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ScienceDirect
Applied Mathematics Letters 21 (2008) 298–302
Applied Mathematics Letters
www.elsevier.com/locate/aml

Positive solutions of nonlinear elliptic equations

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Received 7 February 2007; accepted 7 February 2007

Abstract

This work deals with the existence of positive solutions of convection–diffusion equations $\Delta u + f(x, u, \nabla u) = 0$ in an exterior domain of $\mathbb{R}^n (n \geq 3)$.

Keywords: Nonlinear; Elliptic; Solution; Subsolution; Supersolution; Schauder fixed point theorem

1. Introduction

We consider the nonlinear second-order elliptic equation

$$(E) \quad \Delta u + f(x, u, \nabla u) = 0. \quad x \in G_A,$$

in an exterior domain $G_A = \{x \in \mathbb{R}^n | |x| \geq A\}$, where $n \geq 3$ and $A > 0$. We try to prove, under a quite general assumption on function f, that the equation (E) has a positive solution in $G_B = \{x \in \mathbb{R}^n | |x| \geq B\}$ for some $B \geq A$, that is, there exists a function $u \in C^2(G_B)$ such that u satisfies (E) at every point $x \in G_B$. A subsolution of (E) is a function u that satisfies $\Delta u + f(x, u, \nabla u) \geq 0$, and a supersolution of (E) is a function u that satisfies $\Delta u + f(x, u, \nabla u) \leq 0$; these are defined similarly.

In 1997, Constantin [1,2] proved the existence of the equation

$$(E^*) \quad \Delta u + p(x, u) + q(|x|)x \cdot \nabla u = 0$$

in the exterior domain G_A as follows:

Theorem A. *Assume that p is locally Hölder continuous in $G_A \times \mathbb{R}$ [3] and q is of C^1. If*

$$0 \leq p(x, t) \leq a(|x|)w(t), \quad t \in \mathbb{R}_+, \; x \in \mathbb{R}^n,$$

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0893-9659/$ - see front matter
doi:10.1016/j.aml.2007.02.026

where $a \in C(\mathbb{R}_+, \mathbb{R}_+)$ *and* $w \in C^1(\mathbb{R}_+, \mathbb{R}_+)$ *with* $w(0) = 0$, *then there is a positive solution* $u(x)$ *to* (E^*) *on* G_B *for some* $B \geq A$ *with* $\lim_{|x|\to\infty} u(x) = 0$ *provided that* q *is bounded and*

$$\int_0^\infty s[a(s) + |q(s)|]\mathrm{d}s < \infty.$$

We shall extend this theorem to a more general result in the next section.

2. Main results

Define $S_B = \{x \in \mathbb{R}^n \,|\, |x| = B\}$ for $B \geq A$.

In order to prove our main result, we need the following excellent lemma; see Noussair and Swanson [5].

Lemma B. *Assume that* f *is locally Hölder continuous in* $G_A \times \mathbb{R} \times \mathbb{R}^n$. *If there are a positive subsolution* w *and a positive supersolution* v *to* (E) *in* G_B *such that* $w(x) \leq v(x)$ *for all* $x \in G_B \cup S_B$, *then* (E) *has a solution* u *in* G_B *satisfying* $w(x) \leq u(x) \leq v(x)$ *in* $G_B \cup S_B$ *and* $u(x) = v(x)$ *on* S_B.

We are now in a position to state and prove our main result.

Theorem C (*Existence Theorem*). *Suppose that* f *is locally Hölder continuous in* $G_A \times \mathbb{R} \times \mathbb{R}^n$ *and satisfies*

$$0 \leq f(x, t, z) \leq k(|x|, t) + g(|x|, x \cdot z) \quad t \in \mathbb{R}_+,\ x \in \mathbb{R}^n, z \in \mathbb{R}^n,$$

where k *and* g *satisfy:*

(A_1) $k \in C(\mathbb{R}_+ \times \mathbb{R}, \mathbb{R}_+)$ *with* $k(\cdot, 0) = 0$ *satisfies a Lipschitz condition with respect to the second variable, that is, there exists a function* $M_1 \in L^1(\mathbb{R}_+; (0, \infty))$ *such that* $|k(a, b)| \leq M_1(a)|b|$ *on* $\mathbb{R}_+ \times [-2, 2]$,

(A_2) $g \in C(\mathbb{R}_+ \times \mathbb{R}, \mathbb{R}_+)$ *with* $g(\cdot, 0) = 0$ *satisfies a Lipschitz condition with respect to the second variable, that is, there exists a function* $M_2 \in L^1(\mathbb{R}_+; (0, \infty))$ *such that* $|g(a, b)| \leq M_2(a)|b|$ *on* $\mathbb{R}_+ \times \mathbb{R}$.

Then there is a positive solution $u(x)$ *to* (E) *on* G_B *for some* $B \geq A$ *with* $\lim_{|x|\to\infty} u(x) = 0$ *if* $\int_0^\infty s[M_1(s) + M_2(s)]\mathrm{d}s < \infty$.

Proof. Let us consider the differential equation

$$(r^{n-1}y')' + r^{n-1}k_0(r, y) + r^n g_0(r, y') = 0, \quad r > 1, \tag{1}$$

where $k_0(a, b) = \begin{cases} k(a, b) & \text{if } b > 0, \\ -k(a, |b|) & \text{if } b \leq 0, \end{cases}$ and $g_0(a, b) = \begin{cases} g(a, b) & \text{if } b > 0, \\ -g(a, |b|) & \text{if } b \leq 0. \end{cases}$

Clearly, k_0 and g_0 still satisfy (A_1) and (A_2). The change of variables

$$r = \beta(s) = \left(\frac{1}{n-2}s\right)^{\frac{1}{n-2}}, \qquad h(s) = sy(\beta(s))$$

transforms (1) into

$$h''(s) + \frac{1}{n-2}\beta'(s)\beta(s)k_0\left(\beta(s), \frac{h(s)}{s}\right) + \frac{\beta(s)^3}{(n-2)^2 s} g_0\left(\beta(s), \frac{(n-2)h'(s)}{\beta(s)} - \frac{h(s)}{\beta(s)^{n-1}}\right) = 0.$$

It follows from (A_1) and (A_2) that, for each $s \in \mathbb{R}_+$, we have

$$\left|k_0\left(\beta(s), \frac{h(s)}{s}\right)\right| \leq M_1(\beta(s))\left|\frac{h(s)}{s}\right| \quad \text{for } \left|\frac{h(s)}{s}\right| \leq 2, \tag{2}$$

and

$$\left|g_0\left(\beta(s), \frac{(n-2)h'(s)}{\beta(s)} - \frac{h(s)}{\beta(s)^{n-1}}\right)\right| \leq M_2(\beta(s))\left\{\frac{(n-2)h'(s)}{\beta(s)} - \frac{h(s)}{\beta(s)^{n-1}}\right\}. \tag{3}$$

From (1)–(3), it is natural to consider

$$h''(s) + \frac{1}{n-2}\beta'(s)\beta(s)M_1(\beta(s))\left|\frac{h(s)}{s}\right| + M_2(\beta(s))\left\{h'(s) - \frac{h(s)}{s}\right\}\beta'(s)\beta(s) = 0.$$

Let

$$b(s) = \frac{1}{n-2}\frac{\beta'(s)\beta(s)M_1(\beta(s))}{s} + \frac{\beta'(s)\beta(s)M_2(\beta(s))}{s}, \quad s \geq 1,$$
$$c(s) = \beta'(s)\beta(s)M_2(\beta(s)), \quad s \geq 1.$$

It follows from $\int_0^\infty s[M_1(s) + M_2(s)]\mathrm{d}s < \infty$ that $\int_1^\infty c(s)\mathrm{d}s < \infty$ and $\int_1^\infty sb(s) < \infty$, which yields $\int_1^\infty \int_t^\infty b(s)\mathrm{d}s\mathrm{d}t < \infty$.

Let $T_0 \geq \max\{1, (n-2)A^{n-2}\}$ satisfy $2\mathrm{e}^{2\int_{T_0}^\infty c(\xi)\mathrm{d}\xi} \int_{T_0}^\infty \int_t^\infty b(s)\mathrm{d}s\mathrm{d}t \leq 1$.

Now, we will show that

$$h''(s) + c(s)h'(s) + b(s)h(s) = 0, \quad s \geq T_0 \tag{4}$$

has a solution $h(s)$ such that $|h(s) - 1| \leq 1$ for all $s \geq T_0$ and $\lim_{s\to\infty} h(s) = 1$.

Consider the Banach space $X = \{x \in C([T_0, \infty), \mathbb{R}) | x(t) \text{ is bounded}\}$ with superemum norm. Let $K = \{x \in X || x(t) - 1| \leq 1, t \geq T_0\}$ and define the operator

$$F : K \to X$$

by

$$Fx(t) = 1 - \int_t^\infty \mathrm{e}^{\int_s^\infty c(\xi)\mathrm{d}\xi} \int_s^\infty \mathrm{e}^{-\int_s^\infty c(\xi)\mathrm{d}\xi} b(r)x(r)\mathrm{d}r\mathrm{d}s, \quad t \geq T_0.$$

Since $0 \leq x(t) \leq 2$ for $x \in K$ and $t \geq T_0$,

$$0 \leq \int_t^\infty \mathrm{e}^{\int_s^\infty c(\xi)\mathrm{d}\xi} \int_s^\infty \mathrm{e}^{-\int_r^\infty c(\xi)\mathrm{d}\xi} b(r)x(r)\mathrm{d}r\mathrm{d}s$$
$$\leq 2\mathrm{e}^{2\int_{T_0}^\infty c(\xi)\mathrm{d}\xi} \int_t^\infty \int_s^\infty b(r)\mathrm{d}r\mathrm{d}s \leq 1, \quad t \geq T_0.$$

Thus $F(K) \subseteq K$.

Next, we prove that F is compact. Let $\{x_n\}_{n=1}^\infty$ be a sequence in K. Define

$$f_n(s) = \mathrm{e}^{\int_s^\infty c(\xi)\mathrm{d}\xi} \int_s^\infty \mathrm{e}^{-\int_r^\infty c(\xi)\mathrm{d}\xi} b(r)x_n(r)\mathrm{d}r, \quad \text{for } s \geq T_0.$$

Then $f_n \in L^1([T_0, \infty), \mathbb{R})$ satisfies $\lim_{p\to\infty} \int_p^\infty |f_n(s)|\mathrm{d}s = 0$ and

$$\int_{T_0}^\infty |f_n(s)|\mathrm{d}s \leq 2\mathrm{e}^{2\int_{T_0}^\infty c(\xi)\mathrm{d}\xi} \int_{T_0}^\infty \int_t^\infty b(s)\mathrm{d}s\mathrm{d}r \leq 1, \quad n \geq 1.$$

By the Lebesgue dominated convergence theorem,

$$\lim_{\delta\to 0} \int_{T_0}^\infty \int_s^{s+\delta} \mathrm{e}^{-\int_r^\infty c(\xi)\mathrm{d}\xi} b(r)\mathrm{d}r\mathrm{d}s = 0$$

and

$$\lim_{\delta\to 0} \int_{T_0}^\infty |\mathrm{e}^{\int_{s+\delta}^\infty c(\xi)\mathrm{d}\xi} - \mathrm{e}^{\int_s^\infty c(\xi)\mathrm{d}\xi}| \int_s^\infty b(r)\mathrm{d}r\mathrm{d}s = 0.$$

Therefore, for any given $\epsilon > 0$, there is a $\gamma > 0$ such that

$$2\mathrm{e}^{\int_{T_0}^\infty c(\xi)\mathrm{d}\xi} \int_{T_0}^\infty \int_s^{s+\delta} \mathrm{e}^{-\int_r^\infty c(\xi)\mathrm{d}\xi} b(r)\mathrm{d}r\mathrm{d}s < \frac{\epsilon}{2}, \quad |\delta| \leq \gamma,$$

and

$$2\mathrm{e}^{\int_{T_0}^\infty c(\xi)\mathrm{d}\xi} \int_{T_0}^\infty |\mathrm{e}^{\int_{s+\delta}^\infty c(\xi)\mathrm{d}\xi} - \mathrm{e}^{\int_s^\infty c(\xi)\mathrm{d}\xi}| \int_s^\infty b(r)\mathrm{d}r\mathrm{d}s < \frac{\epsilon}{2}, \quad |\delta| \leq \gamma.$$

Since $0 \le x_n(t) \le 2$ for all $t \ge T_0$ and $n > 1$, the previous choice of γ enables us to deduce that

$$\begin{aligned}\int_{T_0}^{\infty} |f_n(s+\delta) - f_n(s)| &\le 2\mathrm{e}^{\int_{T_0}^{\infty} c(\xi)\mathrm{d}\xi} \int_{T_0}^{\infty} |\mathrm{e}^{\int_{s+\delta}^{\infty} c(\xi)\mathrm{d}\xi} - \mathrm{e}^{\int_{s}^{\infty} c(\xi)\mathrm{d}\xi}| \int_{s}^{\infty} b(r)\mathrm{d}r\mathrm{d}s \\ &\quad + 2\mathrm{e}^{\int_{T_0}^{\infty} c(\xi)\mathrm{d}\xi} \int_{T_0}^{\infty} |\mathrm{e}^{\int_{s+\delta}^{\infty} c(\xi)\mathrm{d}\xi} - \mathrm{e}^{\int_{s}^{\infty} c(\xi)\mathrm{d}\xi}| \int_{s}^{\infty} b(r)\mathrm{d}r\mathrm{d}s \\ &< \epsilon, \quad n \ge 1, |\delta| < \gamma.\end{aligned}$$

By Riesz's theorem (see [4]), the sequence $\{f_n\}_{n=1}^{\infty}$ is compact in $L^1([T_0, \infty), \mathbb{R})$.

It follows from

$$Fx_n(t) = 1 - \int_t^{\infty} f_n(s)\mathrm{d}s, \quad t \ge T_0,\ n \ge 1$$

that $\{Fx_n\}_{n=1}^{\infty}$ is compact in K. This implies that F is a compact mapping.

By the Schauder fixed point theorem, the mapping F has a fixed point $h \in K$. It is easy to verify that h is a nonnegative solution of (4) in $[T_0, \infty)$ and satisfies $\lim_{s\to\infty} h(s) = 1$.

Take $T_1 > T_0$ so that $h(s) > 0$ for $s \ge T_1$ and let $B = (\frac{1}{n-2}T_1)^{\frac{1}{n-2}} \ge A$. Define $v(x) = y(r) = \frac{h(s)}{s}$ for $r = |x| \ge B$, where $r = \beta(s)$.

Since $\lim_{s\to\infty} h(s) = 1$, $\lim_{|x|\to\infty} v(x) = 0$.

Hence, $v(x) > 0$ on $S_B \cup G_B$ and

$$\begin{aligned}\Delta v + f(x, v(x), \nabla v(x)) &\le (r^{n-1}y')' + r^{n-1}k(r, y) + r^n g(r, y') \\ &= h''(s) + \frac{1}{n-2}\beta'(s)\beta(s)k_0\left(\beta(s), \frac{h(s)}{s}\right) \\ &\quad + \frac{\beta(s)^3}{(n-2)^2 s} g_0\left(\beta(s), \frac{(n-2)h'(s)}{\beta(s)} - \frac{h(s)}{\beta(s)^{n-1}}\right) \\ &\le h''(s) + c(s)h'(s) + b(s)h(s) = 0, \quad r \ge B\end{aligned}$$

which implies that v is a supersolution of (E) on G_B.

Clearly, $w(x) \equiv 0$ satisfies

$$\Delta w(x) + f(x, w(x), \nabla w(x)) \ge 0, \quad x \in G_B.$$

By the Lemma B we see that (E) has a solution $u(x)$ in G_B with $w(x) \le u(x) \le v(x)$ for $|x| > B$ and $u(x) = v(x)$ for $|x| = B$.

Finally, we will show that u is positive. We choose a positive number $k > \frac{n}{2B^2}$.

For any given $\epsilon > 0$, we define

$$u_\epsilon = \inf_{x\in S_B}\{u(x)\} + \epsilon \mathrm{e}^{-k|x|^2}, \quad x \in S_B \cup G_B,$$

where $u(x)$ is a solution of (E) in G_B. If $x \in G_B$, then it follows from

$$\begin{aligned}(\Delta u_\epsilon)(x) &= \epsilon(4k^2|x|^2 - 2kn)\mathrm{e}^{-k|x|^2} \\ &> 0 \ge -f(x, u, \nabla u) \\ &= (\Delta(u + \epsilon \mathrm{e}^{-kB^2}))(x),\end{aligned}$$

that $(\Delta(u + \epsilon \mathrm{e}^{-kB^2} - u_\epsilon))(x) < 0$.

On the other hand, by using the fact that $|x| \ge B$, we get

$$u(x) + \epsilon \mathrm{e}^{-kB^2} - u_\epsilon(x) \ge 0, \quad x \in G_B.$$

Since $u(x) \ge 0$ on G_B and $u_\epsilon(x)$ is bounded on G_B, the function

$$z_\epsilon(x) = u(x) + \epsilon \mathrm{e}^{-kB^2} - u_\epsilon(x), \quad x \in G_B \cup S_B$$

has a finite infimum in $G_B \cup S_B$.

For any $C > B$,

$$\inf z_\epsilon(x) = \min z_\epsilon(x) \quad \text{on } G_{BC} = \{x | B \le |x| \le C\}.$$

If there exists a $x_0 \in \{x | B < |x| \le C\}$ with $z_\epsilon(x_0) = \min_{x\in G_{BC}}\{z_\epsilon(x)\}$, then $(\Delta z_\epsilon)(x_0) \ge 0$, which is a contradiction. Thus $\min_{x\in G_{BC}} z_\epsilon(x)$ lies on $\{x | |x| = B\}$ for all $C > B$. It follows from

$$\inf_{x\in G_B\cup S_B} z_\epsilon(x) = \min_{x\in S_B} z_\epsilon(x) \ge 0$$

that $u_\epsilon(x) \le u(x) + \epsilon \mathrm{e}^{-kB^2},\ x \in G_B \cup S_B$. Letting in the previous relation $\epsilon \to 0$, we get

$$u(x) \ge \inf_{x\in S_B} u(x) = \inf_{x\in S_B} v(x) = y(B) = \frac{h((n-2)B^{n-2})}{(n-2)B^{n-2}} = \frac{h(T_1)}{T_1} > 0, \quad x \in G_B$$

and this shows that $u(x)$ is positive in G_B.

It follows from $u(x) \le v(x)$ for $|x| \ge B$ and $\lim_{|x|\to\infty} v(x) = 0$ that $\lim_{|x|\to\infty} u(x) = 0$. This completes the proof. □

Remark. $f(x, u, \nabla u) = p(x, u) + q(|x|)x \cdot \nabla u$, where p is locally Hölder continuous in $G_A \times \mathbb{R}$ satisfying

$$0 \le p(x, t) \le a(|x|)w(t), \quad t \in \mathbb{R}_+,\ x \in \mathbb{R}^n.$$

Here $a \in C(\mathbb{R}_+, \mathbb{R}_+)$, $w \in C^1(\mathbb{R}_+, \mathbb{R}_+)$ with $w(0) = 0$, q is of C^1 with a bound and $\int_0^\infty s[a(s) + |q(s)|]\mathrm{d}s < \infty$. Moreover, if $k(|x|, t) = a(|x|)w(t)$ and $g(|x|, x \cdot z) = q(|x|)x \cdot z$, then our Theorem C is reduced to Theorem A.

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