

# On the control of defined-benefit pension plans

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## Abstract

Conventionally, contribution rates for defined-benefit pension plans have been set with reference to funding levels without making allowance for current market interest rates: for example, on one-year bonds where rates of return on fund assets are not independent from one year to the next. We consider how to make use of market information to reduce contribution rate volatility. The purpose of this paper is to provide a model for determining an appropriate contribution rate for defined benefit pension plans under a model where interest rates are stochastic and rates of return are random.

We extend previous work in two ways. First, we introduce a model for short-term interest rates, which can be used to help control contribution-rate volatility. Second, we model three assets rather than the usual one (cash, bonds and equities) to allow comparison of different asset strategies. We develop formulae for unconditional means and variances. We then discuss how variability can be controlled most efficiently by setting contribution rates with reference to current funding levels and interest rates.

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## 1. Introduction

A variety of factors that influence the volatility of the funding level and the contribution rate of a define-benefit (DB) pension plan including: the amortization strategy (Cairns, 1994; Dufresne, 1989; Bowers et al., 1979); the amortization period (Dufresne, 1988, 1989; Haberman, 1994; Cairns, 1994; Cairns and Parker, 1997); frequency of valuation (Cairns, 1994; Haberman, 1993); and the delay period (Balzer and Benjamin, 1980; Zimbidis and Haberman, 1993). The main purpose of this paper is to develop further the approach to setting contribution rates as a means of reducing the variance of the funding level and contribution rate under DB plans. The choice of spread period for surplus and deficit is one of the most important ways of control of the stability of the pension plan (see, for example, Dufresne, 1988, 1989; Haberman, 1994; Cairns and Parker, 1997). In this paper, we aim to extend the spread period contribution model and take advantage of the current market information about interest rates to reduce further the variance of the funding level and contribution rate.

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A pension plan's trustees are responsible for choosing long-term investment advice and the actuary is normally required to advise the trustees and/or the employers. Thus, actuaries are essential for advising trustees on a variety of possible investment strategies and for making sensible comments and suggestions on the implementation of the distribution of assets for each plan in order to match its anticipated liabilities. The aggregate investment return rate of the pension fund has been investigated on a model with independent and identically distributed (i.i.d.) returns (Dufresne, 1988, 1989), an AR time-series model (Mandl and Mazurova, 1996; Haberman, 1994; Cairns and Parker, 1997), and an MA time-series model (Haberman, 1997; Bédard, 1999). The plausible term structure of AR and MA time series models was considered by Chang (2000). These aggregate-return models take the investment strategy as given exogenously and model the returns on the fund as a univariate times series. In an attempt to make the approach to investments more realistic we explicitly allow for several assets in the portfolio. Thus, instead of using an aggregate return rate of the pension plan, we consider a more general investment model where the pension plan's return is a combination of numbers of the return on the individual assets.

In this paper we extend previous work to include three assets rather than just one: cash, long bonds and equities. Their returns are underpinned in a coherent way by a model for the one-year, risk-free interest rate and with appropriate correlations between different asset classes. Section 2 describes the basic details of the model and proposes a simple method for setting the contribution rate which accounts for both the current funding level (as normal) and current interest rates (new). With this model we are able to derive formulae for unconditional (that is long-run) means and variances of the funding level and for the contribution rate. In Section 3, we discuss how the contribution strategy can be used to control most effectively variability in the funding level and in the contribution rate itself. Here we reintroduce and extend the concept of efficient contribution strategies.

In Section 4, we build a super efficient region which minimizes the variance of contribution rate based upon specific funding constraints and discuss the optimal investment and contribution strategies.

## 2. A discrete-time model pension plan

We assume that we have three assets: a one-year bond (cash); a long-dated bond; and an equity asset. The log-return on cash between times  $t - 1$  and  $t$  is  $y(t - 1)$ . The log-return rate on the bond is  $\delta_b(t)$ , and the log-return on the equity is  $\delta_e(t)$ . Thus, investments of 1 at time  $t - 1$  will grow to  $e^{y(t-1)}$ ,  $e^{\delta_b(t)}$  or  $e^{\delta_e(t)}$ , respectively. We will further assume that  $y(t)$  follows the AR(1) process

$$y(t) = y + \phi(y(t - 1) - y) + \sigma_y Z_y(t) \quad (2.1)$$

where the  $Z_y(t)$  are independent and identically distributed (i.i.d.) standard normal random variables. This is similar to a discrete-time version of the Vasicek (1977) model. Excess returns on the equity asset,  $\Delta_e(t) = \delta_e(t) - y(t - 1)$ , are assumed to be i.i.d. and normally distributed with a mean greater than zero (that is, a positive risk premium). Similarly, the excess returns on a long-dated bond,  $\Delta_b(t) = \delta_b(t) - y(t - 1)$ , are also assumed to be i.i.d. and normally distributed with mean greater than zero. Thus,

$$\Delta_b(t) = \delta_b(t) - y(t - 1) = \Delta_b + \sigma_{by} Z_y(t) + \sigma_b Z_b(t) \quad (2.2)$$

$$\Delta_e(t) = \delta_e(t) - y(t - 1) = \Delta_e + \sigma_{ey} Z_y(t) + \sigma_{eb} Z_e(t) + \sigma_e Z_{\delta_e}(t) \quad (2.3)$$

where the  $Z_\delta(t)$ ,  $Z_y(t)$  and  $Z_b(t)$  are  $N(0, 1)$  random variables that are independent of one another and i.i.d. through time. Both  $\sigma_{ey}$  and  $\sigma_{by}$  will normally be negative since if the short-term interest rate,  $y(t)$ , goes up, then the prices of long-term bonds or equities typically go down and vice versa. The  $\sigma_b Z_b(t)$  term allows us to use, in effect, a two-factor interest-rate model since it allows for a degree of independence from one-year bonds.

Since we are considering a one-year bond, the return from  $t - 1$  up to  $t$  is known at time  $t - 1$  whereas the return on equities and bonds are only known at time  $t$ . This explains the use of  $y(t - 1)$  for the return on the one-year bond for  $t - 1$  to  $t$  rather than  $y(t)$ . In contrast, the unknown  $\Delta_b(t)$  and  $\Delta_e(t)$  are used to reflect the unknown elements of returns on the long-dated bonds and equities. In particular, bond prices at time  $t$  depend upon the new one-year rate of interest at  $t$ ,  $y(t)$ , through their dependence on  $\sigma_{by} Z_y(t)$ . The extent to which unanticipated returns on equities ( $\Delta_e(t)$ ) reflect unanticipated changes in  $y(t)$  appears in the parameter  $\sigma_{ey}$  with further equity specific risk being reflected through  $\sigma_{\delta_e}$  and  $Z_e(t)$ . Further correlation with long bonds is reflected through the parameter  $\sigma_{eb}$ .

Suppose we invest a proportion  $p_1$  of the pension fund in equities,  $p_2$  in long-term bonds and the remaining assets in one-year bonds. We will assume that the return on the fund from  $t - 1$  to  $t$  can be written as

$$1 + i(t) = \exp[y(t - 1) + p_1 \Delta_e(t) + p_2 \Delta_b(t) + \rho(p_1, p_2)] \tag{2.4}$$

where  $\rho(p_1, p_2) = (1/2)p_1 v_{ee} + (1/2)p_2 v_{bb} - (1/2)(p_1^2 v_{ee} + 2p_1 p_2 v_{eb} + p_2^2 v_{bb})$ ,  $v_{ee} = \text{Var} \Delta_e(t)$ ,  $v_{bb} = \text{Var} \Delta_b(t)$  and  $v_{eb} = \text{Cov}(\Delta_e(t), \Delta_b(t))$ .

Now we can remark that this formula for  $1 + i(t)$  gives an approximation to the return on a buy-and-hold strategy. However, we can give the formula for  $1 + i(t)$  a stronger justification. First, let us assume that the market operates in continuous time, and that the market is complete in the usual sense of derivative pricing. Then we can show (see Appendix B) that  $1 + i(t)$  as given for any  $p_1$  and  $p_2$  can be replicated given 1 at time  $t - 1$  provided we follow a suitable hedging strategy. The function  $\rho(p_1, p_2)$  is a second-order adjustment which ensures that the model is arbitrage free, and we can remark further that if  $p_1 = 1$  and  $p_2 = 0$  or if  $p_1 = 0$  and  $p_2 = 1$  then  $\rho(p_1, p_2) = 0$ , which implies that  $1 + i(t) = \exp[y(t - 1) + \Delta_e(t)]$  and  $\exp[y(t - 1) + \Delta_b(t)]$ , respectively.

We use the following additional notation which assumes that we have a stable membership in the pension plan with no salary increases (or we use the total salary roll as the unit of currency):

$F(t)$  is the fund size at  $t$ ;  $C(t)$  the contribution rate at  $t$ ;  $B$  the benefit outgo at the start of each year (assumed constant);  $i_v$  the actuarial valuation interest rate; AL the actuarial liability (assumed constant); NC the normal contribution rate consistent with AL and  $i_v$ .

Stability of the membership with no salary increases means that the actuarial liability does not change over time. Consistency between NC and AL thus means that

$$\Rightarrow \text{AL} = (1 + i_v)(\text{AL} + \text{NC} - B) \Rightarrow \text{NC} = B - d_v \text{AL} \tag{2.5}$$

where  $d_v = 1 - v_v$  and  $v_v = (1 + i_v)^{-1}$ .

Annual contributions,  $C(t)$ , are allowed to depend not just upon the current funding level (as is normal) but also on the current level of interest rates. The particular form we use is

$$C(t) = \text{NC} + k_1(\text{AL} - F(t)) + k_2 \frac{e^{y'} - e^{y(t)}}{e^{y(t)}} \tag{2.6}$$

where  $k_1, k_2$  and  $y'$  are the key control factors. If  $k_2 = 0$  then we revert to the classical case (see, for example, Cairns and Parker, 1997; Haberman, 1994, 1997). In a continuous-time model with  $y(t)$  constant and only one asset class, Cairns (2000) proved that this contribution strategy using the spread method is superior (mathematically optimal) to other approaches (such as the amortization of losses method used in North America).

The purpose of introducing the  $k_2$  term is to allow adjustment for future expected returns. For example, if  $y(t)$  is currently high then we might feel that contributions could be lower than would otherwise be the case because of higher expected returns than normal in the next few years. We will see later if this term allows us to reduce variability.

Given  $C(t)$  we have the usual dynamics for  $F(t)$ :

$$F(t) = (1 + i(t))[F(t - 1) + C(t - 1) - B]$$

We now take into account the earlier expression for  $(1 + i(t))$  and work backwards recursively to get (see, for example, Cairns and Parker, 1997):

**Lemma 2.1.**

$$F(t) = (\theta_v - k_2) \sum_{s=0}^{\infty} (1 - k_1)^s \exp(S_y(t, s) + S_p(t, s)) + k_2 e^{y'} \sum_{s=0}^{\infty} (1 - k_1)^s \exp(S_y(t, s) - y(t - 1 - s) + S_p(t, s))$$

where  $\theta_v = (k_1 - d_v)\text{AL}$  provided  $k_1$  has been chosen so that this sum converges.

Within this expression, first,

$$S_y(t, s) = \sum_{j=0}^s y(t-1-j) = (s+1)y + \sum_{j=1}^{s+1} \frac{(1-\phi^j)\sigma_y}{1-\phi} Z_y(t-j) + \sum_{j=s+2}^{\infty} \frac{\phi^{j-s-1}(1-\phi^{s+1})\sigma_y}{1-\phi} Z_y(t-j)$$

$$\Rightarrow S_y(t, 0) - y(t-1) = 0$$

and for  $s \geq 1$

$$S_y(t, s) - y(t-1-s) = S_y(t, s-1) = sy + \sum_{j=1}^s \frac{(1-\phi^j)\sigma_y}{1-\phi} Z_y(t-j) + \sum_{j=s+1}^{\infty} \frac{\phi^{j-s}(1-\phi^s)\sigma_y}{1-\phi} Z_y(t-j).$$

(The latter equality is, of course, zero if we define  $\sum_{j=1}^s (\cdot) \equiv 0$  when  $s = 0$ .) Second,

$$S_p(t, s) = \sum_{j=0}^s p_1 \Delta_e(t-j) + \sum_{j=0}^s p_2 \Delta_b(t-j) + (s+1)\rho(p_1, p_2)$$

$$= (s+1)\alpha_0 + \alpha_1 \sum_{j=0}^s Z_y(t-j) + \alpha_2 \sum_{j=0}^s Z_e(t-j) + \alpha_3 \sum_{j=0}^s Z_b(t-j)$$

where  $\alpha_0 = p_1 \Delta_e + p_2 \Delta_b + \rho(p_1, p_2)$ ;  $\alpha_1 = p_1 \sigma_{ey} + p_2 \sigma_{by}$ ;  $\alpha_2 = p_1 \sigma_e$  and  $\alpha_3 = p_1 \sigma_{eb} + p_2 \sigma_b$ .

**Theorem 2.2.** *The unconditional expected values and the variances of the fund size and contribution rate are as follows:*

(a) 
$$E[F(t)] = (\theta_v - k_2)\Psi_1 + k_2\Psi_2$$

$$E[C(t)] = NC + k_1(AL - E[F(t)]) + k_2(e^{y'} e^{-y+(1/2)(\sigma_y^2/1-\phi^2)} - 1)$$

where

$$\Psi_1 = \sum_{s=0}^{\infty} (1-k_1)^s \exp((s+1)(y+\alpha_0) + (1/2)V_1(s))$$

$$\Psi_2 = e^{y'} \sum_{s=0}^{\infty} (1-k_1)^s \exp((s+1)(y+\alpha_0) - y + (1/2)V_2(s))$$

and

$$V_1(s) = \text{Var}(S_y(t, s) + S_p(t, s)); V_2(s) = \text{Var}(S_y(t, s-1) + S_p(t, s)).$$

Thus,  $E[F(t)]$  and  $E[C(t)]$  are both linear functions of  $k_2$  but nonlinear functions of  $k_1$ .

(b) 
$$\text{Var}[F(t)] = h_2 k_2^2 + h_1 k_2 + h_0 \tag{2.7}$$

$$\text{Var}[C(t)] = a_2 k_2^2 + a_1 k_2 + a_0 \tag{2.8}$$

where

$$h_0 = \sum_{r,s=0}^{\infty} (1-k_1)^{r+s} C_1(r, s) + e^{2y'} \sum_{r,s=0}^{\infty} (1-k_1)^{r+s} C_3(r, s) - 2e^{y'} \sum_{r,s=0}^{\infty} (1-k_1)^{r+s} C_2(r, s)$$

$$h_1 = -2\theta_v \sum_{r,s=0}^{\infty} (1-k_1)^{r+s} C_1(r, s) + 2\theta_v e^{y'} \sum_{r,s=0}^{\infty} (1-k_1)^{r+s} C_2(r, s)$$

$$h_2 = \theta_v^2 \sum_{r,s=0}^{\infty} (1-k_1)^{r+s} C_1(r, s)$$

Table 1  
Values for  $AL(i_v)$  (Eq. (2.9)) for different values of  $i_v$ , with the corresponding normal contribution rates  $NC(i_v)$  (Eq. (2.5))

$i_v$	$AL(i_v)$	$NC(i_v)$
0.02	644.87	27.36
0.03	579.73	23.11
0.04	525.39	19.79
0.05	479.66	17.16
0.06	440.85	15.05

$$a_0 = k_1^2 h_0 + e^{2y'} \text{Var}(e^{-y(t)}) + 2k_1 e^{y'} \sum_{r=0}^{\infty} (1 - k_1)^r C_4(r) - 2k_1 e^{2y'} \sum_{r=0}^{\infty} (1 - k_1)^r C_5(r)$$

$$a_1 = k_1^2 h_1 - 2k_1 e^{y'} \theta_v \sum_{r=0}^{\infty} (1 - k_1)^r C_4(r)$$

$$a_2 = k_1^2 h_2$$

and

$$C_1(r, s) = \text{Cov}(e^{S_y(t,r)+S_p(t,r)}, e^{S_y(t,s)+S_p(t,s)})$$

$$C_2(r, s) = \text{Cov}(e^{S_y(t,r)-y(t-1-r)+S_p(t,r)}, e^{S_y(t,s)+S_p(t,s)})$$

$$C_3(r, s) = \text{Cov}(e^{S_y(t,r)-y(t-1-r)+S_p(t,r)}, e^{S_y(t,s)-y(t-1-s)+S_p(t,s)})$$

$$C_4(r) = \text{Cov}(e^{S_y(t,r)+S_p(t,r)}, e^{-y(t)})$$

$$C_5(r) = \text{Cov}(e^{S_y(t,r)-y(t-1-r)+S_p(t,r)}, e^{-y(t)}).$$

For a proof of this result and more detailed formulae for these functions, see Appendix A. In these expressions note that  $\psi_1, \psi_2, h_0, h_1, h_2, a_0, a_1, a_2$  are all functions of  $k_1$  but not of  $k_2$ .

For the actuarial liability we will assume a simple model (as in Cairns and Parker, 1997) where:

- there is one member at each of ages 25–64;
- each year one new member aged 25 joins the plan;
- no deaths or other decrements before age 65;
- on retirement at age 65 each member receives a benefit of  $B = 40$  which accrues uniformly over the 40 years of service.

Thus, the accrued or past-service liability, when the valuation rate of interest is  $i_v$ , is

$$AL = AL(i_v) = \sum_{x=25}^{64} (x - 25)(1 + i_v)^{x-65} = \left( 40 - \frac{(1 - v_v^{40})}{i_v} \right) \frac{1 + i_v}{i_v} \tag{2.9}$$

where

$$v_v = \frac{1}{1 + i_v}.$$

Sample values for  $AL(i_v)$  are given in Table 1.

### 3. Optimal strategies for the contribution rate

In this section, we will discuss how to make the best use of current market interest rates to control variability. We have previously specified in Eq. (2.6) how current interest rates can be used in a simple fashion to adjust the contribution

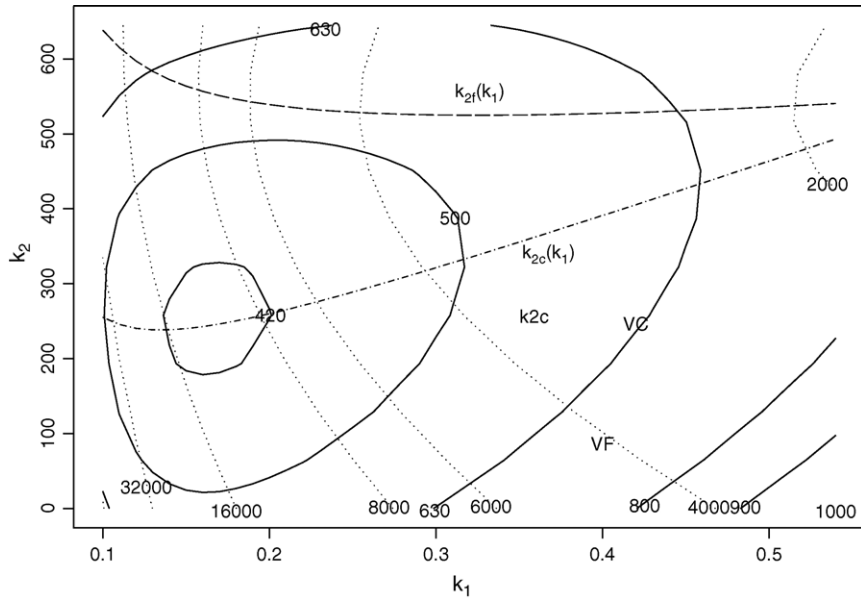


Fig. 1. Contour plot of  $\text{Var}[F(t)]$  (dotted lines, contours at the levels  $\text{VF} = 2000, 4000, 6000, 8000, 16,000$ , and  $32,000$ ) and  $\text{Var}[C(t)]$  (solid lines, contours at the levels  $\text{VC} = 400, 500, 600, 800$ , and  $900$ ) for different  $k_2$  and  $k_1$  with  $p_1 = 0.4$  (equities) and  $p_2 = 0.3$  (bonds). Also plotted are  $k_{2f}(k_1)$  (long dashed line) and  $k_{2c}(k_1)$  (dot-dashed line). Parameter values are  $y = 0.03, \Delta_e = 0.02, \Delta_b = 0.01, \phi = 0.7, \sigma_e = 0.12, \sigma_{by} = -0.05, \sigma_{ey} = -0.03, \sigma_y = 0.03, \sigma_{eb} = 0.02, \sigma_b = 0.03, y' = 0.0309$  and  $i_v = 0.02$ .

rate through the term  $k_2(e^{y'-y(t)} - 1)$ . The question of how to make best use of current interest rates then comes down to choosing the best value for  $k_2$ .

Now we can note that, given  $k_1$ , the variances of both  $F(t)$  and  $C(t)$  are quadratic in  $k_2$  (Eqs. (2.7) and (2.8)). It follows that the values

$$k_{2f} = k_{2f}(k_1) = \frac{-h_1}{2h_2}$$

$$k_{2c} = k_{2c}(k_1) = \frac{-a_1}{2a_2}$$

minimise, respectively, the variances of  $F(t)$  and  $C(t)$ .

In Fig. 1, we plot contours for  $\text{Var}[F(t)]$  and  $\text{Var}[C(t)]$  over a range of values for  $k_1$  and  $k_2$  in the case where  $p_1 = 0.4$  and  $p_2 = 0.3$ . By superimposing one set of contours on the other we are able to compare simultaneously the effect of  $k_1$  and  $k_2$  on the two variances. First suppose that  $k_2 = 0$  (the old method for determining  $C(t)$ ). The minimum value for  $\text{Var}[C(t)]$  is a little over 500 when  $k_1$  is around 0.16. Minimising over  $k_2$  as well clearly delivers substantial reductions in the variances. For example, if the objective is to minimize  $\text{Var}[C(t)]$ , then by the  $k_1$  approach (minimize  $\text{Var}[C(t)]$  over  $k_1$  with  $k_2 = 0$ ) we have  $\text{Var}[F(t)] \approx 24,000$  and  $\text{Var}[C(t)] \approx 500$ . By the  $k_2$  approach (minimize  $\text{Var}[C(t)]$  over  $k_1$  and  $k_2$ ), we have  $\text{Var}[F(t)] \approx 12,000$  (a reduction of about 50%) and  $\text{Var}[C(t)] \approx 400$  (a reduction of about 20%) when  $k_1 = 0.17$  and  $k_2 = 250$ .

Depending on what the plan objectives and constraints are, we will have different strategies for  $k_1$  and  $k_2$ . One example might be the imposition of a constraint that  $\text{Var}[F(t)]$  is less than 6000. From Fig. 1, we see that this imposes a constraint that  $k_1$  must exceed about 0.2 (and then only when  $k_2$  lies between 500 and 600). Then, given  $k_1$  it is always optimal to choose  $k_2$  between the lines for  $k_{2f}(k_1)$  and  $k_{2c}(k_1)$  (since there is always a value in this interval which can reduce both variances compared with values of  $k_2$  outside). A second example might specify the value of  $k_1$  (for example, an amortization factor based on the average future working lifetime) with minimisation over  $k_2$  only. Then it will always be efficient to choose a value of  $k_2$  between  $k_{2c}(k_1)$  and  $k_{2f}(k_1)$ . Any value outside this range can be improved upon (that is both  $\text{Var}[C(t)]$  and  $\text{Var}[F(t)]$  can be reduced) by changing  $k_2$  to

a suitable point between  $k_{2c}(k_1)$  and  $k_{2f}(k_1)$ . We define the region between the lines  $k_{2f}$  and  $k_{2c}$  as the *efficient region*.

To be more precise, for a fixed value of  $k_1$ , we define  $k_2^* = \min(k_{2f}, 2k_{2c})$  and  $k_2^\diamond = \min(k_{2c}, 2k_{2f})$ .  $\text{Var}[F(t)]$  and  $\text{Var}[C(t)]$  are quadratic functions of  $k_2$ , achieving their minima at  $k_{2f}$  and  $k_{2c}$ , respectively by definition.

If  $k_{2f} > k_{2c}$ , choose any  $\hat{k}_2 \in [k_{2c}, k_2^*]$ . Since  $0 \leq \hat{k}_2 \leq k_{2f}$ , we have

$$\text{Var}[F(t)]_{k_2=0} \geq \text{Var}[F(t)]_{\hat{k}_2} \geq \text{Var}[F(t)]_{k_{2f}}$$

and since  $k_{2c} \leq \hat{k}_2 \leq 2k_{2c}$ , we have

$$\text{Var}[C(t)]_{k_2=0} = \text{Var}[C(t)]_{2k_{2c}} \geq \text{Var}[C(t)]_{\hat{k}_2} \geq \text{Var}[C(t)]_{k_{2c}}.$$

Hence  $k_2 = \hat{k}_2$  achieves a simultaneous reduction in both  $\text{Var}[F(t)]$  and  $\text{Var}[C(t)]$  from their values at  $k_2 = 0$  in the case of  $k_{2f} > k_{2c}$ .

If  $k_{2c} > k_{2f}$ , choose any  $\hat{k}_2 \in [k_{2f}, k_2^\diamond]$ . Since  $0 \leq \hat{k}_2 \leq k_{2c}$ , we have

$$\text{Var}[C(t)]_{k_2=0} \geq \text{Var}[C(t)]_{\hat{k}_2} \geq \text{Var}[C(t)]_{k_{2c}}$$

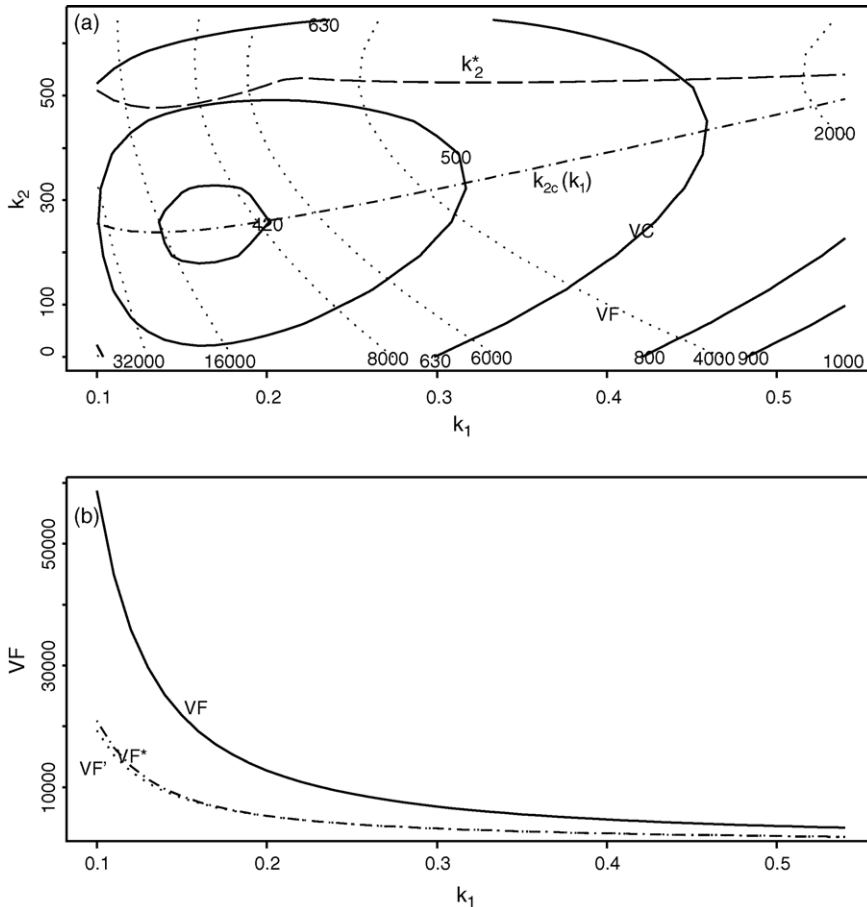


Fig. 2. (a) (Top) Contour plot of  $\text{Var}[F(t)]$  (dotted lines) and  $\text{Var}[C(t)]$  (solid lines) for different  $k_2$  and  $k_1$  when  $p_1 = 0.4$  and  $p_2 = 0.3$ . Also plotted are  $k_2^*$  (long dashed line) and  $k_{2c}$  (short dashed line). (b) (Bottom) Values of  $\text{Var}[F(t)]$  when  $k_2 = 0$  (VF, solid line),  $k_2 = k_2^*$  (VF\*, dot-dashed line), and  $k_2 = k_{2f}(k_1)$  (VF', dotted line), all corresponding to the first graph. Parameter values:  $y = 0.03$ ,  $\Delta_e = 0.02$ ,  $\Delta_b = 0.01$ ,  $\phi = 0.7$ ,  $\sigma_e = 0.12$ ,  $\sigma_{by} = -0.05$ ,  $\sigma_{ey} = -0.03$ ,  $\sigma_y = 0.03$ ,  $\sigma_{eb} = 0.02$ ,  $\sigma_b = 0.03$ ,  $y' = 0.0309$  and  $i_v = 0.02$ .

and since  $k_{2f} \leq \hat{k}_2 \leq 2k_{2f}$ , we have

$$\text{Var}[F(t)]_{k_2=0} = \text{Var}[F(t)]_{2k_{2f}} \geq \text{Var}[F(t)]_{\hat{k}_2} \geq \text{Var}[F(t)]_{k_{2f}}.$$

Hence  $k_2 = \hat{k}_2$  achieves a simultaneous reduction in both  $\text{Var}[F(t)]$  and  $\text{Var}[C(t)]$  from their values at  $k_2 = 0$  in the case of  $k_{2c} > k_{2f}$ .

If  $k_{2c} = k_{2f}$ , then  $\hat{k}_2 = k_{2c} = k_{2f}$  is the best strategy for reducing both  $\text{Var}[F(t)]$  and  $\text{Var}[C(t)]$ , simultaneously.

These ideas are illustrated in Fig. 2. In the top graph (a), we have plotted  $k_{2c}$  and  $k_2^*$ . Given  $k_1$ , any value of  $k_2$  between  $k_{2c}$  and  $k_2^*$  will reduce both  $\text{Var}F(t)$  and  $\text{Var}C(t)$  relative to  $k_2 = 0$ . However, in some cases ( $k_1 < 0.24$ )  $\text{Var}F(t)$  can be reduced further by increasing  $k_2$  from  $k_2^*$  to  $k_{2f}$  (Fig. 2, bottom (b)).

Corresponding to Fig. 2(a) and (b) gives us the graphs of  $\text{Var}[F(t)]$  when  $k_2 = 0$ , the minimum  $\text{Var}[F(t)]$  subject to  $\text{Var}[C(t)] \leq \text{Var}[C(t)]_{k_2=0}$  and the minimum unconstrained  $\text{Var}[F(t)]$ . We see from Fig. 2(b) that  $\text{VF}^*$  ( $\text{Var}F(t)$  at  $k_2^*$ ) is not much different from  $\text{VF}'$  ( $\text{Var}F(t)$  at  $k_{2f}$ ) and that it can give us the rate of minimum  $\text{Var}[F(t)]$  subject to  $\text{Var}[C(t)] \leq \text{Var}[C(t)]_{k_2=0}$ . We can note that this small difference allows us achieve a significant reduction in  $\text{Var}[C(t)]$  for only a small deterioration in  $\text{Var}[F(t)]$  when we move from  $k_{2f}(k_1)$  to  $k_2^*(k_1)$ . More generally, within this efficient region ( $0 < k_2 < k_2^*$ ) we then can choose optimal values for  $k_1, k_2, p_1$  and  $p_2$  according to different objective functions and constraints.

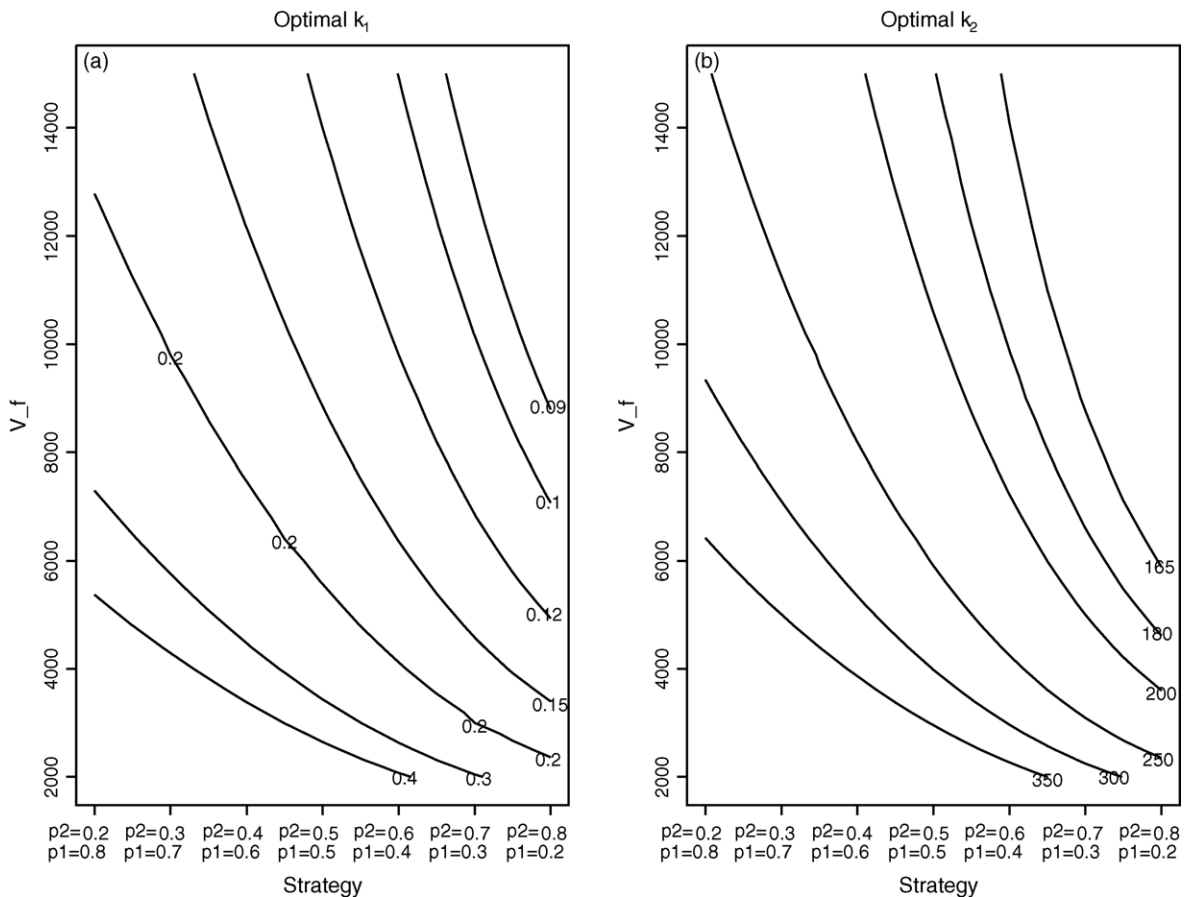


Fig. 3. Contour plots for the optimal values of  $k_1$  (left-hand plot (a)) and  $k_2$  (right-hand plot (b)) for the problem *minimise*  $\text{Var}C(t)$  subject to  $\text{Var}F(t) = V_f$  and for specified asset strategies  $(p_1, p_2)$ .  $p_1$  is the proportion in equities and  $p_2$  is the proportion in bonds,  $p_1 + p_2 = 1$ . Parameter values are  $y = 0.03, \Delta_e = 0.02, \Delta_b = 0.01, \phi = 0.7, \sigma_e = 0.12, \sigma_{by} = -0.05, \sigma_{ey} = -0.03, \sigma_y = 0.03, \sigma_{eb} = 0.02, \sigma_b = 0.03, y' = 0.0309$  and  $i_v = 0.04$ .



#### 4. Optimal investment and contribution strategies

In this section we will consider optimization when there are specific objectives and constraints put in place. In the previous discussion we were concerned only with minimisation of the Variance of  $F(t)$  or  $C(t)$ . As the basis for what follows we will start by investigating the problem:

$$\text{minimize over } k_1 \text{ and } k_2 : \text{Var}[C(t)], \quad \text{subject to } \text{Var}[F(t)] = V_f$$

and for specified values of  $p_1$  (equities) and  $p_2$  (bonds).

In Fig. 3, we have plotted contours for the optimal values of  $k_1$  (left-hand plot) and  $k_2$  (right-hand plot). In this plot we have restricted ourselves to asset strategies where  $p_1 + p_2 = 1$  (that is, zero investment in cash). For example, when we require  $\text{Var}F(t) = V_f = 8000$  with  $p_1 = 0.4$  and  $p_2 = 0.6$ , the optimal value for  $k_1$  is about 0.13, and the optimal value for  $k_2$  is about 190.

In Fig. 4, we show what the consequences are of using these optimal values for  $k_1$  and  $k_2$  for the chosen values of  $V_f$ ,  $p_1$  and  $p_2$ . For these inputs we have calculated the values of  $\text{Var}[C(t)]$ ,  $E[C(t)]$  and  $E[F(t)]$ . Contours for each of these variables are shown in Fig. 4. First, (solid lines) we can see that  $\text{Var}[C(t)]$  decreases as we move from left to right. This reflects the fact that we are investing more in bonds and less in equities. For the same reason, however,  $E[C(t)]$  is increasing from left to right, since bonds are low return as well as low risk. The impact of this is less marked on  $E[F(t)]$ , which at first is surprising. However, we can see from Fig. 3 that  $k_1$  is closely linked to the constrained value of  $V_f$ : the lowest values of  $\text{Var}[F(t)]$  can only be achieved by amortizing surplus or deficit as quickly as possible (that is, by having  $k_1$  close to 1). The same high values of  $k_1$  mean that  $E[F(t)]$  will be close to the actuarial liability  $AL = 525$  (Table 1, for  $i_v = 0.04$ ).

**Example 1.** Suppose the objective function is to minimize  $\text{Var}[C(T)]$  with the constraint that  $\text{Var}[F(T)]$  is less than 8000. From Fig. 3 we can see that  $k_1$  must greater than around 0.1 (that is, the amortization period should be less than about 11 years). If the required  $\text{Var}[C(T)]$  can not be more than 200, then Fig. 4 indicates that the investment strategy cannot allocate more than 45% to equities. If we further require that  $E[C(T)]$  can not be more than 4, then we become restricted to an approximately triangular region in Fig. 4. This region indicates that we must invest between 38 and 45% in equities and  $k_1$  should be between 0.13 and 0.25.

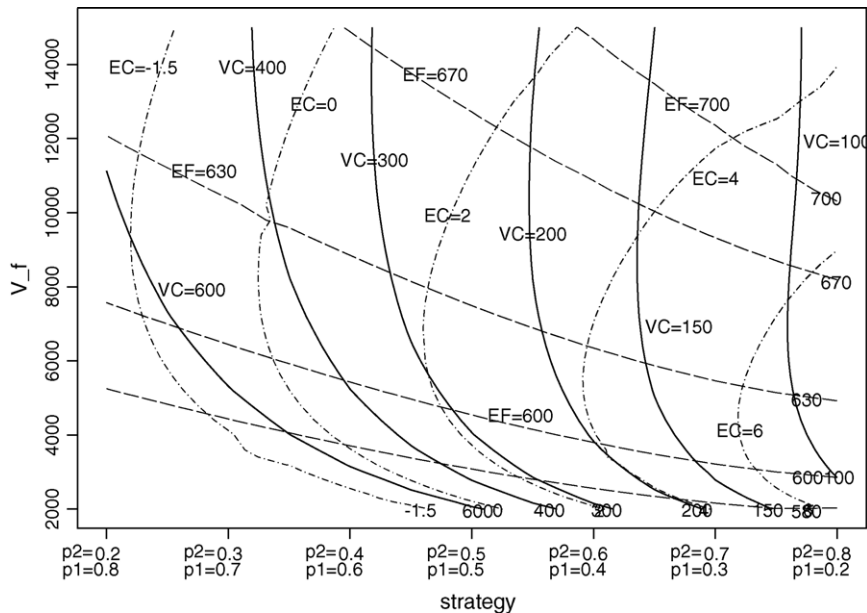


Fig. 4. Contours for  $\text{Var}[C(t)]$  (solid lines),  $E[C(t)]$  (dot-dashed lines) and  $E[F(t)]$  (long-dashed lines) as a function of  $V_f$ ,  $p_1$  and  $p_2$  and assuming that the optimal values for  $k_1$  and  $k_2$  are being used for each  $(V_f, p_1, p_2)$ . Parameter values are  $y = 0.03$ ,  $\Delta_c = 0.02$ ,  $\Delta_b = 0.01$ ,  $\phi = 0.7$ ,  $\sigma_e = 0.12$ ,  $\sigma_{by} = -0.05$ ,  $\sigma_{ey} = -0.03$ ,  $\sigma_y = 0.03$ ,  $\sigma_{cb} = 0.02$ ,  $\sigma_b = 0.03$ ,  $y' = 0.0309$  and  $i_v = 0.04$ .

**Example 2.** If we wish to obtain an optimal  $\text{Var}[C(T)]$  under the control that  $E[C(T)]$  is between 0 and 4,  $\text{Var}[F(T)]$  is less than 8000, and  $E[F(T)]$  is more than 600, the available region in Fig. 4 would be shaped approximately like a trapezium, with the an equity holding of between 38 and 67%, and with  $\text{Var}[F(T)]$  between 3800 and 8000. The minimum  $\text{Var}[C(t)]$  would be about 160 at the top right corner of this trapezium (where  $\text{Var}[F(T)] = 8000$  and  $E[C(T)] = 4$ ). Our optimal strategy then is to invest 38% in equities and the rest in bonds, and to set  $k_1 = 0.13$  (equivalent with  $i_v = 0.04$  to an amortization period of about 8 years). This gives, as remarked above,  $\text{Var}[F(T)] = 8000$  and  $E[C(T)] = 4$ . If, instead, we wish to restrict  $\text{Var}[C(T)]$  to be not more than 300 and  $\text{Var}[F(T)]$  to 8000 and seek for the smallest  $E[C(T)]$ , we will obtain an optimal  $E[C(T)] \approx 1.5$ . Our optimal strategy is to invest 56% in equities and 44% in bonds with the amortization period near to 5 years ( $k_1 = 0.18$ ) and  $\text{Var}[F(T)] = 8000$ .

**Example 3.** If our constraint is that the amortization period must not be more than 7 years (that is, we require  $k_1$  to be larger than 0.16), and  $E[C(T)]$  is less than 4, in order to minimize  $\text{Var}[C(T)]$ , our optimal strategy will be to invest about 40% in equities and the rest in bonds with the optimal  $\text{Var}[C(T)]$  equal to about 180.

## 5. Conclusions

In this paper we have investigated a model for defined-benefit pension plans which incorporates a Vasicek type of model for the short-term interest rate and three assets: cash, bonds and equities. We have proposed a simple method for adjusting the contribution rate to account for the current level of interest rates as well as the usual adjustment for the current funding level. Using this model we have derived formulae for the unconditional moments of the funding level and the contribution rate.

A number of illustrative examples have been given which demonstrate that the new adjustment to the contribution rate, taking account of current interest rates, does improve stability significantly, particularly where there is a strong degree of persistence in interest rates. The approach therefore indicates that the standard approach to liability valuation using an artificial valuation interest rate can be improved upon by making an adjustment for market conditions. What we have not done here is to look at direct methods for valuing liabilities using the current term-structure of interest rates. This is a topic for further investigation.

We have developed further the notion of efficient regions for various subsets of the control parameters  $k_1, k_2, i_v$  and  $(p_1, p_2)$  depending on different constraints and objectives. These are regions that we can move into to reduce the variances of both  $F(t)$  and  $C(t)$ .

## Acknowledgement

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## Appendix A. Proof of Theorem 2.2

(a)(i) Recall that

$$\begin{aligned}
 F(t) = & (\theta_v - k_2) \sum_{s=0}^{\infty} (1 - k_1)^s \exp(S_y(t, s) + S_p(t, s)) \\
 & + k_2 e^{y' t} \sum_{s=0}^{\infty} (1 - k_1)^s \exp(S_y(t, s) - y(t - 1 - s) + S_p(t, s)).
 \end{aligned} \tag{A.1}$$

For notational convenience write

$$X_s = \exp(S_y(t, s) + S_p(t, s))$$

and

$$Y_s = \exp(S_y(t, s) - y(t - 1 - s) + S_p(t, s)).$$

Then

$$E[F(t)] = (\theta_v - k_2) \sum_{s=0}^{\infty} (1 - k_1)^s E[X_s] + k_2 e^{y'} \sum_{s=0}^{\infty} (1 - k_1)^s E[Y_s]$$

with

$$E[X_s] = \exp \left[ (s + 1)(y + \alpha_0) + \frac{1}{2} V_1(s) \right]$$

$$E[Y_s] = \exp \left[ (s + 1)(y + \alpha_0) - y + \frac{1}{2} V_2(s) \right]$$

where

$$V_1(s) = \text{Var}(S_y(t, s) + S_p(t, s))$$

$$V_2(s) = \text{Var}(S_y(t, s) - y(t - 1 - s) + S_p(t, s))$$

(a)(ii) Next recall that

$$C(t) = \text{NC} + k_1(\text{AL} - F(t)) + k_2(e^{y'-y(t)} - 1). \tag{A.2}$$

Hence,

$$E[C(t)] = \text{NC} + k_1(\text{AL} - E[F(t)]) + k_2(E[e^{y'-y(t)}] - 1)$$

$$= \text{NC} + k_1(\text{AL} - E[F(t)]) + k_2(E[e^{y'-y+(1/2)\gamma_y(0)}] - 1)$$

where

$$\gamma_y(s) = \text{Cov}[y(t), y(t - s)] = \frac{\sigma_y^2 \phi^s}{1 - \phi^2}.$$

(b)(i) From Eq. (A.1) we also have

$$\text{Var}[F(t)] = (\theta_v - k_2)^2 \sum_{r,s=0}^{\infty} (1 - k_1)^{r+s} C_1(r, s) + 2(\theta_v - k_2)k_2 e^{y'} \sum_{r,s=0}^{\infty} (1 - k_1)^{r+s} C_2(r, s)$$

$$+ k_2^2 e^{2y'} \sum_{r,s=0}^{\infty} (1 - k_1)^{r+s} C_3(r, s)$$

where

$$C_1(r, s) = \text{Cov}(X_r, X_s)$$

$$C_2(r, s) = \text{Cov}(Y_r, X_s)$$

$$C_3(r, s) = \text{Cov}(Y_r, Y_s);$$

Expressions for  $C_1$ ,  $C_2$  and  $C_3$  are given below. Finally we separate out terms involving  $k_2$  and  $k_2^2$  to get  $\text{Var}[F(t)] = h_0 + h_1 k_2 + h_2 k_2^2$  as in the statement of the theorem.

(b)(ii) From (A.2) we can deduce that

$$\text{Var}[C(t)] = k_1^2 \text{Var}[F(t)] - 2k_1 k_2 e^{y'} \text{Cov}[F(t), e^{-y(t)}] + k_2^2 e^{2y'} \text{Var}[e^{-y(t)}]$$

$$= k_1^2 (h_0 + h_1 k_2 + h_2 k_2^2) - 2k_1 k_2 e^{y'}$$

$$\times \left( (\theta_v - k_2) \sum_{s=0}^{\infty} (1 - k_1)^s C_4(s) + k_2 e^{y'} \sum_{s=0}^{\infty} (1 - k_1)^s C_5(s) \right) + k_2^2 e^{2y'} \text{Var}[e^{-y(t)}]$$

where

$$C_4(s) = \text{Cov}[X_s, e^{-y(t)}]$$

$$C_5(s) = \text{Cov}[Y_s, e^{-y(t)}]$$

Rearranging this we get

$$\text{Var}[C(t)] = a_0 + a_1 k_2 + a_2 k_2^2$$

where

$$a_0 = k_1^2 h_0$$

$$a_1 = k_1^2 h_1 - 2k_1 e^{y'} \theta_v \sum_{s=0}^{\infty} (1 - k_1)^s C_4(s)$$

$$a_2 = k_1^2 h_2 + 2k_1 e^{y'} \sum_{s=0}^{\infty} (1 - k_1)^s C_4(s) - 2k_1 e^{2y'} \sum_{s=0}^{\infty} (1 - k_1)^s C_5(s) + k_2^2 e^{2y'} \text{Var}[e^{-y(t)}]$$

To calculate these moments more explicitly we need to work out the  $V_i$ 's and the  $C_i$ 's.

$$\begin{aligned} C_1(r, s) &= \text{Cov}(e^{S_y(t,r)+S_p(t,r)}, e^{S_y(t,s)+S_p(t,s)}) \\ &= E[e^{S_y(t,r)+S_p(t,r)+S_y(t,s)+S_p(t,s)}] - E[e^{S_y(t,r)+S_p(t,r)}] \times E[e^{S_y(t,s)+S_p(t,s)}] \\ &= \exp((r+s+2)(y+\alpha_0) + \frac{1}{2}W_1(r, s)) - \exp((r+1)(y+\alpha_0) \\ &\quad + \frac{1}{2}V_1(r)) \exp((s+1)(y+\alpha_0) + \frac{1}{2}V_1(s)) \end{aligned}$$

$$\begin{aligned} C_2(r, s) &= \text{Cov}(e^{S_y(t,r-1)+S_p(t,r)}, e^{S_y(t,s)+S_p(t,s)}) \\ &= E[e^{S_y(t,r-1)+S_p(t,r)+S_y(t,s)+S_p(t,s)}] - E[e^{S_y(t,r-1)+S_p(t,r)}] \times E[e^{S_y(t,s)+S_p(t,s)}] \\ &= \exp((r+s+2)(y+\alpha_0) - y + \frac{1}{2}W_2(r, s)) - \exp((r+1)(y+\alpha_0) - y \\ &\quad + \frac{1}{2}V_2(r)) \exp((s+1)(y+\alpha_0) + \frac{1}{2}V_1(s)) \end{aligned}$$

$$\begin{aligned} C_3(r, s) &= \text{Cov}(e^{S_y(t,r-1)+S_p(t,r)}, e^{S_y(t,s-1)+S_p(t,s)}) \\ &= E[e^{S_y(t,r-1)+S_p(t,r)+S_y(t,s-1)+S_p(t,s)}] - E[e^{S_y(t,r-1)+S_p(t,r)}] \times E[e^{S_y(t,s-1)+S_p(t,s)}] \\ &= \exp((r+s+2)(y+\alpha_0) - 2y + \frac{1}{2}W_3(r, s)) - \exp((r+1)(y+\alpha_0) - y \\ &\quad + \frac{1}{2}V_2(r)) \exp((s+1)(y+\alpha_0) - y + \frac{1}{2}V_2(s)) \end{aligned}$$

$$\begin{aligned} C_4(r) &= \text{Cov}(e^{S_y(t,r)+S_p(t,r)}, e^{-y(t)}) = E[e^{S_y(t,r)+S_p(t,r)-y(t)}] - E[e^{S_y(t,r)+S_p(t,r)}] \times E[e^{-y(t)}] \\ &= \exp((r+1)(y+\alpha_0) - y + \frac{1}{2}W_4(r, s)) - \exp((r+1)(y+\alpha_0) + \frac{1}{2}V_1(r)) \exp(-y + \frac{1}{2}\gamma_y(0)) \end{aligned}$$

$$\begin{aligned} C_5(r) &= \text{Cov}(e^{S_y(t,r-1)+S_p(t,r)}, e^{-y(t)}) = E[e^{S_y(t,r-1)+S_p(t,r)-y(t)}] - E[e^{S_y(t,r-1)+S_p(t,r)}] \times E[e^{-y(t)}] \\ &= \exp((r+1)(y+\alpha_0) - 2y + \frac{1}{2}W_5(r, s)) - \exp((r+1)(y+\alpha_0) - y + \frac{1}{2}V_2(r)) \exp(-y + \frac{1}{2}\gamma_y(0)) \end{aligned}$$

where

$$V_1(s) = \text{Var}(S_y(t, s) + S_p(t, s))$$

$$V_2(s) = \text{Var}(S_y(t, s-1) + S_p(t, s))$$

$$W_1(r, s) = \text{Var}(S_y(t, r) + S_y(t, s) + S_p(t, r) + S_p(t, s))$$

$$W_2(r, s) = \text{Var}(S_y(t, r - 1) + S_y(t, s) + S_p(t, r) + S_p(t, s))$$

$$W_3(r, s) = \text{Var}(S_y(t, r - 1) + S_y(t, s - 1) + S_p(t, r) + S_p(t, s))$$

$$W_4(s) = \text{Var}(S_y(t, s) + S_p(t, s) - y(t))$$

$$W_5(s) = \text{Var}(S_y(t, s - 1) + S_p(t, s) - y(t))$$

These formulae for the  $C_k$  exploit the normality of  $y(t)$ ,  $S_y(t, r)$ , etc.

We now derive each of these five functions:

$$\begin{aligned} V_1(s) = \text{Var}(S_y(t, s) + S_p(t, s)) &= (\alpha_2^2 + \alpha_3^2)(s + 1) + \alpha_1^2(s + 1) + \frac{2\alpha_1\sigma_y}{1 - \phi} \left( s - \frac{\phi(1 - \phi^s)}{1 - \phi} \right) \\ &+ \frac{\sigma_y^2}{(1 - \phi)^2} \left( s - \frac{2\phi(1 - \phi^s)}{1 - \phi} + \frac{\phi^2(1 - \phi^{2s})}{1 - \phi^2} \right) + \left( \frac{(1 - \phi^{s+1})\sigma_y}{1 - \phi} \right)^2 + \frac{\sigma_y^2}{(1 - \phi)^2} (1 - \phi^{s+1})^2 \frac{\phi^2}{1 - \phi^2} \end{aligned}$$

If  $s = 0$ , then  $V_2(s) = \alpha_1^2 + \alpha_2^2 + \alpha_3^2$ .

Suppose  $s \geq 1$ . Then:

$$\begin{aligned} V_2(s) = \text{Var}(S_y(t, s - 1) + S_p(t, s)) &= (\alpha_2^2 + \alpha_3^2)(s + 1) + \alpha_1^2(s + 1) + \frac{2\alpha_1\sigma_y}{1 - \phi} \left( s - \frac{\phi(1 - \phi^s)}{1 - \phi} \right) \\ &+ \frac{\sigma_y^2}{(1 - \phi)^2} \left( s - \frac{2\phi(1 - \phi^s)}{1 - \phi} + \frac{\phi^2(1 - \phi^{2s})}{1 - \phi^2} \right) + \frac{\sigma_y^2}{(1 - \phi)^2} (1 - \phi^s)^2 \frac{\phi^2}{1 - \phi^2} \end{aligned}$$

For  $W_1(r, s)$ , if  $r = s$  then  $W_1(r, s) = 4V_1(s)$ .

Suppose  $s \geq r$ . Let

$$Q_1(r, s) = S_y(t, r) + S_y(t, s) + S_p(t, r) + S_p(t, s)$$

Then

$$\begin{aligned} Q_1(r, s) &= S_y(t, r) + S_y(t, s) + S_p(t, r) + S_p(t, s) \\ &= 2\alpha_1 Z_y(t) + \sum_{j=1}^r \left( \frac{2(1 - \phi^j)\sigma_y}{1 - \phi} + 2\alpha_1 \right) Z_y(t - j) + \left( \frac{2(1 - \phi^{r+1})\sigma_y}{1 - \phi} + \alpha_1 \right) Z_y(t - r - 1) \\ &+ \sum_{j=r+2}^s \left( \frac{(\phi^{j-r-1}(1 - \phi^{r+1}) + 1 - \phi^j)\sigma_y}{1 - \phi} + \alpha_1 \right) Z_y(t - j) \\ &+ \frac{(\phi^{s-r}(1 - \phi^{r+1}) + 1 - \phi^{s+1})\sigma_y}{1 - \phi} Z_y(t - s - 1) \\ &+ \sum_{j=s+2}^{\infty} \left( \frac{\phi^{j-r-1}(1 - \phi^{r+1}) + \phi^{j-s-1}(1 - \phi^{s+1})}{1 - \phi} \sigma_y \right) Z_y(t - j) + 2\alpha_2 \sum_{j=0}^r Z_e(t - j) \\ &+ \alpha_2 \sum_{j=r+1}^s Z_e(t - j) + 2\alpha_3 \sum_{j=0}^r Z_b(t - j) + \alpha_2 \sum_{j=r+1}^s Z_b(t - j) \end{aligned}$$

Thus:

$$\begin{aligned} W_1(r, s) = \text{Var}(Q_1(r, s)) &= (\alpha_2^2 + \alpha_3^2)(4 + s + 3r) + 4\alpha_1^2(r + 1) \\ &+ \frac{8\alpha_1\sigma_y}{1 - \phi} \left( r - \frac{\phi(1 - \phi^r)}{1 - \phi} \right) + \frac{4\sigma_y^2}{(1 - \phi)^2} \left( r - \frac{2\phi(1 - \phi^r)}{1 - \phi} + \frac{\phi^2(1 - \phi^{2r})}{1 - \phi^2} \right) \\ &+ \left( \frac{2(1 - \phi^{r+1})\sigma_y}{1 - \phi} + \alpha_1 \right)^2 + \alpha_1^2(s - r - 1) \end{aligned}$$

$$\begin{aligned}
 &+ \frac{2\alpha_1\sigma_y}{1-\phi} \left( s-r-1 + \frac{\phi(1-\phi^{s-r-1})}{1-\phi} - \frac{2\phi^{r+2}(1-\phi^{s-r-1})}{1-\phi} \right) \\
 &+ \frac{\sigma_y^2}{(1-\phi)^2} \left( s-r-1 + 2(1-2\phi^{r+1})\frac{\phi(1-\phi^{s-r-1})}{1-\phi} \right) \\
 &+ \frac{\sigma_y^2}{(1-\phi)^2} \left( (1-4\phi^{r+1} + 4\phi^{2r+2})\frac{\phi^2(1-\phi^{2(s-r-1)})}{1-\phi^2} \right) + \frac{\phi^2\sigma_y^2}{1-\phi^2} \frac{(1+\phi^{s-r} - 2\phi^{s+1})^2}{(1-\phi)^2}
 \end{aligned}$$

Suppose  $s \geq r$ . Then:

$$\begin{aligned}
 Q_2(r, s) &= S_y(t, r-1) + S_y(t, s) + S_p(t, r) + S_p(t, s) = 2\alpha_1 Z_y(t) + \sum_{j=1}^r \left( \frac{2(1-\phi^j)\sigma_y}{1-\phi} + 2\alpha_1 \right) Z_y(t-j) \\
 &+ \sum_{j=r+1}^s \left( \frac{(\phi^{j-r}(1-\phi^r) + 1-\phi^j)\sigma_y}{1-\phi} + \alpha_1 \right) Z_y(t-j) + \frac{(\phi^{s-r+1}(1-\phi^r) + 1-\phi^{s+1})\sigma_y}{1-\phi} Z_y(t-s_1) \\
 &+ \sum_{j=s+2}^{\infty} \left( \frac{\phi^{j-r}(1-\phi^r) + \phi^{j-s-1}(1-\phi^{s+1})}{1-\phi} \sigma_y \right) Z_y(t-j) + 2\alpha_2 \sum_{j=0}^r Z_e(t-j) \\
 &+ \alpha_2 \sum_{j=r+1}^s Z_e(t-j) + 2\alpha_3 \sum_{j=0}^r Z_b(t-j) + \alpha_2 \sum_{j=r+1}^s Z_b(t-j)
 \end{aligned}$$

$$\begin{aligned}
 W_2(r, s) &= \text{Var}(Q_2(r, s)) = (\alpha_2^2 + \alpha_3^2)(4 + s + 3r) + 4\alpha_1^2(r + 1) + \frac{8\alpha_1\sigma_y}{1-\phi} \left( r - \frac{\phi(1-\phi^r)}{1-\phi} \right) \\
 &+ \frac{4\sigma_y^2}{(1-\phi)^2} \left( r - \frac{2\phi(1-\phi^r)}{1-\phi} + \frac{\phi^2(1-\phi^{2r})}{1-\phi^2} \right) \\
 &+ \alpha_1^2(s-r) + \frac{2\alpha_1\sigma_y}{1-\phi} \left( s-r + (1-2\phi^r)\frac{\phi(1-\phi^{s-r})}{1-\phi} \right) \\
 &+ \frac{\sigma_y^2}{(1-\phi)^2} \left( s-r + 2(1-2\phi^r)\frac{\phi(1-\phi^{s-r})}{1-\phi} + (1-4\phi^r + 4\phi^{2r})\frac{\phi^2(1-\phi^{2(s-r)})}{1-\phi^2} \right) \\
 &+ \left( \frac{(\phi^{s-r+1}(1-\phi^r) + 1-\phi^{s+1})\sigma_y}{1-\phi} \right)^2 + \frac{\sigma_y^2}{(1-\phi)^2} (1 + \phi^{s-r+1} - 2\phi^{s+1})^2 \frac{\phi^2}{1-\phi^2}
 \end{aligned}$$

Suppose  $s = r$ . Then:

$$\begin{aligned}
 Q_2(r, s) &= S_y(t, s-1) + S_y(t, s) + 2S_p(t, s) = 2\alpha_1 Z_y(t) + \sum_{j=1}^s \left( \frac{2(1-\phi^{j+1})\sigma_y}{1-\phi} + 2\alpha_1 \right) Z_y(t-j) \\
 &+ \left( \frac{(\phi(1-\phi^s) + 1-\phi^{s+1})\sigma_y}{1-\phi} \right) Z_y(t-s-1) \\
 &+ \sum_{j=s+2}^{\infty} \left( \frac{\phi^{j-s+1}(1-\phi^{s+1}) + \phi^{j-s}(1-\phi^s)}{1-\phi} \sigma_y \right) Z_y(t-j) + 2\alpha_2 \sum_{j=0}^s Z_e(t-j) + 2\alpha_3 \sum_{j=0}^s Z_b(t-j)
 \end{aligned}$$

$$W_2(r, s) = \text{Var}(Q_2(r, s)) = (\alpha_2^2 + \alpha_3^2)(4 + 4s) + 4\alpha_1^2(s + 1) + \frac{8\alpha_1\sigma_y}{1-\phi} \left( s - \frac{\phi(1-\phi^s)}{1-\phi} \right)$$

$$\begin{aligned}
 & + \frac{4\sigma_y^2}{(1-\phi)^2} \left( s - \frac{2\phi(1-\phi^s)}{1-\phi} + \frac{\phi^2(1-\phi^{2s})}{1-\phi^2} \right) + \left( \frac{(\phi(1-\phi^s) + 1 - \phi^{s+1})\sigma_y}{1-\phi} \right)^2 \\
 & + \frac{\sigma_y^2}{(1-\phi)^2} (1 + \phi - 2\phi^{s+1})^2 \frac{\phi^2}{(1-\phi^2)}
 \end{aligned}$$

Suppose  $r \geq s$ . Then:

$$\begin{aligned}
 Q_2(r, s) &= S_y(t, r-1) + S_y(t, s) + S_p(t, r) + S_p(t, s) \\
 &= 2\alpha_1 Z_y(t) + \sum_{j=1}^s \left( \frac{2(1-\phi^{j+1})\sigma_y}{1-\phi} + 2\alpha_1 \right) Z_y(t-j) + \left( \frac{2(1-\phi^{s+1})\sigma_y}{1-\phi} + \alpha_1 \right) Z_y(t-s_1) \\
 &+ \sum_{j=s+2}^r \left( \frac{(\phi^{j-s-1}(1-\phi^{s+1}) + 1 - \phi^j)\sigma_y}{1-\phi} + \alpha_1 \right) Z_y(t-j) \\
 &+ \sum_{j=r+1}^{\infty} \left( \frac{\phi^{j-r}(1-\phi^r) + \phi^{j-s-1}(1-\phi^{s+1})}{1-\phi} \sigma_y \right) Z_y(t-j) + 2\alpha_2 \sum_{j=0}^s Z_e(t-j) \\
 &+ \alpha_2 \sum_{j=s+1}^r Z_e(t-j) + 2\alpha_3 \sum_{j=0}^s Z_b(t-j) + \alpha_3 \sum_{j=s+1}^r Z_b(t-j)
 \end{aligned}$$

$$\begin{aligned}
 W_2(r, s) = \text{Var}(Q_2(r, s)) &= (\alpha_2^2 + \alpha_3^2)(4 + r + 3s) + 4\alpha_1^2(s+1) + \frac{8\alpha_1\sigma_y}{1-\phi} \left( s - \frac{\phi(1-\phi^s)}{1-\phi} \right) \\
 &+ \frac{4\sigma_y^2}{(1-\phi)^2} \left( s - \frac{2\phi(1-\phi^s)}{1-\phi} + \frac{\phi^2(1-\phi^{2s})}{1-\phi^2} \right) + \left( \frac{2(1-\phi^{s+1})\sigma_y}{1-\phi} + \alpha_1 \right)^2 + \alpha_1^2(r-s-1) \\
 &+ \frac{2\alpha_1\sigma_y}{1-\phi} \left( r-s-1 + (1-2\phi^{s+1}) \frac{\phi(1-\phi^{r-s-1})}{1-\phi} \right) \\
 &+ \frac{\sigma_y^2}{(1-\phi)^2} \left( r-s-1 + 2(1-2\phi^{s+1}) \frac{\phi(1-\phi^{r-s-1})}{1-\phi} \right) \\
 &+ \frac{\sigma_y^2}{(1-\phi)^2} \left( (1-4\phi^{s+1} + 4\phi^{2s+2}) \frac{\phi^2(1-\phi^{2(r-s-1)})}{1-\phi^2} \right) + \frac{\sigma_y^2}{(1-\phi)^2} (1 + \phi^{r-s-1} - 2\phi^r)^2 \frac{\phi^2}{(1-\phi^2)}
 \end{aligned}$$

Suppose  $r = 0$ . Then:

$$\begin{aligned}
 Q_2(r, s) &= S_y(t, s) + S_p(t, 0) + S_p(t, s) = 2\alpha_1 Z_y(t) + \sum_{j=1}^s \left( \frac{(1-\phi^j)\sigma_y}{1-\phi} + \alpha_1 \right) Z_y(t-j) \\
 &+ \left( \frac{(1-\phi^{s+1})\sigma_y}{1-\phi} \right) Z_y(t-s-1) + \sum_{j=s+2}^{\infty} \left( \frac{\phi^{j-s-1}(1-\phi^{s+1})}{1-\phi} \sigma_y \right) Z_y(t-j) + 2\alpha_2 Z_e(t) \\
 &+ \alpha_2 \sum_{j=1}^s Z_e(t-j) + 2\alpha_3 Z_b(t) + \alpha_3 \sum_{j=1}^s Z_b(t-j)
 \end{aligned}$$

$$\begin{aligned}
 W_2(r, s) = \text{Var}(Q_2(r, s)) &= (\alpha_2^2 + \alpha_3^2)(s+4) + 4\alpha_1^2 + \alpha_1^2 s + \frac{2\alpha_1\sigma_y}{1-\phi} \left( s - \frac{\phi(1-\phi^s)}{1-\phi} \right) \\
 &+ \frac{\sigma_y^2}{(1-\phi)^2} \left( s - \frac{2\phi(1-\phi^s)}{1-\phi} + \frac{\phi^2(1-\phi^{2s})}{1-\phi^2} \right) + \left( \frac{(1-\phi^{s+1})\sigma_y}{1-\phi} \right)^2 \\
 &+ \frac{\sigma_y^2}{(1-\phi)^2} (1 - \phi^{s+1})^2 \frac{\phi^2}{(1-\phi^2)}
 \end{aligned}$$

Now consider  $W_3(r, s)$  Suppose  $s \geq r$ . Then:

$$\begin{aligned}
 Q_3(r, s) &= S_y(t, r - 1) + S_y(t, s - 1) + S_p(t, r) + S_p(t, s) \\
 &= 2\alpha_1 Z_y(t) + \sum_{j=1}^r \left( \frac{2(1 - \phi^{j+1})\sigma_y}{1 - \phi} + 2\alpha_1 \right) Z_y(t - j) \\
 &\quad + \sum_{j=r+1}^s \left( \frac{(\phi^{j-r}(1 - \phi^r) + 1 - \phi^j)\sigma_y}{1 - \phi} + \alpha_1 \right) Z_y(t - j) \\
 &\quad + \sum_{j=s+1}^{\infty} \left( \frac{\phi^{j-r}(1 - \phi^r) + \phi^{j-s}(1 - \phi^s)}{1 - \phi} \sigma_y \right) Z_y(t - j) + 2\alpha_2 \sum_{j=0}^r Z_e(t - j) \\
 &\quad + \alpha_2 \sum_{j=r+1}^s Z_e(t - j) + 2\alpha_3 \sum_{j=0}^r Z_b(t - j) + \alpha_3 \sum_{j=r+1}^s Z_b(t - j) \\
 W_3(r, s) &= \text{Var}(Q_3(r, s)) = (\alpha_2^2 + \alpha_3^2)(4 + s + 3r) + 4\alpha_1^2(r + 1) + \frac{8\alpha_1\sigma_y}{1 - \phi} \left( r - \frac{\phi(1 - \phi^r)}{1 - \phi} \right) \\
 &\quad + \frac{4\sigma_y^2}{(1 - \phi)^2} \left( r - \frac{2\phi(1 - \phi^r)}{1 - \phi} + \frac{\phi^2(1 - \phi^{2r})}{1 - \phi^2} \right) + \alpha_1^2(s - r) \\
 &\quad + \frac{2\alpha_1\sigma_y}{1 - \phi} \left( s - r + (1 - 2\phi^r) \frac{\phi(1 - \phi^{s-r})}{1 - \phi} \right) \\
 &\quad + \frac{\sigma_y^2}{(1 - \phi)^2} \left( s - r + 2(1 - 2\phi^r) \frac{\phi(1 - \phi^{s-r})}{1 - \phi} + (1 - 4\phi^r + 4\phi^{2r}) \frac{\phi^2(1 - \phi^{2(s-r)})}{1 - \phi^2} \right) \\
 &\quad + \frac{\sigma_y^2}{(1 - \phi)^2} (1 + \phi^{s-r} - 2\phi^s)^2 \frac{\phi^2}{(1 - \phi^2)}
 \end{aligned}$$

Suppose  $r = s$ . Then:

$$W_3(r, s) = 4V_2(s)$$

If  $r = s = 0$ ,  $W_3(r, s) = 4\alpha_1^2 + 4(\alpha_2^2 + \alpha_3^2)$ .

Suppose  $r = 0$ . Then:

$$Q_3(r, s) = S_y(t, s - 1) + S_p(t, 0) + S_p(t, s)$$

$$\begin{aligned}
 Q_3(r, s) &= S_y(t, s - 1) + S_p(t, 0) + S_p(t, s) = 2\alpha_1 Z_y(t) + \sum_{j=1}^{s-1} \left( \frac{(1 - \phi^j)\sigma_y}{1 - \phi} + \alpha_1 \right) Z_y(t - j) \\
 &\quad + \sum_{j=s+1}^{\infty} \left( \frac{\phi^{j-s}(1 - \phi^s)}{1 - \phi} \sigma_y \right) Z_y(t - j) + 2\alpha_2 Z_e(t) + \alpha_2 \sum_{j=1}^s Z_e(t - j) + 2\alpha_3 Z_b(t) + \alpha_3 \sum_{j=1}^s Z_b(t - j)
 \end{aligned}$$

$$\begin{aligned}
 W_3(r, s) &= \text{Var}(Q_3(r, s)) = (\alpha_2^2 + \alpha_3^2)(s + 4) + 4\alpha_1^2 + \alpha_1^2 s + \frac{2\alpha_1\sigma_y}{1 - \phi} \left( s - \frac{\phi(1 - \phi^s)}{1 - \phi} \right) \\
 &\quad + \frac{\sigma_y^2}{(1 - \phi)^2} \left( s - \frac{2\phi(1 - \phi^s)}{1 - \phi} + \frac{\phi^2(1 - \phi^{2s})}{1 - \phi^2} \right) + \frac{\sigma_y^2}{(1 - \phi)^2} (\phi - \phi^s)^2 \frac{\phi^2}{(1 - \phi^2)}
 \end{aligned}$$

$$y(t) = y + \sum_{j=0}^{\infty} \phi^j \sigma_y Z_y(t - j)$$



Consider:

$$\begin{aligned}
 Q_4(s) &= S_y(t, s) + S_p(t, s) - y(t) = sy + (s + 1)\alpha_0 + (\alpha_1 - \sigma_y)Z_y(t) \\
 &+ \sum_{j=1}^s \left( \frac{(1 - \phi^j)\sigma_y}{1 - \phi} - \phi^j\sigma_y + \alpha_1 \right) Z_y(t - j) + \left( \frac{(1 - \phi^{s+1})\sigma_y}{1 - \phi} - \phi^{s+1}\sigma_y \right) Z_y(t - j) \\
 &+ \sum_{j=s+2}^{\infty} \left( \frac{\phi^{j-s-1}(1 - \phi^{s+1})}{1 - \phi} \sigma_y - \phi^j\sigma_y \right) Z_y(t - j) + \alpha_2 \sum_{j=0}^s Z_e(t - j) + \alpha_3 \sum_{j=0}^s Z_b(t - j)
 \end{aligned}$$

$$\begin{aligned}
 W_4(s) = \text{Var}(Q_4(s)) &= (\alpha_1 - \sigma_y)^2 + \alpha_1^2 s + \frac{\phi^2(1 - \phi^{2s})}{1 - \phi^2} \sigma_y^2 + \frac{\sigma_y^2}{(1 - \phi)^2} \left( s - \frac{2\phi(1 - \phi^s)}{1 - \phi} + \frac{\phi^2(1 - \phi^{2s})}{1 - \phi^2} \right) \\
 &- 2\alpha_1\sigma_y \frac{\phi(1 - \phi^s)}{1 - \phi} + \frac{2\alpha_1\sigma_y}{1 - \phi} \left( s - \frac{\phi(1 - \phi^s)}{1 - \phi} \right) - \frac{2\sigma_y^2}{1 - \phi} \left( \frac{\phi(1 - \phi^s)}{1 - \phi} - \frac{\phi^2(1 - \phi^{2s})}{1 - \phi^2} \right) \\
 &+ \left( \frac{(1 - \phi^{s+1})\sigma_y}{1 - \phi} - \phi^{s+1}\sigma_y \right)^2 + \frac{\sigma_y^2}{(1 - \phi)^2} (1 - \phi^{s+1})^2 \frac{\phi^2}{(1 - \phi)^2} \\
 &+ \sigma_y^2 \frac{\phi^{2s+4}}{1 - \phi^2} - \frac{2\sigma_y^2}{1 - \phi} (\phi^{s-1} - \phi^{2s}) \frac{\phi^4}{1 - \phi^2} + (\alpha_2^2 + \alpha_3^2)(s + 1)
 \end{aligned}$$

Consider, for  $s \geq 0$ :

$$\begin{aligned}
 Q_5(s) &= S_y(t, s - 1) + S_p(t, s) - y(t) = (s - 1)y + (s + 1)\alpha_0 + (\alpha_1 - \sigma_y)Z_y(t) \\
 &+ \sum_{j=1}^s \left( \frac{(1 - \phi^j)\sigma_y}{1 - \phi} - \phi^j\sigma_y + \alpha_1 \right) Z_y(t - j) + \sum_{j=s+1}^{\infty} \left( \frac{\phi^{j-s}(1 - \phi^s)}{1 - \phi} \sigma_y - \phi^j\sigma_y \right) Z_y(t - j) \\
 &+ \alpha_2 \sum_{j=0}^s Z_e(t - j) + \alpha_3 \sum_{j=0}^s Z_b(t - j)
 \end{aligned}$$

$$\begin{aligned}
 W_5(s) = \text{Var}(Q_5(s)) &= (\alpha_1 - \sigma_y)^2 + \alpha_1^2 s + \frac{\phi^2(1 - \phi^{2s})}{1 - \phi^2} \sigma_y^2 + \frac{\sigma_y^2}{(1 - \phi)^2} \left( s - \frac{2\phi(1 - \phi^s)}{1 - \phi} + \frac{\phi^2(1 - \phi^{2s})}{1 - \phi^2} \right) \\
 &- 2\alpha_1\sigma_y \frac{\phi(1 - \phi^s)}{1 - \phi} + \frac{2\alpha_1\sigma_y}{1 - \phi} \left( s - \frac{\phi(1 - \phi^s)}{1 - \phi} \right) - \frac{2\sigma_y^2}{1 - \phi} \left( \frac{\phi(1 - \phi^s)}{1 - \phi} - \frac{\phi^2(1 - \phi^{2s})}{1 - \phi^2} \right) \\
 &+ \frac{\sigma_y^2}{(1 - \phi)^2} (1 - \phi^s)^2 \frac{\phi^2}{(1 - \phi)^2} + \sigma_y^2 \frac{\phi^{2s+2}}{1 - \phi^2} - \frac{2\sigma_y^2}{1 - \phi} (\phi^s - \phi^{2s}) \frac{\phi^2}{1 - \phi^2} + (\alpha_2^2 + \alpha_3^2)(s + 1)
 \end{aligned}$$

If  $s = 0$ , then:

$$Q_5(s) = S_p(t, 0) - y(t + 1)$$

$$W_5(s) = \text{Var}(S_p(t, 0) - y(t)) = (\alpha_1 - \sigma_y)^2 + \sigma_y^2 \frac{\phi^2}{1 - \phi^2} + \alpha_2^2 + \alpha_3^2$$

### Appendix B. Asset returns

In this appendix we will demonstrate that the asset returns for a mixed investment strategy involving the one-year cash account, bonds and equities can reasonably be given by Eq. (2.4). We will do this by discussing a continuous-time model that runs in the background, even though decisions in the pensions model are only made on an annual basis. To this end we will introduce the following notation. Let  $S_t(u)$  represent the value at  $u$ , for  $t - 1 \leq u \leq t$ , of an investment

of 1 at time  $t - 1$  in asset  $i$ , for  $i = 0, 1, 2, 3$ . Any coupon or dividend income on the assets are assumed to be reinvested in the same asset meaning that the total return process  $S_i(u)$  represents the price of a *tradeable asset*.

Asset 0 is the zero-coupon bond maturing at time  $t$ , so that  $S_0(u) = P(u, t)/P(t - 1, t)$  and  $P(s, T)$  is the usual notation for the price at  $s$  of a zero-coupon bond maturing at  $T$ . Asset 1 is the equity account, so that, in line with Section 2 we have  $S_1(t) = \exp[y(t - 1) + \Delta_e(t)]$ . Asset 2 is the bond account, with  $S_2(t) = \exp[y(t - 1) + \Delta_b(t)]$ . Asset 3 is an additional asset which cannot be replicated using assets 0–2. We know that such assets exist since we have three independent sources of risk and the minimum requirement, then, for a market to be complete is that there are four tradeable assets.

We will assume in our background model that the market is complete, implying that any derivative payment at  $t$  can be replicated over the interval  $t - 1$  to  $t$  using  $S_0(u)$  to  $S_3(u)$ .

We now recall the Fundamental Theorem of Asset Pricing that asserts

- (a) The market is arbitrage-free if and only if there exists a martingale measure  $Q$ .
- (b) The market is complete if and only if there exists a *unique* martingale measure  $Q$ .

Instead of using the instantaneous cash account as the numeraire we will use  $S_0(u)$ . We thus seek a measure  $\hat{Q}$  under which the prices of all tradeable assets discounted by  $S_0(u)$  are martingales. If such a  $\hat{Q}$  exists then the assumption of completeness made above ensures that all claims at  $t$  can be replicated and therefore have a well-defined value at  $t - 1$ .

We construct our model as follows. Let  $\hat{S}_i(t) = S_i(t)/S_0(t)$  be the discounted asset prices using  $S_0(t)$  as the numeraire. Let  $\hat{Z}_y(t)$ ,  $\hat{Z}_b(t)$  and  $\hat{Z}_e(t)$  be i.i.d. standard normal random variables under a measure  $\hat{Q}$  equivalent to the real-world measure  $P$ . In our construction of the model we will assume that the prices of all tradeable assets discounted by  $S_0(t)$  under  $\hat{Q}$  are martingales. Specifically we assume that

$$\begin{aligned}\hat{S}_1(t) &= \exp[\Delta_e(t)] \\ \hat{S}_2(t) &= \exp[\Delta_b(t)]\end{aligned}$$

where

$$\Delta_e(t) = \sigma_{ey}\hat{Z}_y(t) + \sigma_{eb}\hat{Z}_b(t) + \sigma_e\hat{Z}_e(t) - \frac{1}{2}v_{ee}$$

$$\Delta_b(t) = \sigma_{by}\hat{Z}_y(t) + \sigma_b\hat{Z}_b(t) - \frac{1}{2}v_{bb}$$

$$v_{ee} = \sigma_{ey}^2 + \sigma_{eb}^2 + \sigma_e^2$$

and

$$v_{bb} = \sigma_{by}^2 + \sigma_b^2.$$

Clearly  $E_{\hat{Q}}[\hat{S}_i(t)|y(t - 1)] = 1$ .

Now consider a derivative security that pays at time  $t$

$$V(t) = \exp[y(t - 1) + p_1\Delta_e(t) + p_2\Delta_b(t) + \rho(p_1, p_2)]$$

where

$$\rho(p_1, p_2) = \frac{1}{2}p_1v_{ee} + \frac{1}{2}p_2v_{bb} - \frac{1}{2}(p_1^2v_{ee} + 2p_1p_2v_{eb} + p_2^2v_{bb})$$

and

$$v_{eb} = \sigma_{ey}\sigma_{by} + \sigma_{eb}\sigma_b.$$

It is straightforward to show that

$$E_{\hat{Q}} \left[ \frac{V(t)}{S_0(t)} \middle| y(t - 1) \right] = 1$$

Earlier in this appendix we assumed, with the help of an additional asset, that the market was complete. It follows that the derivative paying  $V(t)$  at  $t$  can be replicated using the four assets  $S_0(u), \dots, S_3(u)$  and that the value of this

derivative is 1 at time  $t - 1$ . In other words a return of

$$1 + i(t) = \exp[y(t - 1) + p_1 \Delta_e(t) + p_2 \Delta_b(t) + \rho(p_1, p_2)]$$

at time  $t$  can be described as a *fair return* at  $t$  on an initial investment of 1 at time  $t - 1$  in relation to 100% investments in equities, bonds or cash.

As a final remark, if we assume that, in continuous time,  $\hat{S}_1(u)$  and  $\hat{S}_2(u)$  have constant volatilities and instantaneous covariance for  $t - 1 < u < t$  then the replicating strategy for  $1 + i(t)$  above maintains constant proportions of  $p_1$  in equities,  $p_2$  in bonds and  $1 - p_1 - p_2$  in the zero-coupon bond maturing at  $t$ . This follows from the observation that  $\hat{V}(u) = E_{\hat{Q}}[V(t)|\mathcal{F}_u] \propto \hat{S}_1(u)^{p_1} \hat{S}_2(u)^{p_2}$ . We can note also that this does not require the use of the third risky asset  $S_3(u)$  which was required for completeness of the larger market.

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