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Legal restrictions and sunspots: A further inquiry on the real-bills doctrine versus the quantity theory debate

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Abstract

This paper argues that economic fluctuations before the Peel's Bank Act were caused by extrinsic random events relating to possible suspension of convertibility of the Bank of England. We show that there are economies in which (i) sunspot equilibria exist in the absence of the legal restrictions espoused by the quantity theory; (ii) the restrictions can be used to eliminate those equilibria; (iii) the proposals of the real-bills doctrine improve the outcomes of the quantity theory restrictions as well as free banking; and (iv) the welfare improvement of the real-bills regime can be accomplished only with the legal restrictions imposed.
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1. Introduction

In the second quarter of the 19th century Britain had witnessed a series of vigorous public debates resulting in the passage of Peel's Bank Act in 1844. Although the Act had profound influence on the development of monetary economics it is not until very recently that inquisitors such as Sargent and Wallace (1982), Smith (1988), Mourmouras and Russel (1992) started rigorous analysis regarding the causes that precipitated the Act. [1] SW have since organized the discussion around two opposing doctrines: the quantity theory of money and the real-bills doctrine. They constructed a model in which fluctuations in the demand and supply for private credits cause price-level fluctuations. Quantity theorists were identified with proponents of legal restrictions [2] that prohibit private banks from issuing small denomination liabilities, and make the central bank a monopoly issuer of currency-like assets. The real-bills doctrinists were identified with proponents of free banking, i.e., unfettered private intermediation. Under these representations, SW argued that the quantity theory restrictions produce Pareto inefficient allocations, while the real-bills doctrine is potentially consistent with Pareto efficiency, if the central bank pursued an open-market discretionary policy meant to vitiate the quantity theory restrictions.

The attributed source of fluctuations by SW was not shared by many advocates of Peel's Act, as they seemed to have believed that those fluctuations were driven by "speculative" forces. In this vein, Smith (1988) has constructed a model of "sunspots"—randomness that has nothing to do with the economic fundamentals. He argued that the kind of legal restrictions embodied in the Act were meant to separate money from credit markets. Smith showed that real economic variables (including price levels) might respond to sunspots, hence, fluctuate excessively in an economy without the legal restrictions; and the occurrence of sunspot equilibria could be precluded once the restrictions were imposed.

Instead of separating money from credit as Sargent–Wallace–Smith claimed, Mourmouras and Russel (1992) argued that Peel's Act was meant to put a quantity limit on the private credit. Following Smith they also construct "nonfundamental" equilibria, and show that the Act could be used to preclude such equilibria. A key feature of the kind of nonfundamental equilibria Mourmouras and Russel considered is that currency might lose value. This distinctive property is contrasted with Smith's sunspot equilibria in which currency never loses value and randomness affecting the economy is a recurrent phenomenon.

Mourmouras and Russel's contribution begs a new question: Why would currency (notes issued by Bank of England) lose value given that Britain had resumed the specie convertibility of notes issued by Bank of England since 1821? In this paper we will argue that although during the two decades before the passage of Peel's Act, the

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[1] Although SW did not mention the Act explicitly, it was clear from Sargent (1987, p. 256, fn. 2) that their 1982 paper was inspired by it.

[2] Except in Cesarano (1994), it is shown that, contrary to some interpretations, M. Friedman's work on the optimum quantity of money does not support the legal restrictions theory but follows a diametrically opposite approach.

Bank of England had never suspended such convertibility, there is documented evidence that the suspension of currency convertibility (into specie) is indeed considered a possibility. In the sunspot equilibria characterized below currency might lose value with some positive probabilities. That will be interpreted as *potential* suspension of currency convertibility.

We model the quantity theory restriction as granting the government a monopoly of note issues. The real-bills doctrine is strictly interpreted as central bank discretionary strategy intended to vitiate the restriction. In a laissez-faire economy absent of the quantity theory restriction, there are sunspot equilibria in which the price level, the stock of inside money, and other equilibrium quantities might display variations not due to relevant "economic fundamentals". It will then be shown that there are economies in which (a) the quantity theory restriction can be used to eliminate the sunspot equilibria; (b) the real-bills doctrine improves, in a unequivocal way, the outcomes of the quantity theory regime as well as the free banking (laissez-faire) regime; and (c) the welfare improvement of the real-bills regime can be accomplished only with the legal restrictions imposed. To rule out fluctuations caused by this possibility, legal restrictions are called. Yet, the restrictions also create their own kind of inefficiency. The real-bills doctrine presents a solution to this inefficiency. Somewhat paradoxically, the prescription of the doctrine vitiates the restrictions and achieves Pareto optimality—all been done in the name of normal bank operations.

This paper proceeds as follows. The basic model is presented in the next section. As SW and Smith, we first examine potential equilibria under free banking. It will be seen that not only sunspot equilibria exist, but most sunspot interpretations in Smith may be preserved in a construct that is similar to (and even simpler than) that of SW. In Section 3, we introduce the legal restriction that has the effect of granting the government a monopoly of note issues. The equilibria under the quantity theory regime and the real-bills regime are then studied. In this section, we discuss several related issues. These include the interaction between the preferences and the effect of legal restrictions, the relationship between dynamic efficiency and legal restrictions. We conclude the paper in Section 4.

2. The model

The model is an infinite-horizon economy with two-period-lived overlapping generations similar to one in Smith (1989). In this economy, all goods are nonstorable. It is assumed that each generation is identical in size and composition, and has two types of individuals: savers and borrowers. It is also assumed that savers constitute half of each generation and borrowers the other half. [3] Agents are identical with respect to their endowments and preferences within each group. A typical saver in

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[3] In Smith (1989), there is a parameter $1 > \lambda > 0$ describing the fraction of savers in each generation. Our assumption amounts to $\lambda = 1/2$. None of Smith's results bears on this simplification. The only difference is that, when imposing the legal restriction, we cannot interpret the restriction as precluding "small" depositors as SW did.

each generation has a strictly positive endowment α when he is young and nothing when he is old. The saver's preference is represented by $\ln c_1 + \ln c_2$. c_j denotes the consumption of the agent in their jth period of life. A typical borrower in each generation has a strictly positive endowment β when he is old but nothing when he is young. The borrower's preference is represented by c_1. It is assumed that

$$\alpha/2 > \beta. \tag{1}$$

Similar to Smith's condition (i), condition (1) here is necessary for the existence of a monetary equilibrium in this overlapping-generations economy.

There is a durable asset called "currency". It is assumed, without loss of generality, that a typical initial old saver holds one unit of currency and purchases goods from contemporary young generation. The latter accept currency in anticipation of future purchases. Following SW, we also assume that there exist zero-cost, competitive intermediaries which bring together borrowers and savers, and call such intermediaries "banks". The equilibrium studied below therefore carries an interpretation of free banking equilibrium.

The fundamentals of the economy described so far are deterministic. Extrinsic uncertainty in the sense of Cass and Shell (1983) is now introduced. It is assumed that there are n distinct states of nature. Each state is indexed by $i \in N = \{1, \ldots, n\}$. The evolution of these states is described by a first-order, time-invariant Markov process whose transition matrix is represented by $[\pi_{ij}]$, where π_{ij} denotes the conditional probability of realization of state j, given state i.

Under the assumptions made above, young borrowers will never accumulate currency, while young savers can either sell goods in exchange for currency, or lend to young borrowers. Let s_t^i denote the price of currency in terms of date-t goods, m_t^i the per capita accumulation of real balances, and x_t^i the amount of goods lent by a representative young saver. One unit of goods lent at t repays R_t^i units at date $t + 1$, so R_t^i is unity plus real interest rate. [4] Conditional on the state at t, savers of each generation maximize

$$\ln(\alpha - m_t^i - x_t^i) + \sum_{j=1}^{n} \pi_{ij} \ln \left[R_t^i x_t^i + \left(\frac{s_{t+1}^j}{s_t^i} \right) m_t^i \right],$$

subject to $m_t^i \geqslant 0$ and $\alpha \geqslant m_t^i + x_t^i$. The necessary and sufficient condition for individual's optimization is

$$\sum_{j=1}^{n} \pi_{ij} \frac{R_t^i}{x_t^i R_t^i + (s_{t+1}^j / s_t^i) m_t^i} = \sum_{j=1}^{n} \pi_{ij} \frac{s_{t+1}^j / s_t^i}{x_t^i R_t^i + (s_{t+1}^j / s_t^i) m_t^i}. \tag{2}$$

Following the convention of SW and Smith, we study the stationary equilibrium of the economy. A stationary equilibrium consists of time-invariant prices and allocations such that all markets clear. Given the logarithmic utility function, per capita saving is

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[4] The subscript denotes the date and the superscript denotes the state of nature.

$$m^i + x^i = \alpha/2. \tag{3}$$

Let μ^i denote the fraction of savings of each saver accumulated in the form of currency, while $1 - \mu^i$ is the fraction of savings in the form of loan. The money market clearing requires

$$m^i \equiv \mu^i \left(\frac{\alpha}{2}\right) = s^i. \tag{4}$$

The RHS of (4) is the supply of real balances per capita and the LHS of (4) is the demand for real balances per capita. The credit market clearing requires

$$x^i \equiv (1 - \mu^i)\left(\frac{\alpha}{2}\right) = \left(\frac{\beta}{R^i}\right), \tag{5}$$

where the RHS is demand for loan of each young borrower. Finally, individual optimization implies that (2) can be rewritten as

$$\sum_{j=1}^{n} \pi_{ij} \frac{s^i R^i}{\beta + s^j} = \sum_{j=1}^{n} \pi_{ij} \frac{s^j}{\beta + s^j}. \tag{6}$$

A preliminary examination of the system of equations (3)–(6) reveals that there are two deterministic solutions. The first is a deterministic monetary equilibrium with constant real balances,

$$s^i = s^* \equiv \alpha/2 - \beta, \quad \forall i. \tag{7}$$

For future references, we will refer to it as the "LF-I" equilibrium. In this equilibrium, savers of the initial old generation consume $\alpha/2$ and savers of each generation thereafter consume $(\alpha/2, \alpha/2)$. Borrowers of the initial old generation consume nothing and borrowers of each generation thereafter consume $(\beta, 0)$. The general price level is constant at $(\alpha/2 - \beta)^{-1}$. The real interest rate on private loans is equal to zero, so is the nominal interest rate.

The second solution corresponds to an equilibrium with $s^i = 0$ all $i \in N$. For future references, we will refer to it as the "LF-II" equilibrium. In this equilibrium, only private credits are in circulation. The initial old savers consume β and all subsequent savers consume $(\alpha/2, \beta)$. The initial old borrowers consume nothing and all subsequent borrowers consume $(\alpha/2, 0)$. The real interest rate is equal to $2\beta/\alpha - 1$, which is negative by (1).

For the deterministic free banking equilibria above, the extrinsic uncertainty does not affect equilibrium prices and allocations. In the rest of this section, we show that they are not the only equilibria. Any other equilibrium, if exists, may be properly called "sunspot equilibrium" according to Cass and Shell. In the following, we first establish a result that rules out a certain class of sunspot equilibria.

Let $\gamma = \text{argmax}_{i \in N} s^i$. It is noted, from (3)–(5), that $s^\gamma > \alpha/2 - \beta$ if and only if $R^\gamma > 1$. In this case, Eq. (6) implies

$$\sum_{j=1}^{n} \pi_{\gamma j} \frac{s^\gamma}{\beta + s^j} < \sum_{j=1}^{n} \pi_{\gamma j} \frac{s^\gamma R^\gamma}{\beta + s^j} = \sum_{j=1}^{n} \pi_{\gamma j} \frac{s^j}{\beta + s^j} \leqslant \sum_{j=1}^{n} \pi_{\gamma j} \frac{s^\gamma}{\beta + s^j},$$

a contradiction. Therefore, $0 \leqslant s^i \leqslant \alpha/2 - \beta$, all i. On the other hand, let $\tau = \text{argmin}_{i \in N} s^i$. By the same reasoning, Eqs. (3)–(5) imply that $0 < s^\tau < \alpha/2 - \beta$ if and only if $R^\tau < 1$, and Eq. (6) implies

$$\sum_{j=1}^{n} \pi_{\tau j} \frac{s^\tau}{\beta + s^j} > \sum_{j=1}^{n} \pi_{\tau j} \frac{s^\tau R^\tau}{\beta + s^j} = \sum_{j=1}^{n} \pi_{\tau j} \frac{s^j}{\beta + s^j} \geqslant \sum_{j=1}^{n} \pi_{\tau j} \frac{s^\tau}{\beta + s^j}.$$

Again, a contradiction. Combining these two contradictions, we have

Theorem 1. *There are no (first-order Markov) stationary sunspot equilibria with $s^i > 0$ for all $i \in N$.* [5]

Theorem 1 directs our attention to solutions of (3)–(6) that the price of money, s^i, may be zero in some, but not all, states of nature. We will show that there is a class of sunspot solutions satisfying (6) with the value of currency being zero in some states of nature.

To recapitulate, note from (3)–(5), Eq. (6) may be rewritten as

$$\frac{\beta s^i}{(\alpha/2) - s^i} = \sum_{j=1}^{n} \frac{\pi_{ij} s^j / \beta + s^j}{\sum_{k=1}^{n} \pi_{ik}/\beta + s^k}. \tag{8}$$

We are ready to state our next result.

Theorem 2. *Suppose there exists an absorbing state 'n'. That is, $\pi_{in} = 0$, $i = 1, \ldots, n - 1$, and $\pi_{nn} = 1$. If, in addition,*

$$1 - \frac{\beta}{\alpha} > \pi_{in}, \quad i = 1, \ldots, n - 1. \tag{9}$$

Then, for a given $[\pi_{ij}]$, Eq. (8) has a unique solution within the class of solutions that $s^* \geqslant s^i > 0$, $i \neq n$ and $s^n = 0$.

Proof. See Appendix A. $\square$

The stochastic process in Theorem 2 is a generalization of the class of uncertainty considered by Blanchard (1979). Works related to this class of uncertainty include, among others, Blanchard and Watson (1982), Obstfeld and Rogoff (1986), and Weil (1987).

Roughly, Theorem 2 says that the extrinsic uncertainty will affect an equilibrium with positive probability, but will not do so eventually with probability one. [6] Given the interpretation that the nth absorbing state is such that the suspension of convertibility is permanent, two points need to be noted here. First, that the absorbing state

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[5] This theorem also appears as the result proved in Appendix A in Smith (1989), who mistakenly stated the theorem without the proviso.

[6] In contrast, Smith (1988, 1989) considered the class of extrinsic uncertainty that will matter *forever*.

in which fiat money loses value is a permanent state of affairs, whereas historically suspensions were (and were meant to be) temporary and were meant to alleviate banks' difficulties meeting abnormally high levels of demand for specie. And, second, that suspensions were triggered by relatively well understood events, such as wars- not by extrinsic uncertainties.

It should be noted that not all sunspot equilibria described in Theorem 2 are interesting. For instance, there are two sets of absorbing states, $\{1,\ldots,n-1\}$ and $\{n\}$. Then, since $\pi_{in} = 0$ all $i \neq n$, $s^i = s^*$ all $i \neq n$ and $s^n = 0$. This sunspot equilibrium is a pure randomization of the LF-I and LF-II equilibria. Sunspots matter but only in the beginning of the horizon. The following example builds on the two sets of absorbing states, and illustrates equilibrium fluctuations that resemble "cycles", if starting from the transient states.

Example. Let $\alpha = 4$, $\beta = 1$, $n = 4$, with the transition probability matrix

$$(\pi_{ij}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0.1 & 0.1 & 0.8 & 0 \\ 0 & 0.8 & 0.1 & 0.1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The prices of currency can be solved as $s^1 = 1 = s^2 = s^3$, and $s^4 = 0$. The (unity plus) real interest rates on inside money are $R^1 = 1 = R^2 = R^3$, and $R^4 = 0.50$. The stocks of inside money are $x^1 = 1 = x^2 = x^3$, and $x^4 = 2$.

For future references, we will refer to the sunspot equilibria established in Theorem 2 as the "LF-III" equilibria. These equilibria have several interesting features. First, as Smith (1988, p. 8) quoted White's (1984) interpretation of some versions of the Currency School, "high prices produce high volume of inside indebtedness". This can be seen as $1 - \mu^i > 1 - \mu^j$ if and only if $1/s^i > 1/s^j$. Second, according to Joplin, quoted by White (1984, p. 100) and cited in Smith (1988, p. 8), "the interest of money, when it is abundant, is not reduced, but the circulation ... is diminished". This is also the case that the market interest rate remains constant, while the stock of inside indebtedness, $1 - \mu^i$, may increase, and the level of real balances, $m^i (=s^i)$, may decrease. Third, our model predicts that the expansion of banking reduces the real interest rate. This can be seen as $1 - \mu^i > 1 - \mu^j$ if and only if $R^i < R^j$. [7] Finally, as Laidler (1984, pp. 150–151) discussed the view of Adam Smith concerning excessive price fluctuation caused by the expansion of unrestricted private banks, the economy exhibited "rising" price levels in general. This can be verified as follows. From (7) and Theorem 2, $(\alpha - \beta)/2 = s^* \geqslant s^i$ all $i \in N$. It then follows that

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[7] This implication is different from Smith (1988, p. 14), but consistent with what SW (1982, pp. 1228–1229) referred to as the social productivity of banking of Adam Smith. It should be noted that Laidler did not concur this interpretation.

$$s^i \geqslant \frac{(\beta/2)s^i}{(\alpha/2) - s^i} = \sum_{j=1}^{n} \pi_{ij}s^j, \quad i = 1, \ldots, n-1.$$

That is, the expected inflation rate, $(s^i/\sum_{j=1}^{n}\pi_{ij}s^j) - 1$, is nonnegative.

3. Equilibrium under legal restrictions

In this section, we assume a legal restriction that forbids banks from issuing liabilities smaller than some minimal denomination in terms of the date-t goods. Denote such a denomination by $\underline{x}$ and suppose that $\underline{x} > \alpha$. [8]

3.1. The quantity theory regime

The legal restriction prevents savers lent to banks. The only asset available to savers is currency. With borrowers now precluded from the market, the model is a standard overlapping-generations model of one good, homogeneous agent. It has been subjected to numerous studies. A unique deterministic monetary equilibrium, which we will be referring to as the "QT-I" equilibrium, can be found as follows. Savers of the initial old generation maximize consumptions. Savers of all following generations maximize

$$\ln(\alpha - m_t^i) + \sum_{j=1}^{n} \pi_{ij} \ln \left[\left(\frac{s_{t+1}^j}{s_t^i} \right) m_t^i \right].$$

Given the logarithmic utility function, per capita saving m_t^i is equal to $\alpha/2$. In equilibrium, savers consume $(\alpha/2, \alpha/2)$ through out their lives. Borrowers consume their endowments $(0, \beta)$. The general price level is constant at $(\alpha/2)^{-1}$, which is lower than the price levels under any free banking equilibria. That the equilibrium is unique is seen as a result of Balasko and Shell (1981). Note that equilibria with zero real balances (i.e., the LF-II and LF-III equilibria) are no longer viable under the legal restriction. This is because savers' utilities in cashless states are negative infinite. Anticipating this, currency will never lose value.

3.2. The real-bills regime

In this regime we consider a central bank discount-window strategy intended to vitiate the legal restriction imposed at the beginning of this section. Specifically, let the government stand ready at each date t to grant safe one-period loans in the form of newly printed currency at a zero nominal interest rate: if someone borrows h units of currency at t, he must pay back h units at $t + 1$. It is assumed that the denomination restriction imposed before is still effective on all private loans, except for this type of government loans.

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[8] This restriction is also implied by a 100% reserve requirement as discussed in SW.

We now show that the LF-I equilibrium is the unique equilibrium under the real-bills regime. To verify this claim, it is noted that, because the government is granting zero interest loans, the real return on currency is $\sum_{j=1}^{n} \pi_{ij} s^j / s^i$. The aggregate supply of currency at date t consists of the supply of existing real balances from the saver s^i, and the supply of real balances from the borrower $(\beta/2)(s^i/\sum_{j=1}^{n} \pi_{ij} s^j)$. The aggregate demand for currency at date t consists of desired saving of the saver $\alpha/2$ and the demand for real balances of the borrower to pay off loans $\beta/2$. The money market equilibrium requires

$$s^i + \frac{\beta s^i}{\sum_{j=1}^{n} \pi_{ij} s^j} = \frac{\alpha}{2} + \beta. \tag{10}$$

We claim that $s^i = \alpha/2$ all $i \in N$ is the unique solution of (10). To see that, let us rewrite (10) as

$$\frac{\alpha}{2} = s^i + \beta \left(\frac{s^i}{\sum_{j=1}^{n} \pi_{ij} s^j} - 1 \right).$$

If $\max_i s^i \equiv s^\gamma > \alpha/2$, then $(s^\gamma/\sum_{j=1}^{n} \pi_{\gamma j} s^j) - 1$ is nonnegative. This implies

$$\frac{\alpha}{2} = s^\gamma + \beta \left(\frac{s^\gamma}{\sum_{j=1}^{n} \pi_{\gamma j} s^j} - 1 \right) > \frac{\alpha}{2}.$$

A contradiction. On the other hand, if $\min_i s^i \equiv s^\tau < \alpha/2$, then $(s^\tau/\sum_{j=1}^{n} \pi_{\tau j} s^j) - 1$ is nonpositive. This implies

$$\frac{\alpha}{2} = s^\tau + \left(\frac{\beta s^\tau}{\sum_{j=1}^{n} \pi_{\tau j} s^j} - 1 \right) < \frac{\alpha}{2}.$$

Again a contradiction. Therefore, $s^i = \alpha/2$ all $i \in N$, and the LF-I equilibrium is the only equilibrium under the real-bills regime.

3.3. Welfare comparisons of regimes

Just as both SW and Smith have concluded, the quantity theory regime and the free banking regime are certainly not Pareto comparable. Savers of each generation are never worse off under the quantity theory regime than under the free banking regime, but borrowers are never better off. It is also clear that the real-bills regime Pareto dominates the quantity theory regime, since borrowers are always better off, while savers are never worse off. In comparison with the free banking regime, the real-bills regime claims its social superiority since it achieves the unique monetary equilibrium which is also Pareto optimal. [9]

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[9] Sproul (1994, 1998) argued that the real bills doctrine has been rejected on the grounds that it places no adequate limits on money creation and therefore gives no safeguard against inflation. He pointed out several major flaws in the criticisms of the real bills doctrine and then suggested it is the dominant theory of money.

4. Conclusions

In many respects, this paper has reached conclusions similar to, but more forthright than, those obtained by SW or Smith. For example, it is the case that those authors had to defend for price-level instability (or even price-level indeterminacy) of the real-bills doctrine, while we can speak, without ambiguity, for its price-level stability. It is also the case that, in terms of Pareto criterion, the real-bills regime holds a clear advantage over free banking and the quantity theory regime. Our results seem to be more conducive to the history of monetary economics.

There is another case in which our conclusions have more to offer. As noted in the introduction, the sunspot interpretation of Smith relies on the assumption that the aggregate borrowing function is increasing in the real interest rate. That assumption is seen as unnecessary. Any economy that generates the coexistence of inside and outside money will presumably have sunspot equilibria that fit our discussion. This suggests that the sunspot interpretation of the historical arguments leading to the passage of Peel's Bank Act is applicable to more general economies than previously thought. [10]

Finally, it is our hope that this paper conveys the message, which may or may not be transparent from the analytical literature, but relevant to the historical as well as on-going debates between the real-bills proponents and the quantity theorists. That is, given the legal restrictions imposed on an economy that have the effect of separating money from credit markets, the real-bills proposals of central bank discretion may have desirable effects. However, the doctrine must not be unduly credited—if it is to be interpreted as a principle for normal banking operations in an otherwise laissez-faire economy.

Acknowledgment

Appendix A

Proof of Theorem 2. For future references, we first define $s(\phi)$ as the solution to

$$\frac{(\beta/2)s}{(\alpha/2) - s} = \phi \geqslant 0.$$

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[10] One caveat in applying the sunspot interpretation of this paper is in order. We have described a class of sunspot equilibria in which sunspots matter only in the transient stage. Certainly, we do not know the length of such a stage, as it depends on the probabilities of reaching the absorbing states.

It is straightforward to show that $s(0) = 0$ and $s(\phi)$ is a strictly increasing function of ϕ. Denote $\mathscr{L}$ the space of bounded sequences in R^{n-1} with the max norm. Define an operator $G = (g^1, \ldots, g^{n-1})$ mapping $\mathscr{L}$ into $\mathscr{L}$ by

$$g^i(f) = \left\{ \sum_{j=1}^{n-1} \pi_{ij} s(f^j) \right\}.$$

If G has a fixed point f, then $(s(f), 0)$ corresponds to a solution to (8).

In the following we show that (i) there exists a finite number $\bar{\psi}$ such that $G(\mathscr{L}(\bar{\psi})) \subset \mathscr{L}(\bar{\psi})$ where $\mathscr{L}(\bar{\psi}) \equiv \{ s \in \mathscr{L} : \bar{\psi} \leqslant s^i < \infty \}$; (ii) G has a unique fixed point in $\mathscr{L}(\bar{\psi})$; and (iii) G has a unique, strictly positive, fixed point in general.

To proceed, first note that the proof of Theorem 1 implies that all solutions of (8) must not be greater than s^*. Denote ψ^* the solution for $s^* = s(\psi^*)$. Without loss of generality, we can restrict our attention to the set $\mathscr{L}(\bar{\psi}) = \{ s \in \mathscr{L} : \bar{\psi} \leqslant s^i \leqslant \psi^* \}$ when locating the fixed point.

Next, consider the greatest lower bound $\bar{\psi}$ of all positive ψ satisfying

$$\frac{1}{\gamma} \leqslant \frac{\alpha/2}{(\beta/2) + (\psi/\gamma)},$$

where $1 \geqslant \gamma = \min_i 1 - \pi_{in} > \beta/\alpha$ from (9). For any $f \in \mathscr{L}(\bar{\psi})$,

$$g^i(f) = \sum_{j=1}^{n-1} \pi_{ij} s(f^j) \geqslant (1 - \pi_{in}) \min_j s(f^j) \geqslant \gamma \left[\frac{\alpha/2}{(\beta/2) + f^\alpha} \right] f^\alpha.$$

Here, we will show that

(I) $\| \ln G(f) - \ln G(h) \| \leqslant \rho \| \ln f - \ln h \|, \quad$ all $f, h \in \mathscr{L}(\bar{\psi}),$

where $\rho \in [0, 1)$ and $\| \cdot \|$ is the Euclidean norm. As shown in Stokey and Lucas (1989, pp. 511–513), this implies that T has a unique fixed point in $\mathscr{L}(\bar{\psi})$.

To establish (I), notice that for any strictly positive sequences f and h, we have

$$\left| \ln \sum_{j=1}^{n-1} s^j(f) - \ln \sum_{j=1}^{n-1} s^j(h) \right| = \left| \ln \sum_{j=1}^{n-1} \frac{s^j(h)}{\sum_{k=1}^{n-1} s^k(h)} \frac{s^j(f)}{s^j(h)} \right| \leqslant \sup_j |\ln s^j(f) - \ln s^j(h)|.$$

Applying this fact to the problem at hand, we find that

$$\| \ln G(f) - \ln G(h) \| \leqslant \sup_j |\ln s^j(f) - \ln s^j(h)|$$
$$\leqslant \sup_j \rho |\ln f^j - \ln h^j| = \rho \| \ln f - \ln h \|,$$

where the first inequality uses the definition of G and the fact established above, and the second uses the definition of $s(f)$ and the mean-value theorem.

(iii) From the choice of $\bar{\psi}$, any $\bar{\psi} \geqslant \psi > 0$ can be used to define $\mathscr{L}(\cdot)$ so that $G\mathscr{L}(\psi) \subset \mathscr{L}(\psi)$. Since $\mathscr{L}(\bar{\psi})$ is a closed, convex subset of $\mathscr{L}(\psi)$, by Corollary 1 of Stokey and Lucas (1989, p. 52), it follows that the fixed point is unique in $\mathscr{L}(\psi)$.

Since the choice of ψ is arbitrary, it follows that (8) has a unique, strictly positive solution. $\square$

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