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COHOMOLOGY OF SUB-HOPF-ALGEBRAS OF THE STEENROD ALGEBRA II

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1. Introduction

In this paper we continue our previous one [4] to study another interesting property of cohomology of sub-Hopf-algebras of the mod 2 Steenrod algebra A.

We recall some notations in [4]. For a sequence $(n_1, n_2, \ldots)$ of non-negative integers (possibly equal to ∞) denote by $A(n_1, n_2, \ldots)$ the $\mathbf{Z}_2$-submodule of A generated by all the Milnor basis elements $\mathrm{Sq}(r_1, r_2, \ldots)$ with $r_i < 2^{n_i}$. It is proved in [2], [3] that $A(n_1, n_2, \ldots)$ is a sub-Hopf-algebra of A if and only if for $i > j \geq 1$, $n_i \geq \min(n_j, n_{i-j} - j)$. In particular $A_d = A(1,2,3,4,\ldots)$ is a sub-Hopf-algebra of A. It can be shown that a sub-Hopf-algebra $C = A(0,\ldots,0, n_t, n_{t+1}, \ldots)$ $(t \geq 1)$ of A is an exterior subalgebra if and only if $n_i \leq t$ all i (see [1]); so any exterior sub-Hopf-algebra of A is also a subalgebra of A_d.

Our main result is the following

Theorem 1.1. *Let $B = A(n_1, n_2, \ldots)$ be any sub-Hopf-algebra of the Steenrod algebra A such that each n_i is finite. Let $\theta \in H^*(B)$. Then θ is non-nilpotent if and only if for some exterior sub-Hopf-algebra C of B the image of θ in $H^*(C)$ is non-nilpotent.*

The result in Theorem 1.1 is stronger than a result in [4] where we proved that if $\theta \in H^*(B)$ is non-nilpotent (B is as in Theorem 1.1) then its image in $H^*(B \cap A_d)$ is also non-nilpotent (for any exterior sub-Hopf-algebra of B is a subalgebra of $B \cap A_d$). It is a conjecture of Adams that Theorem 1.1 is true for the Steenrod algebra itself (private communication). The result in Theorem 1.1 gives evidence that the "Adams conjecture" is probably true. I would like to express my sincere thanks to Professor J. F. Adams for the correspondence in which he suggested to me his conjecture. With Theorem 1.1 Adams conjecture is equivalent to the conjecture given in [4].

In the next section we recall some technical results from [4] and infer from them some further technical results. In Section 3 we use the results obtained in Section 2 and a key theorem in [4] concerning the nilpotency of certain cohomology classes of sub-Hopf-algebras of the Steenrod algebra to complete the proof of Theorem 1.1.

2. Some technical lemmas

Let Γ be a connected locally finite cocommutative Hopf algebra over $\mathbf{Z}_2$. Let Λ be a normal sub-Hopf-algebra of Γ such that $\Gamma/\Gamma\bar{\Lambda} = E[\bar{x}]$ where $\bar{\Lambda}$ is the augmentation ideal of Λ and $\bar{x}$ is the image in $\Gamma/\Gamma\bar{\Lambda}$ of a homogeneous indecomposable element x of Γ. Let $\alpha \in H^{1,*}(\Gamma)$ be the class corresponding to x. Let $i^*: H^*(\Gamma) \to H^*(\Lambda)$ be the induced homomorphism in cohomology of the inclusion $i: \Lambda \to \Gamma$. In [4] we prove the following lemma.

Lemma 2.1. (a) $\operatorname{Ker} i^* =$ *ideal of* $H^*(\Gamma)$ *generated by* α.

(b) *Let* $\theta \in H^*(\Gamma)$ *be an element such that both* $\theta\alpha \in H^*(\Gamma)$ *and* $i^*(\theta)$ *are nilpotent. Then* θ *is also nilpotent.*

We shall infer from Lemma 2.1 a technical result (Lemma 2.3) that will be needed in the proof of Theorem 1.1 in Section 3. If a pair (Γ, Λ) of Hopf algebras is as in the situation in Lemma 2.1 then we say that Γ is obtained from Λ by adding one generator x.

Lemma 2.2. *Suppose* Γ *is obtained from* Λ_i *by adding one generator* $x_i \in \Gamma$, $i = 1, 2$. *Let* $\alpha_i \in H^{1,*}(\Gamma)$ *be the class corresponding to* x_i. *Assume that* $\alpha_1\alpha_2$ *is nilpotent. Let* $\theta \in H^*(\Gamma)$ *be a non-nilpotent element. Then one of the images of* θ *in* $H^*(\Lambda_1)$ *and* $H^*(\Lambda_2)$ *is non-nilpotent.*

Proof. Let θ_i be the image of θ in $H^*(\Lambda_i)$, $i = 1, 2$. Suppose both θ_1 and θ_2 are nilpotent. So there is an integer k such that $\theta_1^k = 0$. By Lemma 2.1 (a) there is some $\beta_1 \in H^*(\Gamma)$ such that $\theta^k = \beta_1\alpha_1$. By assumption $\alpha_1\alpha_2$ is nilpotent. So $\theta^k\alpha_2$ is nilpotent. Since θ_2^k is also nilpotent, by Lemma 2.1 (b), we see θ^k is nilpotent which is contradictory to our assumption that θ is non-nilpotent. Thus one of θ_1, θ_2 is non-nilpotent. This proves the lemma.

Now we suppose given two sub-Hopf-algebras Λ, Λ' of Γ that satisfy the following conditions.

(1) Either (a) there is a finite sequence $\Gamma = \Gamma_0 \supset \Gamma_1 \supset \cdots \Gamma_n = \Lambda$ or (b) there is an infinite sequence $\Gamma = \Gamma_0 \supset \Gamma_1 \supset \cdots \supset \Lambda$ such that each Γ_i is obtained from Γ_{i+1} by adding one generator $x_i \in \Gamma_i$ and such that $\lim_i H^*(\Gamma_i) = H^*(\Lambda)$.

(2) Γ is obtained from Λ' by adding one generator $y \in \Gamma$ with $y \in \Gamma_i$ all i. (So each Γ_i is obtained from $\Gamma_i \cap \Lambda'$ by adding the generator y).

(3) Let α_i, β_i be the classes in $H^*(\Gamma_i)$ corresponding to $x_i, y \in \Gamma_i$ respectively, $i \geq 0$. We assume that each $\alpha_i \cdot \beta_i \in H^*(\Gamma_i)$ is nilpotent.

Lemma 2.3. *Under these assumptions, suppose $\theta \in H^*(\Gamma)$ is a non-nilpotent element such that its image in $H^*(\Lambda')$ is nilpotent. Then the image of θ in $H^*(\Lambda)$ is non-nilpotent.*

Proof. Let $\theta(i)$, $\theta(\Lambda)$, $\theta(\Lambda')$, $\theta(i, \Lambda')$ be the image of θ in $H^*(\Gamma_i)$, $H^*(\Lambda)$, $H^*(\Lambda')$, $H^*(\Gamma_i \cap \Lambda')$ respectively. We first show that each $\theta(i)$ is non-nilpotent. We prove this by induction on i. The result is true for $i = 0$ since $\theta = \theta(0)$. Suppose $i > 0$ and suppose the result is true for $i - 1$. By assumption (3) $\alpha_{i-1}\beta_{i-1} \in H^*(\Gamma_{i-1})$ is nilpotent. Also by assumption $\theta(\Lambda')$ is nilpotent and hence so is $\theta(i-1, \Lambda')$. By our inductive hypothesis $\theta(i-1)$ is non-nilpotent. Therefore, by applying Lemma 2.2 to the triple $(\Gamma_{i-1}, \Gamma_i, \Gamma_{i-1} \cap \Lambda')$, $\theta(i)$ is also non-nilpotent. This completes our induction.

We want to show that $\theta(\Lambda)$ is non-nilpotent. Suppose not, say $\theta(\Lambda)^t = 0$ for some integer $t \geq 1$. By assumption (1) there is some integer q such that $(\theta(q))^t = 0$ in $H^*(\Gamma_q)$. This is absurd because by the work above $\theta(q)$ is non-nilpotent. This completes the proof of the lemma.

3. The Proof of Theorem 1.1

We will need the following theorem concerning the nilpotency of certain cohomology classes of sub-Hopf-algebras of the Steenrod algebra to complete the proof of Theorem 1.1. This theorem is the key result in [4] to prove the main theorems there.

Theorem 3.1. *Let $B = A(n_1, \ldots, n_m, n_{m+1}, \ldots)$ be a sub-Hopf-algebra of the Steenrod algebra A such that $n_i < \infty$ for $i \leq m - 1$. Suppose $[\xi_l^{2^\lambda}]$ and $[\xi_m^{2^\mu}]$ are cocycles in the cobar construction $\bar{c}(B^*)$ such that $l \leq m$ and $l \leq \mu$. Then the class $[\xi_l^{2^\lambda}][\xi_m^{2^\mu}]$ is a nilpotent element of $H^*(B)$.*

Now let B be as in Theorem 1.1. To prove Theorem 1.1 it suffices to prove that if $\theta \in H^*(B)$ is non-nilpotent then there is a certain exterior sub-Hopf-algebra C of B such that the image of θ in $H^*(C)$ is non-nilpotent. (The other direction is obvious.) We shall hereafter refer *the main conclusion of Theorem* 1.1 *for the pair* (B, θ) to the above statement. The following is obvious and will be used a couple of times later.

Lemma 3.2. *Let (B, θ) be as in Theorem 1.1 and let B' be a sub-Hopf-algebra of B such that the image θ' of θ in $H(B')$ is non-nilpotent. Suppose the main conclusion of Theorem 1.1 is true for (B', θ') then it also holds for (B, θ).*

Let us first prove a particular case of Theorem 1.1.

Proposition 3.3. *Let $B = A(n_1, n_2, \ldots)$ be a sub-Hopf-algebra of the Steenrod algebra A such that the set $\{n_1, n_2, \ldots\}$ is bounded. Let $\theta \in H^*(B)$ be a non-nilpotent element. Then the main conclusion of Theorem 1.1 is true for (B, θ).*

Proof. Let l be the first integer such that $n_l > 0$. Suppose $n_l > 1$. Construct a finite sequence $B = B_0 \supset B_1 \supset \cdots B_{n_l-1}$ of sub-Hopf-algebras by $B_i = A(0, \ldots, 0, n_l - i, n_{l+1}, \ldots)$, $0 \leq i \leq n_l - l$. B_i is obtained from B_{i+1} by adding the generator $P_l^{n_l-i-1} \in B_i$, $0 \leq i \leq n_l - l - 1$. By Theorem 3.1 the class $[\xi_l^{2n_l-i-1}]$ of $H^*(B_i)$ corresponding to $P_l^{n_l-i-1}$ is nilpotent because $n_l - i - 1 \geq l$. So by applying Lemma 2.1 (b) a finite of times we see the image of $\theta \in H^*(B = B_0)$ in $H^*(B_{n_l-l})$ is still non-nilpotent. Therefore, to prove the proposition, we may assume $n_l \leq l$ (by Lemma 3.2).

Let $k = \max\{n_j\}$ and let $\nu(B) = \max\{0, k - l\}$. We prove the proposition by induction on $\nu(B)$ (under the assumption $n_l \leq l$). If $\nu(B) = 0$ then $k \leq l$. In this case B is itself an exterior sub-Hopf-algebra of A. So the result is true for $\nu(B) = 0$. Suppose $\nu(B) > 0$ and suppose the result is true for all pairs (B', θ') with $\nu(B')$ defined and $\nu(B') < \nu(B)$. Let B_i be defined as in the preceding paragraph with $0 \leq i \leq n_l$. Let $\theta(i)$ be the image of θ in $H^*(B_i)$. If $\theta(n_l)$ is non-nilpotent then, since $\nu(B_{n_l}) = k - l - 1 = \nu(B) - 1 < \nu(B)$, by our induction hypothesis the result is true for $(B_{n_l}, \theta(n_l))$. So by Lemma 3.2 the result is also true for (B, θ). Therefore we may assume that for some r, $0 \leq r < n_l$, $\theta(r)$ is non-nilpotent but $\theta(r+1)$ is nilpotent. (Note that $\theta(0) = \theta$ is assumed to be non-nilpotent.) Let $\{n_{m_1}, n_{m_2}, \ldots\}$ be the subset of $\{n_l, n_{l+1}, \ldots\}$ with $m_1 < m_2 < \cdots$ such that $n_{m_j} = k$. Note that $m_1 > l$. Construct a sequence $B_r = \Gamma_0 \supset \Gamma_1 \supset \cdots \supset \Gamma_\infty$ of sub-Hopf-algebras as follows.

$$\Gamma_0 = B_r = A(0, \ldots 0, n_l - r, \ldots, \underset{m_1}{k}, \ldots, \underset{m_2}{k}, \ldots),$$

$$\Gamma_1 = A(0, \ldots, 0, n_l - r, \ldots, \underset{m_1}{k-1}, \ldots, \underset{m_2}{k}, \ldots),$$

$$\vdots$$

$$\Gamma_s = A(0, \ldots, 0, n_l - r, \ldots, \underset{m_1}{k-1}, \ldots, \underset{m_2}{k-1}, \ldots, \underset{m_s}{k-1}, \ldots, \underset{m_{s+1}}{k}, \ldots, \underset{m_{s+2}}{k}, \ldots),$$

$$\vdots$$

$$\Gamma_\infty = A(0, \ldots, 0, n_l - r, \ldots, \underset{m_1}{k-1}, \ldots, \underset{m_2}{k-1}, \ldots).$$

This sequence of sub-Hopf-algebras is finite or infinite depending on the set $\{m_j\}$ is finite or infinite; in either case we have $\nu(\Gamma_\infty) = k - l - 1 = \nu(B_r = \Gamma_0) - 1 < \nu(B_r) = \nu(B)$. Each Γ_i is obtained from Γ_{i+1} by adding the generator $P_{m_{i+1}}^{k-1} \in \Gamma_i$ ($P_{m_{i+1}}^{k-1}$ is an indecomposable element of Γ_i because $n_l - r < n_l \leq l < k - 1$). Also $P_l^{n_l-r-1} \in \Gamma_i$ all i. It is clear that $\lim_i H^*(\Gamma_i) = H^*(\Gamma_\infty)$. Since $\nu(B) = k - l > 0$, by Theorem 3.1 the class $[\xi_l^{2n_l-r-1}]\,[\xi_{m_{i+1}}^{2k-1}]$ is a nilpotent element of $H^*(\Gamma_i)$ all i. By the assumption above, $\theta(r) \in H^*(B_r)$ is non-nilpotent but $\theta(r+1) \in H^*(B_{r+1})$ is nilpotent. Therefore, by applying Lemma 2.3 to the sequence $\{\Gamma_i\}$ and the sub-Hopf-algebras $A=\Gamma_\infty$, $A'=B(r+1)$, we see the image θ_∞ of $\theta(r)$ in $H^*(\Gamma_\infty)$ (which is precisely the image of θ in $H^*(\Gamma_\infty)$) is non-nilpotent. Since $\nu(\Gamma_\infty)<\nu(B)$, by our induction hypothesis, the result is true for $(\Gamma_\infty, \theta_\infty)$. Again by Lemma 3.2 the result is also true for (B,θ). This completes the proof of Proposition 3.3.

Again let (B,θ) be as in Theorem 1.1. We impose on (B,θ) a condition described as follows. Suppose $B=A(0,\ldots,0,n_l,n_{l+1},\ldots)$ with $n_l>0$. Let $B'=A(0,\ldots,0,n_l-1,n_{l+1},\ldots)$. B is obtained from B' by adding the generator $P_l^{n_l-1}$. The condition to be imposed on (B,θ) is

(*) The image θ' of θ in $H^*(B')$ is nilpotent

Proposition 3.4. *Suppose (B,θ) satisfies (*). For each $k\geq 1$ let $A_{(k)}=A(k,k,k,\ldots)$ and $B_{(k)}=B\cap A_{(k)}$. Then there is some $p\geq 1$ such that the image $\theta(p)$ of θ in $H^*(B_{(p)})$ is non-nilpotent.*

To see the implication of Proposition 3.4 note that if $B_{(p)}=A(0,\ldots,0,n_l',n_{l+1}',\ldots)$ then $\{n_j'\}$ is a bounded set. So by Lemma 3.2 and Proposition 3.3 the main conclusion of Theorem 1.1 is true for (B,θ) (provided the condition (*) is satisfied).

Proof of Proposition 3.4. We may assume $B\neq B_{(k)}$ all k. Choose any integer p such that $p\geq n_l$ and $p\geq l$. We show that $\theta(p)$ is non-nilpotent. Let $\{m_1,m_2,\ldots\}$ be the subset of $\{l,l+1,\ldots\}$ with $m_1<m_2<\cdots$ such that $n_{m_j}>p$. Since $B\neq B_{(k)}$ all k the set $\{m_j\}$ is infinite. Construct a sequence $B=\Gamma_0\supset\Gamma_1\supset\cdots\supset\Gamma_\infty$ of sub-Hopf-algebras as follows.

$$\Gamma_0=B=A(0,\ldots,0,n_l,\ldots,n_{m_1},\ldots,n_{m_2},\ldots),$$

$$\Gamma_1=A(0,\ldots,0,n_l,\ldots,n_{m_1}-1,\ldots,n_{m_2},\ldots),$$

$$\vdots$$

$$\Gamma_{n_{m_1}-p}=A(0,\ldots,0,n_l,\ldots,\underset{m_1}{p},\ldots,n_{m_2},\ldots),$$

$$\Gamma_{n_{m_1}-p+1}=A(0,\ldots,0,n_l,\ldots,\underset{m_1}{p},\ldots,n_{m_2}-1,\ldots,n_{m_3},\ldots),$$

$$\vdots$$

$$\Gamma_{n_{m_1}-p+n_{m_2}-p}=A(0,\ldots,0,n_l,\ldots,\underset{m_1}{p},\ldots,\underset{m_2}{p},\ldots,n_{m_3},\ldots),$$

$$\vdots$$

$$\Gamma_\infty=A(0,\ldots,0,n_l,\ldots,\underset{m_1}{p},\ldots,\underset{m_2}{p},\ldots).$$

Note that $\Gamma_\infty=B_{(p)}$. Each $\Gamma_i(0\leq i<\infty)$ is obtained from Γ_{i+1} by adding one generator $P_{t_i}^{s_i}$ with $l\leq s_i$, $l<t_i$. Also $P_l^{n_l-1}\in\Gamma_i$ all i. It is clear that $\lim_{s,i}H^*(\Gamma_i)=H^*(\Gamma_\infty=B_{(p)})$. By Theorem 3.1 the class $[\xi^{2^{n_l-r-1}}]\,[\xi_{t_i}^{2^{s_i}}]$ is a nilpotent element of $H^*(\Gamma_i)$ all i. By assumption (B,θ) satisfies (*). So the image θ' of θ in $H^*(B')$ is

nilpotent where $B' = A(0,\ldots,0,n_l-1,n_{l+1},\ldots)$. Again applying Lemma 2.3 to the sequence $\{\Gamma_i\}$and the sub-Hopf-algebras $\Lambda = B_{(p)} = \Gamma_\infty$. $\Lambda' = B'$ we see $\theta(p)$ is non-nilpotent. This proves the proposition.

Finally we prove

Proposition 3.5. *Let (B,θ) be as in Theorem* 1.1. *Then there is a sub-Hopf-algebra B' of B such that the image θ' of θ in $H^*(B')$ is non-nilpotent and (B',θ') satisfies the condition* (*).

Proof. Let $B = A(0,\cdots,0,n_l,n_{l+1},\ldots)$ where $n_l > 0$. Whithout loss of generality we assume each $n_j > 0$. Construct a sequence $B = B_0 \supset B_1 \supset \cdots$ of sub-Hopf-algebras of B as follows.

$$\begin{aligned}
B = B_0 &= A(0,\ldots,0,\ n_l,\ n_{l+1},\ n_{l+2},\ldots),\\
B_1 &= A(0,\ldots,0,\ n_l-1,\ n_{l+1},\ n_{l+2},\ldots),\\
&\ \vdots\\
B_{n_l} &= A(0,\ldots,0,\ 0,\ n_{l+1},\ n_{l+2},\ldots),\\
&\ \vdots\\
B_{n_l+1} &= A(0,\ldots,0,\ 0,\ n_{l+1}-1,\ n_{l+2},\ldots),\\
&\ \vdots\\
B_{n_l+n_{l+1}} &= A(0,\ldots,0,\ 0,\ 0,\ n_{l+2},n_{l+3},\ldots),\\
&\ \vdots
\end{aligned}$$

Let θ_i be the image of θ in $H^*(B_i)$. It is clear that $\theta_k = 0$ for all large k. So we can find an integer t such that θ_t is non-nilpotent but θ_{t+1} is nilpotent. Then (B_t,θ_t) satisfies the condition (*). This proves the proposition.

Given any pair (B,θ) as in Theorem 1.1. From Propositions 3.5, 3.4, 3.3 we see there is a sub-Hopf-algebra B' of B such that the image θ' of θ in $H^*(B')$ is non-nilpotent and such that the main conclusion of Theorem 1.1 is true for (B',θ'). By Lemma 3.2 the result is also true for (B,θ). This completes the proof of Theorem 1.1.

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