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ELSEVIER Applied Mathematics and Computation 169 (2005) 366–387
APPLIED MATHEMATICS AND COMPUTATION
www.elsevier.com/locate/amc

Blow-up solutions of nonlinear differential equations

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Abstract

We consider the initial value problems for second order nonlinear differential equations of the form

$$\left(|u'|^{m-2}u'\right)' = u^p$$

and the system

$$u_i'' = f_i(u_1, u_2), \quad i = 1, 2.$$

By using the energy method, some blow-up properties such as the life span, blow-up rates and blow-up constants are given and the asymptotic behavior of the global solution is also discussed.

Keywords: Blow-up solution; Global solution

1. Introduction

We shall consider the initial value problem for second order scalar differential equation of the form

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0096-3003/$ - see front matter
doi:10.1016/j.amc.2004.09.085

$$\left(|u'|^{m-2}u'\right)' = u^p, \tag{1.1}$$

where $m \geqslant 2$ and $p > m - 1$, and the system of two ordinary differential equations

$$\begin{cases} u'' = f(u, v), \\ v'' = g(u, v). \end{cases} \tag{1.2}$$

These problems are occurred in the study of non-Newtonian fluid theory by Esteban and Vazquez [1] and Herrero and Vazquez [2]. When $m = 2$, the scalar equation is described by the Calligraphic process in Li [3]. Some asymptotic behavior and oscillation results for the scalar differential equation of the second order are given by O'Regan [4], Wang and Gao [5], Bobisub and O'Regan [6], and Yang [7] and regularity results is recently obtained by Lin [8].

In this paper, we shall discuss the blow-up properties, such as the life span, the blow-up rates, blow-up constants, and the asymptotic behavior of the global solution by using the energy method. Some interesting properties of the solutions for (1.1) are found in Li [3] when $m = 2$. Here we shall consider the more general equations and extend the results of [3] to the case $m \geqslant 2$.

The content of this paper is divided in two parts. In the first part, we study the Eq. (1.1). We first give some fundamental lemmas and notations in Section 2.1, which will be used later. Then the asymptotic behavior of the global solution is discussed in Section 2.2. The life span of the local solutions is estimated in Section 2.3. The blow-up rates and blow-up constants are given in Section 2.4. In Section 2.5, some properties of the life span are obtained. In the second part, we discuss the system (1.2). Some fundamental Lemmas are derived in Section 3.1. The existence of blow-up solution and the upper bound for the life span are given in Section 3.2. We also investigate a particular system of the form

$$\begin{cases} u'' = u^p v^{p+1}, \\ v'' = v^p u^{p+1}, \end{cases} \quad p > 1 \tag{1.3}$$

and give more blow-up properties under various conditions. Note that when the forcing form in (2.1) is replaced by $|u|^p u$, $p > m - 1$, we can get the similar result for the blow-up properties of the corresponding solutions.

2. On the scalar differential equation

In this section we shall consider the second order differential equation of the form

$$\begin{cases} (|u'|^{m-2}u')' = u^p, \\ u(0) = u_0, \quad u'(0) = u_1, \end{cases} \tag{2.1}$$

where $m \geqslant 2$ and $p > m - 1$.

For $p < m - 1$, we have the nonuniqueness of the solution. Thus we only consider the case $p > m - 1$ (for the special case $p = m - 1$, it can be similarly discussed). Since the forcing term is locally Lipschitz, the local existence and uniqueness of the problem (2.1) can be proved by Banach fixed point theorem. Hereafter we shall discuss the behavior and the properties of solutions and find the estimate for the life span through energy method.

2.1. Fundamental lemmas

Definition. A function $g : \mathbb{R} \to \mathbb{R}$ blows up means that g exists only in finite time, that is, there is a finite number T^* such that

$$\lim_{t \to T^*} g(t)^{-1} = 0$$

and a function $g: R \to R$ has a blow-up rate $q > 0$ if

$$\lim_{t \to T^*} (T^* - t)^q g(t) = \beta.$$

In this case, β is called the blow-up constant of g.

Let $u \in C^1[0, T)$. Define the energy function

$$E(t) = (m-1)|u'(t)|^m - \alpha u(t)^{p+1},$$

where $\alpha = \frac{m}{p+1}$.

Theorem 2.1. *If $u \in C^2(0, T) \cap C^0[0, T)$ is the classical solution of the problem (2.1) with the life span T, then we have*

$$E(t) = E(0) = (m-1)|u_1|^m - \alpha u_0^{p+1}, \quad \forall t \in [0, T). \tag{2.2}$$

Proof. From (2.1), we have

$$(m-1)|u'(t)|^{m-2}u''(t) = u(t)^p. \tag{2.3}$$

By multiplying (2.3) on both sides with $u'(t)$ and then integrating it from 0 to t, we get (2.2). □

Let

$$a(t) = u(t)^2, \quad t \geqslant 0$$

and

$$J(t) = a(t)^{-\frac{p-m+1}{2m}}, \quad t \geqslant 0.$$

By some elementary calculations, we have the following lemmas.

Lemma 2.2. *Suppose that $u \in C^2(0,T) \cap C^0[0,T)$ is a classical solution of the problem (2.1), then we have following identities*:

$$(i) \ \ |u'(t)|^{m-2}a''(t) = \frac{2(p+m+1)}{m}|u'(t)|^m - \frac{2(p+1)}{m(m-1)}E(0). \tag{2.4}$$

$$(ii) \ \ (m-1)(p+1)|u'(t)|^{m-2}a'(t) - (m-1)(p+1)|u_1|^{m-2}a'(0)$$
$$= 2(\beta-1)(p-m+1)\int_0^t u(r)^{p+1}\,\mathrm{d}r + 2(p+1)E(0)t. \tag{2.5}$$

$$(iii) \ \ (m-1)|J'(t)|^{m-2}J''(t) = k_1^{\beta-m}J(t)^{\beta-1}. \tag{2.6}$$

$$(iv) \ \ (m-1)|J'(t)|^m = (m-1)|J'(0)|^m - \frac{p-m+1}{p+1}k_1^{\beta-m}(J(0)^\beta - J(t)^\beta), \tag{2.7}$$

where $\beta = \frac{m(p+1)}{p-m+1}$ *and* $k_1 = \left(\frac{(p+1)(p-m+1)^{m-1}}{m^m}E(0)\right)^{\frac{1}{\beta-m}}$.

Proof. (i) Note that $2u(t)u''(t) = a''(t) - 2u'(t)^2$. From (2.3), we have

$$2mu(t)^{p+1} = m(m-1)|u'(t)|^{m-2}a''(t) - 2m(m-1)|u'(t)|^{m-2}(u'(t))^2.$$

By (2.2), we get

$$2(m-1)(p+1)|u'(t)|^m - 2(p+1)E(0)$$
$$= m(m-1)|u'(t)|^{m-2}a''(t) - 2m(m-1)|u'(t)|^m$$

and (2.4) follows at once.

(ii) By integrating the identity (2.4) from 0 to t, we obtain

$$2(m-1)(p+m+1)\int_0^t |u'(r)|^m\,\mathrm{d}r$$
$$= m(m-1)\int_0^t |u'(r)|^{m-2}a''(r)\,\mathrm{d}r + 2(p+1)E(0)t.$$

Since $u'(t)^2 = \frac{1}{2}a''(t) - u(t)u''(t)$ and by (2.3), we have

$$(m-1)(p+1)\int_0^t |u'(r)|^{m-2}a''(r)\,\mathrm{d}r$$
$$= 2(p+m+1)\int_0^t u(r)^{p+1}\,\mathrm{d}r + 2(p+1)E(0)t. \tag{2.8}$$

Note that

$$\int_0^t |u'(r)|^{m-2}a''(r)\,\mathrm{d}r = \left[|u'(r)|^{m-2}a'(r)\right]_0^t - \int_0^t 2(m-2)\left(|u'|^{m-2}u''u\right)(r)\,\mathrm{d}r$$
$$= \left[|u'(r)|^{m-2}a'(r)\right]_0^t - \frac{2(m-2)}{m-1}\int_0^t u(r)^{p+1}\,\mathrm{d}r. \tag{2.9}$$

Hence, by (2.8) and (2.9), we arrive at (2.5).

(iii) Let $q = \frac{p-m+1}{2m}$. By differentiating $J(t)$ twice and multiplying $(m-1)|u'(t)|^{m-2}$ on both sides, we have

$$(m-1)|u'(t)|^{m-2}J''(t) = -q(m-1)|u'(t)|^{m-2}a(t)^{-q-2}$$
$$\times\left(a(t)a''(t) - (q+1)a'(t)^2\right). \tag{2.10}$$

Multiplying (2.4) on both sides with $a(t)$, we get the following identity:

$$|u'(t)|^{m-2}\left(a(t)a''(t) - (q+1)a'(t)^2\right)$$
$$= |u'(t)|^{m-2}\left(\frac{(p+m+1)}{2m} - (q+1)\right)a'(t)^2 - \frac{2}{\alpha(m-1)}E(0)a(t). \tag{2.11}$$

By (2.10) and (2.11), we obtain

$$(m-1)|u'(t)|^{m-2}J''(t) = \frac{p-m+1}{\alpha m}E(0)J(t)^{\frac{p+m+1}{p-m+1}}. \tag{2.12}$$

On the other hand, by the definitions of $J(t)$, we get

$$|u'(t)|^{m-2} = \left(\frac{m}{p-m+1}\right)^{m-2}|J'(t)|^{m-2}J(t)^{-\frac{(p+1)(m-2)}{p-m+1}}. \tag{2.13}$$

Combining (2.12) and (2.13), the assertion (2.6) holds.

(iv) (2.7) is easily obtained by integrating (2.6) from 0 to t. □

Lemma 2.3. *Let C_1 and C_2 be any real constants. Suppose that $u \in C^2(R^+)$ is a nonnegative function satisfying the following differential inequality*:

$$u'' + C_1u' + C_2u \leqslant 0, \quad \forall t \geqslant 0,$$

$$u(0) = 0, \quad u'(0) = 0,$$

then u must be a zero function.

Proof. We first consider the case (i) $C_1 = 0$. If $C_2 < 0$, let $C_2 = -k^2$ with $k \neq 0$. Then we have $u''(t) - k^2u(t) \leqslant 0$. Multiplying e^{kt} on both sides of the above inequality and then integrating from 0 to t, we have $u'(t) \leqslant ku(t)$, $\forall t \geqslant 0$. By integration again, $u(t) \leqslant 0$, $\forall t \geqslant 0$. Therefore, u is a zero function. If $C_2 \geqslant 0$, we have $u'' \leqslant 0$. Hence we get $u'(t) \leqslant 0$ and thus $u(t) = 0$ in R^+.

In the case (ii) $C_1 \neq 0$, let $v(t) = e^{\frac{tC_1}{2}}u(t)$, then we have

$$v''(t) - \left(\left(\frac{C_1}{2}\right)^2 - C_2\right)v(t) \leqslant 0.$$

By case (i), u is a zero function in R^+. □

Lemma 2.4. *If $f(t)$ and $g(t,r)$ are continuous with respect to their variables and the limit $\lim_{t\to T}\int_0^{f(t)} g(t,r)\mathrm{d}r$ exists, then*

$$\lim_{t\to T}\int_0^{f(t)} g(t,r)\,\mathrm{d}r = \int_0^{f(T)} g(T,r)\,\mathrm{d}r.$$

2.2. Asymptotic behavior of global solutions

In this section, we shall consider asymptotic properties of global solutions of the problem (2.1).

Theorem 2.5. *If $u \in C^2(0,T) \cap C^0[0,T)$ is a classical solution of the problem (2.1) with $u_0 = 0$ and $u_1 = 0$ then u must be null.*

Proof. Since $u_0 = u_1 = 0$, by Theorem 2.1, we have

$$(m-1)|u'(t)|^m = \frac{m}{p+1}u(t)^{p+1}. \tag{2.14}$$

And by (2.4), we get

$$|u'(t)|^{m-2}a''(t) = \frac{2(p+m+1)}{(m-1)(p+1)}a(t)^{\frac{p+1}{m}}. \tag{2.15}$$

Let

$$s_1 := \sup\{t \geqslant 0 : a(t) \leqslant 1\}. \tag{2.16}$$

Since $a(0) = a'(0) = 0$, the supremum exists. By using (2.14)–(2.16), we obtain $a''(t) \leqslant (p+m+1)a(t)$ in $[0,s_1]$. Lemma 2.3 implies that $a(t) = 0$ in $[0,s_1]$, Hence $u(t) = 0$ and $a'(t) = 0$ in $[0,s_1]$. Continuing this process we get the nullity of u on $[0,T)$. □

Remark. If $E(0) = 0$ and $a'(0) = 0$, then $u_0 = u_1 = 0$ and u must be null.

Theorem 2.6. *If $u \in C^2(0,\infty) \cap C^0[0,\infty)$ is a classical solution of the problem (2.1) with $E(0) = 0$ and $a'(0) < 0$, then*

$$\lim_{t\to\infty} t^{\frac{2m}{p-m+1}} a(t) = a(0)^{\frac{p+m+1}{p-m+1}} \left(-\frac{p-m+1}{2m} a'(0) \right)^{-\frac{2m}{p-m+1}}. \tag{2.17}$$

Proof. From (2.12) with $E(0) = 0$, we get $J''(t) = 0$. By integrating it once and twice from 0 to t, we have

$$J'(t) = -\frac{p-m+1}{2m} a(0)^{-\frac{p+m+1}{2m}} a'(0) \tag{2.18}$$

and

$$J(t) = J(0) - \frac{p-m+1}{2m} a(0)^{-\frac{p+m+1}{2m}} a'(0)t \tag{2.19}$$

or

$$a(t) = a(0)^{\frac{p+m+1}{p-m+1}} \left(a(0) - \frac{p-m+1}{2m} a'(0)t \right)^{-\frac{2m}{p-m+1}}. \tag{2.20}$$

Since $a'(0) < 0$, from (2.18) and (2.19), we see that $J'(t) \geqslant 0$ and $J(t) \geqslant 0$ $\forall t \geqslant 0$. And (2.17) follows at once from (2.20). □

2.3. Estimates for the life span of blow-up solutions

In this section, we shall find an upper bound for the life span of blow-up solutions under two different cases: (I) $E(0) \leqslant 0$ and (II) $E(0) > 0$. If the negativity of the solution happens, p should be a positive rational number. We say that p is odd (even) if $p = \frac{r}{s}$, $(r,s) = 1$, r is odd (even) and s is odd.

Case (I) $E(0) \leqslant 0$.

Theorem 2.7. *Let $u \in C^2(0,T) \cap C^0[0,T)$, $T \leqslant +\infty$, be a classical solution of the problem (2.1) with life span T. If $E(0) \leqslant 0$, then T is bounded. Furthermore, we have the following estimates.*

(*i*) *If $E(0) < 0$ and $a'(0) \geqslant 0$, then*

$$T \leqslant T_1^* = \frac{m(m-1)^{\frac{1}{m}}}{p-m+1} \int_0^{J(0)} \frac{\mathrm{d}r}{\sqrt[m]{\alpha + E(0)r^\beta}}. \tag{2.21}$$

(*ii*) *If $E(0) < 0$ and $a'(0) < 0$, then*

$$T \leqslant T_2^* = \frac{m(m-1)^{\frac{1}{m}}}{p-m+1} \left(\int_0^{k_2} \frac{\mathrm{d}r}{\sqrt[m]{\alpha + E(0)r^\beta}} + \int_{J(0)}^{k_2} \frac{\mathrm{d}r}{\sqrt[m]{\alpha + E(0)r^\beta}} \right), \tag{2.22}$$

where $\alpha = \frac{m}{p+1}$, $\beta = \frac{m(p+1)}{p-m+1}$ *and* $k_2 = \left(\frac{-\alpha}{E(0)}\right)^{\frac{1}{\beta}}$.

(*iii*) *If* $E(0) = 0$ *and* $a'(0) > 0$, *then*

$$T \leqslant T_3^* = \frac{2m}{p-m+1}\frac{a(0)}{a'(0)}. \tag{2.23}$$

Proof. We first claim that $a(0) > 0$ under the condition $E(0) < 0$. If not, then $a(0) = 0$, that is, $u_0 = 0$. Thus

$$E(0) = (m-1)|u_1|^m \geqslant 0.$$

This contradicts to $E(0) < 0$.

(i) By (2.4), we have the inequality

$$m(m-1)|u'(t)|^{m-2}a''(t) \geqslant -2(p+1)E(0). \tag{2.24}$$

Since $E(0) < 0$, from (2.24), we have $a''(t) \geqslant 0\ \forall t \geqslant 0$. By the assumption that $a'(0) \geqslant 0$, we then obtain $a'(t) \geqslant 0$, $\forall t \geqslant 0$ or $J'(t) \leqslant 0$, $\forall t \geqslant 0$. Therefore, by taking mth root in (2.7) and simplifying the sum of the first two terms, and the third term in the mth root to $\frac{m(p-m+1)}{(m-1)(p+1)}k_1^{\beta-m}$ and $\frac{(p-m+1)^m}{(m-1)m^m}E(0)$ respectively, we have

$$\frac{J'(t)}{\sqrt[m]{\frac{\alpha+E(0)J(t)^\beta}{m-1}}} = -\frac{p-m+1}{m}. \tag{2.25}$$

Now we claim that there exists T_1^* such that $J(T_1^*) = 0$. Indeed, by (2.6), we have $J''(t) \leqslant 0$. By integrating $J''(t)$ from 0 to t, we obtain

$$J(t) \leqslant J(0)\left(1 - \frac{p-m+1}{2m}\frac{a'(0)}{a(0)}t\right).$$

Thus there exists a finite number $T_1^* \leqslant \frac{2m}{p-m+1}\frac{a(0)}{a'(0)}$ such that $J(T_1^*) = 0$.

On the other hand, by integrating (2.25) from 0 to T_1^* and letting $r = J(t)$, we get

$$\int_0^{J(0)} \frac{\mathrm{d}r}{\sqrt[m]{\frac{1}{(m-1)}(\alpha + E(0)r^\beta)}} = \frac{p-m+1}{m}T_1^*.$$

Hence we get (2.21).

(ii) Since $E(t) < 0$, by (2.24), we see that $a(t)$ is a nonnegative convex function. By the assumption that $a'(0) < 0$ we can find a unique finite number t_0, which is the critical point of a, such that

$$\begin{cases} a'(t) < 0, & \forall t \in [0, t_0), \\ a'(t_0) = 0, & \\ a'(t) > 0, & \forall t \in (t_0, \infty). \end{cases} \tag{2.26}$$

Note that $a(t_0) > 0$. If not, then $a(t_0) = 0$, thus

$$E(0) = E(t_0) = (m-1)|u'(t_0)|^m \geqslant 0.$$

This is a contradiction to the assumption $E(0) < 0$.

Hence we see that $a(t) > 0 \quad \forall t \geqslant 0$, $u'(t_0) = 0$, $E(0) = -\alpha u(t_0)^{p+1}$ and $J(t_0)^\beta = \frac{-m}{(p+1)E(0)} = \frac{-\alpha}{E(0)}$. By the definition of $J'(t)$ and the positivity of $a(t)$, (2.26) implies that

$$\begin{cases} J'(t) > 0, & \forall t \in [0, t_0), \\ J'(t_0) = 0, & \\ J'(t) < 0, & \forall t \in (t_0, \infty) \end{cases}$$

and from (2.6), we get

$$J'(t) = \frac{p-m+1}{m} \sqrt[m]{\frac{1}{(m-1)}(\alpha + E(0)J(t)^\beta)} \quad \forall t \in [0, t_0] \tag{2.27}$$

and

$$J'(t) = -\frac{p-m+1}{m} \sqrt[m]{\frac{1}{(m-1)}(\alpha + E(0)J(t)^\beta)} \quad \forall t \in (t_0, \infty). \tag{2.28}$$

Since $J''(t) \leqslant 0$ and $J'(t) < 0 \ \forall t > t_0$, J is monotone decreasing in (t_0, ∞). Thus there exists a $T_2^* > t_0$ such that $J(T_2^*) = 0$.

By integrating (2.27) from 0 to t_0, we have

$$t_0 = \frac{m}{p-m+1} \int_{J(0)}^{k_2} \frac{\mathrm{d}r}{\sqrt[m]{\frac{1}{(m-1)}(\alpha + E(0)r^\beta)}}. \tag{2.29}$$

And integrating (2.28) from t_0 to T_2^* we get

$$T_2^* - t_0 = \frac{m}{p-m+1} \int_{J(T_2^*)}^{J(t_0)} \frac{\mathrm{d}r}{\sqrt[m]{\frac{1}{(m-1)}(\alpha + E(0)r^\beta)}}. \tag{2.30}$$

Hence by (2.29) and (2.30), we get (2.22).

(iii) If $E(0) = 0$, from (2.6) we get $J''(t) = 0 \ \forall t \geqslant 0$. By integration twice from 0 to t, we have

$$J(t) = J(0)\left(1 - \frac{p-m+1}{2m}\frac{a'(0)}{a(0)}t\right). \tag{2.31}$$

Since $a'(0) > 0$, there exists T_3^* such that $J(T_3^*) = 0$, (2.23) is then obtained. □

Case (II) $E(0) > 0$.

Theorem 2.8. *Let* $u \in C^2(0,T) \cap C^0[0,T)$, $T \leqslant +\infty$, *be a classical solution of the problem* (2.1). *If* $E(0) > 0$, *then* T *is bounded. Furthermore, an upper bound for* T *is estimated.*

(*i*) *If* $|a'(0)|^m > \frac{2^m}{m-1}E(0)a(0)^{\frac{m}{2}}$, *then we have*

$$T \leqslant T_4^* = \frac{m(m-1)^{\frac{1}{m}}}{p-m+1}\int_0^{J(0)} \frac{\mathrm{d}r}{\sqrt[m]{\alpha + E(0)r^{\beta}}}. \tag{2.32}$$

(*ii*) *If* $|a'(0)|^m = \frac{2^m}{m-1}E(0)a(0)^{\frac{m}{2}}$ *and*

(*a*) *if* $u_1 > 0$, *then we have*

$$T \leqslant T_5^* = \frac{m(m-1)^{\frac{1}{m}}}{p-m+1}\int_0^{\infty} \frac{\mathrm{d}r}{\sqrt[m]{\alpha + E(0)r^{\beta}}}; \tag{2.33}$$

(*b*) *if* $u_1 < 0$, *and* p *is odd, then we have*

$$T \leqslant T_6^* = \frac{m(m-1)^{\frac{1}{m}}}{p-m+1}\int_0^{\infty} \frac{\mathrm{d}r}{\sqrt[m]{\alpha + E(0)r^{\beta}}}; \tag{2.34}$$

(*c*) *if* $u_1 < 0$ *and* p *is even, then there exist a critical point* t_1 *and a null point* z_1 *of* u *such that*

$$T \leqslant T_7^* = z_1 + T_5^*, \tag{2.35}$$

where

$$t_1 = \int_0^{-u(t_1)} \frac{\mathrm{d}r}{\sqrt[m]{|u_1|^m - \frac{\alpha}{(m-1)}r^{p+1}}} \tag{2.36}$$

and

$$z_1 = 2t_1, \tag{2.37}$$

here $-u(t_1) = \left(\frac{(m-1)|u_1|^m}{\alpha}\right)^{\frac{1}{p+1}}$.

(*iii*) *If* $|a'(0)|^m < \frac{2^m}{m-1}E(0)a(0)^{\frac{m}{2}}$, p *is even, and*

(*a*) *if* $a'(0) > 0$ *then there exist a critical point* t_2 *and a null point* z_2 *such that*

$$T \leqslant T_8^* = z_2 + T_5^*, \tag{2.38}$$

where

$$t_2 = C(m,p)\int_{\tilde{J}(0)}^{k_3} \frac{\mathrm{d}r}{\sqrt[m]{1 - \left(\frac{p-m+1}{m}\right)^{m-1}k_1^m r^{p+1}}} \tag{2.39}$$

and

$$z_2=C(m,p)\left(\int_{\tilde{J}(0)}^{k_3}\frac{\mathrm{d}r}{\sqrt[m]{1-\left(\frac{p-m+1}{m}\right)^{m-1}k_1^m r^{p+1}}}+\int_0^{k_3}\frac{\mathrm{d}r}{\sqrt[m]{1-\left(\frac{p-m+1}{m}\right)^{m-1}k_1^m r^{p+1}}}\right), \tag{2.40}$$

here $k_3=\left(\frac{m}{(p-m+1)k_1}\right)^{\alpha}$, $C(m,p)=m^{-1}(p-m+1)^{\frac{m-1}{m}}(m-1)^{\frac{1}{m}}(p+1)^{\frac{1}{m}}$ *and* $\tilde{J}(0)=k_1^{-\alpha}a(0)^{\frac{1}{2}}$.

(*b*) *if* $a'(0)\leqslant 0$, *then there exists a null point* z_3 *such that*

$$T\leqslant T_9^*=z_3+T_5^*, \tag{2.41}$$

where

$$z_3=C(m,p)\int_0^{\tilde{J}(0)}\frac{\mathrm{d}r}{\sqrt[m]{1-\left(\frac{p-m+1}{m}\right)^{m-1}k_1^m r^{p+1}}} \tag{2.42}$$

and $C(m,p)$ *is given in* (*iii*)(*b*).

Remark. By (2.2) with $u_1 = 0$, we have

$$|a'(0)|^m-\frac{2^m}{m-1}E(0)a(0)^{\frac{m}{2}}=\frac{m2^m}{(m-1)(p+1)}u_0^{p+1}|u_0|^m.$$

Hence the sign of $|a'(0)|^m-\frac{2^m}{m-1}E(0)a(0)^{\frac{m}{2}}$ is determined by the sign of u_0.

Proof of Theorem 2.8. We set

$$\widetilde{E}(t):=(m-1)k_1^m|J'(t)|^m-\frac{p-m+1}{p+1}(k_1J(t))^{\beta}, \tag{2.43}$$

then by (2.7), we have

$$\widetilde{E}(t)=\widetilde{E}(0). \tag{2.44}$$

By some calculations, we have

$$\widetilde{E}(t)=(m-1)k_1^m\left(\frac{p-m+1}{2m}\right)^m a(t)^{-\frac{p+m+1}{2}}\left(|a'(t)|^m-\frac{2^m}{m-1}E(0)a(t)^{\frac{m}{2}}\right)$$

or

$$\widetilde{E}(t)=k_1^m\frac{(p-m+1)^m}{m^{m-1}(p+1)}|u(t)|^{-(p+1)}u(t)^{p+1}. \tag{2.45}$$

(*i*) From the assumption $|a'(0)|^m>\frac{2^m}{m-1}E(0)a(0)^{\frac{m}{2}}$, (2.44) implies that $\widetilde{E}(t)>0$. And by (2.45), we get

$$u(t)^{p+1} > 0 \quad \forall t \geqslant 0. \tag{2.46}$$

Hence by (2.45), we have

$$\widetilde{E}(t) = \widetilde{E}(0) = \frac{(p-m+1)^m}{m^{m-1}(p+1)} k_1^m = \frac{m(p-m+1)}{(p+1)^2 E(0)} k_1^\beta.$$

By (2.5) and (2.46), we get

$$|u'(t)|^{m-2} a'(t) \geqslant |u_1|^{m-2} a'(0) + \frac{2}{m-1} E(0)t. \tag{2.47}$$

Now we claim that $a'(0) \geqslant 0$. Suppose not, then $a'(0) < 0$. From (2.47), we see that $a'(t) \geqslant 0$ for large t. Let $s_0 > 0$ be the first number such that $a'(s_0) = 0$, by (2.5), we get

$$(m-1)(p+1)|u'(t)|^{m-2} a'(t) = 2(p+1)E(0)(t-s_0) + 2(\beta-1)(p-m+1)\int_{s_0}^{t} u(r)^{p+1}\,\mathrm{d}r \quad \forall t \geqslant s_0. \tag{2.48}$$

By (2.46), we have

$$\begin{cases} a'(t) < 0, & \text{for } t \in (0, s_0), \\ a'(s_0) = 0, & \\ a'(t) > 0, & \text{for } t \in (s_0, \infty), \end{cases}$$

and we also have $a(s_0) > 0$ by (2.46). Hence, $u'(s_0) = 0$. Therefore, by using (2.2) and (2.46) again, we obtain that

$$(p+1)E(0) = -mu(s_0)^{p+1} < 0.$$

This contradicts to the assumption $E(0) > 0$. Hence we get $a'(0) \geqslant 0$. By using the same arguments as in the proof of Theorem 2.7, we get (2.32).

(*ii*)-(*a*) In this case, we have $u_0 = 0$, hence $a(0) = 0$ and $a'(0) = 0$.

We shall claim that $a'(t) > 0$ $\forall t > 0$. Suppose not, there exists some $t^* > 0$ such that $a'(t^*) = 0$. Let $\bar{t} > 0$ be the first number such that $a'(\bar{t}) = 0$ and $u(t) > 0$ in $(0, \bar{t})$. By the positivity of u_1 and (2.5), we have

$$(m-1)(p+1)|u'(\bar{t})|^{m-2} a'(\bar{t}) = 2(p+1)E(0)\bar{t} + 2(\beta-1)(p-m+1)\int_0^{\bar{t}} u(r)^{p+1}\,\mathrm{d}r. \tag{2.49}$$

The left hand side of (2.49) is zero while the right hand side is positive. It leads to a contradiction.

Hence $J'(t) < 0$ $\forall t > 0$. By (2.7), for any $\tilde{t} > 0$, we have

$$J'(t) = -\sqrt[m]{|J'(\tilde{t})|^m - \frac{(p-m+1)^m}{m^m(m-1)} E(0)(J(\tilde{t})^\beta - J(t)^\beta)}, \quad \forall t \geqslant \tilde{t} \tag{2.50}$$

and

$$\lim_{\tilde{t}\to 0}\left(|J'(\tilde{t})|^m - \frac{(p-m+1)^m}{m^m(m-1)}E(0)J(\tilde{t})^{\frac{m(p+1)}{p-m+1}}\right) = \frac{m(p-m+1)^m}{(m-1)(p+1)m^m}. \qquad (2.51)$$

By (2.50) and (2.51), (2.33) is obtained.

(*ii*)-(*b*) From (2.45), since p is odd, we have (2.46). By similar arguments as in (*ii*)-(*a*), we can get (2.34).

(*ii*)-(*c*) We first claim that there is a critical point of u, that is, u is not strictly monotone decreasing in $[0,\infty)$. Suppose not, since $u_0 = 0$ and $u_1 < 0$ and from (2.2), we get

$$-u(t)\left(\frac{\alpha}{(m-1)|u_1|^m}\right)^{\frac{1}{p+1}} \leqslant 1 \quad \forall t \geqslant 0. \qquad (2.52)$$

From (2.2), we also have

$$u'(t) = -\sqrt[m]{|u_1|^m + \frac{\alpha}{(m-1)}u(t)^{p+1}} \quad \forall t \geqslant 0.$$

By integrating above equality from 0 to t and using (2.52), we have

$$\begin{aligned} t &\leqslant \left(\frac{m-1}{\alpha}\right)^{\frac{1}{p+1}}|u_1|^{-\frac{p-m+1}{p+1}}\int_0^1 \frac{\mathrm{d}s}{\sqrt[m]{1-s^{p+1}}} \\ &< \alpha^{-\frac{1}{p+1}}(m-1)^{\frac{1}{p+1}}(-u_1)^{-\frac{p-m+1}{p+1}}\int_0^1 \frac{\mathrm{d}s}{\sqrt[m]{1-s^m}}. \end{aligned}$$

Since $\int_0^1 \frac{\mathrm{d}s}{\sqrt[m]{1-s^m}} = \Gamma\left(1+\frac{1}{m}\right)\Gamma\left(1-\frac{1}{m}\right) < \infty$, it leads to a contradiction for large t. Therefore u must have a critical point $t_1 > 0$.

To calculate the critical point t_1, we start from (2.2) and have

$$u(t_1) = -\left(\frac{E(0)}{\alpha}\right)^{\frac{1}{p+1}} = -\left(\frac{(m-1)|u_1|^m}{\alpha}\right)^{\frac{1}{p+1}}.$$

By integrating (2.52) from 0 to t_1, we get the identity (2.36).

Next we show that u has a null point. Since p is even, u' is monotone increasing for $t \geqslant t_1$. Suppose that $u(t) < 0$ for all $t > 0$, By (2.2), we have

$$\int_{-u(t)}^{-u(t_1)} \frac{\mathrm{d}r}{\sqrt[m]{|u_1|^m - \frac{\alpha}{m-1}r^{p+1}}} = t - t_1. \qquad (2.53)$$

The above integral is less than a constant. Hence it is impossible for sufficiently large t.

From (2.53) we get

$$z_1 = t_1 + \int_0^{-u(t_1)} \frac{\mathrm{d}r}{\sqrt[m]{|u_1|^m - \frac{\alpha}{m-1} r^{p+1}}} = 2t_1.$$

Since $u(z_1) = 0$, $u'(z_1) > 0$ and $E(0) > 0$, then (2.35) holds by using the result in case (*ii*)-(*a*).

(*iii*) Let $\tilde{u}(t) = k_1 J(t)$, by (2.6) $\tilde{u}$ satisfies (1.1) with replacing p by $\beta - 1$. Let $\tilde{a}(t) = \tilde{u}(t)^2$ and $\tilde{J}(t) = \tilde{a}(t)^{-\frac{m}{2(p-m+1)}}$, we have $\tilde{a}(t) = k_1^2 J(t)^2$, and

$$\tilde{a}'(0) = -k_1^2 \left(\frac{m}{p-m+1} \right) a(0)^{-\frac{p+1}{p-m+1}} a'(0).$$

Since $|a'(0)|^m < \frac{2^m}{m-1} E(0) a(0)^{\frac{m}{2}}$ and p is even, by the above remark after (2.42), we have $u_0 < 0$. Thus

$$\widetilde{E}(0) = -k_1^m \frac{(p-m+1)^m}{(p+1)m^{m-1}} < 0.$$

In the case (*iii*)-(*a*), since $a'(0) > 0$, $\tilde{a}'(0) < 0$. By Theorem 2.7, there exists z_2 such that $\tilde{J}(z_2) = 0$ and the assertions (2.39) and (2.40) follow at once. Since $u(z_2) = 0$ and $u'(z_2) > 0$, by Theorem 2.8 in $[z_2, T)$, we have (2.38).

In the case (*iii*)-(*b*), we have $\tilde{a}'(0) \geqslant 0$. By similar arguments as in (*iii*)-(*a*), we obtain (2.41). □

2.4. Blow-up rates and blow-up constants

Theorem 2.9. *Let $u \in C^2(0,T) \cap C^0[0,T)$ be a classical solution of the problem (2.1). If one of the following assumptions holds*:

(*i*) $E(0) < 0$.
(*ii*) $E(0) = 0$, $a'(0) > 0$.
(*iii*) $E(0) > 0$.

Then we have the blow-up rates $\frac{2m}{p-m+1}$, $\frac{p+m+1}{p-m+1}$ and $\frac{m(p+1)}{p-m+1}$ of a, a', and $|u'|^{m-2}a''$ respectively and the blow-up constants K_1, K_2, K_3 of a, a', and $|u'|^{m-2}a''$ respectively. More precisely, we have, for $i \in \{1,2,3,4,5,6,7,8,9\}$,

$$\lim_{t \to T_i^*} (T_i^* - t)^{\frac{2m}{p-m+1}} a(t) = K_1, \tag{2.54}$$

$$\lim_{t \to T_i^*} (T_i^* - t)^{\frac{p+m+1}{p-m+1}} a'(t) = K_2, \tag{2.55}$$

and

$$\lim_{t\to T_i^*}(T_i^*-t)^{\frac{m(p+1)}{p-m+1}}|u'(t)|^{m-2}a''(t)=K_3, \tag{2.56}$$

where

$$K_1 := m^{\frac{2(m-1)}{p-m+1}}(m-1)^{\frac{2}{p-m+1}}(p+1)^{\frac{2}{p-m+1}}(p-m+1)^{-\frac{2m}{p-m+1}},$$

$$K_2 := 2m^{\frac{p+m-1}{p-m+1}}(p+1)^{\frac{2}{p-m+1}}(m-1)^{\frac{2}{p-m+1}}(p-m+1)^{-\frac{p+m+1}{p-m+1}},$$

and

$$K_3 := 2(p+m+1)m^{\frac{(m-1)(p+1)}{p-m+1}}(p+1)^{\frac{m}{p-m+1}}(m-1)^{\frac{m}{p-m+1}}(p-m+1)^{-\frac{m(p+1)}{p-m+1}}.$$

Proof. (*i*) If $a'(0)\geqslant 0$, from (2.25), by using (2.21) we can get

$$\int_0^{J(t)}\frac{1}{T_1^*-t}\frac{\mathrm{d}r}{\sqrt[m]{\frac{1}{m-1}(\alpha+E(0)r^\beta)}}=\frac{p-m+1}{m}\quad\forall t\geqslant 0. \tag{2.57}$$

Let $(T_1^*-t)s=r$ in (2.57) and let $t\to T_1^*$, we have

$$\lim_{t\to T_1^*}\frac{1}{\sqrt[m]{\frac{\alpha}{m-1}}}\frac{J(t)}{T_1^*-t}=\frac{p-m+1}{m}. \tag{2.58}$$

(2.58) is equivalent to (2.54) for $i=1$.

If $a'(0)<0$, by integrating (2.28) from t to T_2^*, we obtain

$$\int_0^{J(t)}\frac{\mathrm{d}r}{\sqrt[m]{\frac{1}{m-1}(\alpha+E(0)r^\beta)}}=\frac{p-m+1}{m}(T_2^*-t)\quad\forall t\geqslant t_0. \tag{2.59}$$

By the similar arguments as above, we get (2.54) for $i=2$.

Furthermore, from (2.25) and (2.28), we find

$$\lim_{t\to T_i^*}J'(t)=-\frac{(p-m+1)m^{-\frac{m-1}{m}}}{(p+1)^{\frac{1}{m}}(m-1)^{\frac{1}{m}}}. \tag{2.60}$$

Therefore for $i=1,2$, we have

$$\lim_{t\to T_i^*}a(t)^{-\frac{p+m+1}{2m}}a'(t)=2m^{\frac{1}{m}}(p+1)^{-\frac{1}{m}}(m-1)^{-\frac{1}{m}},$$

by using (2.54) and (2.55) is obtained.

From (2.54) and (2.55), we get

$$\lim_{t\to T_i^*}(T_1^*-t)^{\frac{m(p+1)}{p-m+1}}u'(t)^m=m^{\frac{mp}{p-m+1}}(p+1)^{\frac{2m}{p-m+1}}(m-1)^{\frac{2m}{p-m+1}}(p-m+1)^{-\frac{(p+1)m}{p-m+1}}. \tag{2.61}$$

Combining (2.4) and (2.61), we obtain (2.56).

(*ii*) If $E(0) = 0$ and $a'(0) > 0$, by using (2.31), we get

$$a(t) = a(0)^{\frac{p+m+1}{p-m+1}}\left(\frac{p-m+1}{2m}a'(0)\right)^{-\frac{2m}{p-m+1}}(T_3^* - t)^{-\frac{2m}{p-m+1}}. \tag{2.62}$$

Therefore, (2.54)–(2.56) with $i = 3$ follow at once from (2.62).

(*iii*). The estimates (2.54)–(2.56) for $i \in \{4,5,6,7,8,9\}$ can be achieved as in case (*i*), so we omit the proofs. □

Theorem 2.10. *Let $u \in C^2(0,T) \cap C^0[0,T)$ be a classical solution of the problem (2.1). If one of the following assumptions holds*:

(*i*) $E(0) > 0$, $|a'(0)|^m = \frac{2^m}{m-1}E(0)a(0)^{\frac{m}{2}}$, $u_1 < 0$ *and p is even.*
(*ii*) $E(0) > 0$, $|a'(0)|^m < \frac{2^m}{m-1}E(0)a(0)^{\frac{m}{2}}$, $a'(0) > 0$.
(*iii*) $E(0) > 0$, $|a'(0)|^m < \frac{2^m}{m-1}E(0)a(0)^{\frac{m}{2}}$, $a'(0) \leqslant 0$.

Then there are null points z_i, $i \in \{1,2,3\}$ such that

$$\lim_{t\to z_i} a(t)(z_i - t)^{-2} = \left(\frac{E(0)}{m-1}\right)^{\frac{2}{m}} \tag{2.63}$$

and

$$\lim_{t\to z_i} a'(t)(z_i - t)^{-1} = -2\left(\frac{E(0)}{m-1}\right)^{\frac{2}{m}}. \tag{2.64}$$

Proof. For $i = 1$, by (2.36) and (2.37) we use the same arguments as in the proof of Theorem 2.9.

For $i = 2,3$, as in the proof of Theorem 2.8 (*iii*), we set $\tilde{u}(t) = k_1 J(t)$, $\tilde{a}(t) = \tilde{u}(t)^2$. By Theorem 2.9, we get

$$\lim_{t\to z_i}\tilde{a}(t)(z_i - t)^{\frac{2(p-m+1)}{m}} = m^{\frac{-2(p-m+1)}{m}}(m-1)^{\frac{2(p-m+1)}{m^2}}(p+1)^{\frac{2(p-m+1)}{m^2}}(p-m+1)^{\frac{2(m-1)(p-m+1)}{m^2}}. \tag{2.65}$$

Thus (2.63) follows at once.

By Theorem 2.9 again, we also have

$$\lim_{t\to z_i}(a(t)(z_i - t)^{-2})^{\frac{-(p+1)}{m}}(z_i - t)^{-1}a'(t) = -2\left(\frac{E(0)}{m-1}\right)^{\frac{-2(p-m+1)}{m^2}}. \tag{2.66}$$

By (2.63) and (2.66), (2.64) is obtained. □

2.5. Properties of the life span of T_1^* and T_3^*

In this section we give some properties of the life span of T_1^* and T_3^*. Since T_1^* depends on four variables, u_0, u_1, p and m, we write $T_1^*(u_0,u_1,p,m)$ instead of T_1^*, and from (2.21), we have

$$T_1^*(u_0,u_1,p,m) = \left(\frac{m^{\frac{m-1}{m}}(m-1)^{\frac{1}{m}}(p+1)^{\frac{1}{m}}}{(p-m+1)}\left(\frac{-m}{(p+1)E(0)}\right)^{\frac{p-m+1}{m(p+1)}}\right) \cdot \left(\int_0^{\left(-\frac{p+1}{m}E(0)\right)^{\frac{p-m+1}{m(p+1)}}a(0)^{-\frac{p-m+1}{2m}}} \frac{\mathrm{d}r}{\sqrt[m]{1-r^{\frac{m(p+1)}{p-m+1}}}}\right). \tag{2.67}$$

We see that

(i) $T_1^*(u_0,u_1,p,m) \to \infty$ as $p-m+1 \to 0$ for each fixed u_0, u_1 and m.
(ii) $T_1^*(u_0,u_1,p,m) \to T_3^*(u_0,u_1,p,m)$ as $E(0) \to 0$.

In particular, when $u_1 \neq 0$, and either $u_0 < 0$ and p is odd, or $u_0 > 0$, we have $a'(0) = 0$ and $E(0) = \frac{-m}{p+1}u_0^{p+1} < 0$. By some calculation, we also have

$$T_1^*(u_0,0,p,m) = m^{\frac{-1}{m}}(m-1)^{\frac{1}{m}}(p+1)^{-\frac{m-1}{m}}u_0^{-\frac{p-m+1}{m}}\frac{\Gamma\left(\frac{m-1}{m}\right)\Gamma\left(\frac{p-m+1}{m(p+1)}\right)}{\Gamma\left(\frac{p}{p+1}\right)}. \tag{2.68}$$

Corollary 2.11. *For fixed m, the life span $T_1^*(u_0,0,p,m)$ has the following properties*:

(*i*) *If $u_0 \geqslant 1$, then the life span $T_1^*(u_0,0,p,m)$ is decreasing in $p \in (m-1,\infty)$.*
(*ii*) *There exists a constant u_0^* such that $T_1^*(u_0,0,p,m)$ decreases in p for $u_0^* \leqslant u_0 < 1$.*
(*iii*) *If $0 < u_0 < u_0^*$, then there exists a p^* such that $T_1^*(u_0,0,p,m)$ is decreasing in $(m-1,p^*)$ and $T_1^*(u_0,0,p,m)$ is increasing in (p^*,∞).*

Proof. Note that

$$\frac{\partial}{\partial p}T_1^*(u_0,0,p,m) = \left(\frac{m^{\frac{m-1}{m}}(m-1)^{\frac{1}{m}}(p+1)^{\frac{1}{m}}u_0^{-\frac{p-m+1}{m}}}{p-m+1}\right) \cdot \left(\int_0^1 \frac{f_1(u_0,p,s)}{\sqrt[m]{1-s^{\frac{m(p+1)}{p-m+1}}}}\,\mathrm{d}s\right), \tag{2.69}$$

where

$$f_1(u_0,p,s)=\frac{1}{m(p+1)}-\frac{1}{p-m+1}-\frac{\ln u_0}{m}\frac{s^{\frac{m(p+1)}{p-m+1}}\ln s}{1-s^{\frac{m(p+1)}{p-m+1}}}$$

for $s\in[0,1]$.

Thus all properties in (*i*)–(*iii*) follow at once from the properties of f_1. □

To find the properties of T_3^*, we start from (2.23), after some computations, we have

$$T_3^*(u_0,u_1,p,m)=\frac{mu_0}{(p-m+1)u_1}. \tag{2.70}$$

Since $E(0) = 0$, we have $|u_1|=\sqrt[m]{\frac{\alpha}{m-1}}u_0^{\frac{p+1}{m}}$. If $u_0 > 0$ and $u_1 > 0$, from (2.70), we obtain

$$T_3^*(u_0,p,m)=m^{\frac{m-1}{m}}(m-1)^{\frac{1}{m}}(p+1)^{\frac{1}{m}}(p-m+1)^{-1}u_0^{-\frac{p-m+1}{m}}.$$

Corollary 2.12. *Let m be fixed. If $u_0 > 0$, $u_1 > 0$, then the life span $T_3^*(u_0,p,m)$ has the following properties*:

(*i*) *For $u_0\geqslant 1$, the life span $T_3^*(u_0,p,m)$ is decreasing in p.*

(*ii*) *There exists a constant u_0^* such that for $u_0^*\leqslant u_0<1$, the life span $T_3^*(u_0,p,m)$ is decreasing in p.*

(*iii*) *For $0<u_0<u_0^*$, there exists a $p^* > m-1$ such that $T_3^*(u_0,p,m)$ is decreasing in $(m-1,p^*)$ and $T_3^*(u_0,p,m)$ is increasing in (p^*,∞).*

Proof. Note that

$$\frac{\partial}{\partial p}T_3^*(u_0,p,m)=m^{\frac{m-1}{m}}(m-1)^{\frac{1}{m}}(p+1)^{\frac{1}{m}}(p-m+1)u_0^{-\frac{p-m+1}{m}}f_2(u_0,p),$$

where

$$f_2(u_0,p)=\frac{1}{m(p+1)}-\frac{1}{p-m+1}-\frac{\ln u_0}{m}.$$

Hence from the properties of $f_2(u_0,p)$, we have our results. □

3. On the system of differential equations

In this section we shall consider the initial value problem for a system of second order ordinary differential equations of the form

$$\begin{cases} u'' = f(u,v), \\ v'' = g(u,v), \\ u(0) = u_0, \quad u'(0) = u_1, \\ v(0) = v_0, \quad v'(0) = v_1, \end{cases} \tag{3.1}$$

under the following assumption:

$$\frac{\partial f}{\partial v} = \frac{\partial g}{\partial u}, \tag{3.2}$$

where f and g are of class C^1.

Remark 1. For brevity, we only consider a system of two equations. In fact, a system of k ($\geqslant 2$) equations can be similarly studied under the Hamiltonian assumptions.

Hereafter we shall discuss the properties of blow-up solutions by using the energy method.

3.1. Fundamental lemmas

Lemma 3.1. *Let (u,v) be a classical solution of the problem (3.1) with the life span T. Define the energy function $E(t)$, $t \geqslant 0$ by*

$$E(t) = u'(t)^2 + v'(t)^2 - 2M(u,v). \tag{3.3}$$

Then

$$E(t) = E(0) \quad \textit{for all } t \geqslant 0, \tag{3.4}$$

where

$$M(u,v) = \int_0^u f(s,v)\,\mathrm{d}s + \int_0^v g(0,s)\,\mathrm{d}s. \tag{3.5}$$

Proof. From (3.5) we see that $\frac{\partial M}{\partial u} = f(u,v)$ and $\frac{\partial M}{\partial v} = g(u,v)$. By differentiating (3.3) and using (3.1), we get $E'(t) = 0$. Therefore $E(t) = E(0)$ for all $t \geqslant 0$. □

Hereafter, we assume that there exists some $q > 0$ such that

$$uf(u,v) + vg(u,v) \geqslant 2(2q+1)M(u,v) \quad \text{for } u,v \in R. \tag{3.6}$$

For example, when $f(u,v) = u^p v^{p+1}$ and $g(u,v) = v^p u^{p+1}$, (3.6) is satisfied when $q = \frac{p}{2}$, $p > 0$.

Let

$$a(t) = u(t)^2 + v(t)^2 \quad \text{and} \quad J(t) = a(t)^{-q}, \qquad t \geqslant 0.$$

Lemma 3.2. *Suppose that (u, v) is a classical solution of the problem (3.1), then we have the following identities.*

$$(i)\ \frac{1}{2}a''(t) - \left(u'(t)^2 + v'(t)^2\right) = u(t)f(u(t), v(t)) + v(t)g(u(t), v(t)). \tag{3.7}$$

$$(ii)\ J''(t) \leqslant 2q(2q+1)E(0)J(t)^{\frac{q+1}{q}}, \quad t \geqslant 0. \tag{3.8}$$

(*iii*) *If $J'(t) \leqslant 0$ for $t \geqslant 0$, then*

$$J'(t)^2 \geqslant J'(0)^2 - 4q^2E(0)J(0)^{\frac{2q+1}{q}} + 4q^2E(0)J(t)^{\frac{2q+1}{q}} \tag{3.9}$$

and if $J'(t) \geqslant 0$ for $t \geqslant 0$, then

$$J'(t)^2 \leqslant J'(0)^2 - 4q^2E(0)J(0)^{\frac{2q+1}{q}} + 4q^2E(0)J(t)^{\frac{2q+1}{q}}. \tag{3.10}$$

Proof. (i) By differentiating $a(t)$ twice and using (3.1), we obtain (3.7).

(ii) Since $J(t) = a(t)^{-q}$, then

$$J''(t) = -qa(t)^{-q-2}\left(a(t)a''(t) - (q+1)a'(t)^2\right). \tag{3.11}$$

By Cauchy–Schwartz inequality, (3.6) and (3.7), we have

$$a(t)a''(t) - (q+1)a'(t)^2 \geqslant -2(2q+1)a(t)E(0). \tag{3.12}$$

Combining (3.11) and (3.12), we obtain (3.8).

(iii) If $J'(t) \leqslant 0$, multiplying (3.8) with $J'(t)$ on both sides and then integrating from 0 to t, we have (3.9). Similarly, we also get (3.10) if $J'(t) \geqslant 0$. □

3.2. Estimates for the life span

Theorem 3.3. *Let (u, v) be a classical solution of the problem (3.1) with the life span T. If $E(0) \leqslant 0$ then T is bounded. Furthermore,*

(*i*) *if $E(0) < 0$ and $a'(0) \geqslant 0$, then*

$$T \leqslant T_1^* = \min\left(\frac{a(0)q}{a'(0)}, \frac{1}{2q}\int_0^{J(0)} \frac{\mathrm{d}r}{\sqrt{k_4 + E(0)r^{k_5}}}\right), \tag{3.13}$$

where

$$k_4 = a(0)^{-(2q+1)}\left(\frac{a'(0)^2}{4a(0)} - E(0)\right) \quad \textit{and} \quad k_5 = \frac{2q+1}{q}; \tag{3.14}$$

(*ii*) *if* $E(0) < 0$ *and* $a'(0) < 0$, *then*

$$T \leqslant T_2^* = \frac{1}{2q}\int_0^{J(\tau^*)} \frac{\mathrm{d}r}{\sqrt{-E(0)a(0)^{-(2q+1)} + E(0)r^{k_2}}} + \tau^*, \tag{3.15}$$

where $\tau^* = \frac{a'(0)}{2(2q+1)E(0)}$;

(*iii*) *if* $E(0) = 0$ *and* $a'(0) > 0$, *then*

$$T \leqslant T_3^* = \frac{a(0)q}{a'(0)}. \tag{3.16}$$

Proof. Note that $a(0) > 0$ under the condition $E(0) < 0$. By (3.7) and (3.3), we get

$$a''(t) \geqslant -2(2q+1)E(0). \tag{3.17}$$

As in the proof of Theorem 2.7, we get the estimates of the life spans in each case. □

Example 3.4. Let (u, v) be a classical solution of the particular system

$$\begin{cases} u'' = u^p v^{p+1}, \\ v'' = v^p u^{p+1}, \\ u(0) = u_0, \quad u'(0) = u_1, \\ v(0) = v_0, \quad v'(0) = v_1, \end{cases} \tag{3.18}$$

with the life span T, where $p > 0$. We have the following results:

Case (i) $E(0) < 0$.
(i-a) If $a'(0) \geqslant 0$, then

$$T \leqslant T_1^* = \min\left\{\frac{2}{p}\frac{a(0)}{a'(0)}, \frac{1}{p}\int_0^{J(0)} \frac{\mathrm{d}r}{\sqrt{k_4 + E(0)r^{k_5}}}\right\}.$$

(i-b) If $a'(0) < 0$, then

$$T \leqslant T_2^* = \frac{1}{p}\int_0^{k_6} \frac{\mathrm{d}r}{\sqrt{-E(0)a(0)^{-(p+1)} + E(0)r^{k_5}}} + \frac{1}{p}\int_{J(0)}^{k_6} \frac{\mathrm{d}r}{\sqrt{-E(0)a(0)^{-(p+1)} + E(0)r^{k_5}}}.$$

Case (ii) $E(0) = 0$.

If $a'(0) > 0$, then

$$T \leqslant T_3^* = \frac{2}{p}\frac{a(0)}{a'(0)}.$$

Case (iii) $E(0) > 0$ and u, v are of the same sign.

(iii-a) If $u_0, v_0 > 0$, then we have

$$T \leqslant T_4^* = \frac{1}{p}\int_0^{J(0)} \frac{\mathrm{d}r}{\sqrt{k_1 + E(0)r^{k_5}}}.$$

(iii-b) If $u_0 = v_0 = 0$, $u_1 > 0$ and $v_1 > 0$, then we have

$$T \leqslant T_5^* = \frac{1}{p}\int_0^{\infty} \frac{\mathrm{d}r}{\sqrt{1 + p^2E(0)r^{k_5}}}.$$

(iii-c) If $u_0 = v_0 = 0$, $u_1 < 0$, $v_1 < 0$ and p is odd, then we have

$$T \leqslant T_6^* = \frac{1}{p}\int_0^{\infty} \frac{\mathrm{d}r}{\sqrt{1 + p^2E(0)r^{k_5}}}.$$

where $E(0) = (u_1^2 + v_1^2) - \frac{2}{p+1}u_0^{p+1}v_0^{p+1}$, $a(0) = u_0^2 + v_0^2$, $a'(0) = 2(u_0u_1 + v_0v_1)$, $J(0) = a(0)^{-\frac{p}{2}}$, $k_4 = a(0)^{-(p+1)}\left(\frac{a'(0)^2}{4a(0)} - E(0)\right)$, $k_5 = \frac{2(p+1)}{p}$ and $k_6 = \left(\frac{-2^{-p}}{(p+1)E(0)}\right)^{\frac{1}{k_5}}$.

Proof. In the case $E(0) \leqslant 0$, we apply Theorem 3.3 , and in the case $E(0) > 0$, we follow the similar arguments as that of Theorem 2.8 to get the assertion. □

References

[1] J.R. Esteban, J.L. Vazquez, On the equation of turbulent in one-dimensional porous media, Nonlinear Analysis 10 (1986) 1303–1325.

[2] M.A. Herrero, J.L. Vazquez, On the propagation properties of a nonlinear degenerate parabolic equation, Communications in Partial Differential Equations 7 (1982) 1381–1402.

[3] M.R. Li, On the differential equation $u'' - u^p = 0$, preprint, National Chengchi University, Taipei, Taiwan, 1999.

[4] D. O'Regan, Some general existence principles and results for $(\varphi(y'))' = q(t)f(t, y, y')$. $(0 < t < 1)$, SIAM Journal on Mathematical Analysis 24 (1993) 648–668.

[5] J.Y. Wang, W.J. Gao, A singular boundary value problem for the one-dimensional p-Laplacian, Journal of Mathematical Analysis and Applications 201 (1996) 851–866.

[6] L.E. Bobisub, D. O'Regan, Existence of positive solutions for singular ordinary differential equations with nonlinear boundary conditions, Proceedings of the American Mathematical Society 124 (1996) 2081–2087.

[7] Z.D. Yang, Existence of positive solutions for a class of singular two point boundary value problems of second order nonlinear equation, Applied Mathematics and Mechanics 17 (1996) 465–476.

[8] J.H. Lin, The regularity of solutions for non-linear differential equation $u'' - u^p = 0$, Master thesis, National Chengchi University, Taipei, Taiwan, 1999.