The Moore-Penrose Inverses of Singular M-Matrices

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ABSTRACT

Suppose M is a real square matrix such that off-diagonal elements of M are nonpositive and all principal minors of M are nonnegative. Necessary and sufficient conditions are given in order that M have a nonnegative Moore-Penrose inverse M^+.

INTRODUCTION

A real square matrix $A=(a_{i,j})$ is called an M-matrix if $a_{i,j}\leqslant 0$ whenever $i\neq j$ and all principal minors of A are positive. Such matrices were introduced in 1937 by Ostrowski [5] and arise in investigations concerning the convergence of iteration processes in linear algebra and spectral properties of matrices. In 1953, Schneider [7,8] extended the M-matrix to the concept of singular M-matrix by establishing some analogues to some results of Ostrowski.

For a real square matrix A with nonpositive off-diagonal elements, it is known that A is an M-matrix if and only if A is nonsingular and $A^{-1}\geqslant 0$. The purpose of this paper is to investigate the Moore-Penrose inverses of singular M-matrices. Section 1 contains the notation and preliminaries. In Sec. 2, a necessary condition for the nonnegativity of the Moore-Penrose inverse of a singular M-matrix is given. In Sec. 3, we characterize all singular M-matrices whose Moore-Penrose inverses are nonnegative.

1. NOTATION AND PRELIMINARIES

Throughout this paper, all matrices considered are real. A square matrix is a real function on $N\times N$, where N is the set of indices $1,2,\ldots,n$, and n is a positive integer. If A is a matrix, we shall denote by $a_{i,j}$ the value of A at (i,j). The tranpose of A will be denoted by A^T, the range of A by $R(A)$, and the null space of A by $N(A)$. The spectral radius of A is the maximum of the moduli of the eigenvalues of A and will be denoted by $\rho(A)$. The determinant of A will be denoted by $\det A$. If $M \subseteq N$ and if A is a matrix on $N \times N$, we define $A(M)$ to be the restriction of A to $M \times M$. $A(M)$ is called the principal submatrix of A, and $\det A(M)$ is called the principal minor of A corresponding to M. A matrix $A=(a_{ij})$ is said to be nonnegative, or $A \geqslant 0$, if $a_{ij} \geqslant 0$ for each (i,j). If $a_{ij}>0$ for each (i,j), we say A is positive, or $A>0$. A vector $X=(x_i)$ in R^N is said to be nonnegative, or $X \geqslant 0$, if $x_i \geqslant 0$ for each $i \in N$. We write $X>0$ if $x_i>0$ for each $i \in N$. If A and B are two matrices, we write $B \geqslant A$ if $B-A \geqslant 0$. We shall denote by Z the class of all real square matrices whose off-diagonal elements are all nonpositive.

The following theorem contains most of the important characterizations of M-matrices.

THEOREM 1.1 [3, Theorem 4.3; 4, Theorem 2.1]. *Suppose* $A \in Z$. *Then the following statements are equivalent*:

(1a) $A=\lambda I-B$, *where* I *is the identity matrix*, $B \geqslant 0$, *and* $\lambda>\rho(B)$, $\rho(B)$ *being a maximal eigenvalue of* B.

(1b) *The real part of each eigenvalue of* A *is positive.*

(1c) *All principal minors of* A *are positive.*

(1d) A^{-1} *exists and* $A^{-1} \geqslant 0$.

(1e) *There exists a vector* $X>0$ *such that* $AX>0$.

Following Fiedler and Ptak [3], we shall denote by K the class of all matrices $A \in Z$ fulfilling one of the conditions in Theorem 1.1. Also, we denote by K_0 the class of all matrices $A \in Z$ which have all principal minors nonnegative. A singular matrix in K_0 is called a singular M-matrix.

The following theorem characterizes a matrix $A \in Z$ which has nonnegative principal minors.

THEOREM 1.2 [3, Theorem 5.1; 4, Theorem 2.1] *Suppose* $A \in Z$. *Then the following statements are equivalent*:

(2a) $A=\lambda I-B$, *where* I *is the identity matrix*, $B \geqslant 0$, *and* $\lambda \geqslant \rho(B)$, $\rho(B)$ *being a maximal eigenvalue of* B.

(2b) *The real part of each eigenvalue of* A *is nonnegative.*

(2c) $A \in K_0$.

2. REDUCIBILITY AND NONNEGATIVITY OF THE MOORE-PENROSE INVERSE

Let A be an arbitrary $m\times n$ matrix. The Moore-Penrose inverse [1] of A is the unique $n\times m$ matrix A^+ satisfying $AA^+A=A$, $A^+AA^+=A^+$, $(AA^+)^T=AA^+$, and $(A^+A)^T=A^+A$. The following results [1] are basic properties of A^+ for an $m\times n$ matrix A.

(2.1) $A^+=A^{-1}$ if A is nonsingular.
(2.2) $(A^T)^+=(A^+)^T$.
(2.3) If U and V are orthogonal matrices, then $(UAV)^+=V^TA^+U^T$.
(2.4) A^+A is the projection on $R(A^T)$ along $N(A^T)$.
(2.5) $R(A^+)=R(A^T)$ and $N(A^+)=N(A^T)$.

A matrix A of order n, $n\geqslant 2$, is said to be reducible if there exists a permutation matrix P such that

$$PAP^T=\begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22}\end{bmatrix},$$

where A_{11} and A_{22} are square submatrices of A. A matrix is irreducible if it is not reducible. In this paper, a one-by-one matrix is also said to be irreducible; a one-by-one matrix is nonsingular (singular) if it is nonzero (zero). We shall use the following well-known results about an irreducible matrix in K_0 throughout this paper.

THEOREM 2.1 [3, Theorems 5.6, 5.7]. *Let $M\in K_0$ be an irreducible matrix of order n.*

(a) *If M is singular, then* $\operatorname{rank} M=n-1$, *and there exists a vector $X>0$ such that $MX=0$.*
(b) *All proper principal minors of M are positive.*

LEMMA 2.2. *Let M be a square matrix with $M^+\geqslant 0$. If Q is a vector in $R(M^T)$ and Y is a vector in $N(M^T)$ such that $MQ-dY\geqslant 0$ for some real number d, then $Q\geqslant 0$.*

Proof. Let $MQ=dY+b$ for some vector $b\geqslant 0$. Since M^+M is the projection on $R(M^T)$ along $N(M^T)$, we have $Q=M^+MQ=M^+b\geqslant 0$. ∎

THEOREM 2.3. *If M is an $n\times n$ $(n\geqslant 2)$ singular irreducible matrix in K_0, then $M^+ \ngeqslant 0$.*

Proof. Partition M as follows:

$$M=\begin{bmatrix} M_{11} & M_{12} \\ M_{21} & m_{22} \end{bmatrix},$$

where $M_{11}\in K$ and $\operatorname{rank} M=\operatorname{rank} M_{11}=n-1$ by Theorem 2.1. There exists a vector $X>0$ such that $M_{11}^TX>0$ by (1e) of Theorem 1.1. Let $Q=M^T\cdot(X^T|0)^T$; then $Q\in R(M^T)$, $Q\ngeqslant 0$, and $Q\neq 0$. Since M is singular and irreducible, $M^TY=0$ for some vector $Y>0$ by Theorem 2.1. Thus, we can find a real number d such that $MQ-dY\geqslant 0$. If $M^+\geqslant 0$, then $Q\geqslant 0$. But this contradicts the fact that Q is nonzero and nonpositive. Hence, $M^+\ngeqslant 0$.

3. CHARACTERIZATIONS OF ELEMENTS OF K_0 WITH NONNEGATIVE MOORE-PENROSE INVERSES

It is obvious, from Theorem 2.3, that a necessary condition for a singular matrix $M\in K_0$ to have $M^+\geqslant 0$ is that M must be reducible.

THEOREM 3.1. *Let $M\in K_0$ be partitioned as follows:*

$$M=\left[\begin{array}{c|ccc} M_1 & & M_2 & \\ \hline & B_{11} & \cdots & B_{1s} \\ 0 & & \ddots & \vdots \\ & 0 & & B_{ss} \end{array}\right],$$

where B_{ii} is singular and irreducible for $i=1,\dots,s$. If $M^+\geqslant 0$, then $B_{ij}=0$ for $i\neq j$.

Proof. There exists a vector $X_i>0$, by Theorem 2.1, such that $B_{ii}{}^TX_i=0$ for $i=1,\dots,s$. We claim $B_{s-1,s}=0$.

Let $Q=M^T\cdot(0|0,\dots,0,X_{s-1}{}^T,0)^T$. Then $Q\leqslant 0$, and all blocks in MQ are nonnegative except possibly the last block $B_{s,s}B_{s-1,s}{}^TX_{s-1}$. Let $X=$

$(0|0,\ldots,0,X_s^T)^T$, then $X \in N(M^T)$. Clearly, there exists a real number d such that $MQ - dX \geqslant 0$ since $X_s > 0$. Hence, $Q \geqslant 0$ by Lemma 2.2. If $B_{s-1,s} \neq 0$, then Q is nonzero and nonpositive, and we get a contradiction. Therefore, $B_{s-1,s} = 0$.

We now assume that $B_{ij} = 0$ for $j = i+1,\ldots,s$, and $i = k+1,\ldots,s-1$, and at least one of B_{kl}, $l = k+1,\ldots,s$, is not zero. Let $Q_1 = M^T \cdot (0|0,\ldots,0,X_k^T,0,\ldots,0)^T$; then Q_1 is a nonzero and nonpositive vector in $R(M^T)$. And $MQ_1 = (V|Y_1^T,\ldots,Y_s^T)^T$, where V is a nonnegative row vector, $Y_j = \sum_{l=k+1}^{s} B_{jl} B_{kl}^T X_k$ for $j = 1,\ldots,k$, and $Y_i = \sum_{l=i}^{s} B_{il} B_{kl}^T X_k$ for $i = k+1,\ldots,s$. Since $Y_h \geqslant 0$ for $h = 1,\ldots,k$, all blocks in MQ_1 are nonnegative except possibly the blocks $Y_{k+1},\ldots,Y_s$. Let $X_1 = (0|0,\ldots,0,X_{k+1}^T,\ldots,X_s^T)^T$; then $X_1 \in N(M^T)$. Thus, we can find a real number d_1 so that $MQ_1 - d_1 X_1 \geqslant 0$. Hence, $Q_1 \geqslant 0$. But this is a contradiction to the fact that Q_1 is nonzero and nonpositive. Therefore, $B_{kl} = 0$ for $l = k+1,\ldots,s$.

Repeating the same process, we finally obtain $B_{ij} = 0$ for $i \neq j$. ■

Corollary 3.2. *If M is a matrix in K_0 such that M is partitioned into the form*

$$M = \begin{bmatrix} B_{11} & \cdots & B_{1s} \\ & \ddots & \vdots \\ 0 & & B_{ss} \end{bmatrix},$$

where B_{ii} is singular and irreducible for $i = 1,\ldots,s$, then $M^+ \geqslant 0$ if and only if $M = 0$.

Proof. The corollary follows from Theorem 3.1. and Theorem 2.2.

Before we proceed, we need the following results about an irreducible M-matrix.

Theorem 3.3 [9, Theorem 3.9]. *If $B \geqslant 0$ is an $n \times n$ matrix, then the following are equivalent:*

(1) *$\alpha > \rho(B)$, and B is irreducible;*
(2) *$\alpha I - B$ is nonsingular, and $(\alpha I - B)^{-1} > 0$.*

We now prove a key theorem in the characterizations of matrices $M \in K_0$ with the property $M^+ \geqslant 0$.

THEOREM 3.4. *Let $M \in K_0$ be partitioned as follows:*

$$M = \begin{bmatrix} M_1 & M_2 & M_3 & M_4 \\ 0 & D & E & F \\ 0 & 0 & A & C \\ 0 & 0 & 0 & B \end{bmatrix},$$

where

$$D = \begin{bmatrix} D_{11} & \cdots & D_{1s} \\ & \ddots & \vdots \\ 0 & & D_{ss} \end{bmatrix}, \qquad E = \begin{bmatrix} E_{1,s+1} & \cdots & E_{1,t} \\ \vdots & & \vdots \\ E_{s,s+1} & \cdots & E_{s,t} \end{bmatrix},$$

$$F = \begin{bmatrix} F_{1,t+1} & \cdots & F_{1,n} \\ \vdots & & \vdots \\ F_{s,t+1} & \cdots & F_{s,n} \end{bmatrix}, \qquad A = \begin{bmatrix} A_{s+1,s+1} & \cdots & A_{s+1,t} \\ & \ddots & \vdots \\ 0 & & A_{t,t} \end{bmatrix},$$

$$\text{and} \quad B = \begin{bmatrix} B_{t+1,t+1} & \cdots & B_{t+1,n} \\ & \ddots & \vdots \\ 0 & & B_{n,n} \end{bmatrix}$$

are such that D_{ii} and B_{jj} are singular and irreducible for $i = 1,\ldots,s$ and $j = t+1,\ldots,n$, and A_{hh} is nonsingular and irreducible for $h = s+1,\ldots,t$. If $M^+ \geqslant 0$, then

(1) *$B_{ij} = 0$ and $D_{ij} = 0$ for $i \neq j$;*
(2) *$E = 0$ and $F = 0$.*

Proof. We first note that $B_{ij} = 0$ for $i \neq j$, by Theorem 3.1. There exist vectors $X_i > 0$ and $X_j > 0$, by Theorem 2.1., such that $D_{ii}{}^T X_i = 0$ for $i = 1,\ldots,s$, and $B_{jj}{}^T X_j = 0$ for $j = t+1,\ldots,n$. We define

$$Z_l = E_{s,l} X_s \qquad \text{for} \quad l = s+1,\ldots,t,$$

and

$$Y_{s+1} = -\left(A_{s+1,s+1}{}^T\right)^{-1} Z_{s+1},$$

$$Y_l = -\left(A_{ll}{}^T\right)^{-1}\left(Z_l + \sum_{j=s+1}^{l-1} A_{jl} Y_j\right) \qquad \text{for} \quad l = s+2,\ldots,t.$$

It is clear that $Z_l \leqslant 0$, since $X_s > 0$ and $E_{s,l} \leqslant 0$ for $l = s+1,\ldots,t$. Now $A_{ll}{}^T$ is an irreducible M-matrix, so $(A_{ll}{}^T)^{-1} > 0$ by Theorem 3.3. This implies $Y_l \geqslant 0$ for $l = s+1,\ldots,t$.

We claim $F_{s,j} = 0$ for $j = t+1,\ldots,n$. Let

$$Q = M^T \cdot \left(0|0,\ldots,0,X_s{}^T|\, Y_{s+1}{}^T,\ldots, Y_t{}^T|0\right)^T.$$

Since

$$E^T \cdot \left(0,\ldots,0,X_s{}^T\right)^T + A^T\left(Y_{s+1}{}^T,\ldots,Y_t{}^T\right)^T = \left(Z_{s+1}{}^T,\ldots,Z_t{}^T\right)^T$$
$$+\left(-Z_{s+1}{}^T,\ldots,-Z_t{}^T\right)^T = 0,$$

we have $Q = (0|0|0|\, W^T)^T$, where

$$W = F^T \cdot \left(0,\ldots,0,X_s{}^T\right)^T + C^T \cdot \left(Y_{s+1}{}^T,\ldots,Y_t{}^T\right)^T$$

$$MQ = \left(W^T M_4{}^T |\, W^T F^T |\, W^T C^T |\, W^T B^T\right)^T$$

and all blocks in MQ are nonnegative except possibly the block BW. Let $X = (0|0|0|X_{t+1}{}^{\mathrm{T}},\ldots,X_n{}^T)^T$; then $X \in N(M^T)$, and thus there exists a real number d such that $MQ - dX \geqslant 0$. By Lemma 2.2, $Q \geqslant 0$. If one of $F_{s,j}$, $j = t+1,\ldots,n$, is not zero, Q will be nonzero and nonpositive, which is a contradiction. Therefore, $F_{s,j} = 0$ for $j = t+1,\ldots,n$.

Next, we claim $E_{s,j} = 0$ for $j = s+1,\ldots,t$. Suppose that one of $E_{s,j}$, $j = s+1,\ldots,t$, is not zero. Let $U_k = E_{s,k}{}^T X_s$ for $k = s+1,\ldots,t$. Then $E_k \leqslant 0$ for $k = s+1,\ldots,t$, and at least one of these E_k is nonzero. We define V_{s+1} and V_k for $k = s+2,\ldots,t$, as follows:

$$V_{s+1} = -\left(A_{s+1,s+1}{}^T\right)^{-1} U_{s+1}, \quad \text{and} \quad V_k = -\left(A_{k,k}{}^T\right)^{-1}\left(U_k + \sum_{j=s+1}^{k-1} A_{j,k}{}^T V_j\right).$$

Now let $V=(V_{s+1}{}^T,\ldots,V_t{}^T)^T$; then V is nonzero and nonnegative.

Case 1: $C^TV\neq 0$. Let

$$Q_1=M^T\cdot\left(0|0,\ldots,0,X_s{}^T|V^T|0\right)^T;$$

then $Q_1\leqslant 0$ and $Q_1\neq 0$, since $E^T\cdot(0,\ldots,0,X_s{}^T)^T+A^T\cdot(V_{s+1}{}^T,\ldots,V_t{}^T)^T=0$. Now

$$MQ_1=\left(V^TCM_4{}^T|V^TCF^T|V^TCC^T|V^TCB^T\right)^T,$$

and all blocks in MQ_1 are nonnegative except possibly the block BC^TV. Let

$$X_{Q_1}=\left(0|0|0|X_{t+1}{}^T,\ldots,X_n{}^T\right)^T;$$

then $X_{Q_1}\in N(M^T)$. Thus, $MQ_1-d_1X_{Q_1}\geqslant 0$ for some real number d_1. By Lemma 2.2, $Q_1\geqslant 0$, which is a contradiction.

Case 2: $C^TV=0$. Let

$$Q_2=M^T\cdot\left(0|0,\ldots,0,X_s{}^T|0|0\right)^T;$$

then $Q_2\leqslant 0$ and $Q_2\neq 0$. Now

$$MQ_2=\left(U^TM_3{}^T|U^TE^T|P_{s+1}{}^T,\ldots,P_t{}^T|0\right)^T,$$

where $U=(U_{s+1}{}^T,\ldots,U_t{}^T)^T$, and $P_k=\sum_{j=k}^t A_{k,j}U_j$ for $k=s+1,\ldots,t$. All blocks in MQ_2 are nonnegative except possibly the blocks $P_{s+1},\ldots,P_t$. Now, let

$$X_{Q_2}=\left(0|0,\ldots,0,X_s{}^T|V^T|0\right)^T;$$

then $X_{Q_2}\in N(M^T)$, since $C^TV=0$. If $U_k=0$, then $P_k=\sum_{j=k+1}^t A_{k,j}U_j\geqslant 0$. If $U_k\neq 0$, then $V_k>0$. It is easy to see that $MQ_2-d_2X_{Q_2}\geqslant 0$ for some real number d_2. By Lemma 2.2, $Q_2\geqslant 0$. We again get a contradiction. Therefore, $E_{s,j}=0$ for $j=s+1,\ldots,t$.

We now prove $D_{s-1,s}=0$. We define $Y_l{}'$ $(\geqslant 0)$ similarly to the way we define $Y_l(\geqslant 0)$, $l=s+1,\ldots,t$, so that

$$E^T\cdot\left(0,\ldots,0,X_{s-1}{}^T,0\right)^T+A^T\left(Y_{s+1}{}'^T,\ldots,Y_t{}'^T\right)^T=0.$$

Let

$$Q_3 = M^{T}\cdot\left(0|0,\ldots,0,X_{s-1}{}^{T},0|Y_{s+1}{}'^{T},\ldots,Y_t{}'^{T}|0\right)^{T},$$

and let

$$X_{Q_3} = \left(0|0,\ldots,0,X_s{}^{T}|0|X_{t+1}{}^{T},\ldots,X_n{}^{T}\right)^{T}.$$

Then $MQ_3 - d_3X_{Q_3} \geqslant 0$ for some real number d_3, and thus $Q_3 \geqslant 0$ by Lemma 2.2. If $D_{s-1,s} \neq 0$, then Q_3 will be nonzero and nonpositive, which is a contradiction. Hence, $D_{s-1,i} = 0$.

By using the same argument as we did to show $F_{s,j} = 0$ and $E_{s,i} = 0$, we can show that $F_{s-1,j} = 0$ and $E_{s-1,i} = 0$ for $i = s+1,\ldots,t$ and $j = t+1,\ldots,n$. Repeating the same process, we finally obtain $B_{i,j} = 0$, $D_{i,j} = 0$ for $i \neq j$, $E = 0$, and $F = 0$. ■

Corollary 3.5. *Let $M \in K_0$ be partitioned as follows:*

$$M = \begin{bmatrix} M_1 & M_2 & M_3 \\ 0 & D & E \\ 0 & 0 & A \end{bmatrix},$$

where A, D, and E are the same as in Theorem 3.4. If $M^{+} \geqslant 0$, then $D_{i,j} = 0$ for $i \neq j$, and $E = 0$.

Corollary 3.6. *Let $M \in K_0$ be partitioned into the form*

$$M = \begin{bmatrix} D & E \\ 0 & A \end{bmatrix},$$

where A, D, and E are the same as those in Theorem 3.4. Then $M^{+} \geqslant 0$ if and only if $D = 0$ and $E = 0$.

Proof. The corollary follows from Corollary 3.5 and Theorem 2.3. ■

Theorem 3.7. *Let $M \in K_0$ be partitioned as follows:*

$$M = \begin{bmatrix} A & C \\ 0 & B \end{bmatrix},$$

where A, B, and C are the same as in Theorem 3.4. *Then* $M^+ \geqslant 0$ *if and only if* $B=0$ *and* $C=0$.

Proof. The proof is similar to that of Theorem 3.4.. ∎

If M is a reducible matrix, then by definition there exists a permutation matrix P such that

$$PMP^T = \begin{bmatrix} M_{11} & \cdots & M_{1n} \\ & \ddots & \vdots \\ 0 & & M_{nn} \end{bmatrix},$$

where M_{ii} is square and irreducible for $i=1,\dots,n$. We will regroup the blocks on the diagonal in the following way:

Suppose that M_{ii}, $i=k,k+1,\dots,l$, is singular, and suppose that $M_{k-1,k-1}$ and $M_{l+1,l+1}$ are nonsingular. Then we group M_{kk}, $M_{k+1,k+1},\dots,M_{l,l}$ together to form a new block on the diagonal and call it D_{kk}. That is,

$$D_{k,k} = \begin{bmatrix} M_{k,k} & \cdots & M_{k,l} \\ & \ddots & \vdots \\ 0 & & M_{l,l} \end{bmatrix},$$

where all blocks on the diagonal are singular, and $M_{k-1,k-1}$ and $M_{l+1,l+1}$ in PMP^T are nonsingular. We perform the same regrouping for nonsingular blocks on the diagonal of PMP^T. Thus, we can rewrite PMP^T in the following form:

$$PMP^T = \begin{bmatrix} D_{1,1} & \cdots & D_{1,t} \\ & \ddots & \vdots \\ 0 & & D_{t,t} \end{bmatrix}, \tag{3.0}$$

where $D_{i,i}$ is a submatrix (of PMP^T) of the form

$$D_{i,i} = \begin{bmatrix} M_{i,i} & \cdots & M_{i,j} \\ & \ddots & \vdots \\ 0 & & M_{j,j} \end{bmatrix},$$

such that either (1) every block on the diagonal of $D_{i,i}$ is singular and every block on the diagonals of $D_{i-1,i-1}$ and $D_{i+1,i+1}$ is nonsingular, or (2) every block on the diagonal of $D_{i,i}$ is nonsingular and every block on the diagonals of $D_{i-1,i-1}$ and $D_{i+1,i+1}$ is singular.

We now characterize all matrices $M \in K_0$ whose $M^+ \geqslant 0$.

THEOREM 3.8. *Let M be a nonzero matrix in K_0. A necessary and sufficient condition for $M^+ \geqslant 0$ is that there exists a permutation matrix P such that*

$$PMP^T = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} \qquad \text{with} \quad A \in K.$$

REMARK. The zero blocks in PMP^T may not be present.

Proof.

Necessity: We assume $M^+ \geqslant 0$. If $M \in K$, then the statement is true. If M is singular, then M must be reducible by Theorem 2.3. Let P_1 be a permutation matrix such that

$$P_1 M P_1{}^T = \begin{bmatrix} D_{1,1} & \cdots & D_{1,t} \\ & \ddots & \vdots \\ 0 & & D_{t,t} \end{bmatrix}$$

is a matrix of the form (3.0).

We will proceed by using the induction on t. We first note $t \geqslant 2$, since if $t=1$, then all blocks on the diagonal of $P_1 M P_1{}^T$ are singular and irreducible, and thus $M=0$ by Corollary 3.2. The necessity part is true for $t=2$ by Corollary 3.6. and Theorem 3.7. We then assume the necessity part is true for $t \leqslant k-1$.

Case 1. Every block on the diagonal of $D_{k,k}$ is nonsingular. We rewrite $P_1 M P_1{}^T$ in the following form:

$$P_1 M P_1{}^T = \begin{bmatrix} M_1 & M_2 & M_3 \\ 0 & D_{k-1,k-1} & D_{k-1,k} \\ 0 & 0 & D_{k,k} \end{bmatrix}.$$

Since $(P_1MP_1^T)^+ = P_1M^+P_1^T \geqslant 0$, we obtain $D_{k-1,k}=0$ by Corollary 3.5. Let P_2 be a permutation matrix such that

$$P_2MP_2^T = \begin{bmatrix} M_1 & M_3 & M_2 \\ 0 & D_{k,k} & 0 \\ 0 & 0 & D_{k-1,k-1} \end{bmatrix}.$$

Thus $D_{k-2,k-2}$ and $D_{k,k}$ are merged into one block. By the induction hypothesis, there exists a permutation matrix P such that

$$PMP^T = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} \quad \text{with} \quad A \in K.$$

Case 2. Every block on the diagonal of $D_{k,k}$ is singular. We rewrite $P_1MP_1^T$ in the following form:

$$P_1MP_1^T = \begin{bmatrix} M_1 & M_2 & M_3 & M_4 \\ 0 & D_{k-2,k-2} & D_{k-2,k-1} & D_{k-2,k} \\ 0 & 0 & D_{k-1,k-1} & D_{k-1,k} \\ 0 & 0 & 0 & D_{k,,k} \end{bmatrix},$$

such that every block on the diagonal of $D_{k-2,k-2}$ is singular and every block on the diagonal of $D_{k-1,k-1}$ is nonsingular.

Since $(P_1MP_1^T)^+ = P_1M^+P_1^T \geqslant 0$, we have $D_{k-2,k-1}=0$ by Theorem 3.4. Hence, there exists a permutation matrix p_3 such that

$$P_3MP_3^T = \begin{bmatrix} M_1 & M_3 & M_2 & M_4 \\ 0 & D_{k-1,k-1} & 0 & D_{k-1,k} \\ 0 & 0 & D_{k-2,k-2} & D_{k-2,k} \\ 0 & 0 & 0 & D_{k,,k} \end{bmatrix}.$$

Thus $D_{k-2,k-2}$ and $D_{k,k}$ are merged into one block. By the induction hypothesis, there exists a permutation matrix P such that

$$PMP^T = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} \quad \text{with} \quad A \in K.$$

Sufficiency: Clearly,

$$(PMP^T)^+ = \begin{bmatrix} A^{-1} & 0 \\ 0 & 0 \end{bmatrix} \geqslant 0 \quad \text{implies} \quad M = P^T \begin{bmatrix} A^{-1} & 0 \\ 0 & 0 \end{bmatrix} P \geqslant 0.$$

∎

For any square matrix of order n, an $n \times n$ matrix X which satisfied $AXA = A$, $XAX = X$, and $AX = XA$ is called the group inverse of A. It is known that the group inverse of a matrix A does not always exist, but when it exists it is unique and is denoted by $A^{\#}$. The existence of $A^{\#}$ is equivalent to the condition that $\operatorname{rank} A = \operatorname{rank} A^2$, which in turn is equivalent to the requirement that $R(A) \cap N(A) = \{0\}$ [1, pp. 162, 165]. There exists a class of matrices such that the group inverse and the Moore-Penrose inverse are the same. We shall call a square matrix A range-Hermitian if $R(A) = R(A^T)$. It is well known that $A^{\#} = A^+$ if and only if A is range-Hermitian [1, p. 164].

For any square matrix A, A and $A^{\#}$ have the same range and the same null space, by the defining equations of $A^{\#}$, if $A^{\#}$ exists. Since $A^{\#}A$ is an idempotent matrix, $A^{\#}A$ is the projection on $R(A^{\#}A)$ along $N(A^{\#}A)$. But $R(A^{\#}A) = R(A^{\#}) = R(A)$; hence $A^{\#}Ax = x$ for x in $R(A)$. Also, we have $A^{\#} = A^{-1}$ if A is nonsingular, $(A^T)^{\#} = (A^{\#})^T$, and $(PAP^T)^{\#} = PA^{\#}P^T$ for any permutation matrix P.

By using the same argument as we did before, we can obtain the same results about $M^{\#}$ as those in Theorem 2.3, Theorem 3.1, Corollary 3.2, Theorem 3.4, Corollary 3.5, Corollary 3.6, Theorem 3.7, and Theorem 3.8. Therefore, we obtain the following equivalent statements.

THEOREM 3.9. *Let M be a nonzero matrix in K_0. The following statements are equivalent*:

(1) $M^+ \geqslant 0$.

(2) *There exists a permutation matrix P such that*

$$PMP^T = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} \qquad \text{with} \quad A \in K.$$

(3) $\operatorname{rank} M = \operatorname{rank} M^2$ *and* $M^{\#} \geqslant 0$.

Furthermore, if one of (1), (2), *and* (3) *holds, then* $M^+ = M^{\#}$.

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