Computers and Mathematics with Applications 56 (2008) 2580–2587

Contents lists available at ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

Existence of positive solutions for second order functional differential equations

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ARTICLE INFO

Article history:
Received 24 April 2008
Accepted 27 May 2008

Keywords:
Second order functional differential equations
Boundary value problems
Positive solution
Krasnoselskii fixed point theorem

ABSTRACT

We afford a existence criterion of positive solutions of a boundary value problem concerning a second order functional differential equation by using the Krasnoselskii fixed point theorem on cones in Banach spaces. Moreover, we also apply our results to establish several existence theorems of multiple positive solutions for some functional differential equations.

1. Introduction

In this paper, we deal with the existence of positive solutions to the functional differential equation

$$u''(t) + F(t, u_t) = 0, \quad t \in (0, 1).$$

The solutions u must satisfy the initial function

$$u(s) = \phi(s), \quad -r \le s \le 0, \text{ for certain given } \phi,$$

and boundary condition of Sturm–Liouville's type

$$(BC) \begin{cases} u(0) = 0, \\ \gamma u(1) + \delta u'(1) = 0, \end{cases}$$

where

$$\gamma, \delta \ge 0 \quad \text{and} \quad \gamma + \delta > 0.$$

Our notations are defined as follows. We denote the set of all real numbers and the set of all nonnegative real numbers by $\mathbb{R}$ and $\mathbb{R}^+$, respectively. For any fixed $r \in \mathbb{R}^+$, let C_r denote the Banach space of all continuous functions $\phi : [-r, 0] \equiv J \to \mathbb{R}$ endowed with the suprenorm

$$\|\phi\|_J = \sup_{s \in J} |\phi(s)|,$$

and let

$$C_{r,0} = \{\psi \in C_r \mid \psi(0) = 0\}.$$

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doi:10.1016/j.camwa.2008.05.023

The notation u_t above denotes a function in C_r defined by

$$u_t(w) = u_t(w;\phi) := \begin{cases} u(t+w) & \text{if } t+w \ge 0, \\ \phi(t+w) & \text{if } t+w \le 0, \end{cases}$$

where the given ϕ is an element of the space $C_{r,0}$.

From now on, we denote our problem as (BVP). Moreover, if $w \in [-r, 0]$ is fixed, by a solution of the (BVP) we mean a function $u \in C^2[0, 1]$ such that u satisfies the boundary condition (BC), and for a given ϕ, the relation

$$u''(t) + F(t, u_t(w;\phi)) = 0$$

holds for all $t \in [0, 1]$.

There has recently been an increased interest in studying boundary value problems for functional differential equations, see, e.g. the books by Hale [1], Kolmanovskii and Myshkis [2] and Henderson [3]. Furthermore, as pointed out in [4], these problems have arisen from problems of physics and variational problems of control theory, as well as from much applied mathematics which appeared early on in the literature [5,6]. We refer the reader to more detailed treatment in the following interesting research [7–19], and the references therein.

In Section 2, we state the key tool for establishing our main results, that is, the well-known Krasnoselkii fixed point theorem [20,21] and give a lemma that will be used to define a positive operator in a cone. Then, in some function space, we construct an appropriate cone on which we apply the fixed point theorem to our positive operator, this yields our existence results. Moreover, some remarks in Section 3 will imply several corollaries of existence of multiple positive solutions, including the reduction to general ordinary differential equations with boundary condition. Finally, in the last section we give an example as an application.

2. Preliminaries and existence results

In order to abbreviate our discussion, throughout this paper, we assume the following assumptions hold:

(C_1) $k(t, s)$ is the Green's function of the differential equation

$$\begin{cases} u'' = 0, \\ (BC); \end{cases}$$

(C_2) $F : [0, 1] \times C_r \to \mathbb{R}^+$ is a continuous functional.

We now state the Krasnoselkii fixed point theorem [20,21] and a useful lemma which are required for the main result.

Theorem A *([20,21]). Let E be a Banach space, and let $K \subset E$ be a cone in E. Assume Ω_1, Ω_2 are open subsets of E with $0 \in \Omega_1$, $\overline{\Omega_1} \subset \Omega_2$, and let*

$$A : K \cap (\overline{\Omega_2} \setminus \Omega_1) \to K$$

be a completely continuous operator such that either

(i) *$\|Au\| \le \|u\|$, $u \in K \cap \partial\Omega_1$ and $\|Au\| \ge \|u\|$, $u \in K \cap \partial\Omega_2$; or*
(ii) *$\|Au\| \ge \|u\|$, $u \in K \cap \partial\Omega_1$ and $\|Au\| \le \|u\|$, $u \in K \cap \partial\Omega_2$.*

Then A has a fixed point in $K \cap (\overline{\Omega_2} \setminus \Omega_1)$.

Lemma 1. *Suppose that $k(t, s)$ is defined as in (C_1). Then, for any p_1, p_2 with $0 \le p_1 < p_2 \le 1$, we have the following results:*

$$\begin{cases} \dfrac{k(t,s)}{k(s,s)} \le 1, & \text{for } t \in [0,1] \text{ and } s \in [0,1], \\ \dfrac{k(t,s)}{k(s,s)} \ge \min\left\{\dfrac{(1-p_2)\gamma+\delta}{\gamma+\delta}, p_1\right\}, & \text{for} t \in [p_1,p_2] \text{ and } s \in [0,1]. \end{cases}$$

Proof. It is well known that

$$k(t,s) = \begin{cases} (\gamma+\delta-\gamma t)s, & 0 \le s \le t \le 1, \\ (\gamma+\delta-\gamma s)t, & 0 \le t \le s \le 1, \end{cases}$$

which implies

$$\frac{k(t,s)}{k(s,s)} = \begin{cases} \dfrac{\gamma+\delta-\gamma t}{\gamma+\delta-\gamma s}, & 0 \le s \le t \le 1, \\ \dfrac{t}{s}, & 0 \le t \le s \le 1. \end{cases}$$

Hence, we obtain the desired results:

$$\frac{k(t,s)}{k(s,s)} \le 1 \quad \text{for } t \in [0,1],$$

and

$$\frac{k(t,s)}{k(s,s)} \ge \begin{cases} \dfrac{(1-p_2)\gamma+\delta}{\gamma+\delta}, & 0 \le s \le t \le p_2, \\ p_1, & p_1 \le t \le s \le 1. \end{cases} \quad \square$$

From Lemma 1, we define a number

$$M = M(p_1,p_2) := \min\left\{\frac{(1-p_2)\gamma+\delta}{\gamma+\delta}, p_1\right\}$$

and next, state and prove our main results.

Theorem 2 (*Existence Result for* $-1 < w \le 0$). *Suppose the following hypotheses hold:*

(H_1) *there exists a positive constant* λ *such that*

$$F(t,\psi) \le \lambda \left(\int_0^1 k(s,s)\mathrm{d}s\right)^{-1}, \quad \text{for } t \in [0,1] \text{ and } \psi \in C_r \text{ with } \|\psi\|_J \le \lambda,$$

and

(H_2) *there exist* p_1, p_2 *with* $0 \le -w \le p_1 < p_2 \le 1$ *and a positive constant* $\eta \ne \lambda$ *such that*

$$F(t,\psi) \ge \eta \left(\int_{p_1}^{p_2} k\left(\frac{p_1+p_2}{2}, s\right)\mathrm{d}s\right)^{-1}, \quad \text{for } t \in [p_1,p_2] \text{ and } \psi \in C_r \text{ with } M\eta \le \|\psi\|_J \le \eta.$$

Then for any given $\phi \in C_{r,0}$ *with* $\|\phi\|_J \le \lambda$, (BVP) *has at least one positive solution* u *such that* $\|u\|$ *between* λ *and* η.

Proof. Without loss of generality, we assume $\lambda < \eta$. It is clear that (BVP) has a solution $u = u(t)$ if and only if u is the solution of the operator equation

$$u(t) = \int_0^1 k(t,s)F(s,u_s(w;\phi))\mathrm{d}s := A_\phi u(t), \quad u \in C[0,1].$$

Let K be a cone in $C_0[0,1] := \{u \in C[0,1] \mid u(0) = 0\}$ defined by

$$K = \left\{u \in C_0[0,1] \mid u(t) \ge 0, \min_{t\in[p_1,p_2]} u(t) \ge M\|u\|\right\}.$$

Following from the definition of K and Lemma 1 we have

$$\begin{aligned} \min_{t\in[p_1,p_2]}(A_\phi u)(t) &= \min_{t\in[p_1,p_2]}\int_0^1 k(t,s)F(s,u_s(w;\phi))\mathrm{d}s \\ &\ge M\int_0^1 k(s,s)F(s,u_s(w;\phi))\mathrm{d}s \\ &\ge M\int_0^1 k(t,s)F(s,u_s(w;\phi))\mathrm{d}s. \end{aligned}$$

Thus, $\min_{t\in[p_1,p_2]}(A_\phi u)(t) \ge M\|Au\|$, which implies $A_\phi K \subset K$. Furthermore, it is easy to check $A_\phi : K \to K$ is completely continuous. To complete the proof, we separate the rest of proof into the following two steps:

Step 1. Let $\Omega_1 := \{u \in K \mid \|u\| < \lambda\}$. It follows from (H_1) and Lemma 1 that for $u \in \partial\Omega_1$,

$$\begin{aligned} (A_\phi u)(t) &= \int_0^1 k(t,s)F(s,u_s(w;\phi))\mathrm{d}s \\ &\le \int_0^1 k(s,s)F(s,u_s(w;\phi))\mathrm{d}s \\ &\le \lambda\left(\int_0^1 k(s,s)\right)^{-1}\left(\int_0^1 k(s,s)\mathrm{d}s\right)\frac{\|u\|}{\lambda} \\ &= \|u\|. \end{aligned}$$

Hence,

$$\|A_\phi u\| \le \|u\| \quad \text{for } u \in \partial\Omega_1 \cap K.$$

Step 2. Let $\Omega_2 := \{u \in K \mid \|u\| < \eta\}$. It follows from the definitions of $\|u\|$ and K that

$$\begin{cases} u(t) \le \|u\| = \eta & \text{for } t \in [0,1], \\ u(t) \ge \min\limits_{t\in[p_1,p_2]} u(t) \ge M\|u\| = M\eta & \text{for } t \in [p_1,p_2], \end{cases}$$

for $u \in \partial\Omega_2$, which implies

$$M\eta \le u(t) \le \eta \quad \text{for } t \in [p_1,p_2].$$

Moreover, it follows from $0 \le -w \le p_1 < p_2 \le 1$ that $s + w \ge 0$ for $s \in [p_1,p_2]$. This implies $u_s(w;\phi) = u(s+w)$ for $s \in [p_1,p_2]$. Hence,

$$\begin{aligned} (A_\phi u)\left(\frac{p_1+p_2}{2}\right) &= \int_0^1 k\left(\frac{p_1+p_2}{2}, s\right) F(s; u_s(w,\phi))\mathrm{d}s \\ &\ge \int_{p_1}^{p_2} k\left(\frac{p_1+p_2}{2}, s\right) F(s; u_s(w,\phi))\mathrm{d}s \\ &\ge \eta\left(\int_{p_1}^{p_2} k\left(\frac{p_1+p_2}{2}, s\right)\mathrm{d}s\right)^{-1}\left(\int_{p_1}^{p_2} k\left(\frac{p_1+p_2}{2}, s\right)\mathrm{d}s\right)\frac{\|u\|}{\eta} \\ &= \|u\|, \end{aligned}$$

which implies

$$\|A_\phi u\| \ge \|u\| \quad \text{for } u \in \partial\Omega_2.$$

Therefore, by Theorem A, we complete this proof. □

Note this given w may not belong to $(-1, 0]$, hence, we can only conclude the following.

Theorem 3 (*Existence Result for $-r \le w \le 0$*). *Suppose the following hypotheses hold:*

(H_1) *there exists a positive constant λ such that*

$$F(t,\psi) \le \lambda\left(\int_0^1 k(s,s)\mathrm{d}s\right)^{-1}, \quad \textit{for } t \in [0,1] \textit{ and } \psi \in C_r \textit{ with } \|\psi\|_J \le \lambda,$$

and

(H_3) *there exist p_1, p_2 with $0 \le p_1 < p_2 \le 1$ and a positive constant $\eta \ne \lambda$ such that*

$$F(t,\psi) \ge \eta\left(\int_{p_1}^{p_2} k\left(\frac{p_1+p_2}{2}, s\right)\mathrm{d}s\right)^{-1}, \quad \textit{for } t \in [p_1,p_2] \textit{ and } \psi \in C_r \textit{with} \|\psi\|_J \le \eta.$$

Then for any given $\phi \in C_{r,0}$ with $\|\phi\|_J \le \min\{\lambda,\eta\}$, (BVP) has at least one positive solution u such that $\|u\|$ between λ and η.

Proof. This proof follows in similar fashion to that of Theorem 2. One just needs to modify Step 2 in the process of the demonstration of Theorem 2 as the following:

Step 2. Let $\Omega_2 := \{u \in K \mid \|u\| < \eta\}$. It follows from the definitions of $\|u\|$ and K that

$$\begin{cases} u(t) \le \|u\| = \eta & \text{for } t \in [0,1], \\ u(t) \ge \min\limits_{t\in[p_1,p_2]} u(t) \ge M\|u\| = M\eta & \text{for } t \in [p_1,p_2], \end{cases}$$

for $u \in \partial\Omega_2$, which implies

$$M\eta \le u(t) \le \eta \quad \text{for } t \in [p_1,p_2].$$

Moreover, for $s \in [p_1,p_2]$,

$$u_s(w;\phi) := \begin{cases} u(s+w) & \text{if } s+w \ge 0, \\ \phi(s+w) & \text{if } s+w \le 0, \end{cases}$$

Which implies,

$$\|u_s(w;\phi)\| \le \eta.$$

Hence,

$$\begin{aligned}(A_\phi u)\left(\frac{p_1+p_2}{2}\right) &= \int_0^1 k\left(\frac{p_1+p_2}{2}, s\right) F(s; u_s(w,\phi))\mathrm{d}s \\ &\geq \int_{p_1}^{p_2} k\left(\frac{p_1+p_2}{2}, s\right) F(s; u_s(w,\phi))\mathrm{d}s \\ &\geq \eta \left(\int_{p_1}^{p_2} k\left(\frac{p_1+p_2}{2}, s\right)\mathrm{d}s\right)^{-1} \left(\int_{p_1}^{p_2} k\left(\frac{p_1+p_2}{2}, s\right)\mathrm{d}s\right)\frac{\|u\|}{\eta} \\ &= \|u\|,\end{aligned}$$

which implies

$$\|A_\phi u\| \geq \|u\| \quad \text{for } u \in \partial\Omega_2. \quad \square$$

3. Applications

Remark 4. Assume that $F(t,\psi)$ satisfies the following property $\mathbb{P}$:

If $\max_{t\in[0,1]} F(t,\psi)$ is unbounded, then, there exists a ϕ with $\|\phi\|_J$ large enough such that for any $\psi \in C_r$ with $\|\psi\|_J \leq \|\phi\|_J$, we have $\max_{t\in[0,1]} F(t,\psi) \leq \max_{t\in[0,1]} F(t,\phi)$.

Given p_1, p_2 with $0 \leq p_1 < p_2 \leq 1$ and let

$$\begin{aligned}\max F_0 &:= \lim_{\|\psi\|_J\to 0} \max_{t\in[0,1]} \frac{F(t,\psi)}{\|\psi\|_J}, \\ \min F_0 &:= \lim_{\|\psi\|_J\to 0} \min_{t\in[p_1,p_2]} \frac{F(t,\psi)}{\|\psi\|_J}, \\ \max F_\infty &:= \lim_{\|\psi\|_J\to \infty} \max_{t\in[0,1]} \frac{F(t,\psi)}{\|\psi\|_J},\end{aligned}$$

and

$$\min F_\infty := \lim_{\|\psi\|_J\to \infty} \min_{t\in[p_1,p_2]} \frac{F(t,\psi)}{\|\psi\|_J}.$$

Since

$$\left(\int_0^1 k(s,s)\mathrm{d}s\right)^{-1} := A = \frac{6(\gamma+\delta)}{\gamma+3\delta},$$

and

$$\left(\int_{p_1}^{p_2} k\left(\frac{p_1+p_2}{2}, s\right)\mathrm{d}s\right)^{-1} := B = \frac{16(\gamma+\delta)}{(p_2-p_1)(L_1L_2+L_3L_4)},$$

where

$$\begin{aligned}&L_1 := p_2+3p_1, \qquad L_2 := 2\gamma - p_1\gamma - p_2\gamma + 2\delta, \\ &L_3 := 4\gamma + 4\delta - 3\gamma p_2 - \gamma p_1, \qquad L_4 := p_1+p_2,\end{aligned}$$

we have the following results:

Suppose that $\max F_0 := C_1 \in [0, A)$. Taking $\epsilon = A - C_1$, there exists a $\lambda_1 > 0$ (λ_1 can be chosen arbitrarily small) such that for any $\psi \in C_r$ with $\|\psi\|_J \leq \lambda_1$, we have

$$\max_{t\in[0,1]} \frac{F(t,\psi)}{\|\psi\|_J} \leq \epsilon + C_1 = A.$$

Hence,

$$F(t,\psi) \leq A\|\psi\|_J \leq A\lambda_1, \quad \text{for } t \in [0,1] \text{ and } \psi \in C_r \text{ with } \|\psi\|_J \in [0,\lambda_1],$$

which satisfies the hypothesis (H_1) of Theorem 2.

Suppose that $\min F_\infty := C_2 \in (\frac{B}{M}, \infty]$. Taking $\epsilon = C_2 - \frac{B}{M} > 0$, there exists an $\eta_1 > 0$ (η_1 can be chosen arbitrarily large) such that for any $\psi \in C_r$ with $\|\psi\|_J \geq M\eta_1$, we have

$$\min_{t\in[p_1,p_2]} \frac{F(t,\psi)}{\|\psi\|_J} \geq -\epsilon + C_2 = \frac{B}{M}.$$

Hence,

$$F(t,\psi) \geq \frac{M}{B}\|\psi\|_J \geq \frac{B}{M}M\eta_1 = B\eta_1, \quad \text{for } t \in [p_1, p_2] \text{ and } \psi \in C_r \text{ with } \|\psi\|_J \in [M\eta_1, \eta_1]$$

which satisfies the hypothesis (H_2) of Theorem 2.

Suppose that $\min F_0 := C_3 \in (\frac{B}{M}, \infty]$. Taking $\epsilon = C_3 - \frac{B}{M} > 0$, there exists an $\eta_2 > 0$ (η_2 can be chosen small enough) such that for any $\psi \in C_r$ with $\|\psi\|_J \leq \eta_2$, we have

$$\min_{t\in[p_1,p_2]} \frac{F(t,\psi)}{\|\psi\|_J} \geq -\epsilon + C_3 = \frac{B}{M}.$$

Hence,

$$F(t,\psi) \geq \frac{B}{M}\|\psi\|_J \geq \frac{B}{M}M\eta_2 = B\eta_2, \quad \text{for any } t \in [p_1, p_2] \text{ and } \psi \in C_r \text{ with } \|\psi\|_J \in [M\eta_2, \eta_2]$$

which satisfies the hypothesis (H_2) of Theorem 2.

Suppose that $\max F_\infty := C_4 \in [0, A)$. Taking $\epsilon = A - C_4 > 0$, there exists a $\theta > 0$ (θ can be chosen arbitrarily large) such that for any $\psi \in C_r$ with $\|\psi\|_J \geq \theta$, we have

$$(*) \quad \max_{t\in[0,1]} \frac{F(t,\psi)}{\|\psi\|_J} \leq \epsilon + C_4 = A.$$

Now we have the following two cases:

Case 1. Assume that $\max_{t\in[0,1]} F(t,\psi)$ is bounded, that is,

$$F(t,\psi) \leq L, \quad \text{for } t \in [0,1] \text{ and } \psi \in C_r.$$

Taking $\lambda_2 = \frac{L}{A}$, hence,

$$F(t,\psi) \leq L = A\lambda_2, \quad \text{for } t \in [0,1] \text{ and } \psi \in C_r \text{ with } \|\psi\|_J \in [0, \lambda_2].$$

Case 2. Assume that $\max_{t\in[0,1]} F(t,\psi) := G_t(\psi)$ is unbounded. Then, by property $\mathbb{P}$, there exists a ϕ with $\|\phi\|_J := \lambda_2 \geq \theta$ such that for any $\psi \in C_r$with$\|\psi\|_J \in [0, \lambda_2]$, we have

$$\max_{t\in[0,1]} F(t,\psi) = G_t(\psi) \leq G_t(\phi) = \max_{t\in[0,1]} F(t,\phi).$$

This implies

$$F(t,\psi) \leq F(t_0,\phi), \quad \text{for } t \in [0,1] \quad \text{and } \psi \in C_r \text{ with } \|\psi\|_J \in [0, \lambda_2].$$

It follows from $\lambda_2 \geq \theta$ and $(*)$ that

$$F(t,\psi) \leq F(t_0,\phi) \leq A\|\phi\|_J = A\lambda_2, \quad \text{for } t \in [0,1] \text{ and } \psi \in C_r \text{ with } \|\psi\|_J \in [0, \lambda_2].$$

By Cases 1 and 2, the hypothesis (H_1) of Theorem 2 is satisfied.

It follows from Remark 4 that the following corollaries hold.

Corollary 5. *Assume that F satisfies property $\mathbb{P}$ and suppose there exist p_1 and p_2 with $0 \leq -w \leq p_1 < p_2 \leq 1$, A and B are defined as in Remark 4. Then in the case*

(H_4) $\max F_0 = C_1 \in [0, A)$ *and* $\min F_\infty = C_2 \in (\frac{B}{M}, \infty]$, *or*
(H_5) $\min F_0 = C_3 \in (\frac{B}{M}, \infty]$ *and* $\max F_\infty = C_4 \in [0, A)$,

we have following corresponding results (i) *and* (ii) *respectively.*

(i) *For any given $\phi \in C_{r,0}$ with $\|\phi\|_J$ small enough, (BVP) has at least one positive solution.*
(ii) *For any given $\phi \in C_{r,0}$, (BVP) has at least one positive solution.*

Proof. It follows from Remark 4 and Theorem 2 that the desired result holds, immediately. □

Corollary 6. *Assume that F satisfies property $\mathbb{P}$ and suppose there exist p_1 and p_2 with $0 \leq -w \leq p_1 < p_2 \leq 1$, A and B are defined as in Remark 4. If the following hypotheses hold:*

(H_6) $\min F_\infty = C_2$, $\min F_0 = C_3 \in (\frac{B}{M}, \infty]$,
(H_7) *there exists $\lambda^* > 0$ such that*

$$F(t,\psi) \leq A\lambda^*, \quad \text{for } t \in [0,1] \text{ and } \psi \in C_r \text{ with } \|\psi\|_J \in [0, \lambda^*],$$

then, for any given $\phi \in C_{r,0}$ with $\|\phi\|_J \leq \lambda^*$, (BVP) has at least two positive solutions u_1 and u_2 such that $0 < \|u_1\| < \lambda^* < \|u_2\|$.

Proof. It follows form Remark 4 that there exist two real numbers η_1 and η_2 satisfying

$$0 < \eta_2 < \lambda^* < \eta_1,$$
$$F(t, \psi) \geq B\eta_1, \quad \text{for } t \in [p_1, p_2] \text{ and } \psi \in C_r \text{ with } \|\psi\|_J \in [M\eta_1, \eta_1],$$

and

$$F(t, \psi) \geq B\eta_2, \quad \text{for } t \in [p_1, p_2] \text{ and } \psi \in C_r \text{ with } \|\psi\|_J \in [M\eta_2, \eta_2].$$

Thus, by Theorem 2, we see for any given $\phi \in C_{r,0}$ with $\|\phi\|_J \in [0, \lambda^*]$, (BVP) has two positive solutions u_1 and u_2 such that $\eta_2 < \|u_1\| < \lambda^* < \|u_2\| < \eta_1$. Hence, we complete this proof. □

Corollary 7. *Assume that F satisfies property $\mathbb{P}$ and suppose there exist p_1 and p_2 with $0 \leq -w \leq p_1 < p_2 \leq 1$, A and B are defined as in Remark 4. If the following hypotheses hold:*

(H_8) $\max F_0 = C_1$, $\max F_\infty = C_4 \in [0, A)$,
(H_9) *there exists $\eta^* > 0$ such that*

$$F(t, \psi) \geq B\eta^*, \quad \text{for } t \in [p_1, p_2] \text{ and } \psi \in C_r \text{ with } \|\psi\|_J \in [M\eta^*, \eta^*],$$

then, for any given $\phi \in C_{r,0}$ with $\|\phi\|_J$ small enough, (BVP) has at least two positive solutions u_1 and u_2 such that $0 < \|u_1\| < \eta^* < \|u_2\|$.

Proof. It follows from Remark 4 that there exist two real numbers λ_1 and λ_2 satisfying

$$0 < \lambda_1 < \eta^* < \lambda_2,$$
$$F(t, \psi) \leq A\lambda_1, \quad \text{for } t \in [0, 1] \text{ and } \psi \in C_r \text{ with } \|\psi\|_J \in [0, \lambda_1],$$
$$F(t, \psi) \leq A\lambda_2, \quad \text{for } t \in [0, 1] \text{ and } \psi \in C_r \text{ with } \|\psi\|_J \in [0, \lambda_2].$$

Thus, by Theorem 2, we see for any given $\phi \in C_{r,0}$ with $\|\phi\|_J \in [0, \lambda_1]$, (BVP) has two positive solutions u_1 and u_2 such that $\lambda_1 < \|u_1\| < \eta^* < \|u_2\| < \lambda_2$. Hence, we complete this proof. □

Remark 8. We note that in the limiting case $r = 0$, C_r is reduced to $\mathbb{R}$. Then (BVP) can be reduced to a general boundary value problem as follows:

$$(BVP^*) \quad \begin{cases} u''(t) + F(t, u(t)) = 0, & t \in (0, 1), \\ (BC) \end{cases}$$

where $F : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^+$ is continuous. It is easy to check that our Theorems can appropriately apply to (BVP^*). Furthermore, in this case, property $\mathbb{P}$ automatically holds for this function $F(t, u)$ on $[0, 1] \times [0, \infty)$. Hence, all corollaries are applicable to (BVP^*). Note that for many source terms, we can easily compute corresponding "$\max F_0$, $\min F_0$, $\max F_\infty$, $\min F_\infty$" in appropriate ranges, for example, $F(t, u) := \frac{e^u - 1}{1+t^2}$ ($\max F_0 = 1$and $\min F_0 = \frac{1}{2}$), $F(t, u) := u + t^2 e^{-u}$ ($\max F_0 = \infty$, $\min F_0 = \max F_\infty = \min F_\infty = 1$).

To illustrate the use of our results, we present the following example.

Example 9. Consider the boundary value problem

$$u''(t) + p(t)\sqrt{u\left(t - \frac{1}{3}\right)} + C = 0, \quad t \in [0, 1],$$

and

$$u(t) = \phi(t), \quad t \in \left[-\frac{1}{3}, 0\right]$$
$$(BC) \begin{cases} \alpha u(0) - \beta u'(0) = 0, \\ \gamma u(1) + \delta u'(1) = 0, \end{cases}$$

where $p(t)$ is a positive continuous function on $[0, 1]$, $C > 0$, $\phi \in C([-\frac{1}{3}, 0], \mathbb{R})$ is arbitrarily given, $\alpha, \beta, \gamma, \delta \geq 0$ and $\rho := \gamma\beta + \alpha\gamma + \alpha\delta > 0$.

Then, we have

$$F(t, \psi) := p(t)\sqrt{\psi\left(-\frac{1}{3}\right)} + C,$$

which implies that F satisfies property $\mathbb{P}$. One can compute

$$\max F_\infty = 0$$

and for any p_1 and p_2 with $0 \le -\frac{1}{3} \le p_1 < p_2 \le 1$,

$$\min F_0 = \infty.$$

Applying Corollary 5 to this example, we can conclude that there is at least one positive solution to this boundary value problem.

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