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Blow-up of solutions for some non-linear wave equations of Kirchhoff type with some dissipation

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Abstract

The initial boundary value problem for non-linear wave equations of Kirchhoff type with dissipation in a bounded domain is considered. We prove the blow-up of solutions for the strong dissipative term $-\Delta u_t$ and the linear dissipative term u_t by the energy method and give some estimates for the life span of solutions. We also show the nonexistence of global solutions with positive initial energy for non-linear dissipative term by Vitillaro's argument.

MSC: 35L35; 35L70

Keywords: Blow-up; Life span; Dissipation; Non-linear wave equations

1. Introduction

We consider the initial boundary value problem for the following non-linear wave equations of Kirchhoff type:

$$u_{tt} - M(\|\nabla u(t)\|_2^2)\Delta u(t) + g(u_t(t)) = f(u(t)) \quad \text{in } \Omega \times [0, \infty), \tag{1.1}$$

$$u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), \quad x \in \Omega, \tag{1.2}$$

$$u(x, t) = 0, \quad x \in \partial\Omega, \quad t > 0, \tag{1.3}$$

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where $\Omega \subset R^N$, $N \geqslant 1$, is a bounded domain with boundary $\partial\Omega$ so that Divergence theorem can be applied, $\Delta \equiv \sum_{j=1}^N \partial^2/\partial x_j^2$ is the Laplace operator, f is a non-linear function, M is a non-negative locally Lipschitz function, and $g(u_t)$ is the strong dissipative term $-\Delta u_t$ or the linear dissipative term u_t or the non-linear dissipative term $|u_t|^{m-2}u_t$ with $m > 2$. We denote $\|\cdot\|_p$ to be L^p-norm, $p \geqslant 2$.

Let u be a solution of (1.1); we define the energy by

$$E(t) = \frac{1}{2}\|u_t(t)\|_2^2 + \frac{1}{2}\overline{M}(\|\nabla u(t)\|_2^2) - \int_\Omega F(u(t))\,\mathrm{d}x, \quad t \geqslant 0, \tag{1.4}$$

where

$$\overline{M}(s) = \int_0^s M(r)\,\mathrm{d}r \quad \text{and} \quad F(s) = \int_0^s f(r)\,\mathrm{d}r.$$

When $M \equiv 1$, for the case of no dissipation (i.e. $g(u_t) \equiv 0$), there is a large literature on global nonexistence and blow-up for solutions with $E(0) < 0$ [1,3,5,7,8]. The interaction between the damping term and the source has been considered by Levine [7–9] for the cases of $g(u_t) = -\Delta u_t$ and $g(u_t) = u_t$. He showed that solutions with $E(0) < 0$ blow-up in finite time. On the other hand, for semi-linear wave equations with nonlinear dissipative terms: $u_{tt} - \Delta u + |u_t|^{\beta-2}u_t = |u|^{\alpha-2}u$, Georgiev and Todorova [2] proved that solutions with large initial data continue to exist globally if $\beta \geqslant \alpha > 2$ and blow-up in finite time if $2 < \beta < \alpha \leqslant (2N - 2/N - 2)$ (if $N \geqslant 3$) with sufficiently negative initial energy (i.e. $E(0) \leqslant -1$). This result was generalized by Levine and Serrin [11], and then by Levine et al. [10]. Vitillaro [18] combined the arguments in [2,11] to extend these results to positive initial energy.

When M is not a constant function, Eq. (1.1) without the damping and source terms is often called the Kirchhoff-type wave equation; it was first introduced by Kirchhoff [6] in order to study the nonlinear vibrations of an elastic string. The nonexistence of the global solutions of quasi-linear equations with damping terms was investigated by many authors [4,13–16]. The works of Ono [14–16] deal with Eq. (1.1) in two cases with $f(u) = |u|^{p-2}u$, $p > 2$. In the first case, for $g(u_t) = -\Delta u_t$ or u_t, he considered $M(s) = a + bs^\gamma$, where $a \geqslant 0, b \geqslant 0, a + b > 0, \gamma > 0$, and $s \geqslant 0$. He showed that the local solutions blow up at finite time with $E(0) \leqslant 0$ by applying the concavity method. Moreover, he combined the so-called potential well method and concavity method to show blow-up properties with $E(0) > 0$. While in the second case, for $g(u_t) = |u_t|^{m-2}u_t$, $m > 2$, he treated $M(s) = bs^\gamma$, where $b > 0$, $\gamma \geqslant 1$, and $s \geqslant 0$. He proved that the local solution is not global when $p > \max(2\gamma + 2, m)$ and $E(0) < 0$.

In this paper, we shall consider the more general problem by replacing $M(s) = a + bs^\gamma$ and $f(u) = |u|^{p-2}u$ with general $M(s)$ and $f(u)$ under some restrictions for $g(u_t) = -\Delta u_t$ or u_t. We use a direct method [12] to obtain the blow-up properties of local solutions for (1.1)–(1.3), and then we extend the result of [15,16] in this case. We also derive the estimates of upper bound of the blow-up time T. On the other hand, for $g(u_t) = |u_t|^{m-2}u_t$ and $f(u) = |u|^{p-2}u$, we apply the argument of [18] to show the blow-up of local solutions for (1.1)–(1.3) with $\|\nabla u_0\|_2 > \lambda_1$ and $E(0) < E_1$, where λ_1 and E_1 will be specified in Remark 4.1. In this way, we can extend the result of [18] to nonconstant $M(s)$ and the result of [14] to general $M(s)$ and to the condition that $E(0) \geqslant 0$. The estimates of upper bound of the blow-up time are also given. The content of this paper is organized as follows. In Section 2, some local existence is given from [14–16]. Section 3 is divided into two subsections. In Section 3.1, we discuss the blow-up properties of (1.1) for $g(u_t) = -\Delta u_t$. The main result is given in Theorem 3.4 which contains the estimates of upper bound of the blow-up time. The analogous result (Theorem 3.7) of (1.1) for $g(u_t) = u_t$ is also obtained in Section 3.2. In Section 4, the nonexistence of global solutions, for $g(u_t) = |u_t|^{m-2}u_t$ and $f(u) = |u|^{p-2}u$, is given in Theorem 4.3. A special case is also considered and the main result is established in Theorem 4.5.

Let us begin by stating the following two lemmas [12], which will be used later.

Lemma 1.1. *Let $\delta > 0$ and $B(t) \in C^2(0, \infty)$ be a nonnegative function satisfying*

$$B''(t) - 4(\delta + 1)B'(t) + 4(\delta + 1)B(t) \geqslant 0. \tag{1.5}$$

If

$$B'(0) > r_2 B(0) + K_0, \tag{1.6}$$

then

$$B'(t) > K_0$$

for $t > 0$, where K_0 is a constant, $r_2 = 2(\delta + 1) - 2\sqrt{(\delta + 1)\delta}$ is the smallest root of the equation

$$r^2 - 4(\delta + 1)r + 4(\delta + 1) = 0.$$

Proof. See [12]. □

Lemma 1.2. *If $J(t)$ is a nonincreasing function on $[t_0, \infty)$, $t_0 \geqslant 0$ and satisfies the differential inequality*

$$J'(t)^2 \geqslant a + bJ(t)^{2+1/\delta}, \quad for\ t \geqslant t_0, \tag{1.7}$$

where $a > 0$, $b \in R$, then there exists a finite time T^* such that

$$\lim_{t \to T^{*-}} J(t) = 0$$

and the upper bound of T^* is estimated, respectively, by the following cases:

(i) *If $b < 0$ and $J(t_0) < \min\{1, \sqrt{a/-b}\}$ then*

$$T^* \leqslant t_0 + \frac{1}{\sqrt{-b}} \ln \frac{\sqrt{\frac{a}{-b}}}{\sqrt{\frac{a}{-b}} - J(t_0)}.$$

(ii) *If $b = 0$, then*

$$T^* \leqslant t_0 + \frac{J(t_0)}{J'(t_0)}.$$

(iii) *If* $b > 0$, *then*

$$T^* \leqslant \frac{J(t_0)}{\sqrt{a}}$$

or

$$T^* \leqslant t_0 + 2^{(3\delta+1)/2\delta} \frac{\delta c}{\sqrt{a}} \{1 - [1 + cJ(t_0)]^{-1/2\delta}\},$$

where $c = (a/b)^{2+1/\delta}$.

Proof. See [12]. □

2. Local existence

We first state local existence results established in [14–16].

Theorem 2.1. *Let the initial data* (u_0, u_1) *belong to* $(H_0^1(\Omega) \cap H^2(\Omega)) \times L^2(\Omega)$, *and let* $f(u)$ *be a nonlinear function such that* $f(0) = 0$ *and*

$$|f(u) - f(v)| \leqslant c(|u|^{p-2} + |v|^{p-2})|u - v|,$$

for $u, v \in R$, *and some constant* c *and*

$$p \leqslant \frac{2N-4}{N-4} \quad (p < \infty \text{ if } N \leqslant 4).$$

Then, there exists a $T = T(\|\triangle u_0\|_2, \|u_1\|_2) > 0$ *such that problem* (1.1) *with* $g(u_t) = -\Delta u_t$ *admits a unique local solution* u *in the class*

$$C^0([0, T); H_0^1(\Omega) \cap H^2(\Omega)) \cap C^1([0, T); L^2(\Omega)),$$

and

$$u_t \in L^2(0, T; H_0^1(\Omega)).$$

Moreover, at least one of the following statements is valid:

(i) $T = \infty$,
(ii) $\|\triangle u(t)\|_2^2 + \|u_t(t)\|_2^2 \to \infty$ *as* $t \to T^-$.

Theorem 2.2. *Let the initial data* (u_0, u_1) *belong to* $(H_0^1(\Omega) \cap H^2(\Omega)) \times H_0^1(\Omega)$ *and* $u_0 \neq 0$, *and let* $f(u)$ *be a nonlinear function such that* $f(0) = 0$ *and*

$$|f(u) - f(v)| \leqslant c(|u|^{p-2} + |v|^{p-2})|u - v|,$$

for $u, v \in R$, *and some constant* c *and*

$$p \leqslant \frac{2N-4}{N-4} \quad (p < \infty \text{ if } N \leqslant 4).$$

Then, there exists a $T = T(\|\triangle u_0\|_2, \|\nabla u_1\|_2) > 0$ *such that problem* (1.1) *with* $g(u_t) = u_t$ *admits a unique local solution u in the class*

$$C^0([0,T]; (H_0^1(\Omega) \cap H^2(\Omega)) \cap C^1([0,T]; H_0^1(\Omega)) \cap C^2([0,T]; L^2(\Omega)).$$

Moreover, at least one of the following statements is valid:

(i) $T = \infty$,
(ii) $\|\triangle u(t)\|_2^2 + \|\nabla u_t(t)\|_2^2 \to \infty$ *as* $t \to T^-$,
(iii) $\|\nabla u(t)\|_2 \to 0$ *as* $t \to T^-$.

Theorem 2.3. *Let the initial data* (u_0, u_1) *belong to* $(H_0^1(\Omega) \cap H^2(\Omega)) \times H_0^1(\Omega)$ *and* $u_0 \neq 0$, *and let*

$$f(u) = |u|^{p-2}u, \quad \text{where } p \leqslant \frac{2N-6}{N-4} \quad (p < \infty \text{ if } N \leqslant 4).$$

Then, there exists a $T = T(\|\triangle u_0\|_2, \|\nabla u_1\|_2) > 0$ *such that problem* (1.1) *with* $g(u_t) = |u_t|^{m-2}u_t$ *for* $m > 2$ *admits a unique local solution u in the class* $W_1 \cap W_2$ *and* $u_t \in L^m((0,T) \times \Omega)$, *where*

$$W_1 = C_w^0([0,T]; (H_0^1(\Omega) \cap H^2(\Omega)) \cap C_w^1([0,T]; H_0^1(\Omega)),$$
$$W_2 = C^0([0,T]; H_0^1(\Omega)) \cap C^1([0,T]; L^2(\Omega)),$$

here the subscript "w" *means the weak continuity with respect to t* [17].

Moreover, at least one of the following statements is valid:

(i) $T = \infty$,
(ii) $\|\triangle u(t)\|_2^2 + \|\nabla u_t(t)\|_2^2 \to \infty$ *as* $t \to T^-$,
(iii) $\|\nabla u(t)\|_2 \to 0$ *as* $t \to T^-$.

3. Blow-up property for $g(u_t) = u_t$ or $-\Delta u_t$

In this section, we will study blow-up phenomena of two problems, where $g(u_t) = u_t$ in Section 3.1 and $g(u_t) = -\Delta u_t$ in Section 3.2. In the sequel, for the sake of simplicity we will omit the dependence on t, when the meaning is clear. In order to state our results, we make the following assumptions:

(A1) there exists a positive constant $\delta > 0$ such that

$$sf(s) \geqslant (2+4\delta)F(s) \quad \text{for all } s \in R,$$

and

$$(2\delta+1)\overline{M}(s) \geqslant M(s)s \quad \text{for all } s \geqslant 0.$$

It is clear that $f(u) = |u|^{p-2}u$, $p \geqslant 2\gamma+2$ and $M(s) = a + bs^\gamma$, where $a \geqslant 0, b \geqslant 0, a+b > 0$, $\gamma > 0$, $s \geqslant 0$ satisfies (A1) with $\gamma/2 \leqslant \delta \leqslant (p-2)/4$.

3.1. $g(u_t) = u_t$

In this subsection we consider Eq. (1.1) with $g(u_t) = u_t$:

$$u_{tt} - M(\|\nabla u\|_2^2)\triangle u + u_t = f(u). \tag{3.1}$$

Definition. A solution u of (3.1), (1.2), (1.3) is called blow-up if there exists a finite time T^* such that

$$\lim_{t\to T^{*-}} \int_\Omega u^2 \,\mathrm{d}x = \infty.$$

From (1.4), the definition of $E(t)$, we see that

Lemma 3.1. *$E(t)$ is a nonincreasing function on $[0, T)$ and*

$$E(t) = E(0) - \int_0^t \int_\Omega g(u_t(x))u_t(x)\,\mathrm{d}x\,\mathrm{d}t. \tag{3.2}$$

Proof. By differentiating (1.4) and using (1.1), we obtain

$$\frac{\mathrm{d}E(t)}{\mathrm{d}t} = -\int_\Omega g(u_t(x))u_t(x)\,\mathrm{d}x.$$

Thus, Lemma 3.1 follows at once.

Now, let u be a solution of (3.1) and define

$$a(t) = \int_\Omega u^2\,\mathrm{d}x + \int_0^t \int_\Omega u^2\,\mathrm{d}x\,\mathrm{d}t, \quad t \geqslant 0. \tag{3.3}$$

Lemma 3.2. *Assume that* (A1) *holds*; *we have*

$$a''(t) - 4(\delta+1)\int_\Omega u_t^2\,\mathrm{d}x \geqslant (-4-8\delta)E(0) + (4+8\delta)\int_0^t \|u_t\|_2^2\,\mathrm{d}t. \tag{3.4}$$

Proof. From (3.3), we have

$$a'(t) = 2\int_\Omega uu_t\,\mathrm{d}x + \|u\|_2^2. \tag{3.5}$$

By (3.1) and Divergence theorem, we obtain

$$\begin{aligned} a''(t) &= 2\int_\Omega u_t^2\,\mathrm{d}x + 2M(\|\nabla u\|_2^2)\int_\Omega u\triangle u\,\mathrm{d}x + 2\int_\Omega f(u)u\,\mathrm{d}x \\ &= 2\int_\Omega u_t^2\,\mathrm{d}x - 2M(\|\nabla u\|_2^2)\int_\Omega |\nabla u|^2\,\mathrm{d}x + 2\int_\Omega f(u)u\,\mathrm{d}x. \end{aligned} \tag{3.6}$$

By (1.4) and (3.2), we have from (3.6)

$$
\begin{aligned}
&a''(t) - 4(\delta+1)\int_\Omega u_t^2\,\mathrm{d}x \\
&\quad \geqslant (-4-8\delta)E(0) + (4+8\delta)\int_0^t \|u_t\|_2^2\,\mathrm{d}s + \int_\Omega 2(f(u)u - (2+4\delta)F(u))\,\mathrm{d}x \\
&\qquad + \left[(2+4\delta)\overline{M}(\|\nabla u\|_2^2) - 2M(\|\nabla u\|_2^2)\int_\Omega |\nabla u|^2\,\mathrm{d}x\right].
\end{aligned}
$$

Therefore from (A1), we obtain (3.4).

Now, we consider three different cases on the sign of the initial energy $E(0)$.

(1) If $E(0) < 0$, then from (3.4), we have

$$a'(t) \geqslant a'(0) - 4(1+2\delta)E(0)t, \quad t \geqslant 0.$$

Thus we obtain $a'(t) > \|u_0\|_2^2$ for $t > t^*$, where

$$t^* = \max\left\{\frac{a'(0) - \|u_0\|_2^2}{4(1+2\delta)E(0)},\ 0\right\}. \tag{3.7}$$

(2) If $E(0) = 0$, then $a''(t) \geqslant 0$ for $t \geqslant 0$.
Furthermore, if $a'(0) > \|u_0\|_2^2$, then $a'(t) > \|u_0\|_2^2,\ t \geqslant 0$.

(3) For the case that $E(0) > 0$, we first note that

$$
\begin{aligned}
2\int_0^t \int_\Omega u u_t\,\mathrm{d}x\,\mathrm{d}t &= \int_0^t \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u^2\,\mathrm{d}x\,\mathrm{d}t \\
&= \int_\Omega u^2\,\mathrm{d}x - \int_\Omega u_0^2\,\mathrm{d}x.
\end{aligned} \tag{3.8}
$$

By Hölder inequality and Young's inequality, we have from (3.8)

$$\int_\Omega u^2\,\mathrm{d}x \leqslant \int_\Omega u_0^2\,\mathrm{d}x + \int_0^t \|u\|_2^2\,\mathrm{d}t + \int_0^t \|u_t\|_2^2\,\mathrm{d}t. \tag{3.9}$$

By Hölder inequality, Young's inequality again and (3.9), we obtain from (3.5)

$$a'(t) \leqslant a(t) + \int_\Omega u_0^2\,\mathrm{d}x + \int_\Omega u_t^2\,\mathrm{d}x + \int_0^t \|u_t\|_2^2\,\mathrm{d}t. \tag{3.10}$$

Hence by (3.4), (3.10), we obtain

$$a''(t) - 4(\delta+1)a'(t) + 4(\delta+1)a(t) + K_1 \geqslant 0,$$

where

$$K_1 = (4+8\delta)E(0) + 4(\delta+1)\|u_0\|_2^2.$$

Let

$$b(t) = a(t) + \frac{K_1}{4(1+\delta)}, \quad t > 0.$$

Then $b(t)$ satisfies (1.5). By (1.6), we see that if

$$a'(0) > r_2 \left[a(0) + \frac{K_1}{4(1+\delta)} \right] + \|u_0\|_2^2, \tag{3.11}$$

then $a'(t) > \|u_0\|_2^2$, $t > 0$.

Consequently, we have

Lemma 3.3. *Assume that* (A1) *holds and that either one of the following statements is satisfied*:

(i) $E(0) < 0$,
(ii) $E(0) = 0$ *and* $a'(0) > \|u_0\|_2^2$,
(iii) $E(0) > 0$ *and* (3.11) *holds*,
then $a'(t) > \|u_0\|_2^2$ *for* $t > t_0$, *where* $t_0 = t^*$ *is given by* (3.7) *in case* (i) *and* $t_0 = 0$ *in cases* (ii) *and* (iii).

Now, we will find the estimate for the life span of $a(t)$.
Let

$$J(t) = (a(t) + (T_1 - t)\|u_0\|_2^2)^{-\delta} \quad \text{for } t \in [0, T_1], \tag{3.12}$$

where $T_1 > 0$ is a certain constant which will be specified later.
Then we have,

$$J'(t) = -\delta J(t)^{1+1/\delta}(a'(t) - \|u_0\|_2^2)$$

and

$$J''(t) = -\delta J(t)^{1+2/\delta} V(t), \tag{3.13}$$

where

$$V(t) = a''(t)(a(t) + (T_1 - t)\|u_0\|_2^2) - (1+\delta)(a'(t) - \|u_0\|_2^2)^2. \tag{3.14}$$

For simplicity of calculation, we denote

$$P = \int_\Omega u^2 \,\mathrm{d}x,$$

$$Q = \int_0^t \|u\|_2^2 \,\mathrm{d}t,$$

$$R = \int_\Omega u_t^2 \,\mathrm{d}x,$$

$$S = \int_0^t \|u_t\|_2^2 \,\mathrm{d}t.$$

From (3.5), (3.8), and Hölder inequality, we obtain

$$a'(t) = 2\int_\Omega u_t u \,\mathrm{d}x + \int_\Omega u_0^2 \,\mathrm{d}x + 2\int_0^t \int_\Omega u u_t \,\mathrm{d}x \,\mathrm{d}t \leqslant 2(\sqrt{RP} + \sqrt{QS}) + \int_\Omega u_0^2 \,\mathrm{d}x. \tag{3.15}$$

By (3.4), we have

$$a''(t) \geqslant (-4 - 8\delta)E(0) + 4(1+\delta)(R+S). \tag{3.16}$$

Thus, from (3.15), (3.16), we obtain

$$V(t) \geqslant [(-4 - 8\delta)E(0) + 4(1+\delta)(R+S)](a(t) + (T_1 - t)\|u_0\|_2^2) - 4(1+\delta)(\sqrt{RP} + \sqrt{QS})^2.$$

And by (3.12) and (3.3), we have

$$V(t) \geqslant (-4 - 8\delta)E(0)J(t)^{-1/\delta} + 4(1+\delta)(R+S)(T_1 - t)\|u_0\|_2^2 + 4(1+\delta)[(R+S)(P+Q) - (\sqrt{RP} + \sqrt{QS})^2].$$

By Schwarz inequality, the last term in the above inequality is nonnegative. Hence we have

$$V(t) \geqslant (-4 - 8\delta)E(0)J(t)^{-1/\delta}, \quad t \geqslant t_0. \tag{3.17}$$

Therefore by (3.13) and (3.17), we obtain

$$J''(t) \leqslant \delta(4 + 8\delta)E(0)J(t)^{1+1/\delta}, \quad t \geqslant t_0. \tag{3.18}$$

Note that by Lemma 3.3, $J'(t) < 0$ for $t > t_0$. Multiplying (3.18) by $J'(t)$ and integrating from t_0 to t, we obtain

$$J'(t)^2 \geqslant \alpha + \beta J(t)^{2+1/\delta} \quad \text{for } t \geqslant t_0,$$

where

$$\alpha = \delta^2 J(t_0)^{2+2/\delta}[(a'(t_0) - \|u_0\|_2^2)^2 - 8E(0)J(t_0)^{-1/\delta}] \tag{3.19}$$

and

$$\beta = 8\delta^2 E(0). \tag{3.20}$$

We observe that

$$\alpha > 0 \quad \text{iff } E(0) < \frac{(a'(t_0) - \|u_0\|_2^2)^2}{8(a(t_0) + (T_1 - t_0)\|u_0\|_2^2)}.$$

Then by Lemma 1.2, there exists a finite time T^* such that $\lim_{t\to T^{*-}} J(t) = 0$ and the upper bound of T^* is estimated, respectively, according to the sign of $E(0)$. This will imply that

$$\lim_{t\to T^{*-}} \left\{ \int_\Omega u^2 \,\mathrm{d}x + \int_0^t \int_\Omega u^2 \,\mathrm{d}x \,\mathrm{d}t \right\} = \infty. \tag{3.21}$$

Theorem 3.4. *Assume that* (A1) *and* (A2) *hold and that either one of the following statements is satisfied*:

(i) $E(0) < 0$,
(ii) $E(0) = 0$ *and* $a'(0) > \|u_0\|_2^2$,
(iii) $0 < E(0) < \frac{(a'(t_0) - \|u_0\|_2^2)^2}{8(a(t_0) + (T_1 - t_0)\|u_0\|_2^2)}$ *and* (3.11) *holds*,

then the solution u blows up at finite time T^* *in the sense of* (3.21).

In case (i),

$$T^* \leqslant t_0 - \frac{J(t_0)}{J'(t_0)}. \tag{3.22}$$

Furthermore, if $J(t_0) < \min\{1, \sqrt{\alpha/-\beta}\}$, we have

$$T^* \leqslant t_0 + \frac{1}{\sqrt{-\beta}} \ln \frac{\sqrt{\frac{\alpha}{-\beta}}}{\sqrt{\frac{\alpha}{-\beta}} - J(t_0)}. \tag{3.23}$$

In case (ii),

$$T^* \leqslant t_0 - \frac{J(t_0)}{J'(t_0)} \tag{3.24}$$

or

$$T^* \leqslant t_0 + \frac{J(t_0)}{\sqrt{\alpha}}. \tag{3.25}$$

In case (iii),

$$T^* \leqslant \frac{J(t_0)}{\sqrt{\alpha}} \tag{3.26}$$

or

$$T^* \leqslant t_0 + 2^{(3\delta+1)/2\delta} \frac{\delta c}{\sqrt{\alpha}} \{1 - [1 + cJ(t_0)]^{-1/2\delta}\}, \tag{3.27}$$

where $c = (\alpha/\beta)^{2+1/\delta}$; here α and β are in (3.19), (3.20).

Note that in case (i), $t_0 = t^*$ is given in (3.7) and $t_0 = 0$ in case (ii) and (iii).

Remark 3.5. The choice of T_1 in (3.12) is possible under some conditions. We shall discuss it as follows:

(i) For the case $E(0) = 0$.

First, we note that the condition $a'(0) > \|u_0\|_2^2$ implies $\int_\Omega u_0 u_1 \, \mathrm{d}x > 0$.

(1) If $2\delta \int_\Omega u_0 u_1 \, \mathrm{d}x - \|u_0\|_2^2 > 0$, by (3.24), we choose

$$T_1 \geqslant -\frac{J(0)}{J'(0)}.$$

This is equivalent to

$$T_1 \geqslant \frac{\|u_0\|_2^2}{2\delta \int_\Omega u_0 u_1 \, \mathrm{d}x - \|u_0\|_2^2}.$$

In particular, we choose T_1 as

$$T_1 = \frac{\|u_0\|_2^2}{2\delta \int_\Omega u_0 u_1 \, \mathrm{d}x - \|u_0\|_2^2}.$$

We then obtain

$$T^* \leqslant \frac{\|u_0\|_2^2}{2\delta \int_\Omega u_0 u_1 \, \mathrm{d}x - \|u_0\|_2^2}.$$

(2) If $2\delta \int_\Omega u_0 u_1 \, \mathrm{d}x - \|u_0\|_2^2 \leqslant 0$, by (3.25), we choose

$$T_1 \geqslant \frac{J(t_0)}{\sqrt{\alpha}}. \tag{3.28}$$

By Hölder inequality, Young's inequality, and from (3.28), we obtain

$$\|u_0\|_2^2 + T_1 \|u_0\|_2^2 \leqslant \delta(\|u_0\|_2^2 + \|u_1\|_2^2) T_1,$$

then

$$T_1 \geqslant \frac{\|u_0\|_2^2}{\|u_1\|_2^2}, \quad \text{if } 0 < \delta \leqslant 1$$

or

$$T_1 \geqslant \frac{\|u_0\|_2^2}{(\delta - 1)\|u_0\|_2^2 + \|u_1\|_2^2}, \quad \text{if } 1 < \delta.$$

In particular, we have

$$T^* \leqslant T_1 = \frac{\|u_0\|_2^2}{\|u_1\|_2^2}, \quad \text{if } 0 < \delta \leqslant 1$$

or

$$T^* \leqslant T_1 = \frac{\|u_0\|_2^2}{(\delta - 1)\|u_0\|_2^2 + \|u_1\|_2^2}, \quad \text{if } 1 < \delta.$$

(ii) For the case $E(0) < 0$,

(1) If $\int_\Omega u_0 u_1 \, \mathrm{d}x > 0$, then $a'(t) > \|u_0\|_2^2$ and $t^* = 0$. Thus T_1 can be chosen as in (i).

(2) If $\int_\Omega u_0u_1\,\mathrm{d}x\leqslant 0$, then $t^*=a'(0)-\|u_0\|_2^2/4(1+2\delta)E(0)$. Thus, by (3.22), we choose $T_1\geqslant t^*-J(t^*)/J'(t^*)$.

(iii) For the case $E(0)>0$,

(1) If $\|u_0\|_2^2<\delta$ and if

$$E(0)<\min\{\kappa_1,\kappa_2\},$$

where

$$\kappa_1=\frac{(1+\delta)(a'(0)-r_2a(0)-(r_2+1)\|u_0\|_2^2)}{r_2(1+2\delta)}$$

$$\kappa_2=\frac{4\left(\int_\Omega u_0u_1\,\mathrm{d}x\right)^2-1}{8\|u_0\|_2^2}\cdot\frac{\delta-\|u_0\|_2^2}{\delta},$$

then we choose T_1 to satisfy

$$\frac{\|u_0\|_2^2}{\delta-\|u_0\|_2^2}\leqslant T_1\leqslant\frac{4\left(\int_\Omega u_0u_1\,\mathrm{d}x\right)^2-8E(0)\|u_0\|_2^2-1}{8E(0)\|u_0\|_2^2},$$

so that

$$\begin{aligned}\alpha&=\delta^2J(0)^{2+2/\delta}[(a'(0)-\|u_0\|_2^2)^2-8E(0)J(0)^{-1/\delta}]\\&\geqslant 1.\end{aligned}$$

In particular, choosing $T_1=\|u_0\|_2^2/\delta-\|u_0\|_2^2$ and by (3.26), we obtain

$$T^*\leqslant\frac{\dfrac{\|u_0\|_2^2}{\delta-\|u_0\|_2^2}}{\sqrt{4\left(\int_\Omega u_0u_1\,\mathrm{d}x\right)^2-8E(0)\,\dfrac{\delta\|u_0\|_2^2}{\delta-\|u_0\|_2^2}}}.$$

(2) If $\|u_0\|_2^2\geqslant\delta$ with $\delta\leqslant 1$ and if

$$E(0)<\min\{\kappa_3,\kappa_4\},$$

where

$$\kappa_3=\frac{(1+\delta)(a'(0)-r_2a(0)-(r_2+1)\|u_0\|_2^2)}{r_2(1+2\delta)},$$

$$\kappa_4=\frac{(\int_\Omega u_0u_1\,\mathrm{d}x)^2(\|u_0\|_2^2+\|u_1\|_2^2-\delta)}{2(2\|u_0\|_2^2+\|u_1\|_2^2-\delta)\|u_0\|_2^2},$$

then we choose

$$T_1=\frac{\|u_0\|_2^2}{\|u_0\|_2^2+\|u_1\|_2^2-\delta},$$

so that $\alpha > 0$. By (3.26), we have

$$T^* \leqslant \frac{\dfrac{\|u_0\|_2^2(2\|u_0\|_2^2 + \|u_1\|_2^2 - \delta)}{\|u_0\|_2^2 + \|u_1\|_2^2 - \delta}}{\delta\sqrt{4\left(\int_\Omega u_0 u_1\,\mathrm{d}x\right)^2 - 8E(0)\dfrac{\|u_0\|_2^2(2\|u_0\|_2^2 + \|u_1\|_2^2 - \delta)}{\|u_0\|_2^2 + \|u_1\|_2^2 - \delta}}}.$$

If $\|u_0\|_2^2 \geqslant \delta$ with $\delta > 1$ and if

$$E(0) < \min\{\chi_1, \chi_2\},$$

where

$$\chi_1 = \frac{(1+\delta)(a'(0) - r_2 a(0) - (r_2+1)\|u_0\|_2^2)}{r_2(1+2\delta)},$$

$$\chi_2 = \frac{(\int_\Omega u_0 u_1\,\mathrm{d}x)^2(\delta(\|u_0\|_2^2 + \|u_1\|_2^2 - 1))}{2(\|u_0\|_2^2 + \delta(\|u_0\|_2^2 + \|u_1\|_2^2 - 1))\|u_0\|_2^2},$$

then we choose

$$T_1 = \frac{\|u_0\|_2^2}{\delta(\|u_0\|_2^2 + \|u_1\|_2^2 - 1)},$$

so that $\alpha > 0$. By (3.26), we have

$$T^* \leqslant \frac{\dfrac{\|u_0\|_2^2(\|u_0\|_2^2 + \delta(\|u_0\|_2^2 + \|u_1\|_2^2 - 1))}{\delta(\|u_0\|_2^2 + \|u_1\|_2^2 - 1)}}{\delta\sqrt{4\left(\int_\Omega u_0 u_1\,\mathrm{d}x\right)^2 - 8E(0)\dfrac{\|u_0\|_2^2(\|u_0\|_2^2 + \delta(\|u_0\|_2^2 + \|u_1\|_2^2 - 1))}{\delta(\|u_0\|_2^2 + \|u_1\|_2^2 - 1)}}}.$$

Example 3.6. We consider the problem

$$u_{tt} - M(\|\nabla u\|_2^2)\Delta u + u_t = f(u),$$

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \Omega,$$

$$u(x,t) = 0, \quad x \in \partial\Omega, \quad t > 0,$$

where $M(s) = 1 + s$, $f(u) = u^3$, $\Omega = (0,1)$, $u_1(x) = d$ and

$$u_0(x) = \begin{cases} 0 & \text{if } 0 \leqslant x < \frac{1}{4}, \\ 3.36 & \text{if } \frac{1}{4} \leqslant x < \frac{1}{2}, \\ 0.2(1-x) & \text{if } \frac{1}{2} \leqslant x \leqslant 1. \end{cases}$$

Then we have

$$E(0) = \frac{d^2}{2} - 7.9559025.$$

Thus

$$0 < d \leqslant d_c \iff E(0) \leqslant 0,$$

where

$$d_c = \sqrt{15.911805} \approx 3.988960391.$$

We also have the following datum:

$$\delta = \tfrac{1}{2}, \quad a(0) = 2.8241, \quad a'(0) = 1.73d + 2.8241.$$

Then by Remark 3.5, the upper bound for the blow-up time in each case is obtained.

(1) For $0 < d \leqslant 3.264855491$, we have

$$T^*(d) \leqslant \frac{2.8241}{d^2}.$$

(2) $3.264855491 < d \leqslant d_c$, we have

$$T^*(d) \leqslant \frac{2.8241}{0.865d - 2.8241}.$$

(3) $d_c < d < 4.0036$, we have

$$T^*(d) \leqslant \frac{5.6482\kappa_5}{\sqrt{2.9929d^2 - 22.5928E(0)\kappa_5}},$$

where $\kappa_5 = 1 + \frac{2.8241}{2.3241+d^2}$.

3.2. $g(u_t) = -\triangle u_t$

In this subsection we consider Eq. (1.1) with $g(u_t) = -\triangle u_t$:

$$u_{tt} - M(\|\nabla u\|_2^2)\triangle u - \triangle u_t = f(u). \tag{3.29}$$

Definition. A solution u of (3.29), (1.2), (1.3) is called blow-up if there exists a finite time T^* such that

$$\lim_{t \to T^{*-}} \int_\Omega |\nabla u|^2 \, \mathrm{d}x = \infty.$$

Let

$$a(t) = \int_\Omega u^2 \, \mathrm{d}x + \int_0^t \int_\Omega |\nabla u|^2 \, \mathrm{d}x \, \mathrm{d}t, \quad t \geqslant 0$$

and

$$J(t) = [a(t) + (T_1 - t)\|\nabla u_0\|_2^2]^{-\delta}, \quad t \in [0, T_1],$$

instead of (3.3) and (3.12).

By the similar way as above, we have the following results.

Theorem 3.7. *Assume that* (A1) *holds and that either one of the following statements is satisfied*:

(i) $E(0) < 0$,
(ii) $E(0) = 0$ *and* $a'(0) > \|\nabla u_0\|_2^2$,
(iii) $0 < E(0) < \frac{(a'(t_0)-\|\nabla u_0\|_2^2)^2}{8(a(t)+(T_1-t_0)\|\nabla u_0\|_2^2)}$ *and* $2\int_\Omega u_0 u_1 \,\mathrm{d}x > r_2[a(0) + \frac{K_2}{4(1+\delta)}]$,

where $K_2 = (4 + 8\delta)E(0) + 4(\delta + 1)\|\nabla u_0\|_2^2$.

Then there exists a finite time T^* such that the solution u of (3.29), (1.2), (1.3) has the following:

$$\lim_{t\to T^{*-}} \left\{\int_\Omega u^2 \,\mathrm{d}x + \int_0^t \int_\Omega |\nabla u|^2 \,\mathrm{d}x \,\mathrm{d}t\right\} = \infty.$$

Thus,

$$\lim_{t\to T^{*-}} \int_\Omega |\nabla u|^2 \,\mathrm{d}x = \infty.$$

Example 3.8. We consider the problem

$$\begin{aligned} &u_{tt} - M(\|\nabla u\|_2^2)\Delta u - \Delta u_t = f(u),\\ &u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \Omega,\\ &u(x,t) = 0, \quad x \in \partial\Omega, \ t > 0, \end{aligned}$$

where $M(s) = 1 + s$, $f(u) = u^3$, $u_0(x) = 20\sin \pi x/5$, $u_1(x) = d > 0$ and $\Omega = (0, 5)$. Then we have

$$E(0) = \frac{d^2}{2} + 20\pi^2 + 400\pi^4 - 75000.$$

Thus

$$0 < d \leqslant d_c \iff E(0) \leqslant 0,$$

where

$$d_c = \sqrt{150000 - (40 + 800\pi^2)\pi^2} \approx 267.73.$$

We also have the following datum:

$$\delta = \tfrac{1}{2}, \quad a(0) = 1000, \quad a'(0) = \frac{400d}{\pi} + 40\pi^2.$$

Then applying Remark 3.5, the upper bound for the blow-up time is obtained.

(1) For $0 < d \leqslant 6.201255336$, we have

$$T^*(d) \leqslant \frac{200}{\lambda^2 d^2},$$

here λ is the Sobolev constant satisfying $\|u\|_2 \leqslant \lambda \|u_x\|_2$, for $u \in H_0^1(0,5)$.

(2) For $6.201255336 < d \leqslant d_c$, we have

$$T^*(d) \leqslant \frac{1000\pi}{200d - 40\pi^4}.$$

(3) For $d_c < d < 299.82$, we have

$$T^*(d) \leqslant \frac{100\pi}{\sqrt{100d^2 - 10E(0)\pi^2}}.$$

4. Blow-up property for $g(u_t) = |u_t|^{m-2}u_t$ and $f(u) = |u|^{p-2}u$, $p > m > 2$

In this section, when $2 < m < p \leqslant 2N/(N-2)$ ($N < \infty$, if $N = 1, 2$), we consider the problem with $g(u_t) = |u_t|^{m-2}u_t$, $f(u) = |u|^{p-2}u$:

$$u_{tt} - M(\|\nabla u\|_2^2)\Delta u + |u_t|^{m-2}u_t = |u|^{p-2}u \tag{4.1}$$

under the following hypothesis:

(A2) M is a nonnegative locally Lipschitz function satisfying

$$M(s) \geqslant m_0 > 0 \quad \text{for all } s \geqslant 0, \tag{4.2}$$

and there exists $m_1 \geqslant 1$ such that

$$m_1 \overline{M}(s) \geqslant M(s)s \quad \text{for all } s \geqslant 0. \tag{4.3}$$

It is clear that $M(s) = a + bs^\gamma$, $a > 0$, $b \geqslant 0$, $\gamma > 0$, $s \geqslant 0$ satisfies assumption (A2) that we made on M.

Remark 4.1. By (4.2), we have

$$E(t) \geqslant \tfrac{1}{2} m_0 \|\nabla u\|_2^2 - \tfrac{1}{p}\|u\|_p^p, \quad t \geqslant 0. \tag{4.4}$$

By Poincaré inequality,

$$E(t) \geqslant G(\|\nabla u\|_2), \quad t \geqslant 0, \tag{4.5}$$

where

$$G(\lambda) = \frac{1}{2} m_0 \lambda^2 - \frac{B_1^p}{p}\lambda^p,$$

here B_1 is the optimal constant of Sobolev imbedding $H_0^1(\Omega) \hookrightarrow L^p(\Omega)$ given by $B_1^{-1} = \inf\{\|\nabla u\|_2 : u \in H_0^1(\Omega), \|u\|_p = 1\}$.

Note that $G(\lambda)$ has the maximum at $\lambda_1 = (\frac{m_0}{B_1^p})^{1/(p-2)}$, and the maximum value is

$$E_1 = G(\lambda_1) = m_0^{p/(p-2)} \left(\frac{1}{2} - \frac{1}{p}\right) B_1^{-2p/(p-2)}. \tag{4.6}$$

Lemma 4.2. *Assume* $2 < m < p \leqslant 2N/(N-2)$ $(N < \infty$, *if* $N = 1, 2)$ *and* $E(0) < E_1$.

(i) *If* $\|\nabla u_0\|_2 < \lambda_1$, *then* $\|\nabla u(t)\|_2 < \lambda_1$, $t \geqslant 0$.
(ii) *If* $\|\nabla u_0\|_2 > \lambda_1$, *then there exists* $\lambda_2 > \lambda_1$, *such that* $\|\nabla u(t)\|_2 \geqslant \lambda_2$, $t \geqslant 0$ *and there exists* $\lambda_3 > B_1\lambda_1$ *such that* $\|u(t)\|_p \geqslant \lambda_3$, $t \geqslant 0$.

Proof. From the definition of $G(\lambda)$, we see that $G(\lambda)$ is increasing in $(0, \lambda_1)$ and decreasing in (λ_1, ∞), and $G(\lambda) \to -\infty$ as $\lambda \to \infty$.

Since $E(0) < E_1$, so there exists λ_2', λ_2 with $\lambda_2' < \lambda_1 < \lambda_2$ such that $G(\lambda_2') = G(\lambda_2) = E(0)$.

(i) when $\|\nabla u_0\|_2 < \lambda_1$, from (4.5), we have

$$G(\|\nabla u_0\|_2) \leqslant E(0) = G(\lambda_2').$$

It implies $\|\nabla u_0\|_2 < \lambda_2'$.

We claim that $\|\nabla u(t)\|_2 \leqslant \lambda_2'$. If not, then there exists $t_0 > 0$ such that $\|\nabla u(t_0)\|_2 > \lambda_2'$. Case (a) if $\lambda_2' < \|\nabla u(t_0)\|_2 < \lambda_2$, then $G(\|\nabla u(t_0)\|_2) > E(0) \geqslant E(t_0)$. It contradicts (4.5). Case (b) if $\|\nabla u(t_0)\|_2 \geqslant \lambda_2$, then by continuity of $\|\nabla u(t)\|_2$, there exists $0 < t_1 < t_0$ such that $\lambda_2' < \|\nabla u(t_1)\|_2 < \lambda_2$, then $G(\|\nabla u(t_1)\|_2) > E(0) \geqslant E(t_1)$. This implies a contradiction.

(ii) when $\|\nabla u_0\|_2 > \lambda_1$, we deduce as before that $\|\nabla u_0\|_2 > \lambda_1$ implies $\|\nabla u(t)\|_2 \geqslant \lambda_2$, for $t \geqslant 0$.

Hence, from (4.2), (4.4), (4.5) and Lemma 3.1, we have

$$\begin{aligned}\frac{1}{p} \int_\Omega |u|^p \,\mathrm{d}x &\geqslant \frac{1}{2} m_0 \int_\Omega |\nabla u|^2 \,\mathrm{d}x - E(0) + \frac{1}{2} \int_\Omega u_t^2 \,\mathrm{d}x \\ &\geqslant \frac{1}{2} m_0 \int_\Omega |\nabla u|^2 \,\mathrm{d}x - E(0) \\ &\geqslant \frac{1}{2} m_0 \lambda_2^2 - G(\lambda_2) \\ &= \frac{B_1^p}{p} \lambda_2^p.\end{aligned}$$

This implies $\|u\|_p \geqslant B_1\lambda_2 > B_1\lambda_1$, $t \geqslant 0$. Set $\lambda_3 = B_1\lambda_2$, then there exists $\lambda_3 > B_1\lambda_1$ and

$$\|u(t)\|_p \geqslant \lambda_3, \quad t \geqslant 0. \qquad \square$$

Theorem 4.3. *If* (A3) *holds*, $p > \max(2m_1, m)$, *then the local solutions of problems* (4.1), (1.2), (1.3) *blow up at a finite time* T *with* $\|\nabla u_0\|_2 > \lambda_1$ *and* $E(0) < E_1$, *where* $0 < T \leqslant \frac{L(0)^{1-\theta}}{c_{10}(\theta-1)}$, *here* $L(0) = (E_1 - E(0))^{1-\alpha} + 2\delta_1 \int_\Omega u_0 u_1 \,\mathrm{d}x$, $0 < \alpha < 1/m - 1/p$, $\theta = 1/(1-\alpha)$, δ_1 *is a small positive constant given in the proof, and* c_{10} *is given in* (4.24).

Proof. Let

$$H(t) = E_1 - E(t), \quad t \geqslant 0. \tag{4.7}$$

By (3.2), we have

$$H'(t) = -E'(t) = \int_\Omega |u_t|^m \,\mathrm{d}x \geqslant 0. \tag{4.8}$$

Thus, we obtain

$$H(t) \geqslant H(0) = E_1 - E(0) > 0, \quad t > 0. \tag{4.9}$$

Let

$$a(t) = \int_\Omega u^2 \,\mathrm{d}x + 2m_1 E_1 t^2, \quad t \in [0, T_0], \tag{4.10}$$

where T_0 will be specified later.

By differentiating (4.10) twice, to obtain

$$a'(t) = 2\int_\Omega u u_t \,\mathrm{d}x + 4m_1 E_1 t, \quad t \in [0, T_0]$$

and

$$a''(t) = 2\int_\Omega u_t^2 \,\mathrm{d}x + 2\int_\Omega u u_{tt} \,\mathrm{d}x + 4m_1 E_1, \quad t \in [0, T_0]. \tag{4.11}$$

By using (4.1) and (4.3), we obtain

$$\begin{aligned} a''(t) \geqslant{}& 2\int_\Omega u_t^2 \,\mathrm{d}x - 2m_1 \overline{M}(\|\nabla u\|_2^2) - 2\int_\Omega |u_t|^{m-2} u_t u \,\mathrm{d}x \\ &+ 2\int_\Omega |u|^p \,\mathrm{d}x + 4m_1 E_1. \end{aligned}$$

From (3.2), (4.4) and (4.7), we obtain

$$a''(t) \geqslant 2(1+2m_1)\int_\Omega u_t^2 \,\mathrm{d}x + 4m_1 H(t) + c_0 \|u\|_p^p - 2\int_\Omega |u_t|^{m-2} u_t u \,\mathrm{d}x, \tag{4.12}$$

where $c_0 = (2p - 4m_1)/p > 0$.

By observing that

$$\begin{aligned} \left| \int_\Omega |u_t|^{m-2} u_t u \,\mathrm{d}x \right| &\leqslant \|u_t\|_m^{m-1} \|u\|_m \\ &\leqslant c_1 \|u\|_p^{1-(p/m)} \|u\|_p^{(p/m)} \|u_t\|_m^{m-1}, \end{aligned} \tag{4.13}$$

where $c_1 = (\mathrm{vol}(\Omega))^{(p-m)/mp}$.

Note that from (4.7)

$$H(t) = E_1 - \left[\frac{1}{2}\|u_t\|_2^2 + \frac{1}{2}\overline{M}(\|\nabla u\|_2^2) - \frac{1}{p}\|u\|_p^p\right].$$

By (4.2), we have

$$H(t) \leqslant E_1 - \frac{1}{2} m_0 \|\nabla u\|_2^2 + \frac{1}{p}\|u\|_p^p.$$

Thus, by using Lemma 4.2 (ii), we have

$$H(t) \leqslant E_1 - \frac{1}{2} m_0 \lambda_1^2 + \frac{1}{p}\|u\|_p^p.$$

And by (4.6), (4.9), we obtain

$$0 < H(0) \leqslant H(t) \leqslant \frac{1}{p}\|u\|_p^p, \quad \text{for } t \in [0, T_0]. \tag{4.14}$$

Note that by (4.14) and using Hölder inequality, we have from (4.13)

$$\left|\int_\Omega |u_t|^{m-2} u_t u \, \mathrm{d}x\right| \leqslant c_2 \|u\|_p^{p/m} H(t)^{1/p-1/m} \|u_t\|_m^{m-1}.$$

By Young′s inequality and (4.8), we obtain

$$\left|\int_\Omega |u_t|^{m-2} u_t u \, \mathrm{d}x\right| \leqslant c_2(\varepsilon^m \|u\|_p^p + \varepsilon^{-m'} H'(t)) H(t)^{-\overline{\alpha}}, \tag{4.15}$$

where $\overline{\alpha} = 1/m - 1/p > 0$, $\varepsilon > 0$, $m' = m/m-1$, $c_2 = c_1 \cdot p^{1/p-1/m}$.

Letting $0 < \alpha < \overline{\alpha}$ and by (4.14), (4.15), we have

$$\left|\int_\Omega |u_t|^{m-2} u_t u \, \mathrm{d}x\right| \leqslant c_2(\varepsilon^m H(0)^{-\overline{\alpha}} \|u\|_p^p + \varepsilon^{-m'} H(0)^{\alpha-\overline{\alpha}} H(t)^{-\alpha} H'(t)). \tag{4.16}$$

Now, we define

$$L(t) = H(t)^{1-\alpha} + \delta_1 a'(t), \quad t \in [0, T_0], \tag{4.17}$$

where δ_1 is a positive constant to be specified later.

From (4.17),

$$L'(t) = (1-\alpha) H(t)^{-\alpha} H'(t) + \delta_1 a''(t), \quad t \in [0, T_0].$$

By (4.12), we obtain

$$\begin{aligned} L'(t) \geqslant{}& (1-\alpha) H(t)^{-\alpha} H'(t) + 2\delta_1(1+2m_1)\|u_t\|_2^2 + 4\delta_1 m_1 H(t) \\ &+ \delta_1 c_o \|u\|_p^p - 2\delta_1 \int_\Omega |u_t|^{m-2} u_t u \, \mathrm{d}x. \end{aligned}$$

And by (4.15), we have

$$L'(t) \geqslant (1-\alpha-2\delta_1 c_2 \varepsilon^{-m'} H(0)^{\alpha-\overline{\alpha}}) H(t)^{-\alpha} H'(t) + \delta_1 (c_0 - 2c_2 \varepsilon^m H(0)^{-\overline{\alpha}}) \times \|u\|_p^p + 2\delta_1 (1+2m_1) \|u_t\|_2^2 + 4m_1 \delta_1 H(t). \tag{4.18}$$

Now, choosing $\varepsilon > 0$ small such that $c_0 - 2c_2 \varepsilon^m H(0)^{-\overline{\alpha}} \geqslant \frac{1}{2} c_0$, and letting $0 < \delta_1 < (1-\alpha)/2c_2^{-1} \varepsilon^{m'} H(0)^{\overline{\alpha}-\alpha}$.

Then (4.18) becomes

$$\begin{aligned} L'(t) &\geqslant \tfrac{1}{2} c_0 \delta_1 \|u\|_p^p + 2\delta_1 (1+2m_1) \|u_t\|_2^2 + 4m_1 \delta_1 H(t) \\ &\geqslant c_3 \delta_1 (\|u\|_p^p + \|u_t\|_2^2 + H(t)), \end{aligned} \tag{4.19}$$

here $c_3 = \min\{c_0/2,\ 2(1+2m_1), 4m_1\}$. Thus $L(t)$ is a nondecreasing function on $[0, T_0]$.

Letting δ_1 be small enough in (4.17), then we have $L(0) > 0$. Hence

$$L(t) > 0, \quad \text{for } t \in [0, T_0].$$

Now set $\theta = 1/(1-\alpha)$. Since $\alpha < \overline{\alpha} < 1$, it is evident that $1 < \theta < 1/(1-\overline{\alpha})$.

Choosing $\delta_1 > 0$ small enough such that $\delta_1 t < (\lambda_3 / B_1 \lambda_1)^{p/\theta}$, for $t \in [0, T_0]$, and using Lemma 4.2(ii), we obtain from (4.17)

$$L(t) \leqslant H(t)^{1-\alpha} + 2\delta_1 \int_\Omega u_t u \, \mathrm{d}x + \frac{4m_1 E_1}{(B_1 \lambda_1)^{p/\theta}} \|u\|_p^{p/\theta}, \quad \text{for } t \in [0, T_0].$$

By Young's and Hölder inequality, it follows that

$$\begin{aligned} L(t)^\theta &\leqslant 2^{\theta-1} \left[H(t) + \left(2\delta_1 \int_\Omega u_t u \, \mathrm{d}x + c_4 \|u\|_p^{p/\theta} \right)^\theta \right] \\ &\leqslant 2^{\theta-1} [H(t) + 2^{\theta-1} (2^\theta \delta_1^\theta \|u_t\|_2^\theta \|u\|_2^\theta + c_4^\theta \|u\|_p^p)] \\ &\leqslant c_5 [H(t) + \|u\|_p^p + \|u_t\|_2^\theta \|u\|_2^\theta], \end{aligned} \tag{4.20}$$

where $c_4 = 4m_1 E_1 / (B_1 \lambda_1)^{\frac{p}{\theta}}$, $c_5 = \max\{2^{\theta-1},\ 2^{3\theta-2} \delta_1^\theta,\ 2^{2(\theta-1)} c_4^\theta\}$.

On the other hand, for $p > 2$, using Hölder inequality we have

$$\|u_t\|_2^\theta \|u\|_2^\theta \leqslant c_6 \|u_t\|_2^\theta \|u\|_p^\theta,$$

here $c_6 = (\mathrm{vol}(\Omega))^{\theta(p-2)/2p}$.

And by Young's inequality, we obtain

$$\|u_t\|_2^\theta \|u\|_2^\theta \leqslant c_7 (\|u\|_p^{\theta\beta_1} + \|u_t\|_2^{\theta\beta_2}), \tag{4.21}$$

where $1/\beta_1 + 1/\beta_2 = 1$, $c_7 = c_7(c_6, \beta_1, \beta_2) > 0$. In particular, we take $\theta\beta_2 = 2$, i.e. $\beta_2 = 2(1-\alpha)$.

Therefore, for α small enough, the numbers β_1 and β_2 are close to 2.

Now choose $\alpha \in (0,\ \min(\overline{\alpha},\ \frac{1}{2} - \frac{1}{p}))$. Then, by (4.14), we obtain

$$\|u\|_p^{\theta\beta_1} = \left[\left(\frac{1}{pH(0)}\right)^{1/p} \|u\|_p\right]^{\theta\beta_1} \left(\frac{1}{pH(0)}\right)^{-\theta\beta_1/p}$$
$$\leqslant c_8 \|u\|_p^p, \tag{4.22}$$

because

$$\theta\beta_1 = \frac{2}{1-2\alpha} < p,$$

where $c_8 = (1/pH(0))^{1-\theta\beta_1/p}$.

Consequently, by (4.20)–(4.22), we have

$$L(t)^\theta \leqslant c_9[H(t) + \|u\|_p^p + \|u_t\|_2^2]. \tag{4.23}$$

Here $c_9 = c_9(c_5, c_7, c_8) > 0$. From (4.19), (4.23), we obtain

$$L'(t) \geqslant c_{10} L(t)^\theta, \quad \theta > 1, \tag{4.24}$$

here $c_{10} = c_3\delta_1/c_9$.

A simple integration of (4.24) over (0,t) then yields

$$L(t) \geqslant (L(0)^{1-\theta} - c_{10}(\theta - 1)t)^{-1/\theta-1}. \tag{4.25}$$

Since $L(0) > 0$, (4.25) shows that L becomes infinite in a finite time $T \leqslant T^* = L(0)^{1-\theta}/c_{10}(\theta - 1)$.

Remark 4.4. (1) T_0 can be chosen so that $T_0 \geqslant T^*$ in (4.10).

(2) When $M = 1$, the result is just the same as Vitillaro [18].

(3) If the condition $M(s) \geqslant m_0 > 0$, for all $s \geqslant 0$ in (A2) does not hold, another type of $M(s)$ can be considered. For example,

$$M(s) = bs^\gamma, \quad b > 0, \ \ \gamma \geqslant 1, \ \ s \geqslant 0,$$

then there exists $m_1 \geqslant 1$ such that

$$m_1 \overline{M}(s) \geqslant M(s)s, \quad \text{for all } s \geqslant 0.$$

Consider the following problem:

$$u_{tt} - b\|\nabla u\|_2^{2\gamma} \Delta u + |u_t|^{m-2} u_t = |u|^{p-2} u, \quad \text{in } \Omega \times [0, \infty), \tag{4.26}$$

$$u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), \quad x \in \Omega, \tag{4.27}$$

$$u(x, t) = 0, \quad x \in \partial\Omega, \ \ t > 0. \tag{4.28}$$

The nonexistence of global solution of (4.26)–(4.28) can be shown by using the same arguments as in Theorem 4.3.

Theorem 4.5. *If* $p > \max(2m_1, m, 2\gamma+2)$, *then there is no global solution of* (4.26)–(4.28) *with* $\|\nabla u_0\|_2 > \lambda_1$ *and* $E(0) < E_1$, *where*

$$\lambda_1 = \left(\frac{b}{B_1^p}\right)^{1/(p-2\gamma-2)}, \quad E_1 = b^{p/(p-2\gamma-2)}\left(\frac{1}{2(\gamma+1)} - \frac{1}{p}\right) B_1^{-2p\gamma-2/p-2\gamma-2},$$

here B_1 *is the optimal constant of Sobolev imbedding* $H_0^1(\Omega) \hookrightarrow L^p(\Omega)$.

References

[1] J. Ball, Remarks on blowup and nonexistence theorems for nonlinear evolutions, Quart. J. Math. Oxford 28 (1977) 473–486.
[2] V. Georgiev, D. Todorova, Existence of solutions of the wave equations with nonlinear damping and source terms, J. Differential Equations 109 (1994) 295–308.
[3] R.T. Glassey, Blow-up theorems for nonlinear wave equations, Math. Z. 132 (1973) 183–203.
[4] Ikehata Ryo, On solutions to some quasilinear hyperbolic equations with nonlinear inhomogeneous terms, Nonlinear Anal. Theory, Methods Appl. 17 (1991) 181–203.
[5] J.B. Keller, On solutions of nonlinear wave equations, Comm. Pure Appl. Math. 10 (1957) 523–530.
[6] G. Kirchhoff, Vorlesungen űber Mechanik, Leipzig, Teubner, 1883.
[7] H.A. Levine, Instability and nonexistence of global solutions of nonlinear wave equation of the form $Du_{tt} = Au + F(u)$, Trans. Amer. Math. Soc. 192 (1974) 1–21.
[8] H.A. Levine, Some additional remarks on the nonexistence of global solutions to nonlinear wave equations, SIAM J. Math. Anal. 5 (1974) 138–146.
[9] H.A. Levine, Nonexistence of global weak solutions to some properly and improperly posed problems of mathematical physics: the method of unbound Fourier coefficients, Math. Ann. 214 (1975) 205–220.
[10] H.A. Levine, S. Ro Park, Global existence and global nonexistence of solutions of the cauchy problems for a nonlinearly damped wave equation, J. Math. Anal. Appl. 228 (1998) 181–205.
[11] H.A. Levine, J. Serrin, Global nonexistence theorems for quasilinear evolution equations with dissipation, Arch. Rational Mech. Anal. 137 (1997) 341–361.
[12] M.R. Li, L.Y. Tsai, Existence and nonexistence of global solutions of some systems of semilinear wave equations, Nonlinear Anal. Theory, Methods Appl. 54 (2003) 1397–1415.
[13] S.A. Messaoudi, Blow-up of solutions for the Kirchhoff equation of q-Laplacian type with nonlinear dissipation, Colloq. Math. 94 (2002) 103–109.
[14] K. Ono, Blowing up and global existence of solutions for some degenerate nonlinear wave equations with some dissipation, Nonlinear Anal. Theory, Methods Appl. 30 (1997) 4449–4457.
[15] K. Ono, Global existence, decay, and blowup of solutions for some mildly degenerate nonlinear Kirchhoff strings, J. Differential Equations 137 (1997) 273–301.
[16] K. Ono, On global existence, asymptotic stability and blowing up of solutions for some degenerate nonlinear wave equations of Kirchhoff type with a strong dissipation, Math. Methods Appl. Sci. 20 (1997) 151–177.
[17] W.A. Strauss, On continuity of functions with values in various Banach spaces, Pacific J. Math. 19 (1966) 543–551.
[18] E. Vitillaro, Global nonexistence theorems for a class of evolution equations with dissipation, Arch. Rational Mech. Anal. 149 (1999) 155–182.