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UNSTABLE MODULES OVER THE STEENROD ALGEBRA

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1. Introduction

Let A denote the mod 2 Steenrod algebra. In [1] Adams and Margolis proved the following

Theorem 1.1. *Let $0 \to M_1 \to M_2 \to M_3 \to 0$ be an exact sequence of A-modules. If any two of these modules are free A-modules then so is the third.*

The purpose of this note is to prove the following "unstable version" of Theorem 1.1.

Theorem 1.2. *Let $0 \to M_1 \to M_2 \to M_3 \to 0$ be an exact sequence of unstable A-modules. If any two of these modules are free unstable A-modules then so is the third.*

Theorem 1.2 settles a homological problem regarding unstable modules over A which arises when we consider Massey and Peterson's unstable Adams spectral sequence [3, 4].

Corollary 1.3. *Let M be an unstable A-module. Then either M is a free unstable A-module or M has infinite unstable homological dimension, i.e. there are no resolutions of M by free unstable A-modules of finite length.*

Corollary 1.3 is an immediate consequence of Theorem 1.2.

Theorem 1.1 was proved in [1] via an elegant criterion for an A-module to be free over A. From some homological considerations one can also prove this theorem by using

Theorem 1.4. *Let M be an A-module and $y \in M$ be any homogeneous eleme· ~uch that the A-submodule of M generated by y is free over A. Then $y \notin I(A)M$ where $I(A)$ is the augmentation ideal of A.*

Theorem 1.4 is an easy theorem since the algebra generated by any finite subset of elements of A is always finite. It is equally true that the unstable version of Theorem 1.4 implies Theorem 1.2.

Theorem 1.5. *Let M be an unstable A-module and $y \in M$ be any homogeneous element such that the A-submodule of M generated by y is a free unstable A-module on the generator y (of dimension* $\dim y$*). Then $y \notin I(A)M$.*

We shall prove Theorem 1.2 by establishing Theorem 1.5.

2. A classical result and its consequences. The proof of Theorem 1.5

Let $B(n)$ be the Z_2-submodule of the mod 2 Steenrod algebra A generated by all the Serre–Cartan basis elements Sq^I with excess $e(I) > n$. It is proved in [5] that $B(n)$ is actually a left A-submodule of A. The module $F(n)$ defined by $(F(n))_i = (A/B(n))_{n-i}$ is called a free unstable A-module on one generator of dimension n. Any direct sum of these are called free unstable A-modules. (See [5] for details.)

Let P^n be the n-fold product of the infinite real projective spaces RP^∞. The (mod 2) cohomology $H^*(P^n)$ of P^n is well known; it is a polynomial algebra over Z_2 generated by $x_1, x_2, \ldots, x_n$ of dimension 1. Let $v_n = x_1 \cdot x_2 \cdots x_n \in H^n(P^n)$. Define an A-map $\phi : A \to H^*(P^n)$ by $\phi(a) = av_n$. The following is well known. (See [5] for example.)

Theorem 2.1. $\mathrm{Im}\phi \cong F(n)$ i.e. $\mathrm{Im}\phi$ *is a free unstable A-module on the generator v_n of dimension n.*

Next we recall from [2] that $A^* = Z_2[\xi_1, \xi_2, \ldots]$ where A^* is the dual Hopf algebra of A. For any finite sequence $I = (i_1, i_2, \ldots, i_n)$ of non-negative integers of length $l(I) = n$ define $\xi(I) = \xi_{i_1} \cdot \xi_{i_2} \cdots \xi_{i_n}$ and $x(I) = x_1^{2^{i_1}} \cdot x_2^{2^{i_2}} \cdots x_n^{2^{i_n}} \in H^*(P^n)$. The following formula expresses av_n for any $a \in A$.

Theorem 2.2. *In $H^*(P^n)$ $av_n = \sum_{l(I)=n} \langle \xi(I), a \rangle x(I)$ for all $a \in A$.*

For the proof of this theorem see [2] or [5]. We shall use a consequence of Theorem 2.1 and Theorem 2.2 to prove Theorem 1.5.

Let $a \in A$, $a \neq 0$. Write $a = \sum_\alpha \mathrm{Sq}^{I_\alpha}$ in terms of the Serre-Cartan basis elements. Assign a number $e(a)$ to a by $e(a) = \min_\alpha e(I_\alpha)$ where $e(I)$ is the excess of an admissible sequence I. If we write $a = \sum_\beta \mathrm{Sq} J_\beta$ in terms of the Milnor basis elements then assign to a another number $w(a)$ by $w(a) = \min_\beta w(J_\beta)$ where if $J = (r_1, r_2, \ldots, r_k, 0, 0, \ldots)$ then $w(J) = r_1 + r_2 + \cdots + r_k$.

Theorem 2.3. *For any $a \in A$, $a \neq 0$, $e(a) = w(a)$.*

Theorem 2.3 follows from Theorem 2.1 and Theorem 2.2. We leave the proof to the reader.

Corollary 2.4. *Let $\{a_1, a_2, \ldots, a_m\}$ be any finite subset of non-zero elements of A and let $n \geq 1$ be any integer. Then there exists $x \in A$ such that $e(x) = n$ and $e(xa_i) = e(x) + e(a_i)$ all i.*

Proof. Let $a_i = \sum_{v_i} \operatorname{Sq} J_{v_i}$, $i = 1, 2, \ldots, m$ where $\operatorname{Sq} J_{v_i}$ are Milnor basis elements. The set of all these basis elements is finite. Hence we can choose an integer t big enough so that if we let $x = \operatorname{Sq}(0, 0, \ldots, 0, \underset{t}{1}, \underset{t+1}{1}, \ldots, \underset{t+n-1}{1}, 0, 0, \ldots)$ then for all i and all v_i, $x \operatorname{Sq} J_{v_i} = \operatorname{Sq}(r_1^{(v_i)}, r_2^{(v_i)}, \ldots, r_{k_{v_i}}^{(v_i)}, 0, 0, \ldots, \underset{t}{1}, \underset{t+1}{1}, \ldots, \underset{t+n-1}{1}, 0, 0, \ldots)$ where $\operatorname{Sq} J_{v_i} = \operatorname{Sq}(r_1^{(v_i)}, r_2^{(v_i)}, \ldots, r_{k_{v_i}}^{(v_i)}, 0, 0, \ldots)$. It is clear that $w(x) = n$ and $w(xa_i) = w(x) + w(a_i)$ all i. Thus by Theorem 2.3 $e(x) = w(x) = n$ and $e(x_i) = w(xa_i) = w(x) + w(a_i) = e(x) + e(a_i)$ all i. Q.E.D.

We use Corollary 2.4 to prove Theorem 1.5.

Proof of Theorem 1.5. Suppose $y \in I(A)M$, say $y = \sum_{i=1}^m a_i y_i$ where y_i are homogeneous elements of M and a_i are homogeneous elements of $I(A)$. We prove this would lead to a contradiction. For each i, $\dim a_i > 0$; hence $\dim y > \dim y_i$ all i. We can find $x \in A$ such that $e(x) = \dim y$ and $e(xa_i) = e(x) + e(a_i)$ by Corollary 2.4. Thus $e(xa_i) = e(x) + e(a_i) > \dim y > \dim y_i$ all i. From the definition of $e(xa_i)$ we see that $xy = \sum_{i=1}^m xa_i y_i = 0$. On the other hand since $e(x) = \dim y$ and $A \cdot y \subset M$ is a free unstable A-module on the generator y it follows that $xy \neq 0$. Thus we have got a contradiction. This proves that $y \notin I(A)M$. Q.E.D.

References

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