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On the Emden–Fowler equation $u'' - |u|^{p-1}u = 0$

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Abstract

In this paper, we work with the ordinary equation $u'' - |u|^{p-1}u = 0$ for some $p > 0$ and obtain some interesting phenomena concerning blow-up, blow-up rate, existence interval, stability, instability, zeros and critical points of solutions to those equations.

Keywords: Estimate; Life-span; Blow-up; Blow-up rate; Stability

1. Introduction

In our papers [2–7] we studied the semi-linear wave equation $\Box u + f(u) = 0$ under some conditions, and found some interesting results on blow-up, blow-up rate and the estimates for the existence interval of solutions, but no information on the singular set. Here, we wish to deal with particular cases in lower-dimensional wave equations. We hope that the experiences gained here will allow us to deal with more general lower-dimensional cases later.

Consider the stationary, one-dimensional semilinear wave equation

$$\begin{cases} u'' - |u|^{p-1}, & u = 0, \\ u(0) = u_0, & u'(0) = u_1. \end{cases} \tag{1.1}$$

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From some calculations one can find that for $p \in (0,1)$, Eq. (1.1) with $u_0 = 0 = u_1$ possesses infinitely many solutions, so the solutions of the above equation in general are not unique. It is clear that these functions $|u|^{p-1}u,\ \ p \geqslant 1$, are locally Lipschitz; hence, by the standard theory, the local existence of classical solutions is applicable to Eq. (1.1).

We discuss problem (1.1) in three parts: "$p > 1$", "$p < 1$" and "the singularity and regularity of solutions".

Part A: Estimates for the existence interval of solutions of (1.1) for $p > 1$

In Section 2, we deal with the estimations for the existence interval of the solutions of (1.1), in Section 3 with the blow-up rate and blow-up constant, in Section 4 with the global existence, critical point and the asymptotic behavior, in Section 5 with the null points (zero) and triviality, and in Section 6 with stability and instability.

1.1. Notation and fundamental lemmas

For a given function u in this work, we use the following abbreviations:

$$a_u(t) = u(t)^2,\ \ E_u(0) = u_1^2 - \frac{2}{p+1}|u_0|^{p+1},\ \ J_u(t) = a_u(t)^{-\frac{p-1}{4}}.$$

Definition. A function $g : \mathbb{R} \to \mathbb{R}$ with a blow-up rate q means that g exists only in finite time; that is, there is a finite number T^* such that the following are valid:

$$\lim_{t \to T^*} g(t)^{-1} = 0 \tag{1.2}$$

and there exists a non-zero $\beta \in \mathbb{R}$; with

$$\lim_{t \to T^*} (T^* - t)^q g(t) = \beta; \tag{1.3}$$

in this case β is called the blow-up constant of g.

According to the uniqueness of the solutions to Eq. (1.1) for $p > 1$, we can rewrite $a_u(t) = a(t)$, $J_u(t) = J(t)$ and $E_u(t) = E(t)$. After some elementary calculations we obtain the following Lemma 1.

Lemma 1. *Suppose that u is the solution of* (1.1); *then, we have*

$$E(t) = u'(t)^2 - \frac{2}{p+1}|u|^{p+1} = E(0), \tag{1.4}$$

$$(p+3)u'(t)^2 = (p+1)E(0) + a''(t), \tag{1.5}$$

$$J''(t) = \frac{p^2-1}{4}E(0)J(t)^{\frac{p+3}{p-1}}, \tag{1.6}$$

$$J'(t)^2 = J'(0)^2 - \frac{(p-1)^2}{4}E(0)J(0)^{\frac{2(p+1)}{p-1}} + \frac{(p-1)^2}{4}E(0)J(t)^{\frac{2(p+1)}{p-1}} \tag{1.7}$$

and

$$a'(t) = a'(0) + 2E(0)t + \frac{2(p+3)}{p+1}\int_0^t |u(r)|^{p+1}\,\mathrm{d}r. \tag{1.8}$$

The following lemmas are easy to prove, so we omit the arguments.

Lemma 2. *Suppose that r and s are real constants and $u \in C^2(\mathbb{R})$ satisfies*

$$u'' + ru' + su \leqslant 0, \quad u \geqslant 0,$$
$$u(0) = 0, \quad u'(0) = 0;$$

then, u must be null, that is, $u \equiv 0$.

Lemma 3. *If $g(t)$ and $h(t,r)$ are continuous with respect to their variables and the limit $\lim_{t\to T}\int_0^{g(t)} h(t,r)\,\mathrm{d}r$ exists, then*

$$\lim_{t\to T}\int_0^{g(t)} h(t,r)\,\mathrm{d}r = \int_0^{g(T)} h(T,r)\,\mathrm{d}r.$$

2. Estimates for the existence intervals

To estimate the existence interval of the solution of Eq. (1.1), we separate this section into three parts: $E(0) < 0$, $E(0) = 0$ and $E(0) > 0$. Here, the existence interval T of u means that u exists and makes sense only in the interval $[0, T)$ so that problem (1.1) possesses the solution $u \in \bar{C}^2(0, T)$.

2.1. Estimates for the existence intervals under $E(0) \leqslant 0$

We deal with two cases, $E(0) < 0$ and $E(0) = 0$, $a'(0) > 0$, in this subsection, but the case $E(0) = 0$ and $a'(0) \leqslant 0$ will be considered in Sections 4 and 5 later. Here we have the following result:

Theorem 4. *If T is the existence interval of the solution u to* (1.1) *with $E(0) < 0$, then T is finite. Further, for $a'(0) \geqslant 0$ we have the estimate*

$$T \leqslant T_1^* = \frac{2}{p-1}\int_0^{J(0)} \frac{\mathrm{d}r}{\sqrt{\frac{2}{p+1} + E(0)r^{\frac{2p+2}{p-1}}}} \tag{2.1}$$

for $a'(0) < 0$,

$$T \leqslant T_2^* = \frac{2}{p-1}\left(\int_0^{\alpha} + \int_{J(0)}^{\alpha}\right)\frac{\mathrm{d}r}{\sqrt{\frac{2}{p+1} + E(0)r^{\frac{2p+2}{p-1}}}}, \tag{2.2}$$

where $\alpha = (\frac{2}{p+1}\frac{-1}{E(0)})^{\frac{p-1}{2p+2}}$. *Furthermore, if* $E(0)=0$ *and* $a'(0)>0$, *then*

$$T \leqslant T_3^* := \frac{4}{p-1}\frac{a(0)}{a'(0)}. \tag{2.3}$$

Proof. For $E(0)<0$, we know that $a(0)>0$; otherwise, we obtain $a(0)=0$, that is, $u_0=0$. Then $E(0)=u_1^2 \geqslant 0$, which contradicts $E(0)<0$. In this situation, we separate the proof of this theorem into two cases: $a'(0)\geqslant 0$ and $a'(0)<0$.

(i) $a'(0)\geqslant 0$. By (1.5) and (1.7) we find that

$$\begin{cases} a'(t) \geqslant a'(0) - (p+1)E(0)t & \forall\, t \geqslant 0, \\ a(t) \geqslant a(0) + a'(0)t - \dfrac{p+1}{2}E(0)t & \forall\, t\geqslant 0, \end{cases} \tag{2.4}$$

$$J'(t) = -\frac{p-1}{2}\sqrt{\frac{2}{p+1} + E(0)J(t)^{\frac{2p+2}{p-1}}} \leqslant J'(0) \quad \forall t \geqslant 0 \tag{2.5}$$

and

$$J(t) \leqslant a(0)^{-\frac{p-1}{4}} - \frac{p-1}{4}a(0)^{-\frac{p+3}{4}}a'(0)t \quad \forall t\geqslant 0.$$

Thus there exists a finite number

$$T_1^*(u_0,u_1,p) \leqslant \frac{4}{p-1}\frac{a(0)}{a'(0)}$$

such that $J(T_1^*(u_0,u_1,p))=0$ and so $a(t)\to\infty$ as $t\to T_1^*(u_0,u_1,p)$. This means that $T\leqslant T_1^*(u_0,u_1,p)$. Now we estimate $T_1^*(u_0,u_1,p)$. By (2.5) and $J(T_1^*(u_0,u_1,p))=0$ we find that

$$\int_{J(t)}^{J(0)} \frac{\mathrm{d}r}{\sqrt{\frac{2}{p+1} + E(0)r^{\frac{2p+2}{p-1}}}} = \frac{p-1}{2}t \quad \forall t\geqslant 0 \tag{2.6}$$

and hence we obtain estimate (2.1).

(ii) $a'(0)<0$. By (2.4), $a'(0)<0$ and the convexity of a we can find a unique finite number $t_0 = t_0(u_0,u_1,p)$ such that

$$\begin{cases} a'(t) < 0 = a'(t_0) & \text{for } t\in(0,t_0), \\ a'(t) > 0 & \text{for } t > t_0 \end{cases} \tag{2.7}$$

and $a(t_0)>0$. If not, then $u(t_0)=0$; thus, $E(0)=E(t_0)=u'(t_0)^2\geqslant 0$. Yet, this is a contradiction to $E(0)<0$. Hence, we conclude that

$$a(t) > 0 \quad \forall t\geqslant 0, \quad u'(t_0)=0, \quad E(0) = -\frac{2}{p+1}u(t_0)^{p+1} \quad \text{and}$$
$$J(t_0)^{\frac{2p+2}{p-1}} = \frac{2}{p+1}\frac{-1}{E(0)}.$$

After arguments similar to step (i), there exists a $T_2^* := T_2^*(u_0, u_1, p)$ such that the existence interval T of u is bounded by T_2^*, that is, $T \leqslant T_2^*$. By an analogous argument, using (2.7), (1.7) and the fact that

$$J(t_0)^{\frac{2p+2}{p-1}} = \frac{2}{p+1}\frac{-1}{E(0)} \quad \text{and} \quad J(T_2^*) = 0,$$

we conclude that

$$J'(t)^2 = -\frac{(p-1)^2}{4}E(0)(J(t_0)^{\frac{2p+2}{p-1}} - J(t)^{\frac{2p+2}{p-1}}) \quad \forall t \geqslant t_0,$$
$$J'(t)^2 = \frac{(p-1)^2}{4}E(0)(J(0)^{\frac{2p+2}{p-1}} - J(t)^{\frac{2p+2}{p-1}}) \quad \forall t \in [0, t_0],$$

$$J'(t) = -\frac{p-1}{2}\sqrt{\frac{2}{p+1} + E(0)J(t)^{\frac{2p+2}{p-1}}} \quad \forall t \geqslant t_0, \tag{2.8a}$$

$$J'(t) = \frac{p-1}{2}\sqrt{\frac{2}{p+1} + E(0)J(t)^{\frac{2p+2}{p-1}}} \quad \forall t \in [0, t_0], \tag{2.8b}$$

$$\int_{J(t)}^{J(t_0)} \frac{\mathrm{d}r}{\sqrt{\frac{2}{p+1} + E(0)r^{\frac{2p+2}{p-1}}}} = \frac{p-1}{2}(t - t_0) \quad \forall t \geqslant t_0, \tag{2.9a}$$

$$\int_{J(0)}^{J(t_0)} \frac{\mathrm{d}r}{\sqrt{\frac{2}{p+1} + E(0)r^{\frac{2p+2}{p-1}}}} = \frac{p-1}{2}t_0 \tag{2.9b}$$

and

$$T_2^* = t_0 + \frac{2}{p-1}\int_0^{(\frac{2}{p+1}\frac{-1}{E(0)})^{\frac{p-1}{2p+2}}} \frac{\mathrm{d}r}{\sqrt{\frac{2}{p+1} + E(0)r^{\frac{2p+2}{p-1}}}}. \tag{2.10}$$

This estimate (2.10) is equivalent to (2.2).

(iii) For $E(0) = 0$, by (1.6) and $a'(0) > 0$ we obtain that $J'(0) < 0$, $J''(t) = 0$ and $J(t) = a(0)^{-\frac{p-1}{4}-1}(a(0) - \frac{p-1}{4}a'(0)t)$ $\forall t \geqslant 0$. Thus, we conclude that

$$a(t) = a(0)^{\frac{p+3}{p-1}}\left(a(0) - \frac{p-1}{4}a'(0)t\right)^{-\frac{4}{p-1}} \quad \forall t \geqslant 0, \tag{2.11}$$

and (2.3) is proved. □

2.2. *Estimates for the existence intervals under* $E(0) > 0$

In this subsection we consider the case $E(0) > 0$, and we have the following blow-up result.

Theorem 5. *If T^* is the existence interval of u which solves problem* (1.1) *with* $E(0) > 0$, *then T^* is finite. Further, in case of $a'(0) > 0$ we have*

$$T^* \leqslant T_4^*(u_0, u_1, p) = \frac{2}{p-1}\int_0^{J(0)} \frac{\mathrm{d}r}{\sqrt{\frac{2}{p+1} + E(0)r^{q+1}}}. \tag{2.12}$$

In the case of $a'(0) = 0$ we have

$$T^* \leqslant T_5^*(u_0, u_1, p) = \frac{2}{p-1}\int_0^{\infty} \frac{\mathrm{d}r}{\sqrt{\frac{2}{p+1} + E(0)r^{q+1}}}, \tag{2.13}$$

where $q = \frac{p+3}{p-1}$. For $a'(0) < 0$ and $z(u_0, u_1, p)$ given by

$$z(u_0, u_1, p) = \int_0^{\sqrt{a(0)}} \frac{\mathrm{d}r}{\sqrt{E(0) + \frac{2}{p+1}r^{p+1}}}, \tag{2.14}$$

is the zero of a. Further we have

$$T^* \leqslant T_6^*(u_0, u_1, p) := (z + T_5^*)(u_0, u_1, p). \tag{2.15}$$

Proof. The case of a zero for u is deferred to Section 5.

(i) For $a'(0) > 0$, by (1.6) we have

$$\begin{cases} kJ''(t) = (kJ(t))^q, \\ kJ(0) = ka(0)^{-\frac{p-1}{4}}, \ \ kJ'(0) = \frac{1-p}{4}\,ka(0)^{-\frac{p+3}{4}}a'(0), \end{cases}$$

where $k := (\frac{p^2-1}{4}\,E(0))^{\frac{p-1}{4}}$ and $q := \frac{p+3}{p-1}$. Now we set

$$\tilde{E}(t) := k^2 J'(t)^2 - \frac{2}{q+1}(kJ(t))^{q+1}; \tag{2.16}$$

from some calculations we see that $\tilde{E}(t)$ is a constant and by using (1.8) we obtain that

$$\tilde{E}(t) = \frac{(p-1)^2}{2p+2}\,k^2 = \tilde{E}(0),$$
$$\frac{(p-1)^2}{2p+2} = J'(t)^2 - \frac{2k^{q-1}}{q+1}\,J(t)^{q+1}, \tag{2.17}$$

$$a'(t) \geqslant a'(0) + 2E(0)t > 0 \quad \forall t \geqslant 0,$$
$$J'(t) < 0 \quad \forall t \geqslant 0, \tag{2.18}$$

$$J'(t) = -\frac{p-1}{2}\sqrt{\frac{2}{p+1} + E(0)J(t)^{q+1}} \quad \forall t \geqslant 0 \tag{2.19}$$

and

$$\int_{J(t)}^{J(0)} \frac{\mathrm{d}r}{\sqrt{\frac{2}{p+1} + E(0)r^{q+1}}} = \frac{p-1}{2} t \quad \forall t \geqslant 0. \tag{2.20}$$

By (2.19), there exists a finite number $T_4^*(u_0, u_1, p)$, such that $J(T_4^*(u_0, u_1, p)) = 0$, and from (2.20) estimate (2.12) follows easily.

(ii) From $a'(0) = 0 = u_0$, $E(0) = u_1^2$ and (1.8) we obtain

$$\begin{aligned} &a'(t) = 2E(0)t + 2q \int_0^t |u(r)|^{p+1}\,\mathrm{d}r \quad \forall t \geqslant 0, \\ &a(t) > 0 \quad \forall t \geqslant 0; \end{aligned} \tag{2.21}$$

thus, $J(t)$ can be defined for each $t > 0$ and $J'(t) < 0$ $\forall t > 0$.

Using (1.6), for each $\check{t} > 0$ we conclude that

$$J'(t) = -\sqrt{J'(\check{t})^2 - \frac{(p-1)^2}{4} E(0)(J(\check{t})^{q+1} - J(t)^{q+1})} \quad \forall t \geqslant \check{t}, \tag{2.22}$$

$$\lim_{\check{t}\to 0} J'(\check{t})^2 - \frac{(p-1)^2}{4} u_1^2 J(\check{t})^{q+1} = \frac{(p-1)^2}{2(p+1)}; \tag{2.23}$$

thus after inducing (2.22) and (2.23) the estimate (2.13) follows.

(iii) For $a'(0) < 0$, by (2.18) we have $a'(t) \geqslant 0$ for large t.

Suppose z is the first positive number t so that $a'(t) = 0$; then $u(z) = 0$. Otherwise, $u'(z) = 0$ and $E(z) = -\frac{2}{p+1}|u(z)|^{p+1} < 0$, which contradicts the assumption $E(0) = E(z) > 0$. After the time $t = z$, same as the procedures given in the proof of (i), using (2.20) we obtain (2.15). □

2.3. *Some properties concerning the existence interval $T_1^*(u_0, u_1, p)$*

In principle, $T_1^*(u_0, u_1, p)$ depends on three variables u_0, u_1 and p. Set

$$c_{k,\ p} := \frac{(p+1)u_1^2}{2u_0^{p+1}};$$

then

$$T_1^*(u_0, u_1, p) = \frac{\sqrt{q+1}}{\sqrt{p-1}}\, u_0^{-\frac{p-1}{2}} (\sqrt[q]{1 - c_{k,\ p}})^{-1} \int_0^{\sqrt[q+1]{1-c_{k,\ p}}} \frac{\mathrm{d}r}{\sqrt{1 - r^{q+1}}}$$

and $\lim_{p\to\infty} T_1^*(u_0, u_1, p) = 0$, $\lim_{p\to 1} T_1^*(u_0, u_1, p) = \infty$, where $q = \frac{p+3}{p-1}$. For convenience, we consider the case $u_1 = 0$,

$$T_1^*(u_0, 0, p) = \frac{\sqrt{\pi}}{\sqrt{2p+2}}\, u_0^{-\frac{p-1}{2}} \frac{\Gamma(\frac{p-1}{2p+2})}{\Gamma(\frac{p}{p+1})}.$$

Using Maple we obtain the graphs of $T_1^*(u_0, 0, p)$ below

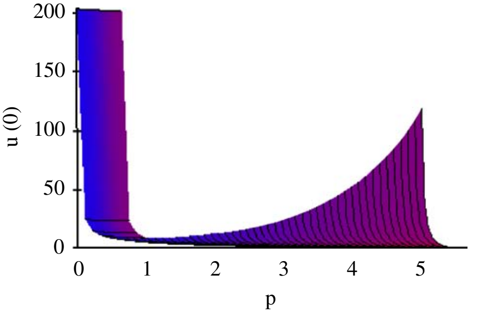

Fig. 1. Graph of $T_1^*(u_0, 0, p)$.

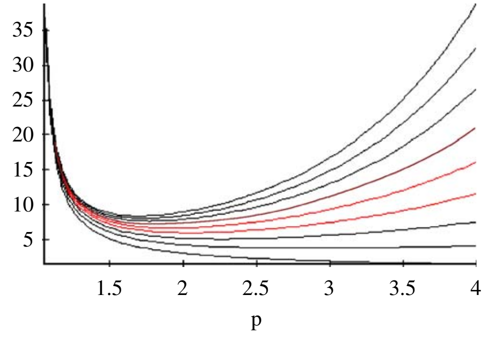

Fig. 2. Graphs of T_1^*, $u_0 \leqslant 1$.

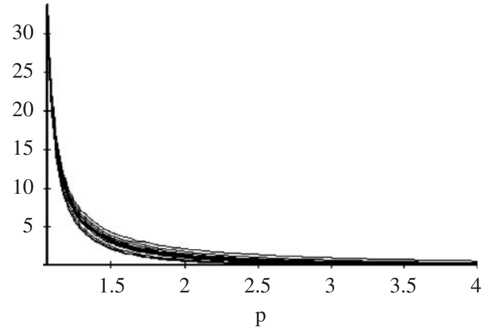

Fig. 3. Graphs of T_1^*, $u_0 > 1$.

The above graphs show the properties of $T_1^*(u_0, 0, p)$ (Figs. 1–3).

(1) there exists a constant u_0^* such that $T_1^*(u_0, 0, p)$ is monotone decreasing in p for $u_0 \in [u_0^*, 1)$;
(2) there is a p_0 such that $T_1^*(u_0, 0, p)$ is decreasing in $(1, p_0)$ and increasing in (p_0, ∞) provided $u_0 \in [0, u_0^*)$;

(3) $T_1^*(u_0, 0, p)$ is differentiable in its variables; and
(4) for $u_0 > 1$ the existence interval $T_1^*(u_0, 0, p)$ is decreasing in p.

We now show the validity of statements (3) and (4) using the monotonicity of $T_1^*(1, 0, p)$ for $u_0 \neq 0$. To prove (1) and (2) we must show the existence of u_0^* with $(\partial/\partial p)\, T_1^*(u_0, 0, p) \leqslant 0$ for $1 > u_0 \geqslant u_0^*$, that is,

$$0 \leqslant \frac{p-1}{p+1}(p+3)\int_0^1 (1-r^{q+1})^{-1/2}\,\mathrm{d}r + 4\int_0^1 (1-r^{q+1})^{-3/2} r^{q+1} \ln r\,\mathrm{d}r + (p-1)^2(\ln u_0)\int_0^1 (1-r^{q+1})^{-1/2}\,\mathrm{d}r,$$

thus the existence of u_0^* can be obtained, provided

$$\frac{p-1}{p+1}(p+3)(r^{q+1}-1) - 4\ln r > 0 \quad \forall r > 1,$$

where $q = (p+3)/(p-1)$. After some calculations it is easy to obtain the above assertion.

It is very difficult to grasp the property of the existence interval $T_1^* := T_1^*(u_0, u_1, p)$, but for fixed initial data we wish to know how the existence interval varies with p, so now we consider the existence interval $T_1^*(0.6, 0.2, p)$ and list the following tables.

p	$T_1^*(0.6, 0.2, p)$
1.001	2001.5
1.004	501.42
1.008	251.42
1.012	168.08

p	$T_1^*(0.6, 0.2, p)$
2	3.4135
2.5	2.7698
3	2.4659
3.6497	2.2644

After some computations we obtain

$$T_1^* = \frac{\sqrt{2p+2}}{p-1}\left(u_0^{p+1} - \frac{p+1}{2}u_1^2\right)^{-\frac{p-1}{2p+2}} \int_0^{\sqrt[q+1]{1-\frac{p+1}{2u_0^{p+1}}u_1^2}} \frac{\mathrm{d}r}{\sqrt{1-r^{q+1}}}.$$

By studying the existence interval T_1^*, we consider its properties with $a'(0) \geqslant 0$ in three cases:

Case 1: $0 < u_0^{p+1} - (p+1)u_1^2/2 < 1$. In this situation we find that

(i) for fixed u_1,

  (5) there exists a constant u_0^* depending on u_1 such that $T_1^*(u_0, u_1, p)$ is monotone decreasing in p for $u_0 \geqslant u_0^*$,
  (6) there is a p_0 so that $T_1^*(u_0, u_1, p)$ decreases in $(1, p_0)$ and increases in (p_0, ∞) provided $u_0 \in [0, u_0^*)$;

(ii) for fixed u_0, the existence interval $T_1^*(u_0, u_1, p)$ decreases in u_1^2.

Case 2: $u_0^{p+1} - (p+1)u_1^2/2 > 1$. The existence interval $T_1^*(u_0, u_1, p)$ decreases in p.

Case 3: $u_0^{p+1} - (p+1)u_1^2/2 = 1$. On the surface

$$\{(u_0, u_1, p) \in \mathbb{R}^3 | u_0^{p+1} - (p+1)u_1^2/2 = 1, \;\; p > 1\},$$

we find that

$$T_1^*(u_0, u_1, p) = T_1^*(u_0, p) = \frac{\sqrt{2p+2}}{p-1} \int_0^{u_0^{-(p-1)/2}} \frac{1}{\sqrt{1 - r^{q+1}}} \, \mathrm{d}r,$$

where $q = (p+3)/(p-1)$ and that $T_1^*(u_0, p)$ is monotone decreasing in u_0 and in p.

3. Blow-up rate and blow-up constant

In this section, we study the blow-up rate and blow-up constant for a, a' and a'' under the conditions in Section 2. We obtained the following results.

Theorem 6. *If u is the solution of problem* (1.1) *with one of the following properties that*:

(i) $E(0) < 0$ *or*
(ii) $E(0) = 0$, $a'(0) > 0$ *or*
(iii) $E(0) > 0$,

then the blow-up rate of a is $4/(p-1)$, and the blow-up constant K_1 of a is $\sqrt[p-1]{4(p-1)^{-4}(p+1)^2}$, that is, for $m = 1, 2, 3, 4, 5, 6$,

$$\lim_{t \to T_m^*} (T_m^* - t)^{\frac{4}{p-1}} a(t) = 2^{\frac{2}{p-1}} (p+1)^{\frac{2}{p-1}} (p-1)^{-\frac{4}{p-1}}. \tag{3.1}$$

The blow-up rate of a' is $(p+3)/(p-1)$, and the blow-up constant K_2 of a' is $2^{\frac{2p}{p-1}}(p+1)^{\frac{2}{p-1}} (p-1)^{-\frac{p+3}{p-1}}$, that is, for $m = 1, 2, 3, 4, 5, 6$,

$$\lim_{t \to T_m^*} (T_m^* - t)^{\frac{p+3}{p-1}} a'(t) = 2^{\frac{2p}{p-1}} (p+1)^{\frac{2}{p-1}} (p-1)^{-\frac{p+3}{p-1}}. \tag{3.2}$$

The blow-up rate of a'' is $(2p+2)/(p-1)$, and the blow-up constant K_3 of a'' is $2^{\frac{2p}{p-1}}(p+1)^{\frac{8}{p-1}}(p-1)^{-\frac{2p+8}{p-1}}(p+3)$, that is, $m = 1, 2, 3, 4, 5, 6$,

$$\lim_{t \to T_m^*} a''(t)(T_m^* - t)^{\frac{2p+2}{p-1}} = 2^{\frac{2p}{p-1}} (p+3)(p+1)^{\frac{2}{p-1}} (p-1)^{-\frac{2p+2}{p-1}}. \tag{3.3}$$

Proof. (i) Under this condition, $E(0)<0,\ a'(0)\geqslant 0$ by (2.1), (2.6) and Lemma 4 we obtain

$$\int_0^{J(t)} \frac{1}{T_1^*-t} \frac{\mathrm{d}r}{\sqrt{\frac{2}{p+1}+E(0)r^{\frac{2p+2}{p-1}}}} = \frac{p-1}{2} \quad \forall t\geqslant 0, \tag{3.4}$$

$$\lim_{t\to T_1^*} \sqrt{\frac{p+1}{2}} \frac{J(t)}{T_1^*-t} = \frac{p-1}{2}. \tag{3.5}$$

This identity (3.5) is equivalent to (3.1) for $m=1$.

For $E(0)<0,\ a'(0)<0$ by (2.9a,b) we also have

$$\int_0^{J(t)} \frac{\mathrm{d}r}{\sqrt{\frac{2}{p+1}+E(0)r^{\frac{2p+2}{p-1}}}} = \frac{p-1}{2}(T_2^*-t) \quad \forall t\geqslant t_0. \tag{3.6}$$

Through Lemma 4 and (3.6), therefore, we obtain (3.1) for $m=2$.

Observing (2.5) and (2.8a,b), we find

$$\lim_{t\to T_m^*} J'(t) = -\frac{p-1}{\sqrt{2p+2}}, \tag{3.7}$$

$$\lim_{t\to T_m^*} a'(t)(T_m^*-t)^{\frac{p+3}{p-1}} = 2^{\frac{2p}{p-1}}(p+1)^{\frac{2}{p-1}}(p-1)^{-\frac{p+3}{p-1}}, \tag{3.8}$$

$$\lim_{t\to T_m^*} u'(t)^2(T_m^*-t)^{\frac{2p+2}{p-1}} = 2^{\frac{2p}{p-1}}(p+1)^{\frac{2}{p-1}}(p-1)^{-\frac{2p+2}{p-1}} \tag{3.9}$$

for $m=1,2$. Using (1.5) and (3.9) for $m=1,2$, we obtain

$$\lim_{t\to T_m^*} a''(t)(T_m^*-t)^{\frac{2p+2}{p-1}} = (p+3)\lim_{t\to T_m^*} u'(t)^2(T_m^*-t)^{\frac{2p+2}{p-1}}. \tag{3.10}$$

Thus, (3.10) and (3.3) are equivalent.

(ii) For $E(0)=0,\ a'(0)>0$, by (2.11) for $m=1,2$, we obtain

$$a(t) = a(0)^{\frac{p+3}{p-1}}\left(\frac{p-1}{4}a'(0)\right)^{-\frac{4}{p-1}} \cdot (T_3^*-t) \quad \forall t\geqslant 0. \tag{3.11}$$

Therefore, estimates (3.1)–(3.3) for $m=3$ follow from (3.11).

(iii) For $E(0)>0$, estimates (3.1)–(3.3) for $m=4, 5, 6$ are similar to the above arguments (i) in the proof of this theorem. $\square$

Now we consider the property of the blow-up constants K_1, K_2 and K_3. We have

$$K_1(p) = 2^{\frac{2}{p-1}}(p+1)^{\frac{2}{p-1}}(p-1)^{-\frac{4}{p-1}},$$
$$K_2(p) = 2^{\frac{2p}{p-1}}(p+1)^{\frac{2}{p-1}}(p-1)^{-\frac{p+3}{p-1}},$$
$$K_3(p) = 2^{\frac{2p}{p-1}}(p+3)(p+1)^{\frac{2}{p-1}}(p-1)^{-\frac{2p+2}{p-1}}.$$

Using Maple we have the graphs of K_1, K_2 and K_3 below (Fig. 4).

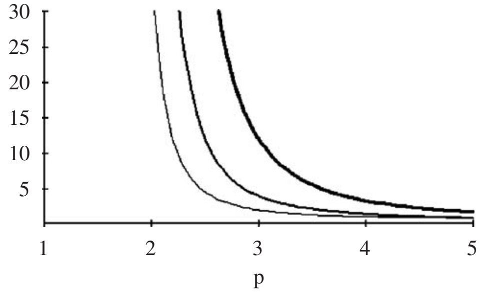

Fig. 4. Graphs of $K_1(p)$ in thin, $K_2(p)$ in medium, $K_3(p)$ in thick.

We see that the graphs, $K_i(p),\ i = 1, 2, 3$, are all decreasing in p, and $K_i(p)$ tends to 1, as p tends to infinity. The monotonicity of these functions can be obtained after showing the following inequalities:

$$\frac{p-1}{p+1} - 2 \leqslant \ln(2p+2) - 2\ln(p-1) \quad \forall p > 1,$$
$$\frac{2p-2}{p+1} + 4\ln(p-1) \leqslant 2\ln 2 + 2\ln(p+1) + p + 3 \quad \forall p > 1,$$
$$\frac{(p-1)^2}{p+3} + \frac{2p-2}{p+1} + 4\ln(p-1) \leqslant 2(\ln 2) + 2\ln(p+1) + 2p + 2 \quad \forall p > 1.$$

These inequalities are easy to verify, so we omit the arguments.

4. Global existence and critical point

In this section we study the following case that $E(0) = 0$ and $a'(0) < 0$.

Here, we consider the global existence of the solutions to problem (1.1) in the following sense:

$$J(t) > 0,\ a'(t)^{-2} > 0,\ a''(t)^{-2} > 0 \quad \forall t \in [0, T],$$

where T is the time that u exists; in other words, in any finite time u does not blow up in C^2 sense, even though u blows up in a finite time in some sense, for example, C^k or L^k for some $k \geqslant 3$.

By Bellman [1, p. 151] every positive proper solution of problem (1.1) has the asymptotic form

$$u(t) \sim ct^{-2/(p-1)}.$$

This result could be obtained and will be explained below only in the case where $E(0) = 0$ and $a'(0) < 0$. Under the condition it is easy to see that $J(t) > 0\ \forall t \in (0, T)$

and

$$a(t) = a(0)^{\frac{p+3}{p-1}} (a(0) - \frac{p-1}{4} a'(0)t)^{\frac{-4}{p-1}} \quad \forall t \in (0, T),$$

$$a'(t)^{-2} = a(0)^{\frac{-2p-6}{p-1}} a'(0)^{-2} \left(a(0) - \frac{p-1}{4} a'(0)t\right)^{\frac{2p+6}{p-1}} > 0 \quad \forall t \in (0, T),$$

$$a''(t)^{-2} = \frac{16}{(p+3)^2} a(0)^{\frac{-2p-6}{p-1}} a'(0)^{-4} \left(a(0) - \frac{p-1}{4} a'(0)t\right)^{\frac{4p+4}{p-1}} > 0 \quad \forall t \in (0, T).$$

Hence we find the limit $\lim_{t\to\infty} a(t) = 0$, $\lim_{t\to\infty} a'(t) = 0$, $\lim_{t\to\infty} a''(t) = 0$ and

$$\lim_{t\to\infty} t^{\frac{4}{p-1}} a(t) = a(0)^{\frac{p+3}{p-1}} \left(\frac{p-1}{-4} a'(0)\right)^{-\frac{4}{p-1}}, \tag{4.1}$$

$$\lim_{t\to\infty} t^{\frac{p+3}{p-1}} a'(t) = a(0)^{\frac{p+3}{p-1}} a'(0) \left(\frac{p-1}{-4} a'(0)\right)^{-\frac{p+3}{p-1}}, \tag{4.2}$$

$$\lim_{t\to\infty} t^{\frac{2p+2}{p-1}} a''(t) = \frac{p+3}{4} a(0)^{\frac{p+3}{p-1}} a'(0)^2 \left(\frac{p-1}{-4} a'(0)\right)^{-\frac{2p+2}{p-1}}. \tag{4.3}$$

Theorem 7. *Suppose that u is the solution of problem* (1.1) *with* $E(0) = 0$ *and* $a'(0) < 0$; *then u can be defined globally and estimates* (4.1)–(4.3) *are valid.*

5. Existence of zero and triviality

In this section, we discuss the triviality of the solution for problem (1.1) in the case where $E(0) = 0$, $a'(0) = 0$.

Proposition. *If u is the solution of problem* (1.1) *with* $p > 1$, $E(0) = 0$ *and* $a'(0) = 0$, *then u must be null.*

Proof. Under the conditions $E(0)=0$, $a'(0)=0$ using (1.5), it is easy to see that $u_0=0=u_1$; herein, the supremum below exists

$$t_1 := \sup\{\alpha : a(t) \leqslant 1 \ \forall t \in [0, \alpha]\},$$

and then

$$(p+1)u'(t)^2 = 2|u(t)|^{p+1} \geqslant 0,$$

$$a''(t) = (p+3)u'(t)^2 = 2\,\frac{p+3}{p+1} \cdot |u(t)|^{p+1} = 2\,\frac{p+3}{p+1} a(t)^{\frac{p+1}{2}}.$$

By Lemma 2 we conclude that

$$a''(t) \leqslant (p+3)a(t), \quad a(t) \equiv 0 \equiv u(t) \quad \text{in } [0, t_1]. \qquad \square$$

Proceeding with these steps we obtain the assertion of this theorem.

For the case where $E(0) > 0 > a'(0)$, we have the result.

Theorem 8. *Suppose that u is the solution to problem* (1.1) *with* $E(0) > 0 > a'(0)$ *and* $z(u_0, u_1, p)$ *given by*

$$z(u_0, u_1, p) = \int_0^{\sqrt{a(0)}} \frac{\mathrm{d}r}{\sqrt{E(0) + \frac{2}{p+1} r^{p+1}}}; \tag{5.1}$$

then $z(u_0, u_1, p)$ *is the zero of a. Further, we have*

$$\lim_{t \to z^-(u_0, u_1, p)} a(t)(z(u_0, u_1, p) - t)^{-2} = E(0)^2, \tag{5.2}$$

$$\lim_{t \to z^-(u_0, u_1, p)} (z(u_0, u_1, p) - t)^{-1} a'(t) = -2E(0)^{3/2}, \tag{5.3}$$

$$\lim_{t \to z^-(u_0, u_1, p)} a''(t) = 2E(0). \tag{5.4}$$

Proof. (1) For $E(0) > 0 > a'(0)$, by (1.4) we obtain that

$$a'(t) = -2\sqrt{E(0)a(t) + \frac{2}{p+1} a(t)^{\frac{p+3}{2}}}, \tag{5.5}$$

$$z(u_0, u_1, p) = \int_0^{a(0)} \frac{\mathrm{d}r}{2\sqrt{E(0)r + \frac{2}{p+1} r^{\frac{p+3}{2}}}}, \tag{5.6a}$$

$$t = \int_{a(t)}^{a(0)} \frac{\mathrm{d}r}{2\sqrt{E(0)r + \frac{2}{p+1} r^{\frac{p+3}{2}}}} \tag{5.6b}$$

and

$$\begin{aligned} z(u_0, u_1, p) &= \int_0^{a(0)} \frac{\mathrm{d}r}{2\sqrt{r}\sqrt{E(0) + \frac{2}{p+1} r^{\frac{p+1}{2}}}} = \int_0^{\sqrt{a(0)}} \frac{\mathrm{d}r}{\sqrt{E(0) + \frac{2}{p+1} r^{p+1}}} \\ &= \left(\frac{p+1}{2}\right)^{\frac{1}{p+1}} E(0)^{\frac{1-p}{2p+2}} \int_0^{(\frac{p+1}{2} E(0))^{\frac{-1}{p+1}} \sqrt{a(0)}} \frac{\mathrm{d}r}{\sqrt{1 + r^{p+1}}}. \end{aligned} \tag{5.7}$$

Thus, (5.1) is proved.

(2) From claim (5.2), by (5.6), (5.7) and Lemma 3 we obtain

$$z(u_0,u_1,p)-t=\int_0^{a(t)}\frac{\mathrm{d}r}{2\sqrt{E(0)r+\frac{2}{p+1}r^{\frac{p+3}{2}}}}$$
$$=\left(\frac{p+1}{2}\right)^{\frac{1}{p+1}}E(0)^{\frac{1-p}{2p+2}}\int_0^{(\frac{p+1}{2}E(0))^{\frac{-1}{p+1}}\sqrt{a(t)}}\frac{\mathrm{d}r}{\sqrt{1+r^{p+1}}},$$

$$(z(u_0,u_1,p)-t)^{-1}\int_0^{(\frac{p+1}{2}E(0))^{\frac{-1}{p+1}}\sqrt{a(t)}}\frac{\mathrm{d}r}{\sqrt{1+r^{p+1}}}=\frac{1}{(\frac{p+1}{2})^{\frac{1}{p+1}}E(0)^{\frac{1-p}{2p+2}}},$$

$$\frac{1}{(\frac{p+1}{2})^{\frac{1}{p+1}}E(0)^{\frac{1-p}{2p+2}}}$$
$$=\lim_{t\to z^-(u_0,u_1,p)}(z(u_0,u_1,p)-t)^{-1}\int_0^{(\frac{p+1}{2}E(0))^{\frac{-1}{p+1}}\sqrt{a(t)}}\frac{\mathrm{d}r}{\sqrt{1+r^{p+1}}}$$
$$=\lim_{t\to z^-(u_0,u_1,p)}(z(u_0,u_1,p)-t)^{-1}\left(\frac{p+1}{2}E(0)\right)^{\frac{-1}{p+1}}\sqrt{a(t)}$$
$$\times\lim_{t\to z^-(u_0,u_1,p)}\int_0^1\frac{\mathrm{d}s}{\sqrt{1+((\frac{p+1}{2}E(0))^{\frac{-1}{p+1}}\sqrt{a(t)}s)^{p+1}}}$$
$$=\lim_{t\to z^-(u_0,u_1,p)}(z(u_0,u_1,p)-t)^{-1}\left(\frac{p+1}{2}E(0)\right)^{\frac{-1}{p+1}}\sqrt{a(t)}. \tag{5.8}$$

Thus we obtain conclusion (5.2).

(3) Using (5.8) and (5.5) we obtain that

$$\lim_{t\to z^-(u_0,u_1,p)}(z(u_0,u_1,p)-t)^{-1}a'(t)$$
$$=-2\lim_{t\to z^-(u_0,u_1,p)}\sqrt{a(t)(z(u_0,u_1,p)-t)^{-2}\left(E(0)+\frac{2}{p+1}a(t)^{\frac{p+1}{2}}\right)}$$
$$=-2E(0)^{\frac{3}{2}}.$$

(4) Applying (1.5), (5.2) and (5.3), we find

$$\lim_{t\to z^-(u_0,u_1,p)}a(t)(z(u_0,u_1,p)-t)^{-2}a''(t)$$
$$=\frac{p+3}{4}\lim_{t\to z^-(u_0,u_1,p)}(a'(t)(z(u_0,u_1,p)-t)^{-1})^2$$
$$-\ (p+1)E(0)\lim_{t\to z^-(u_0,u_1,p)}a(t)(z(u_0,u_1,p)-t)^{-2}$$
$$=2E(0)^3.$$

Hence (5.4) is proved. □

6. Stability and instability

We now consider the applications of the above theorems to the stability theory for the problem

$$\begin{cases} u''(t) = |u(t)|^{p-1}u(t), \\ u(0) = \varepsilon_1, u'(0) = \varepsilon_2. \end{cases} \tag{$*$}$$

We say that problem $(*)$ is stable under condition F, if any nontrivial global solution $u \in C^2(\mathbb{R}^+)$ of $(*)$ under the condition F satisfies

$$\|u\|_{C^2} \to 0 \text{ for } |\varepsilon_1| + |\varepsilon_2| \to 0.$$

According to Theorems 4–8 we have the following result.

Corollary 9. *Problem* $(*)$ *with* $p > 1$ *is stable under* $E_u(0) = 0,\ \varepsilon_1\varepsilon_2 < 0$ *and unstable under one of the following*:

$$E_u(0) < 0, \tag{i}$$

$$E_u(0) = 0 < \varepsilon_1\varepsilon_2, \tag{ii}$$

$$E_u(0) > 0. \tag{v}$$

Theorems 4–8 may be summarized in the following tables:

Energy	$E(0) < 0$	$E(0) = 0$	$E(0) > 0$
T	(i) $a'(0) \geqslant 0$, $T \leqslant T_1^*$. (ii) $a'(0) < 0$, $T \leqslant T_2^*$.	(i) $a'(0) > 0$, $T \leqslant T_3^*$. (ii) $a'(0) < 0$, $T = \infty$. (iii) $a'(0) = 0$, $T = \infty, u \equiv 0$.	(i) $a'(0) > 0$, $T \leqslant T_4^*$. (ii) $a'(0) < 0$, $T \leqslant z + T_5^*$. (iii) $a'(0) = 0$, $T \leqslant T_5^*$.
Rn, Kn	$n + \frac{4}{p-1}$, Kn	$n + \frac{4}{p-1}$, Kn	$n + \frac{4}{p-1}$, Kn
Zero	Non	$a'(0) = 0, u \equiv 0$	$a'(0) < 0, z$

where T := Life-span, Rn := Blow-up rate for $a^{(n)}$, Kn := Blow-up constant for $a^{(n)}$, $n = 0, 1, 2$, and

$$T_1^* = \frac{2}{p-1}\int_0^{a(0)^{-(p-1)/4}} \frac{\mathrm{d}r}{\sqrt{\frac{2}{p+1} + E(0)r^{(2p+2)/(p-1)}}}, \quad T_3^* = \frac{4}{p-1}\,\frac{a(0)}{a'(0)},$$

$$T_2^* = \frac{2}{p-1}\left(\int_0^{\alpha} + \int_{J(0)}^{\alpha}\right)\frac{\mathrm{d}r}{\sqrt{\frac{2}{p+1} + E(0)r^{(2p+2)/(p-1)}}}, \alpha = \left(-\frac{2}{(p+1)E(0)}\right)^{\frac{p-1}{2p+2}},$$

$$T_4^* = \frac{2}{p-1}\int_0^{a(0)^{-(p-1)/4}} \frac{\mathrm{d}r}{\sqrt{\frac{2}{p+1} + E(0)r^{(2p+2)/(p-1)}}} = T_5^*,$$

$$z(u_0, u_1, p) = \int_0^{\sqrt{a(0)}} \frac{\mathrm{d}r}{\sqrt{E(0) + \frac{2}{p+1}\, r^{p+1}}}, \quad K1 = 2^{\frac{2}{p-1}}(p+1)^{\frac{2}{p-1}}(p-1)^{-\frac{4}{p-1}},$$

$$K2{:=}2^{\frac{2p}{p-1}}(p+1)^{\frac{2}{p-1}}(p-1)^{-\frac{p+3}{p-1}}, \quad K3 := 2^{\frac{2p}{p-1}}(p+3)(p+1)^{\frac{2}{p-1}}(p-1)^{-\frac{2p+2}{p-1}}.$$

Part B: Null, critical point and asymptotic behavior at infinity of solutions for Eq. (1.1) under $p < 1$

Before studying the properties of solutions for the differential equation (1.1) we gather some results in the situation where $E_u(0) = 0$.

(i) For $u_0 > 0$ and $u_1 > 0$, we have

$$u(t) = \left(u_0^{\frac{1-p}{2}} + \frac{1-p}{2}\sqrt{\frac{2}{p+1}}t\right)^{\frac{2}{1-p}}$$

and

$$t^{\frac{2}{p-1}} u(t) \to \left(\frac{1-p}{2}\sqrt{\frac{2}{p+1}}\right)^{\frac{2}{1-p}} \quad \text{as } t \to \infty.$$

(ii) For $u_0 > 0$ and $u_1 < 0$, the solutions of (1.1) can be given as

$$u_c(t) = \begin{cases} \left(u_0^{\frac{1-p}{2}} - \frac{1-p}{2}\sqrt{\frac{2}{p+1}}\,t\right)^{\frac{2}{1-p}}, & t \in [0, T_0], \\ 0, & t \in [T_0, T_0{+}c], \\ \pm\left(\frac{(1-p)^2}{2p+2}\right)^{\frac{1}{1-p}} (t-T_0-c)^{\frac{2}{1-p}}, & t \geqslant T_0 + c, \end{cases}$$

where c is any positive real number and $T_0 = \frac{2}{1-p}\sqrt{\frac{p+1}{2} u_0^{1-p}}$, and also

$$t^{\frac{2}{p-1}} u(t) \to \left(\frac{1-p}{2}\sqrt{\frac{2}{p+1}}\right)^{\frac{2}{1-p}} \quad \text{as } t \to \infty.$$

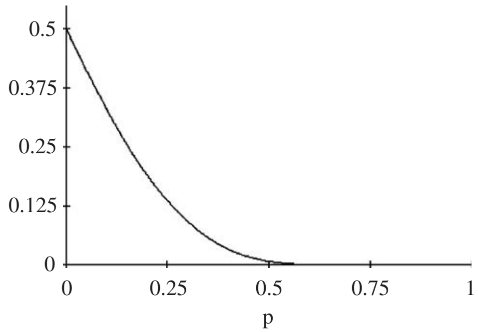

Fig. 5. Graph of $(\frac{1-p}{2}\sqrt{\frac{2}{p+1}})^{\frac{2}{1-p}}$.

(iii) For $u_0 < 0$ and $u_1 > 0$, the solutions of (1.1) can be given as (Fig. 5).

$$u_c(t) = \begin{cases} u_0\left(1 - \dfrac{1-p}{2}\sqrt{\dfrac{2}{p+1}}(-u_0)^{\dfrac{p-1}{2}}t\right)^{\dfrac{2}{1-p}}, & t \in [0, T_1], \\ 0, & t \in [T_1, T_1 + c], \\ \pm\left(\dfrac{(1-p)^2}{2p+2}\right)^{\frac{1}{1-p}}(t - T_1 - c)^{\frac{2}{1-p}}, & t \geqslant T_1 + c, \end{cases}$$

where c is any positive real number and $T_1 = \frac{2}{1-p}\sqrt{\frac{p+1}{2}(-u_0)^{1-p}}$, and also

$$t^{\frac{2}{p-1}}|u(t)| \to \left(\frac{1-p}{2}\sqrt{\frac{2}{p+1}}\right)^{\frac{2}{1-p}} \quad \text{as } t \to \infty.$$

(iv) For $u_0 < 0$ and $u_1 < 0$, the solutions of (1.1) can be given as

$$u(t) = u_0\left(1 + \frac{1-p}{2}\frac{u_1}{u_0}t\right)^{\frac{2}{1-p}}$$

and also

$$t^{\frac{2}{p-1}}u(t) \to -\left(\frac{1-p}{2}\sqrt{\frac{2}{p+1}}\right)^{\frac{2}{1-p}} \quad \text{as } t \to \infty.$$

7. Null point and asymptotic behavior at infinity of the solutions for Eq. (1.1) under $E_u(0) > 0$

In this section, we discuss the case $E_u(0) > 0$ and obtain the following result concerning the null point (zero) and asymptotic behavior at infinity of the solutions for Eq. (1.1):

Theorem 10. *Suppose that T^* is the existence interval of u of the solution of problem* (1.1) *with $E_u(0)>0$ and $u_0^2>0$. Then for*

(1) $u_0>0$ *and* $u_1<0$, *there exists a constant* Z_0 *so that* $T^*\leqslant Z_0$ *and* $\lim_{t\to Z_0^-}u(t)=0$, $\lim_{t\to Z_0^-}u'(t)=-\sqrt{E_u(0)}$ *and* $\lim_{t\to Z_0^-}u'''(t)^{-1}=0$. *Moreover,*

$$Z_0=\int_0^{u_0}\frac{\mathrm{d}r}{\sqrt{E_u(0)+\frac{2}{p+1}r^{p+1}}},\tag{7.1}$$

$$\lim_{t\to Z_0^-}u'''(t)(Z_0-t)^{1-p}=-pE_u(0)^{\frac{p}{2}}.\tag{7.2}$$

(2) $u_0>0$ *and* $u_1>0$,

$$\lim_{t\to\infty}u(t)t^{-\frac{2}{1-p}}=\left(\frac{1-p}{2}\sqrt{\frac{2}{p+1}}\right)^{\frac{2}{1-p}}.\tag{7.3}$$

(3) $u_0<0$ *and* $u_1>0$, *there exists a constant* Z_1 *so that* $T^*\leqslant Z_1$ *and* $\lim_{t\to Z_1^-}u(t)=0$, $\lim_{t\to Z_1^-}u'(t)=\sqrt{E_u(0)}$ *and also* $\lim_{t\to Z_1^-}u'''(t)^{-1}=0$. *Moreover,*

$$Z_1=\int_0^{u_0}\frac{\mathrm{d}r}{\sqrt{E_u(0)+\frac{2}{p+1}r^{p+1}}},\tag{7.4}$$

$$\lim_{t\to Z_1^-}u'''(t)(Z_1-t)^{1-p}=pE_u(0)^{\frac{p}{2}}.\tag{7.5}$$

(4) $u_0<0$ *and* $u_1<0$,

$$\lim_{t\to\infty}u(t)t^{-\frac{2}{1-p}}=-\left(\frac{1-p}{2}\sqrt{\frac{2}{p+1}}\right)^{\frac{2}{1-p}}.\tag{7.6}$$

Proof. (1) For $u_0>0$ and $u_1<0$, after some calculations we obtain

$$u'(t)=-\sqrt{E_u(0)+\frac{2}{p+1}|u|(t)^{p+1}}\leqslant-\sqrt{\frac{2}{p+1}|u|(t)^{p+1}}\quad\forall t\in[0,T^*),$$

$$u(t)\leqslant\left(u_0^{\frac{1-p}{2}}-\frac{1-p}{2}\sqrt{\frac{2}{p+1}}t\right)^{\frac{2}{1-p}}\qquad\forall t\in[0,T^*);\tag{7.7}$$

thus there exists a constant Z_0 so that $T^*\leqslant Z_0$ and $\lim_{t\to Z_0}u(t)=0$.

By (7.7) and Lemma 3 we conclude that $\lim_{t\to Z_0^-} u'(t) = -\sqrt{E_u(0)}$ and

$$t = \int_{u(t)}^{u_0} \frac{\mathrm{d}r}{\sqrt{E_u(0) + \frac{2}{p+1} r^{p+1}}} \quad \forall t \in [0, T^*),$$

$$Z_0 = \lim_{t\to Z_0} \int_{u(t)}^{u_0} \frac{\mathrm{d}r}{\sqrt{E_u(0) + \frac{2}{p+1} r^{p+1}}} = \int_0^{u_0} \frac{\mathrm{d}r}{\sqrt{E_u(0) + \frac{2}{p+1} r^{p+1}}},$$

$$\lim_{t\to Z_0^-} u'''(t)(t - Z_0)^{1-p} = p \lim_{t\to Z_0^-} \left(\frac{u(t)}{t - Z_0}\right)^{p-1} u'(t) = pE_u(0)^{\frac{p}{2}}.$$

Therefore (7.1) and (7.2) are proved.

(2) For $u_0 > 0$ and $u_1 > 0$ we have

$$u'(t) = \sqrt{E_u(0) + \frac{2}{p+1} u(t)^{p+1}} \geqslant \sqrt{\frac{2}{p+1} u(t)^{p+1}} \quad \forall t \geqslant 0,$$

$$u(t)^{\frac{1-p}{2}} \geqslant u_0^{\frac{1-p}{2}} + \frac{1-p}{2} \sqrt{\frac{2}{p+1}}\, t \quad \forall t \geqslant 0. \tag{7.8}$$

On the other hand,

$$u'(t) \leqslant \sqrt{\frac{2}{p+1}} \left(u(t) + \left(\frac{p+1}{2} E_u(0)\right)^{\frac{1}{p+1}}\right)^{\frac{p+1}{2}} \quad \forall t \geqslant 0,$$

$$\left(u(t) + \sqrt[p+1]{\frac{p+1}{2} E_u(0)}\right)^{\frac{1-p}{2}} := w(t) \leqslant w(0) \frac{1-p}{2} \sqrt{\frac{2}{p+1}}\, t \quad \forall t \geqslant 0. \tag{7.9}$$

From (7.8) and (7.9), estimate (7.3) follows.

(3) Similar to the above arguments we can obtain results (7.4)–(7.6). □

8. Critical point and asymptotic behavior at infinity of the solutions for Eq. (1.1) under $E_u(0) < 0$

In this section we discuss the case $E_u(0) < 0$. Similar to the above arguments proving Theorem 10 we have the following result on critical point and asymptotic behavior at infinity of the solutions for Eq. (1.1):

Theorem 11. *Suppose that u is a solution of problem* (1.1) *with* $E_u(0) < 0$. *Then for*

(1) $u_0 > 0, u_1 > 0$,

$$\lim_{t\to\infty} u(t)t^{-\frac{2}{1-p}} = \left(\frac{1-p}{2} \sqrt{\frac{2}{p+1}}\right)^{\frac{2}{1-p}} := AZ(p); \tag{8.1}$$

(2) $u_0 > 0,\ u_1 < 0$, *there exists a constant* Z_2 *so that* $\lim_{t\to Z_2} u'(t) = 0$ *and*

$$Z_2 = \sqrt[p+1]{\frac{p+1}{2}}(-E_u(0))^{\frac{1-p}{2p+2}} \int_1^{(\frac{p+1}{-2}\ E_u(0))^{\frac{-1}{p+1}} u_0} \frac{\mathrm{d}r}{\sqrt{r^{p+1}-1}}; \tag{8.2}$$

(3) $u_0 < 0,\ u_1 < 0$,

$$\lim_{t\to\infty} u(t)t^{-\frac{2}{1-p}} = -\left(\frac{1-p}{2}\sqrt{\frac{2}{p+1}}\right)^{\frac{2}{1-p}}; \tag{8.3}$$

(4) $u_0 < 0,\ u_1 > 0$, *there exists a constant* Z_3 *so that* $\lim_{t\to Z_3} u'(t) = 0$ *and*

$$Z_3 = \sqrt[p+1]{\frac{p+1}{2}}(-E_u(0))^{\frac{1-p}{2p+2}} \int_1^{(\frac{p+1}{-2}\ E_u(0))^{\frac{-1}{p+1}} u_0} \frac{\mathrm{d}r}{\sqrt{r^{p+1}+1}}. \tag{8.4}$$

Proof. (1) *For* $u_0 > 0$ *and* $u_1 > 0$, after some calculations we obtain that

$$u'(t) \leqslant \sqrt{\frac{2}{p+1}\, u(t)^{p+1}} \quad \forall t \geqslant 0, \tag{8.5}$$

$$u(t) \leqslant \left(u_0^{\frac{1-p}{2}} + \frac{1-p}{2}\sqrt{\frac{2}{p+1}}\, t\right)^{\frac{2}{1-p}} \quad \forall t \geqslant 0, \tag{8.6}$$

$$u'(t) \geqslant \sqrt{\frac{2}{p+1}\left(u(t) - \left(\frac{p+1}{2}|E_u(0)|\right)^{\frac{1}{p+1}}\right)^{p+1}} \quad \forall t \geqslant 0$$

and

$$\left(u(t) - \sqrt[p+1]{\frac{p+1}{2}|E_u(0)|}\right)^{\frac{1-p}{2}} := w(t) \leqslant w(0) + \frac{1-p}{2}\sqrt{\frac{2}{p+1}}\,t \quad \forall t \geqslant 0. \tag{8.7}$$

Together with (8.6) and (8.7) we obtain (8.1).

(2) For $u_0 > 0,\ u_1 < 0$, we have

$$u'(t) \geqslant -\sqrt{\frac{2}{1+p}}\, u(t)^{\frac{p+1}{2}},$$

$$u(t)^{\frac{1-p}{2}} \geqslant u_0^{\frac{1-p}{2}} - \frac{1-p}{2}\sqrt{\frac{2}{1+p}}\, t,$$

$$u'(t) \leqslant -\sqrt{\frac{2}{p+1}}\left(u(t) - \left(-\frac{p+1}{2}\, E_u(0)\right)^{\frac{1}{p+1}}\right)^{\frac{p+1}{2}} \tag{8.8}$$

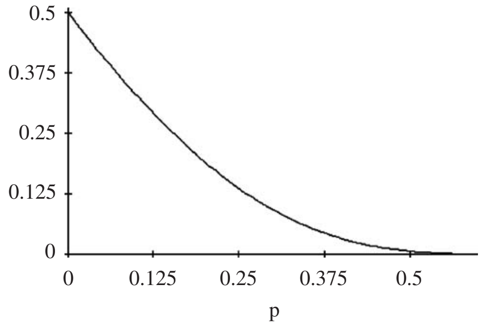

Fig. 6. Graph of $AZ(p)$, $p \in [0, 0.6]$.

and

$$\left(u(t) - \sqrt[p+1]{\frac{p+1}{2}|E_u(0)|}\right)^{\frac{1-p}{2}} = w(t) \leqslant w(0) - \frac{1-p}{2}\sqrt{\frac{2}{p+1}}t; \tag{8.9}$$

thus there exists a constant Z_2 so that

$$u(Z_2) = \left(-\frac{p+1}{2}\,E_u(0)\right)^{\frac{1}{p+1}} \tag{8.10}$$

and $\lim_{t\to Z_2} u'(t) = 0$. By (8.8), (8.10) and Lemma 3 we conclude that

$$t = \int_{u(t)}^{u_0} \frac{\mathrm{d}r}{\sqrt{E_u(0) + \frac{2}{p+1}r^{p+1}}} \quad \forall t \in [0, T^*),$$

$$Z_2 = \int_{(-\frac{p+1}{2}\,E_u(0))^{\frac{1}{p+1}}}^{u_0} \frac{\mathrm{d}r}{\sqrt{E_u(0) + \frac{2}{p+1}\,r^{p+1}}}. \tag{8.11}$$

Estimates (8.11) and (8.2) are equivalent.

(3) Similar to the above arguments it results in estimates (8.3) and (8.4). $\square$

Property of $AZ(p)$:

We have seen that $AZ(p) = (\frac{1-p}{2}\sqrt{\frac{2}{p+1}})^{\frac{2}{1-p}}$ and the graph using Maple (Figs. 6 and 7). As the graph indicates, $AZ(p)$ is decreasing in p, since

$$\frac{\mathrm{d}}{\mathrm{d}p}\left(\frac{1-p}{2}\sqrt{\frac{2}{p+1}}\right)^{\frac{2}{1-p}} = \frac{\sqrt{2}}{1-p}\sqrt{\frac{1}{1+p}}\left(\sqrt{\frac{2}{p+1}}\frac{1-p}{2}\right)^{\frac{2}{1-p}-1} \times \left(\ln\sqrt{\frac{2}{p+1}}\left(\frac{1-p}{2}\right) - \frac{p+3}{2(p+1)}\right)$$

and then $\frac{\mathrm{d}AZ(p)}{\mathrm{d}p} \leqslant 0$ for all $p \in (0, 1)$.

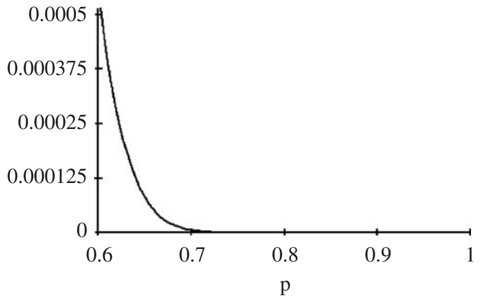

Fig. 7. Graph of $AZ(p)$, $p \geqslant 0.6$.

Part C: Regularity of solutions to problem (1.1) with $p > 1$ and the blow-up constants of $u^{(n)}$

In this section, we study the blow-up behavior of $u^{(n)}$ and the regularity of the solution u of the nonlinear equation (1.1) as $p > 1$. If u blows up at finite time T^*, $|u(t)|$ becomes very large in the neighborhood of T^*, and $u(t)$ retains the same sign in the neighborhood of T^*; thus we study the above-mentioned phenomena only for the *positive* solutions.

9. Regularity of solution to Eq. (1.1), $p \in \mathbb{N}$

In this section, we study the regularity of the positive solution u of the nonlinear equation (1.1) as $p \in \mathbb{N}$. Using (1.4) we have

$$u'(t)^2 = E(0) + \frac{2}{p+1} u(t)^{p+1}, \tag{9.1}$$

where $E(0) = u_1^2 - \frac{2}{p+1} u_0^{p+1}$.

9.1. Regularity of solution to Eq. (1.1) with $p \in \mathbb{N}$

Now, considering the regularity of the positive solution u of problem (1.1) with $p \in \mathbb{N}$, we have the following results:

Theorem 12. *If u is the positive solution of problem* (1.1) *with the existence interval T^* and $p \in \mathbb{N}$, then $u \in C^q(0, T^*)$ for any $q \in \mathbb{N}$ and*

$$u^{(2n)} = \sum_{i=0}^{[(\frac{C_{n0}}{p+1})]} E_{ni} u^{C_{ni}}, \tag{9.2}$$

$$u^{(2n+1)} = \sum_{i=0}^{[(\frac{C_{n0}}{p+1})]} E_{ni} C_{ni} u^{C_{ni}-1} u' = \sum_{i=0}^{[(\frac{C_{n0}}{p+1})]} O_{ni} u^{C_{ni}-1} u' \tag{9.3}$$

for a positive integer n, where $[(\frac{C_{n0}}{p+1})]$ *denotes the Gaussian integer number of* $\frac{C_{n0}}{p+1}$,

$$C_{ni} = (n-i)(p+1) - 2n + 1,\ O_{ni} = E_{ni}C_{ni},\ E_{00} = 1,$$

$$\begin{aligned} E_{n0} &= O_{(n-1)0}\left[\left(\frac{2}{p+1}(C_{(n-1)0}-1)+1\right)\right] \\ &= E_{(n-1)0}C_{(n-1)0}\left[\left(\frac{2}{p+1}(C_{(n-1)0}-1)+1\right)\right], \end{aligned}$$

$$\begin{aligned} E_{n(n-1)} &= O_{(n-1)(n-2)}(C_{(n-1)(n-2)}-1)E(0) \\ &= E_{(n-1)(n-2)}C_{(n-1)(n-2)}(C_{(n-1)(n-2)}-1)E(0) \end{aligned}$$

and

$$\begin{aligned} E_{nk} &= O_{(n-1)(k-1)}(C_{(n-1)(k-1)}-1)E(0) + O_{(n-1)k}\left[\left(\frac{2}{p+1}(C_{(n-1)k}-1)+1\right)\right] \\ &= E_{(n-1)(k-1)}C_{(n-1)(k-1)}(C_{(n-1)(k-1)}-1)E(0) \\ &\quad + E_{(n-1)k}C_{(n-1)k}\left[\left(\frac{2}{p+1}(C_{(n-1)k}-1)+1\right)\right], \end{aligned}$$

for a positive integer k and $0 < k < n$.

Proof. Let v_n be the nth derivative of u, that is, $v_n := u^{(n)}$; then $v_0^n = u^n$, $v_0 = u$, $v_1 = u'$, $v_2 = u''$, $v_1^2 = (u')^2$. Now let us use mathematical induction to prove (9.2). When $n = 1$, we have

$$v_2 = \sum_{i=0}^{[(\frac{C_{10}}{p+1})]} E_{1i}u^{C_{1i}} = E_{10}u^{C_{10}} = v_0^p$$

and

$$\begin{aligned} &C_{00} = (0-0)(p+1) - 2\times 0 + 1 = 1,\ C_{10} = p, \\ &E_{10} = E_{00}C_{00}\left[\left(\frac{2}{p+1}(C_{00}-1)+1\right)\right] = 1. \end{aligned}$$

Suppose that $n \in \mathbb{N}$ and $v_{2n} = \sum_{i=0}^{[(\frac{C_{n0}}{p+1})]} E_{ni}\cdot v_0^{C_{ni}}$. Then by (9.1) we obtain

$$v_{2n+1} = \sum_{i=0}^{[(\frac{C_{n0}}{p+1})]} E_{ni}C_{ni}v_0^{C_{ni}-1}v_1,$$

$$v_{2n+2} = \sum_{i=0}^{[(\frac{C_{n0}}{p+1})]} E_{ni}C_{ni}v_0^{C_{ni}-1}v_2 + \sum_{i=0}^{[(\frac{C_{n0}}{p+1})]} E_{ni}C_{ni}(C_{ni}-1)v_0^{C_{ni}-2}v_1^2,$$

$$
\begin{aligned}
v_{2n+2} = & \sum_{i=0}^{[(\frac{C_{n0}}{p+1})]} O_{ni}\left[\left(\frac{2}{p+1}(C_{ni}-1)+1\right)\right] v_0^{C_{ni}+p-1} \\
& + \sum_{i=0}^{[(\frac{C_{n0}}{p+1})]} O_{ni}(C_{ni}-1)E(0)v_0^{C_{ni}-2} \\
= & \sum_{i=0}^{[(\frac{C_{n0}}{p+1})]} O_{ni}\left[\left(\frac{2}{p+1}(C_{ni}-1)+1\right)\right] v_0^{C_{(n+1)i}} \\
& + \sum_{i=0}^{[(\frac{C_{n0}}{p+1})]} O_{ni}(C_{ni}-1)E(0)v_0^{C_{(n+1)(i+1)}} \\
= & \, O_{n0}\left[\left(\frac{2}{p+1}(C_{n0}-1)+1\right)\right] v_0^{C_{(n+1)0}} + O_{n0}(C_{n0}-1)E(0)v_0^{C_{(n+1)1}} \\
& + O_{n1}\left[\left(\frac{2}{p+1}(C_{n1}-1)+1\right)\right] v_0^{C_{(n+1)1}} + O_{n1}(C_{n1}-1)E(0)v_0^{C_{(n+1)2}} \\
& + O_{n2}\left[\left(\frac{2}{p+1}(C_{n2}-1)+1\right)\right] v_0^{C_{(n+1)2}} + \cdots \\
& + O_{n[(\frac{C_{n0}}{p+1})]}(C_{n[(\frac{C_{n0}}{p+1})]}-1)E(0)v_0^{C_{(n+1)([(\frac{C_{n0}}{p+1})]+1)}}.
\end{aligned}
$$

Hence

$$
v_{2n+2} = \sum_{i=0}^{[(\frac{C_{(n+1)0}}{p+1})]} E_{(n+1)i} \cdot v_0^{C_{(n+1)i}},
$$

which completes the induction steps, and we obtain (9.2). Using (9.2), we obtain (9.3). □

9.2. *The properties concerning $u^{(n)}$*

Drawing the graphs of the $u^{(n)}$ is not easy, so in this section we choose a special index $p = 2$.

We consider only the properties of the solution u for the equation

$$
\begin{cases} u'' = u^2, \\ u(0) = 1, \quad u'(0) = \sqrt{2/3}, \end{cases}
$$

to the case $E(0) = 0$. The solution of the above equation can be solved explicitly

$$
u(t) = \frac{6}{(\sqrt{6}-t)^2}
$$

and this yields the graphs of u, u', u'', $u^{(3)}$ and $u^{(4)}$ below (Fig. 8).

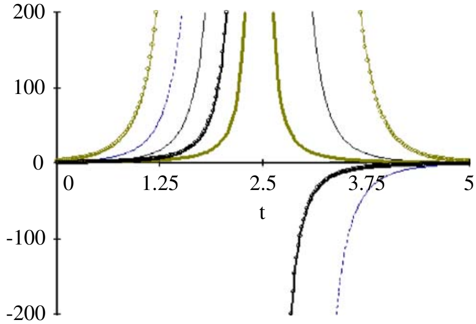

Fig. 8. Graphs of u in thick solid lines, u' in medium dots, u'' in thin solid, $u^{(3)}$ in thin dash, and $u^{(4)}$ in thin dots.

With the aid of a graph with Maple we find that the nth derivative $u^{(n)}$ is smooth and that the blow-up rate of $u^{(n)}$ is increasing in n. Here we do not give a rigorous proof; we will illustrate this in Section 11.

10. Regularity of solution to Eq. (1.1), $p \in \mathbb{Q} - \mathbb{N}$

According to the preceding section we obtain that the positive solution $u \in C^q(0, T)$ of (1.1) with $p \in \mathbb{N}$ for any $q \in \mathbb{N}$. In this section, we reconsider Eq. (1.1) with $p \in \mathbb{Q} - \mathbb{N}$.

Obviously, if we obviate the possibility of $u(t) = 0$, we have the following results:

Except the null points of u, $u^{(q)}$ is differentiable for all $q \in \mathbb{N}$. We have

Theorem 13. *If u is the positive solution of problem* (1.1) *with* $E(0) > 0$, $a'(0) \geqslant 0$, $p \in \mathbb{Q} - \mathbb{N}$, $p \geqslant 1$, *then* $u \in C^q(0, T)$ *for any* $q \in \mathbb{N}$. *Further, we have*

$$u^{(2n)}(t) = \sum_{i=0}^{n-1} E_{ni} u^{C_{ni}}(t), \tag{10.1}$$

$$u^{(2n+1)}(t) = \sum_{i=0}^{n-1} E_{ni} C_{ni} u^{C_{ni}-1}(t) u'(t) = \sum_{i=0}^{n-1} O_{ni} u^{C_{ni}-1}(t) u'(t). \tag{10.2}$$

Proof. Same as the procedures given in the proof of Theorem 12, let us prove (10.1) and (10.2) through mathematical induction. If z is the null point (zero) of u, then

$$\lim_{t \to z} u^{c_{ni}}(t)^{-1} = 0$$

for

$$i > \frac{n(p-1)+1}{p+1} = \frac{C_{n0}}{p+1}$$

since $C_{ni} < 0$, for $i > \frac{C_{n0}}{p+1}$. By Theorem 5, we know that u has a null point only in the case $a'(0) < 0$. Hence, we conclude that $u \in C^q(0, T)$ for any $q \in \mathbb{N}$. $\square$

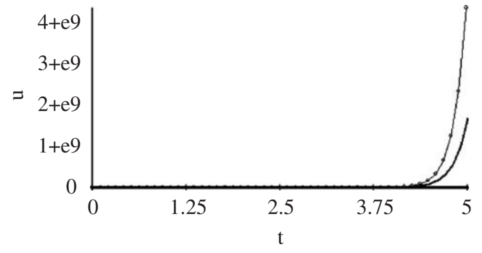

Fig. 9. $u'' = u^2$, $u(0) = -1$ with $u'(0) = 1$ in dots $u'(0) = -1$ in line.

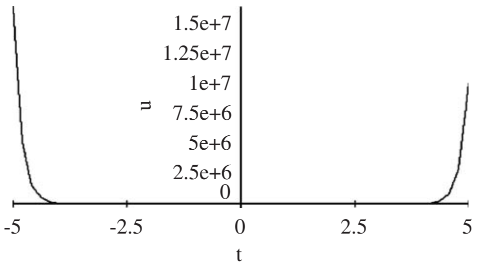

Fig. 10. $u'' = u^2$, $u(0) = 0$, $u'(0) = -1$.

Similarly, by the same arguments above, we also have a result as follows:

Theorem 14. *If u is the positive solution of problem* (1.1) *with* $p \in \mathbb{Q} - \mathbb{N}$, $p \geqslant 1$, $E(0) > 0$ *and* $a'(0) < 0$, *then* $u \in C^{[(p)]+2}(0, T)$, *where* $[(p)]$ *indicates the Gaussian integer number of* p. *Further*, *we have*

$$u^{(2n)}(t) = \sum_{i=0}^{n-1} E_{ni} u^{C_{ni}}(t) \quad \textit{for } n \leqslant \left[\left(\frac{p}{2}\right)\right] + 1, \tag{10.3}$$

$$\begin{aligned} u^{(2n+1)}(t) &= \sum_{i=0}^{n-1} E_{ni} C_{ni} u^{C_{ni}-1}(t) u'(t) \\ &= \sum_{i=0}^{n-1} O_{ni} u^{C_{ni}-1}(t) u'(t) \quad \textit{for } n \leqslant \left[\left(\frac{p}{2}\right)\right] + 1. \end{aligned} \tag{10.4}$$

Proof. Same as the proof of Theorem 13, we also obtain identities (10.3) and (10.4). By Theorem 5, we know that u has a null point (zero) in the case $a'(0) < 0$. (Figs. 9 and 10). If $z(u_0, u_1, p)$ is the null point of u, then

$$\lim_{t \to z^-(u_0, u_1, p)} u^{-c_{ni}}(t) = 0 \text{ for } C_{ni} < 0.$$

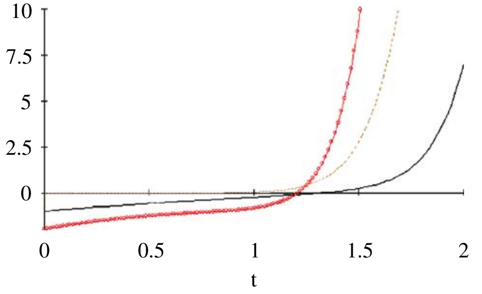

Fig. 11. Graphs of u in solid, u' in dash, u'' in dots.

Hence, for $a'(0) < 0$, we should find the range of n with $C_{ni} \geqslant 0$ as $i = n - 1$, and then $u^{(2n)}$ exists only in such a situation. Here

$$C_{ni} = (p+1)(n-i) - 2n + 1.$$

Let $C_{n(n-1)} = (p+1)(n-(n-1)) - 2n + 1 \geqslant 0$; then we obtain that $n \leqslant \frac{p}{2} + 1$. Since n is an integer, we have $n \leqslant \left[\left(\frac{p}{2}\right)\right] + 1$.

Now $u^{(2n)}$ exists for $n \leqslant [(\frac{p}{2})] + 1$ in the case of $a'(0) < 0$; thus we obtain that $u \in C^{[(p)]+2}(0.T)$. □

Example 10.1. Here we wish to draw the graphs of $u^{(n)}$ for $p \in \mathbb{Q} - \mathbb{N}$, but it is not easy, so we choose a special index $p = \frac{7}{3}$. We consider the properties of the solution u to the case $E(0) > 0$ for the equation

$$\begin{cases} u'' = u^{\frac{7}{3}}, \\ u(0) = -1, \ \ u'(0) = 1. \end{cases}$$

Since the solution of the above equation cannot be solved explicitly, we solve this ODE numerically. We have the graphs of u, u', u'', $u^{(3)}$ $u^{(4)}$ and $u^{(5)}$ below.

By Theorem 4, we know that $u \in C^4(0, T)$. With the help of the graph with Maple, we find the null point of u (Fig. 11) $t_0 \sim 1.4$ and $u^{(5)}(t)$ goes to infinity as t tends to 1.4 (Fig. 12). From the graph we know that $u^{(5)}(t)$ does not exist at $t = t_0$. The blow-up rate of $u^{(n)}$ is increasing in n. This will be illustrated in the next section.

11. The blow-up rate and blow-up constant for $u^{(n)}$

Finding out the blow-up rate and blow-up constant of $u^{(n)}$ of Eq. (1.1) is our main result:

Theorem 15. *If u is the solution of problem* (1.1) *with one of the following properties*:

(i) $E(0) < 0$ *or*
(ii) $E(0) = 0, \ a'(0) > 0$ *or*
(iii) $E(0) > 0$,

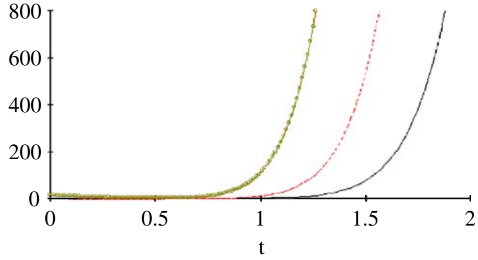

Fig. 12. Graphs of $u^{(3)}$ in solid, $u^{(4)}$ in dash, $u^{(5)}$ in dots.

then the blow-up rate of $u^{(2n)}$ *is* $\frac{2}{p-1}+2n$, *and the blow-up constant of* $u^{(2n)}$ *is* $|E_{n0}(\frac{\sqrt{2(P+1)}}{p-1})^{\frac{2}{p-1}+2n}|$, *that is, for* $n \in \mathbb{N}$, $m \in \{1,2,3,4,5,6\}$,

$$\lim_{t\to T_m^*} u^{(2n)}(t)(T_m^*-t)^{\frac{2}{p-1}+2n} = (\pm 1)^{C_{n0}} E_{n0}\left(\frac{\sqrt{2(P+1)}}{p-1}\right)^{\frac{2}{p-1}+2n} := K_{2n}. \quad (11.1)$$

The blow-up rate of $u^{(2n+1)}$ *is* $\frac{2}{p-1}+2n+1$, *and the blow-up constant of* $u^{(2n+1)}$ *is*

$$\left|E_{n0}C_{n0}\sqrt{\frac{2}{p+1}}\left(\frac{\sqrt{2(P+1)}}{p-1}\right)^{\frac{2}{p-1}+2n+1}\right|,$$

that is, for $n \in \mathbb{N}$, $m \in \{1,2,3,4,5,6\}$,

$$\begin{aligned}&\lim_{t\to T_m^*} u^{(2n+1)}(t)(T_m^*-t)^{\frac{2}{p-1}+2n}\\ &\quad = (\pm)^{C_{n0}} E_{n0}C_{n0}\sqrt{\frac{2}{p+1}}\left(\frac{\sqrt{2(P+1)}}{p-1}\right)^{\frac{2}{p-1}+2n+1} := K_{2n+1},\end{aligned} \quad (11.2)$$

where

$$\begin{aligned}C_{n0} &= (p-1)n+1,\\ E_{n0} &= \Pi_{i=0}^{n-1}\left[\frac{2(p-1)^2 i^2+(p-1)i}{p+1}+(p-1)i+1\right].\end{aligned}$$

Proof. Under condition (i), $E(0)<0$, $a'(0)\geqslant 0$ by (2.6) and (2.1), we obtain

$$\int_0^{J(t)} \frac{1}{T_1^*-t}\frac{\mathrm{d}r}{\sqrt{\frac{2}{p+1}+E(0)r^{\frac{2p+2}{p-1}}}} = \frac{p-1}{2} \quad \forall t\geqslant 0. \quad (11.3)$$

Using Lemma 3 and (2.6), we obtain $\lim_{t\to T_1^*}\sqrt{\frac{p+1}{2}\frac{J(t)}{T_1^*-t}} = \frac{p-1}{2}$; in other words,

$$\lim_{t\to T_1^*} a(t)(T_1^*-t)^{\frac{4}{p-1}} = \left(\frac{\sqrt{2p+2}}{p-1}\right)^{\frac{4}{p-1}}, \tag{11.4}$$

and then

$$\lim_{t\to T_1^*} u(t)(T_1^*-t)^{\frac{2}{p-1}} = \pm\left(\frac{\sqrt{2p+2}}{p-1}\right)^{\frac{2}{p-1}}. \tag{11.5}$$

Here $C_{ni} = p+(n-1-i)(p+1)-2(n-1)$; hence, we have $C_{ni} > C_{nj}$ as $i<j$. From (10.1) and (11.5), it follows that

$$\lim_{t\to T_1^*} u^{(2n)}(t)(T_1^*-t)^{\frac{2}{p-1}\times C_{n0}} = (\pm 1)^{C_{n0}} E_{n0}\left(\frac{\sqrt{2p+2}}{p-1}\right)^{\frac{2}{p-1}\times C_{n0}}.$$

Since $\frac{2}{p-1}\times C_{n0} = \frac{2}{p-1}+2n$, we obtain (11.1) for $m=1$.

By (2.5), (11.4) and (10.2) we find that

$$\lim_{t\to T_1^*} J'(t) = -\frac{p-1}{\sqrt{2p+2}}, \tag{11.6}$$

$$\frac{2\sqrt{2}}{\sqrt{p+1}} = \lim_{t\to T_1^*}(a(t)(T_1^*-t)^{\frac{4}{p-1}})^{-\frac{p-1}{4}-1}\cdot \lim_{t\to T_1^*} a'(t)(T_1^*-t)^{\frac{4}{p-1}\times\frac{p+3}{4}},$$

$$\lim_{t\to T_1^*} u'(t)(T_1^*-t)^{\frac{2}{p-1}+1} = \pm\sqrt{\frac{2}{p+1}}\left(\frac{\sqrt{2p+2}}{p-1}\right)^{\frac{2}{p-1}+1} \tag{11.7}$$

and

$$\begin{aligned}
&\lim_{t\to T_1^*} u^{(2n+1)}(t)(T_1^*-t)^{\frac{2}{p-1}C_{n0}+1}\\
&\quad= \lim_{t\to T_1^*}\sum_{i=0}^{n-1} E_{ni}C_{ni}u^{C_{ni}-1}(t)\cdot u'(t)\cdot(T_1^*-t)^{\frac{2}{p-1}C_{n0}+1}\\
&\quad= \lim_{t\to T_1^*} E_{n0}C_{n0}u^{C_{n0}-1}(t)\cdot u'(t)\cdot(T_1^*-t)^{\frac{2}{p-1}C_{n0}+1}\\
&\quad= \lim_{t\to T_1^*} E_{n0}C_{n0}u^{C_{n0}-1}(t)\cdot(T_1^*-t)^{\frac{2}{p-1}C_{n0}-1}\cdot u'\cdot(T_1^*-t)^{\frac{2}{p-1}+1}\\
&\quad= (\pm)^{C_{n0}}E_{n0}C_{n0}\sqrt{\frac{2}{p+1}}\left(\frac{\sqrt{2p+2}}{p-1}\right)^{\frac{2}{p-1}C_{n0}+1};
\end{aligned}$$

thus (11.2) for $m=1$ is proved.

For $E(0) < 0,\ a'(0) < 0$, by (2.9a,b) we have

$$\int_0^{J(t)} \frac{\mathrm{d}r}{(T_2^* - t)\sqrt{\frac{2}{p+1} + E(0)r^{\frac{2p+2}{p-1}}}} = \frac{p-1}{2} \quad \forall t \geqslant t_0. \tag{11.8}$$

Using Lemma 3, (11.8) and (10.1), therefore, we obtain estimate (11.1) for $m = 2$, and by (2.8a,b) we obtain estimate (11.2) for $m = 2$. (See Appendix A.2.)

Under (ii), $E(0) = 0,\ a'(0) > 0$, we have

$$a(t) = a(0)^{\frac{p+3}{p-1}} \left(\frac{p-1}{4} a'(0)(T_3^* - t) \right)^{-\frac{4}{p-1}} \quad \forall t \geqslant 0. \tag{11.9}$$

In view of (11.9) and (10.1), we obtain estimate (11.1) for $m = 3$. Also, we have

$$J'(t) = J'(0)\ \forall t \geqslant 0 \text{ and } \lim_{t \to T_1^*} a(t)^{-\frac{p-1}{4} - 1} a'(t) = -\frac{p-1}{4} a(0)^{-\frac{p-1}{4} - 1} a'(0).$$

By (11.9) and (10.2), estimate (11.2) for $m = 3$ is completely proved. □

Under (iii), the proofs of estimates (11.1) and (11.2) for $m = 4, 5, 6$ are similar to the above ones; we omit the arguments. □

Theorem 16. *If u is the solution of problem* (1.1) *with $E(0) > 0$ and $a'(0) < 0$, then we have*

$$\lim_{t \to z^-(u_0, u_1, p)} u^{(2n)}(t)(z(u_0, u_1, p) - t)^{-C_{n(n-1)}} = (\pm)^{C_{n(n-1)}} E_{n(n-1)} E(0)^{\frac{C_{n(n-1)}}{2}} \tag{11.10}$$

and

$$\lim_{t \to z^-(u_0, u_1, p)} u^{(2n+1)}(t)(z(u_0, u_1, p) - t)^{-C_{n(n-1)}+1} = E_{n(n-1)} C_{n(n-1)} E(0)^{C_{n(n-1)}-1} \tag{11.11}$$

for $n \in \mathbb{N}$, where z is the null point (*zero*) *of u and*

$$C_{n(n-1)} = p - 2n + 2,$$
$$E_{n(n-1)} = \Pi_{i=0}^{n-1} (p - 2i + 2)(p - 2i + 1) E(0)^{n-1}.$$

Proof. For $E(0) > 0$ *and* $a'(0) < 0$, we have

$$\begin{aligned}
&\lim_{t\to z^-(u_0,u_1,p)} u^{(2n)}(t)(z(u_0,u_1,p)-t)^{-C_{n(n-1)}} \\
&\quad = \lim_{t\to z^-(u_0,u_1,p)} \sum_{i=0}^{n-1} E_{ni} u^{C_{ni}}(t)(z(u_0,u_1,p)-t)^{-C_{n(n-1)}} \\
&\quad = \lim_{t\to z^-(u_0,u_1,p)} E_{n(n-1)} u^{C_{n(n-1)}}(t)(z(u_0,u_1,p)-t)^{-C_{n(n-1)}} \\
&\quad = (\pm 1)^{C_{n(n-1)}} E_{n(n-1)} E(0)^{\frac{C_{n(n-1)}}{2}}.
\end{aligned}$$

Therefore, (11.10) is proved.

From (10.2), we obtain that

$$\begin{aligned}
&\lim_{t\to z^-(u_0,u_1,p)} u^{(2n+1)}(t)(z(u_0,u_1,p)-t)^{-C_{n(n-1)}+1} \\
&\quad = \lim_{t\to z^-(u_0,u_1,p)} \sum_{i=0}^{n-1} E_{ni} C_{ni} u^{C_{ni}-1}(t) u'(t)(z(u_0,u_1,p)-t)^{-C_{n(n-1)}+1} \\
&\quad = \lim_{t\to z^-(u_0,u_1,p)} E_{n(n-1)} C_{n(n-1)} u^{C_{n(n-1)}-1}(t) u'(t)(z(u_0,u_1,p)-t)^{-C_{n(n-1)}+1} \\
&\quad = E_{n(n-1)} C_{n(n-1)} E(0)^{C_{n(n-1)}}.
\end{aligned}$$

Thus, (11.11) is obtained. □

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References

[1] R. Bellman, Stability theory for nonlinear ordinary differential equations.
[2] M.R. Li, Estimates for the life-span of solutions for semilinear wave equations, Proceedings of the Workshop on Differential Equations V, January 10–11, 1997, National Tsing-Hua University, Hsinchu, Taiwan, pp. 129–138.
[3] M.R. Li, On the semilinear wave equations, Taiwanese J. Math. 2 (3) (1998) 329–345.
[4] M.R. Li, Estimates for the life-span of positive solutions of some semilinear wave equations in 2-dimensional bounded domains $\Box u = u^p$, Proceedings of the Workshop on Differential Equations VII, May 1999, National Chung-Hsing University, Taichung, Taiwan, pp. 119–134.
[5] M.R. Li, Estimates for the life-span of solutions of semilinear wave equations, preprint.
[6] M.R. Li, Long-Yi Tsai, On a system of nonlinear wave equations, Taiwanese J. Math. 7 (4) (2003) 557–573.
[7] M.R. Li, Long-Yi Tsai, Existence and nonexistence of global solutions of some systems of semilinear wave equations, Nonlinear Anal. 54 (2003) 1397–1415.