is more general than the KM model.

By adjusting the spread K, the initial price of the FESC can be set to zero and trading becomes a fair game. This fair rate K may be called the FESC swap rate and is provided by:

$$K = \frac{1 - \delta \sum_{k=1}^{n} L(t_0, t_{k-1})P(t_0, t_k) - P(t_0, t_n)}{\delta \sum_{k=1}^{n} P(t_0, t_k)} \qquad (30)$$

The fair equity swap rate K is determined only through the current term structure of interest rates and is not related to the equity price process;

[7]The fixed year fraction $\delta = 1$ is given by KM (2001).

this economic conclusion is identical to CR (1998) and KM (2001) in the case of constant notional principal. However, the determinants of the FESC swap rate K are quite different from those within the HJM model. The K in Equation (30) is determined by the forward LIBOR rates, which are market-observable, whereas those under the HJM model are market-nonobservable instantaneous forward rates.

Pricing Floating-for-Equity Swaps With Variable Notional Principal

The cash flow stream of a floating-for-equity swap with variable notional principal (FESV) is given as follows:

$$
\begin{aligned}
&\text{At time } t_1 &:&\quad \{\$1\}\{[S(t_1)/S(t_0) - 1] - \delta[L(t_0, t_0) + K]\}\\
&\text{At time } t_2 &:&\quad \{S(t_1)/S(t_0)\}\{[S(t_2)/S(t_1) - 1] - \delta[L(t_1, t_1) + K]\}\\
&\text{At time } t_3 &:&\quad \{S(t_2)/S(t_0)\}\{[S(t_3)/S(t_2) - 1] - \delta[L(t_2, t_2) + K]\}\\
&\quad\vdots & &\qquad\qquad\vdots\\
&\text{At time } t_n &:&\quad \{S(t_{n-1})/S(t_0)\}\{[S(t_n)/S(t_{n-1}) - 1] - \delta[L(t_{n-1}, t_{n-1}) + K]\}
\end{aligned}
$$

The approximate pricing formula of a FESV is presented in Theorem 2 and the proof is given in Appendix B.

Theorem 2. The approximate pricing formula of an FESV.

Under the extended LIBOR market model, the approximate price of an FESV at time s, $t_0 \leq s \leq t_1$, is given by

$$
\begin{aligned}
FESV = \frac{S(s)}{S(t_0)}\Bigg\{ n - \sum_{k=2}^{n}\Bigg[(1+\delta K)\frac{P(s,t_k)}{P(s,t_{k-1})}\phi(s;t_{k-1},t_k) \\
+ \delta L(s,t_{k-1})\frac{P(s,t_k)}{P(s,t_{k-1})}\phi(s;t_{k-1},t_k)\rho(s;t_{k-1},t_k)\Bigg]\Bigg\} \\
-(1+\delta L(t_0,t_0)+\delta K)P(s,t_1), \qquad (31)
\end{aligned}
$$

where $\phi(s; t_{k-1}, t_k)$ and $\rho(s; t_{k-1}, t_k)$ are defined by

$$
\phi(s;t_{k-1},t_k) = \exp\left(\int_s^{t_{k-1}} (\zeta(u) + \overline{\sigma}_s(u,t_{k-1}))\cdot(\overline{\sigma}_s(u,t_{k-1}) - \overline{\sigma}_s(u,t_k))\,du\right),
$$

$$
\rho(s;t_{k-1},t_k) = \exp\left(\int_s^{t_{k-1}} \gamma(u,t_{k-1})\cdot(\zeta(u) + \overline{\sigma}_s(u,t_{k-1}))\,du\right),
$$

and $\overline{\sigma}_s(u, \cdot)$ is approximated by Equation (B4) in Appendix B. The accuracy of this approximation is reported in Appendix C.

There are two correlation terms, $\phi(s; t_{k-1}, t_k)$ and $\rho(s; t_{k-1}, t_k)$, in the pricing formula. It is easy to observe from the proof in Appendix B that $\phi(s; t_{k-1}, t_k)$ implicitly represents the correlation between the discount factor and the return from the variable principal during the period $[t_0, t_{k-1}]$ for $k = 2, 3, \ldots, n$. Similarly, the correlation between the LIBOR rate $L(t, t_{k-1})$ and the stock price process is specified implicitly through $\rho(s; t_{k-1}, t_k)$, for $k = 2, 3, \ldots, n$, and these correlations arise due to the floating interest rate payments. Despite the fact that the economic conclusion is similar to that of KM (2001), the determinants of our correlations differ from theirs.

If the floating rates become a constant rate, i.e., $L(t, t_{k-1}) = R \ \forall t \in [s, t_{k-1}]$ for $k = 1, 2, \ldots, n$, the spread $K = 0$ and $\delta = 1$ (in conformance with that of KM (2001)), then Equation (31) degenerates to the pricing formula of the fixed-for-equity swap with variable notional principal presented by KM (2001).

The fair equity swap rate with variable notional principal at the contract initiation is as follows:

$$K = \frac{A}{\delta \sum_{k=1}^{n} \frac{P(t_0, t_k)}{P(t_0, t_{k-1})} \phi(t_0; t_{k-1}, t_k)} \tag{32}$$

where

$$A = n - \sum_{k=1}^{n} \frac{P(t_0, t_k)}{P(t_0, t_{k-1})} \phi(t_0;\ t_{k-1},\ t_k)[1 + \delta L(t_0,\ t_{k-1})\rho(t_0;\ t_{k-1},\ t_k)]$$

Unlike the case of the FESC, the stock price process also has an impact on K via $\phi(t_0; t_{k-1}, t_k)$ and $\rho(t_0; t_{k-1}, t_k)$ because the final payoff is compounded by the percentage change in the stock price due to the variable notional principal. In addition, the advantage of the extended BGM model is that the LIBOR rates are market-observable. We can directly observe the initial LIBOR rates from the market. Compared with the Gaussian HJM case, as given in KM (2001) and others, our model makes it easier to calibrate the parameters associated with the model and to calculate the correlation terms, $\phi(s; t_{k-1}, t_k)$ and $\rho(s; t_{k-1}, t_k)$ for $k = 2, 3, \ldots, n$.

As mentioned above, the extended LIBOR model has a log-normal volatility structure, thereby leading to the positive LIBOR rates, which avoid the pricing error arising from the negative rate with a positive probability in the Gaussian HJM model.

CALIBRATION AND NUMERICAL EXAMPLES

The rates described in the LIBOR market model are the forward LIBOR rates underlying the caps and floors that are actively traded in financial markets. The market data can be employed to calibrate the parameters in the model. In this section, we first present the calibration procedure and then provide some numerical examples for Theorem 2.

We use the mechanism presented by Rebonato (1999) to engage in a simultaneous calibration of the extended LIBOR market model according to the percentage volatilities and to the correlation matrix of the underlying forward LIBOR rates and the stock index. We assume that there are $n - 1$ forward LIBOR rates in a m-factors framework. The steps to calibrate the parameters are given as follows.

First, following the work of Brigo and Mercurio (2001), we assume that $L(t, \cdot)$ has a piecewise-constant instantaneous total volatility structure that depends solely on the time-to-maturity. The elements in Table I, which specify the instantaneous total volatility applied to each period for each rate, can be calculated from the market data. A detailed computational process is also presented in Hull (2003). In addition, we also assume that the stock index has a piecewise-constant instantaneous total volatility structure. The elements in Table II can be calculated from the on-the-run option prices in the market. However, because the duration of stock options is usually shorter than one year, the market-obtainable elements in Table II are not usually sufficient for pricing equity swaps. This problem may be resolved by using the implied (or historical) volatility of the underlying stock index, while assuming that the term structure of the volatility is flat, i.e., $\zeta(t) = \zeta$ for $t \in (t_0, t_n]$.

TABLE I
Instantaneous Volatilities of $L(t, \cdot)$

Instant. Total Vol.	*Time* $t \in (t_0, t_1]$	$(t_1, t_2]$	$(t_2, t_3]$	$\cdots$	$(t_{n-2}, t_{n-1}]$
Fwd Rate: $L(t, t_1)$	$v_{1,1}$	Dead	Dead	$\cdots$	Dead
$L(t, t_2)$	$v_{2,1}$	$v_{2,2}$	Dead	$\cdots$	Dead
$\vdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
$L(t, t_{n-1})$	$v_{n-1,1}$	$v_{n-1,2}$	$v_{n-1,3}$	$\cdots$	$v_{n-1,n-1}$

TABLE II
Instantaneous Volatilities of the Stock Index

Instant. Total Vol.	*Time* $t \in (t_0, t_1]$	$(t_1, t_2]$	$(t_2, t_3]$	$\cdots$	$(t_{n-1}, t_n]$
Fwd Rate: $S(t)$	ζ_1	ζ_2	ζ_3	$\cdots$	ζ_n

Next, we use the historical price data of the forward LIBOR rates and the underlying stock index to derive a full-rank $n \times n$ instantaneous-correlation matrix Γ. Thus, Γ is a positive-definite and symmetric matrix and can be written as

$$\Gamma = H\,\Lambda\,H',$$

where H is a real orthogonal matrix and Λ is a diagonal matrix. Let $A \equiv H\Lambda^{1/2}$ and thus, $AA' = \Gamma$, so that we can find a suitable m-rank ($m \ll n$) matrix B, such that $\Gamma^B = BB'$ is a m-rank correlation matrix and can be used to mimic the market correlation matrix Γ.

The advantage of this is that we may replace the n-dimensional original Brownian motion $dW(t)$ with $BdZ(t)$ where $dZ(t)$ is an m-dimensional Brownian motion. In other words, we change the market correlation structure

$$dW(t)dW(t)' = \Gamma dt$$

to a modeled correlation structure

$$BdZ(t)(BdZ(t))' = BdZ(t)dZ(t)'B' = BB'\,dt = \Gamma^B dt.$$

The remaining problem is how to choose a suitable matrix B. Rebonato (1999) proposed the following form for ith row of B:

$$b_{i,k} = \begin{cases} \cos\theta_{i,k}\prod_{j=1}^{k-1}\sin\theta_{i,j} & \text{if } k = 1, 2, \ldots, m-1, \\ \prod_{j=1}^{k-1}\sin\theta_{i,j} & \text{if } k = m, \end{cases}$$

for $i = 1, 2, \ldots, n$. By finding a $\hat{\theta}$ that solves the following optimization problem

$$\min_{\theta} \sum_{i,j=1}^{n} |\Gamma^B_{i,j} - \Gamma_{i,j}|^2,$$

and substituting $\hat{\theta}$ into B, we obtain a suitable matrix $\hat{B}$ such that $\Gamma^B(= \hat{B}\hat{B}')$ is an approximate correlation matrix for Γ.

$\hat{B}$ can be used to distribute the instantaneous total volatility to each Brownian motion without changing the amount of the instantaneous total volatility. That is,

$$v_{i,j}(\hat{B}(n, 1), \hat{B}(i, 2), \ldots, \hat{B}(i, m)) = (\gamma_1(t, t_i), \gamma_2(t, t_i), \ldots, \gamma_m(t, t_i))$$

$$\zeta(\hat{B}(n, 1), \hat{B}(n, 2), \ldots, \hat{B}(n, m)) = (\zeta_1(t), \zeta_2(t), \ldots, \zeta_m(t))$$

where $i = 1, 2, \ldots, n-1$ and $t \in (t_{j-1}, t_j]$, for each $j = 1, 2, \ldots, n$.

Under the assumption that the instantaneous total volatility structures are piecewise-constant, the above procedure represents a general calibration method without a constraint on choosing the number of factors. Via the distributing matrix $\hat{B}$, the individual instantaneous volatility applied to each Brownian motion, can be derived for each process. With these data calibrated from the market correlation matrix and volatilities, we can employ a Monte Carlo simulation to price any associated interest rate derivatives. Besides, the data can also be used to calculate the price of the FESC and the FESV derived in Theorems 1 and 2.

Based on the actual market data as shown in Tables C2–C4 in Appendix C and the calibration procedure described above, we present some numerical examples for Theorem 2 in Table III. Compared with the

TABLE III
The 5-Year and 10-Year FESV Prices

Date		*3/31/06*	*12/30/05*	*9/30/05*	*6/30/05*
5-year	Thm 2	−0.0077	−0.0078	−0.0079	−0.0081
	MC	−0.0075	−0.0073	−0.0082	−0.0084
	(s.e.)	(0.0021)	(0.0021)	(0.0021)	(0.0021)
10-year	Thm 2	−0.0156	−0.0161	−0.0162	−0.0166
	MC	−0.0142	−0.0165	−0.0151	−0.0158
	(s.e.)	(0.0031)	(0.0031)	(0.0031)	(0.0031)
Date		*3/31/05*	*12/31/04*	*9/30/04*	*6/30/04*
5-year	Thm 2	−0.0080	−0.0086	−0.0093	−0.0091
	MC	−0.0077	−0.0090	−0.0086	−0.0085
	(s.e.)	(0.0021)	(0.0021)	(0.0021)	(0.0021)
10-year	Thm 2	−0.0163	−0.0178	−0.0192	−0.0178
	MC	−0.0167	−0.0176	−0.0183	−0.0175
	(s.e.)	(0.0031)	(0.0032)	(0.0031)	(0.0031)

Note. The prices of the 5-year and 10-year FESV reset annually are presented in this table. They are priced to different quarterly dates over the past 2 years. The flat volatility of S&P500 is assumed to be 20%. The spread K = 15bp. The simulation is based on 50,000 paths.

Monte Carlo simulation, the accuracy of the approximate pricing formula in Theorem 2 obtained by using the most recent 2-year data, is good.

CONCLUSION

This study advances a general model, the extended BGM model, for pricing equity swaps and applies this model to price floating-for-equity swaps with either constant or variable notional principals. We have derived the extended BGM model under the Martingale measure by assuming that both LIBOR rates and stock price processes follow a log-normal volatility structure and have imposed the drift restriction for no-arbitrage in the extended HJM model.

Because the forward LIBOR rates have a log-normal volatility structure, the rates are positive. Because the LIBOR rate is market-observable and its related derivatives, such as caps and swaptions, are widely traded in the market, we can inverse these market quantities to calibrate the parameters associated with the extended BGM model. Moreover, because the LIBOR rates are simply compounded, the extended BGM model is suitable for pricing swaps with a paid-in-arrears feature.

With the above-mentioned advantages, the extended LIBOR market model is a general model for pricing swaps, especially floating-for-equity types. The pricing formula of the floating-for-equity swaps, with either constant or variable notional principals, has been presented and analyzed. We have also discussed the calibration of model parameters. The extended LIBOR market model may also be employed to price other types of swaps, such as two-way equity swaps and capped-equity swaps. These will be examined in future research.

APPENDIX A

Proof of Theorem 1

Applying the martingale pricing method, the price of the FESC at time s, $t_0 \leq s \leq t_1$, is the sum of the discounted value of the expected future cash flows, i.e.,

$$V_{FESC} = \sum_{k=1}^{n} B(s) E^{Q}\left(\frac{\left(\frac{S(t_k)}{S(t_{k-1})} - (1 + \delta(L(t_{k-1}, t_{k-1}) + K))\right)}{B(t_k)} \middle| \mathcal{F}_s \right)$$

$$= B(s)E^Q\left(\left.\frac{\left[\frac{S(t_1)}{S(t_0)} - (1 + \delta(L(t_0, t_0) + K))\right]}{B(t_1)}\right| \mathcal{F}_s\right)$$

$$+ \sum_{k=2}^{n} B(s)E^Q\left(\left.\frac{\left[\frac{S(t_k)}{S(t_{k-1})} - (1 + \delta(L(t_{k-1}, t_{k-1}) + K))\right]}{B(t_k)}\right| \mathcal{F}_s\right)$$

$$= \frac{S(s)}{S(t_0)} - (1 + \delta(L(t_0, t_0) + K))P(s, t_1)$$

$$+ \sum_{k=2}^{n} B(s)E^Q\left(\left.\frac{\frac{S(t_k)}{S(t_{k-1})}}{B(t_k)}\right| \mathcal{F}_s\right)$$

$$-(1 + \delta K)\sum_{k=2}^{n} B(s)E^Q\left(\left.\frac{1}{B(t_k)}\right| \mathcal{F}_s\right)$$

$$-\sum_{k=2}^{n} \delta B(s)E^Q\left(\left.\frac{L(t_{k-1}, t_{k-1})}{B(t_k)}\right| \mathcal{F}_s\right).$$

Using the law of iterated expectation and the changing-numéraire mechanism, the elements of the first two summations are derived as follows:

$$B(s)E^Q\left(\left.\frac{\frac{S(t_k)}{S(t_{k-1})}}{B(t_k)}\right| \mathcal{F}_s\right) = B(s)E^Q\left(\left.\frac{1}{S(t_{k-1})}E^Q\left(\left.\frac{S(t_k)}{B(t_k)}\right|\mathcal{F}_{t_{k-1}}\right)\right| \mathcal{F}_s\right)$$

$$= P(s,\, t_{k-1}),\ \text{for } k = 2, \ldots, n.$$

and

$$B(s)E^Q\left(\left.\frac{1}{B(t_k)}\right|\mathcal{F}_s\right) = P(s,\, t_k).$$

Because $\{L(t, t_{k-1}): 0 \leq t \leq t_{k-1}\}$ is a Martingale under Q_{t_k}, the elements in the third summation are derived as follows:

$$B(s)E^{Q}\left(\frac{L(t_{k-1}, t_{k-1})}{B(t_k)}\middle|\mathcal{F}_s\right) = P(s, t_k)E^{Q_{t_k}}\left(L(t_{k-1}, t_{k-1})\middle|\mathcal{F}_s\right)$$

$$= P(s, t_k)L(s, t_{k-1}).$$

Arranging the above results, the pricing formula of the FESC is

$$\frac{S(s)}{S(t_0)} - \delta\sum_{k=2}^{n} L(s, t_{k-1})P(s, t_k) - \delta L(t_0, t_0)P(s, t_1)$$

$$- P(s, t_n) - \delta K\sum_{k=1}^{n} P(s, t_k).$$

Q.E.D.

APPENDIX B

Proof of Theorem 2

Similar to the proof of Theorem 1, we also adopt the Martingale pricing method to price the FESV. The price of the FESV at time s, $t_0 \leq s \leq t_1$, is the sum of the discounted value of the expected future cash flows, i.e.,

$$V_{FESV} = \sum_{k=1}^{n} B(s)E^{Q}\left(\frac{\frac{S(t_{k-1})}{S(t_0)}\left[\frac{S(t_k)}{S(t_{k-1})} - (1 + \delta L(t_{k-1}, t_{k-1}) + \delta K\right]}{B(t_k)}\middle|\mathcal{F}_s\right)$$

$$= B(s)E^{Q}\left(\frac{\left[\frac{S(t_1)}{S(t_0)} - (1 + \delta L(t_0, t_0) + \delta K)\right]}{B(t_1)}\middle|\mathcal{F}_s\right)$$

$$+ \sum_{k=2}^{n} B(s)E^{Q}\left(\frac{\frac{S(t_{k-1})}{S(t_0)}\left[\frac{S(t_k)}{S(t_{k-1})} - (1 + \delta L(t_{k-1}, t_{k-1}) + \delta K\right]}{B(t_k)}\middle|\mathcal{F}_s\right)$$

$$= \frac{S(s)}{S(t_0)} - (1 + \delta L(t_0, t_0) + \delta K)P(s, t_1) + \frac{1}{S(t_0)}\sum_{k=2}^{n}\underbrace{B(s)E^{Q}\left(\frac{S(t_k)}{B(t_k)}\middle|\mathcal{F}_s\right)}_{\text{(I)}}$$

$$- \frac{(1+\delta K)}{S(t_0)} \sum_{k=2}^{n} \underbrace{B(s)E^{Q}\left(\frac{S(t_{k-1})}{B(t_k)} \middle| \mathcal{F}_s\right)}_{\text{(II)}}$$

$$- \frac{\delta}{S(t_0)} \sum_{k=2}^{n} \underbrace{B(s)E^{Q}\left(\frac{S(t_{k-1})L(t_{k-1},t_{k-1})}{B(t_k)} \middle| \mathcal{F}_s\right)}_{\text{(III)}}. \tag{B1}$$

We solve, respectively, parts I, II, and III as given below.

$$\begin{aligned} \text{I} &= B(s)E^{Q}\left(\frac{S(t_k)}{B(t_k)} \middle| \mathcal{F}_s\right) \\ &= S(s). \end{aligned}$$

Then, part II in Equation (B1) is derived as follows:

$$\begin{aligned} \text{II} &= B(s)E^{Q}\left(\frac{S(t_{k-1})}{B(t_k)} \middle| \mathcal{F}_s\right) \\ &= B(s)E^{Q}\left(\frac{S(t_{k-1})P(t_{k-1},t_k)}{B(t_{k-1})} \middle| \mathcal{F}_s\right) \\ &= E^{Q_{t_{k-1}}}\left(S(t_{k-1})P(t_{k-1},t_k) \middle| \mathcal{F}_s\right)P(s,t_{k-1}) \\ &= E^{Q_{t_{k-1}}}\left(\frac{S(t_{k-1})}{P(t_{k-1},t_{k-1})}\frac{P(t_{k-1},t_k)}{P(t_{k-1},t_{k-1})} \middle| \mathcal{F}_s\right)P(s,t_{k-1}). \end{aligned} \tag{B2}$$

To find the expectation in Equation (B2), define $\frac{S(t)}{p(t,\, t_{k-1})} = X(t)$ and $\frac{P(t,\, t_k)}{P(t,\, t_{k-1})} = Y(t)$. Under the measure $Q_{t_{k-1}}$, both $\{X(t)\}_{t \in [0, t_{k-1}]}$ and $\{Y(t)\}_{t \in [0,\, t_{k-1}]}$ are Martingales with the following dynamics:

$$\begin{aligned} \frac{dX(t)}{X(t)} &= [\zeta(t) + \sigma^*(t,t_{k-1})] \cdot dW_{t_{k-1}}(t), \\ \frac{dY(t)}{Y(t)} &= [\sigma^*(t,t_{k-1}) - \sigma^*(t,t_k)] \cdot dW_{t_{k-1}}(t). \end{aligned}$$

Defining $Z(t) = X(t)Y(t)$ and using Itô's lemma to $Z(t)$, we have

$$\begin{aligned} \frac{dZ(t)}{Z(t)} &= [\zeta(t) + \sigma^*(t,t_{k-1})] \cdot [\sigma^*(t,t_{k-1}) - \sigma^*(t,t_k)]dt \\ &\quad + [\zeta(t) - \sigma^*(t,t_k) + 2\sigma^*(t,t_{k-1})] \cdot dW_{t_{k-1}}(t). \end{aligned} \tag{B3}$$

According to the definition of the bond volatility process $\{\sigma^*(t, T)\}_{t\in[s,T]}$ in Equation (24), $\{\sigma^*(t, T)\}_{t\in[s,T]}$ is not deterministic. Thus, the stochastic differential Equation (B3) is not solvable and the distribution of $Z(t_{k-1})$ is unknown. However, given any fixed initial time s, we can approximate $\sigma^*(t, T)$ by $\overline{\sigma}_s(t, T)$, which is defined by

$$\overline{\sigma}_s(t, T) = \begin{cases} \displaystyle\sum_{k=1}^{[\delta^{-1}(T-t)]} \frac{\delta L(s, T-k\delta)\gamma(t, T-k\delta)}{1+\delta L(s, T-k\delta)} & t \in [0, T-\delta] \ \& \ T-\delta > 0, \\ 0 & \text{otherwise.} \end{cases} \tag{B4}$$

where $0 \leq s \leq t \leq T$. It means that the calendar time of the process $\{L(t, T)\}_{t\in[s,T]}$ in (B4) is frozen at its initial time s and thus the process $\{\overline{\sigma}_s(t, T)\}_{t\in[s,T]}$ becomes deterministic. This is the Wiener chaos order 0 approximation which BGM (1997) used to price swaptions. The accuracy of (B4) is examined in Appendix C.

Substituting $\overline{\sigma}_s(t, T)$ for $\sigma(t, T)$, Equation (B3) can be rewritten as:

$$\begin{aligned} \frac{dZ(t)}{Z(t)} &= [\zeta(t) + \overline{\sigma}_s(t, t_{k-1})] \cdot [\overline{\sigma}_s(t, t_{k-1}) - \overline{\sigma}_s(t, t_k)]\,dt \\ &\quad + [\zeta(t) - \overline{\sigma}_s(t, t_k) + 2\overline{\sigma}_s(t, t_{k-1})] \cdot dW_{t_{k-1}}(t). \end{aligned} \tag{B5}$$

In this method, the drift and volatility terms in (B5) are deterministic, so we can solve equation (B5) and find the approximate distribution of $Z(t_{k-1})$.

Solving the stochastic differential equation (B5) and substituting its solution into the expectation in (B2), we get part II as follows:

$$\begin{aligned} \text{II} &= Z(s)\exp\left(\int_s^{t_{k-1}} [\zeta(u) + \overline{\sigma}_s(u, t_{k-1})] \cdot [\overline{\sigma}_s(u, t_{k-1}) - \overline{\sigma}_s(u, t_k)]\,du\right) P(s, t_{k-1}) \\ &= S(s)\,\frac{P(s, t_k)}{P(s, t_{k-1})}\,\phi(s; t_{k-1}, t_k), \end{aligned}$$

where

$$\begin{aligned} &\phi(s; t_{k-1}, t_k) \\ &\quad = \exp\left(\int_s^{t_{k-1}} (\zeta(u) + \overline{\sigma}_s(u, t_{k-1})) \cdot (\overline{\sigma}_s(u, t_{k-1}) - \overline{\sigma}_s(u, t_k))\,du\right). \end{aligned}$$

Observing Equation (B2), $\phi(s; t_{k-1}, t_k)$ implicitly represents the correlation between the discount factor and the return from the variable principal during the period $[t_0, t_{k-1}]$.

Finally, we derive part III in Equation (B1) as follows:

$$\begin{aligned}
\text{III} &= B(s)E^{Q}\left(\frac{S(t_{k-1})L(t_{k-1},t_{k-1})}{B(t_k)}\middle|\mathcal{F}_s\right) \\
&= E^{Q_{t_{k-1}}}\left(S(t_{k-1})L(t_{k-1},t_{k-1})P(t_{k-1},t_k)\middle|\mathcal{F}_s\right)P(s,t_{k-1}) \\
&= E^{Q_{t_{k-1}}}\left(\frac{S(t_{k-1})}{P(t_{k-1},t_{k-1})}\frac{P(t_{k-1},t_k)}{P(t_{k-1},t_{k-1})}L(t_{k-1},t_{k-1})\middle|\mathcal{F}_s\right)P(s,t_{k-1}) \\
&= E^{Q_{t_{k-1}}}\left(Z(t_{k-1})L(t_{k-1},t_{k-1})\middle|\mathcal{F}_s\right)P(s,t_{k-1}). \qquad \text{(B6)}
\end{aligned}$$

Under Q_{t_k}, the dynamics of $L(t, t_{k-1})$ is

$$dL(t, t_{k-1}) = L(t, t_{k-1})\gamma(t, t_{k-1}) \cdot dW_{t_k}(t).$$

Using Girsanov's theorem, the dynamics of $L(t, t_{k-1})$ under $Q_{t_{k-1}}$ is as follows:

$$\begin{aligned}
dL(t,t_{k-1}) = {}& L(t,t_{k-1})\gamma(t,t_{k-1}) \cdot (\sigma^*(t,t_k) - \sigma^*(t,t_{k-1}))\,dt \\
&+ L(t,t_{k-1})\gamma(t,t_{k-1}) \cdot dW_{t_{k-1}}(t).
\end{aligned}$$

Defining $f(t = Z(t)L(t, t_{k-1})$ and using Itô's lemma, we have

$$\begin{aligned}
\frac{df(t)}{f(t)} &= [\gamma(t,t_{k-1}) \cdot (\sigma^*(t,t_k) - \sigma^*(t,t_{k-1})) \\
&\quad + \gamma(t,t_{k-1}) \cdot (\zeta(t) - \sigma^*(t,t_k) + 2\sigma^*(t,t_{k-1})) \\
&\quad + (\zeta(t) + \sigma^*(t,t_{k-1})) \cdot (\sigma^*(t,t_{k-1}) - \sigma^*(t,t_k))]\,dt \\
&\quad + [\zeta(t) - \sigma^*(t,t_k) + 2\sigma^*(t,t_{k-1}) + \gamma(t,t_{k-1})] \cdot dW_{t_{k-1}}(t) \\
&= \gamma(t,t_{k-1}) \cdot (\zeta(t) + \sigma^*(t,t_{k-1}))\,dt \\
&\quad + (\zeta(t) + \sigma^*(t,t_{k-1})) \cdot (\sigma^*(t,t_{k-1}) - \sigma^*(t,t_k))\,dt \\
&\quad + (\zeta(t) - \sigma^*(t,t_k) + 2\sigma^*(t,t_{k-1}) + \gamma(t,t_{k-1})) \cdot dW_{t_{k-1}}(t). \qquad \text{(B7)}
\end{aligned}$$

Similarly, the drift and volatility terms in (B7) are stochastic and we cannot solve the distribution of $f(t_{k-1})$. By substituting $\overline{\sigma}_s(u, \cdot)$, defined in Equation (B4), for $\sigma^*(u, \cdot)$, (B7) can be rewritten as

$$\frac{df(t)}{f(t)} = \gamma(t, t_{k-1}) \cdot (\zeta(t) + \overline{\sigma}_s(t, t_{k-1}))\,dt$$
$$+ (\zeta(t) + \overline{\sigma}_s(t, t_{k-1})) \cdot (\overline{\sigma}_s(t, t_{k-1}) - \overline{\sigma}_s(t, t_k))\,dt$$
$$+ (\zeta(t) - \overline{\sigma}_s(t, t_k) + 2\overline{\sigma}_s(t, t_{k-1}) + \gamma(t, t_{k-1})) \cdot dW_{t_{k-1}}(t). \quad \text{(B8)}$$

Now, the drift and volatility terms in Equation (B8) are deterministic and the approximate distribution of $f(t_{k-1})$ can be solved.

After solving $f(t_{k-1})$ in Equation (B8) and taking it into the expectation of Equation (B6), we get part III as follows:

$$\text{III} = S(s)\frac{P(s, t_k)}{P(s, t_{k-1})} L(s, t_{k-1})\,\phi\,(s; t_{k-1}, t_k)\rho(s; t_{k-1}, t_k),$$

where $\rho(s; t_{k-1,}\, t_k)$ is defined by

$$\rho(s; t_{k-1}, t_k) = \exp\left(\int_s^{t_{k-1}} \gamma(u, t_{k-1}) \cdot (\zeta(u) + \overline{\sigma}_s(u, t_{k-1}))\,du\right).$$

With the above results, the price of the FESV is given by

$$\frac{S(s)}{S(t_0)}\Bigg\{ n - \sum_{k=2}^{n}\Bigg[(1 + \delta K)\frac{P(s, t_k)}{P(s, t_{k-1})}\phi\,(s; t_{k-1}, t_k)$$
$$+ \delta L(s, t_{k-1})\frac{P(s, t_k)}{P(s, t_{k-1})}\phi\,(s; t_{k-1}, t_k)\rho(s; t_{k-1}, t_k)\Bigg]\Bigg\}$$
$$- (1 + \delta L(t_0, t_0) + \delta K)P(s, t_1).$$

Q.E.D.

APPENDIX C

Examining the Accuracy of the Approximation in Equation (B4)

Because the quotient $\delta L(t, t_i)/(1 + \delta L(t, t_i))$ has low variance under the forward measure Q_{t_i+1}, the calendar time of the process $L(t, \cdot)$ in Equation (B4) could be frozen at its initial time. This argument first appears in Brace et al. (1997). It was developed further in Brace, Dun, and Barton (1998) and formalized by Brace and Womersley (2000). The approximation also appears in Schlögl (2002). Here, we use Monte Carlo simulation to examine the accuracy of the approximation.

Tables C2–4 are drawn from the DataStream database. Via the calibration procedure described in the Calibration and Numerical Examples section 2 to calibrate the model parameters from the market data in Tables C2–C4, we employ Monte Carlo simulation to simulate the stochastic process of the LIBOR rates in two cases. Case 1 is implemented without the approximation assumption, i.e. $L(t, \cdot)$ in Equation (B4) is not frozen at its initial time. On the other hand, Case 2 is implemented with the approximation assumption.

Table C1 presents the result. By observing Table C1, the percentage of relative error becomes larger as the maturity of the LIBOR rate gets longer. Moreover, the percentage of relative error grows larger if the term structure of the volatility becomes larger. Based on the simulation result, the approximation technique seems robust over the past 2 years. Even on the date September 30, 2004, when the LIBOR rates were most volatile, the percentage relative error of the realized LIBOR rate, $L(9, 9)$, is only 3.5487% (the largest percentage relative error), which is still acceptable.

We have justified the approximation by Monte Carlo simulation. The result supports the accuracy of the approximate formula in Theorem 2.

TABLE C1
The Percentage Relative Errors

%	3/31/06	12/30/05	9/30/05	6/30/05	3/31/05	12/31/04	9/30/04	6/30/04
1	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
2	0.0013	0.0022	0.0036	0.0045	0.0044	0.0131	0.0344	0.0458
3	0.0101	0.0179	0.0240	0.0331	0.0278	0.0861	0.1902	0.1368
4	0.0337	0.0616	0.0750	0.1074	0.0827	0.2463	0.4850	0.2729
5	0.0700	0.1274	0.1498	0.2138	0.1785	0.4346	0.7617	0.3956
6	0.1516	0.2747	0.3122	0.4389	0.3212	0.8289	1.3732	0.6574
7	0.2697	0.4875	0.5363	0.7498	0.5234	1.3061	2.0273	0.8942
8	0.4270	0.7702	0.8208	1.1347	0.7816	1.8231	2.7688	1.1984
9	0.6377	1.1370	1.1875	1.6200	1.0998	2.4021	3.5487	1.5072

Note. If the LIBOR rates in Case 1 are viewed as the benchmark, the percentage of relative errors of the rates in Case 2 caused by the approximation assumption, are presented in Table III. Each result is based on 100,000 sample paths.

TABLE C2
Cap Volatilities Quoted in the U.S. Market

%	3/31/06	12/30/05	9/30/05	6/30/05	3/31/05	12/31/04	9/30/04	6/30/04
1	9.93	11.3	13.93	14.39	13.92	19.38	25.53	35.01
2	13.6	15.62	17.31	19.18	18.31	24.56	31.1	31.7
3	15.21	17.81	19.13	21.35	19.81	26.44	31.86	28.84
4	16.06	18.96	19.89	22.33	20.23	26.53	31.1	26.97
5	16.56	19.48	20.31	22.72	20.24	26.08	29.79	25.4
7	17.01	19.97	20.46	22.6	19.82	24.81	27.58	22.92
10	16.98	19.8	19.85	21.58	18.76	22.19	24.37	20.21

Note. The quoted volatilities of the caps in the U.S. market over the past 2 years are presented quarterly in this table. The data for years 6, 8, and 9 can be obtained by an interpolation technique.

TABLE C3
Initial Forward LIBOR Rates and Initial S&P500 Index

%	*3/31/06*	*12/30/05*	*9/30/05*	*6/30/05*	*3/31/05*	*12/31/04*	*9/30/04*	*6/30/04*
S&P500	1294.83	1248.29	1228.81	1191.33	1180.59	1211.92	1114.58	1140.84
0	5.478	4.976	4.562	3.970	3.917	3.162	2.505	2.491
1	5.456	5.075	4.899	4.207	4.804	3.946	3.602	4.088
2	5.449	5.007	4.809	4.317	5.017	4.269	4.196	4.905
3	5.532	5.064	4.868	4.408	5.155	4.631	4.644	5.427
4	5.627	5.118	4.926	4.536	5.288	4.989	4.999	5.805
5	5.651	5.118	5.022	4.630	5.387	5.269	5.308	6.109
6	5.658	5.143	5.053	4.710	5.544	5.469	5.581	6.311
7	5.698	5.168	5.117	4.874	5.587	5.730	5.764	6.417
8	5.738	5.313	5.242	4.946	5.689	5.866	5.945	6.580
9	5.851	5.300	5.325	5.069	5.843	6.072	6.096	6.708
10	5.831	5.382	5.410	5.118	5.772	6.092	6.188	6.745

Note. The forward LIBOR rates and S&P500 index in the U.S. market over the past 2 years are represented quarterly in this table. The rates are obtained from the associated bond prices derived from the zero curves obtained from the DataStream database.

TABLE C4
Forward LIBOR Rate Correlation

	1	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*Stock*
1	1	0.9821	0.9637	0.8911	0.8365	0.8264	0.7375	0.6668	0.5901	0.7265
2	0.9821	1	0.9933	0.9309	0.8922	0.9072	0.8392	0.7761	0.7087	0.6456
3	0.9637	0.9933	1	0.9425	0.9109	0.9373	0.879	0.8235	0.7605	0.6136
4	0.8911	0.9309	0.9425	1	0.7578	0.9116	0.8668	0.8206	0.7549	0.5279
5	0.8365	0.8922	0.9109	0.7578	1	0.9117	0.8841	0.8471	0.8179	0.4718
6	0.8264	0.9072	0.9373	0.9116	0.9117	1	0.9771	0.9505	0.9162	0.3927
7	0.7375	0.8392	0.879	0.8668	0.8841	0.9771	1	0.9721	0.9556	0.284
8	0.6668	0.7761	0.8235	0.8206	0.8471	0.9505	0.9721	1	0.9576	0.1916
9	0.5901	0.7087	0.7605	0.7549	0.8179	0.9162	0.9556	0.9576	1	0.122
Stock	0.7265	0.6456	0.6136	0.5279	0.4718	0.3927	0.284	0.1916	0.122	1

Note. One-year data (April 1, 2005 through March 31, 2006) are used to calculate this correlation matrix.

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