VALUATION OF FLOATING RANGE NOTES IN A LIBOR MARKET MODEL

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This study derives an approximate pricing formula of floating range notes (FRNs) within the multifactor LIBOR market model (LMM) framework. The LMM features the ease for calibration procedure, and the resulting pricing formula is more tractable. In addition, since the underlying rate of FRNs is usually the LIBOR rate, the pricing of the FRNs under the LMM is more direct and full of intuition. © 2008 Wiley Periodicals, Inc. Jrl Fut Mark 28:697–710, 2008

INTRODUCTION

The primary purpose of this study is to provide a general pricing formula for floating range notes (FRNs) in the framework of the LIBOR market model (LMM), which is more intuitive and tractable than other interest rate models. An FRN is a variety of floating-rate notes that entitles the holder to receive (or pay) periodically an interest payment at the end of each period. The payment is calculated by multiplying the interest rate specified at the start of each period

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Received April 2006; Accepted June 2007

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The Journal of Futures Markets, Vol. 28, No. 7, 697–710 (2008)

Published online in Wiley InterScience (www.interscience.wiley.com).
DOI: 10.1002/fut.20310

by the number of days that a reference interest rate (most commonly the LIBOR rate) lies within a specified range. At maturity, the principal of the note is paid back along with the last interest payment.

In contrast with the traditional floating-rate notes, FRNs offer higher rates via the sale of embedded digital options. An investor can increase his interest return by investing in FRNs when he has a keen view that the involved interest rates will stay within a range. It is worth noting that the payment of FRNs depends not only on the floating rate observed at the start of each period but also on the daily level of the reference interest rates, and this feature complicates the valuation procedure.

Some earlier researches have been conducted on the pricing of FRNs. Turnbull (1995) derived a pricing formula for FRNs by using the one-factor Gaussian HJM model (Heath, Jarrow, & Morton, 1992). In the same model and in a more intuitive way, Navatte and Quittard-Pinon (1999) derived a pricing formula for FRNs by employing the changing-numéraire technique. To enhance the calibration capacity to the interest rate covariance matrix obtained in the market, Nunes (2004) generalized the aforementioned results to a formula under the multifactor Gaussian HJM model. Eberlein and Kluge (2006) also provided a pricing formula for FRNs in the Lévy term-structure model of Eberlein and Raible (1999).

The interest rate models used in the previous articles are based on the HJM framework, which relies on the specification of the instantaneous forward rates. The instantaneous forward rates are market-non-observable and abstract, which leads to a complicated transformation from the instantaneous forward rates to the LIBOR rates. Moreover, the pricing formulas of widely traded interest rate derivatives, such as caps, floors, swaptions, etc., based on the Gaussian HJM model are not consistent with market practice. This results in some difficulties in the calibration procedure. Accordingly, pricing FRNs in the LMM framework should be a better choice.

The LMM was developed by Brace, Gatarek, and Musiela (1997; BGM), Miltersen, Sandmann, and Sondermann (1997), and Musiela and Rutkowski (1997). The LMM is widely used by practitioners because of the advantage that the cap pricing formula in the LMM framework is Black's formula, which is consistent with market practice and makes the calibration procedure easier. Furthermore, the underlying interest rates in FRNs are usually the LIBOR rates, and hence pricing FRNs in the LMM is more straightforward.

As examined in Rogers (1996), the Gaussian HJM term-structure model has an important theoretical limitation: The rate can attain negative values with positive probability, which may cause some pricing error in many cases. In contrast, one advantage of adopting the LMM is that the underlying LIBOR rates are positive, which avoids the problem.

Navatte and Quittard-Pinon (1999) and Nunes (2004) assumed that the bounds of ranges in FRNs can vary daily. However, from a theoretical point of view, each embedded digital option with different bounds should have a different spread to compensate for the abandoned payoff. Therefore, the spreads in FRNs were allowed to vary daily to make the pricing formula more general for practical implementation.

This article is organized as follows. The second section briefly reviews the LMM and some useful techniques for the approximate log-normalization and the drift-adjustment when the measure is changed. In the third section, the contracts of a general FRN and some preliminary financial products are introduced and priced. The conclusions are made in the fourth section.

PRESENTATION OF THE LMM

The LMM and some useful techniques for the approximate log-normalization and the drift-adjustment while the measure is changed are reviewed in this section.

The fact that trading takes place continuously in time over an interval $[0, \mathcal{T}]$, $0 < \mathcal{T} < \infty$, is assumed. The uncertainty is described by the filtered spot martingale probability space $(\Omega, F, Q, \{F_t\}_{t \in [0,\mathcal{T}]})$ and an m-dimensional independent standard Brownian motion $W(t) = (W_1(t), W_2(t), \ldots, W_m(t))$ is defined on it. The flow of information accruing to all the agents in the economy is represented by the filtration $\{F_t\}_{t \in [0,\mathcal{T}]}$ that satisfies the usual hypotheses.[1] Note that Q denotes the domestic spot martingale probability measure. The notations are listed as follows.

$P(t, T)$ is the time t price of a zero-coupon bond (ZCB) paying one dollar at time T, $L(t, T; \delta_T)$ is the forward LIBOR rate contracted at time t and applied to the period $[T, T + \delta_T]$ with $0 \leq t \leq T \leq T + \delta_T \leq \mathcal{T}$, QT is the forward martingale measure with respect to the numéraire $P(\cdot, T)$.

The relationship between $L(t, T; \delta_T)$ and $P(t, T)$ can be expressed as follows:

$$L(t, T; \delta_T) = \frac{1}{\delta_T}(P(t, T) - P(t, T + \delta_T))/P(t, T + \delta_T). \tag{1}$$

On the basis of the results of HJM (1992), BGM model interest rate behavior in terms of the forward LIBOR rates, their results are specified briefly as follows.

Assumption 1 (*The LIBOR Rate Dynamics Under The Measure Q*): The dynamics of the LIBOR rate $L(t, T; \delta_T)$ under the spot martingale measure Q is given as follows:

[1] The filtration $\{F_t\}_{t \in [0,\mathcal{T}]}$ is right continuous and F_0 contains all the Q-null sets of F.

$$\frac{dL(t, T; \delta_T)}{L(t, T; \delta_T)} = \gamma(t, T)\sigma(t, T + \delta_T)dt + \gamma(t, T)dW(t) \tag{2}$$

where $0 \leq t \leq T \leq T$, $\sigma(t, \cdot)$ is defined as follows:

$$\sigma(t, T) = \begin{cases} \displaystyle\sum_{j=1}^{\lfloor \delta^{-1}(T-t) \rfloor} \frac{\delta L(t,T - j\delta;\delta)}{1 + \delta L(t,T - j\delta;\delta)} \gamma(t, T - j\delta) & t \in [\tau, T - \delta] \\ & \& \; T - \delta > 0, \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

where δ is some designated length of time,[2] $\lfloor \delta^{-1}(T - t) \rfloor$ denotes the greatest integer that is less than $\delta^{-1}(T - t)$, and the deterministic function $\gamma: R_+^2 \rightarrow R^m$ is bounded and piecewise continuous.

Similar to the multifactor pricing formula used in Nunes (2004), a multifactor LMM to price FRNs was adopted. For greater flexibility, the number of random shocks, m, is not precisely designated but rather depends on the simplicity and the accuracy required by the user.[3] In practice, if the duration of an FRN is short, may be shorter than one year, one-factor model is enough to specify the variability of interest rates. As the duration is lengthened, using of a three-factor model, i.e. $m = 3$, which captures the shift and twist of the entire forward rate curve, is suggested. The first two random shocks can be interpreted, respectively, as the short-term and long-term factors causing the shift of different maturity ranges on the term structure. The correlation between the short-term and long-term forward rates is specified by the third random shock.

According to the derivation procedure of the LMM in BGM (1997), $\{\sigma(t, T)\}_{t\in[0,T]}$ stands for the volatility process of the bond price $P(t, T)$. Based on the definition of the bond volatility process (3), $\{\sigma(t, T + \delta_T)\}_{t\in[0,T+\delta_T]}$ in (2) is found to be stochastic rather than deterministic. Thus, the stochastic differential equation (2) is not solvable and the distribution of $L(T, T; \delta_T)$ is unknown. However, given a fixed initial time, τ, $\sigma(t, T)$ can be approximated by $\overline{\sigma}^{\tau}(t,T))$, which is defined by

$$\left\{ \overline{\sigma}^{\tau}(t, T) = \begin{cases} \displaystyle\sum_{j=1}^{\lfloor \delta^{-1}(T-t) \rfloor} \frac{\delta L(\tau,T - j\delta;\delta)}{1 + \delta L(\tau,T - j\delta;\delta)} \gamma(t, T - j\delta), & t \in [\tau, T - \delta] \\ & \& \; T - \delta > 0 \\ 0 & \text{otherwise} \end{cases} \right\} \tag{4}$$

where $0 \leq t \leq T \leq T$. It means that the calendar time of the process

[2] For ease of computation of Equation (3), $\delta > 0$ (for example, $\delta = 0.25$) can be fixed.

[3] For more details regarding the performance of one- and multi-factor models, please refer to Driessen, Klaassen, and Melenberg (2003) and Rebonato (1999).

$\{L(t, T{-}j\delta; \delta)\}_{t\in[0,T-j\delta]}$ in (4) is frozen at its initial time τ and thus the process $\{\overline{\sigma}^{\tau}(t, T)\}_{t\in[\tau,T]}$ becomes deterministic. By substituting $\overline{\sigma}^{\tau}(t, T+\delta)$ for $\sigma(t, T+\delta)$ in the drift term of (2), the drift and the volatility terms in (2) will be deterministic; hence, it can be solved and the approximate distribution of $L(T, T; \delta_T)$ was found to be log-normally distributed. This argument is the Wiener chaos order 0 approximation, which is first used by BGM (1997) for pricing interest rate swaptions. It was developed further in Brace, Dun, and Barton (1998) and formalized by Brace and Womersley (2000). The approximation also appeared in Schlögl (2002). The accuracy of this approximation is shown to be very accurate.

Proposition 1 (The Approximate LIBOR Rate Dynamica Under the Measure Q): The approximate dynamics of the LIBOR rate $L(t,T;\delta_T)$ under the spot martingale measure Q is given as follows:

$$\frac{dL(t, T; \delta_T)}{(L(t, T; \delta_T)} = \gamma(t, T)\overline{\sigma}^{\tau}(t, T+\delta)dt + \gamma(t, T)dW(t) \tag{5}$$

where $\tau \leq t \leq T \leq T$.

The following proposition specifies the general rule under which the LIBOR rate dynamics changes after the change of the underlying measure. This rule is useful for deriving the pricing formulas of FRNs.

Proposition 2 (The Draft Adjustment Technique in Different Measure): The dynamics of the forward LIBOR rate $L(t, S, \delta_S)$ under an arbitrary forward martingale measure Q^T, where $T \geq S$, is given as follows:

$$\frac{dL(t, S, \delta_S)}{L(t, S, \delta_S)} = \gamma(t, S)(\overline{\sigma}^{\tau}(t, S+\delta_S) - \overline{\sigma}^{\tau}(t, T))dt + \gamma(t, S))dW(t) \tag{6}$$

where $\tau \leq t \leq S$.[4]

The rates described in the LMM are the forward LIBOR rates underlying the caps and floors that are actively traded in financial markets; thus, the market data can be employed to calibrate the parameters in the model. There are many different calibration methodologies. One approach is to calibrate the model covariance matrix by simultaneously fitting the caps and swaptions (e.g. BGM, 1997; Gaterek, 2000; Hull, 2000). Another approach takes the historical interest rate correlation matrix as an input and engages in a simultaneous

[4] $W(t)$ was employed to denote an independent m-dimensional standard Brownian motion under an arbitrary measure without causing any confusion.

calibration of the LMM to the percentage volatilities and to the historical correlation matrix of the underlying forward LIBOR rates (e.g. Pedersen, 1998; Rebonato, 1999; Wu, 2002). See Brigo and Mercurio (2001) for more details.

Having briefly introduced the LMM, the pricing formulas of FRNs are derived in the following section.

VALUATION OF FRNs

The main purpose of this section is to derive the approximate closed-form pricing formulas for FRNs in the LMM. Before introducing a general FRN contract, some notations are defined. τ is the current time and the considered time flow is $0 \leq T_0 < t < T_1 < T_2 < \cdots < T_n$, $C(\tau, T_i)$ is the time τ value of the FRN ith coupon paid at time T_i, $K_{ij}^U(K_{ij}^L)$ is the upper (lower) bound of the range employed at time T_{ij}, D_j is the number of days in the year that contains the period $(T_i, T_{i+1}]$, where $(T_i, T_{i+1}]$ denotes the period from date T_i (excluding this date) up to and including date T_{i+1}, n_0 is the number of days for the period $(\tau, T_1]$, N_i is the number of days for the period $(T_i, T_{i=1}]$, T_{0j} is the date $\tau + j$ for $j = 1, 2, \ldots, n_0$ $(T_{0,n0}=T_1)$, T_{ij} is the date $T_i + j$ for $j = 1, 2, \ldots, N_i$ and $i = 1, 2, \ldots, n-1$ $(T_{\mathrm{i},N0} = T_\mathrm{i} = 1)$, Δ_{ij} is the spread employed for the date T_{ij}, Γ_{ij} is the length of the compounding period (in years) of the reference interest rate observed at time T_{ij} and T_{ij}^* the date $T_{ij} + \Gamma_{ij}$.

From now on, the third argument in $L(t, T_{ij}; \Gamma_{ij})$ is discarded, namely Γ_{ij}, with the hope of not causing any confusion, and still each LIBOR rate has its own compounding period.

Consider an FRN with reset dates $\{T_0, T_1, \ldots, T_{n-1}\}$ *and* payment dates $\{T_1, \ldots, T_n\}$, where T_0 is the latest reset date, T_1 is the next reset date, and T_n is the expiry date. The coupon payments of this FRN is defined as follows.

Definition 1: For the FRN, the coupon at date T_1 is defined to be

$$C(T_1, T_1) = \left[V(T_0, \tau) + \sum_{j=1}^{n_0} (L(T_0, T_0) + \Delta_{0j}) \times I(T_{oj}) \right] / D_0 \tag{7}$$

where $V(T_0, \tau)$ denotes the realized-payoff amount in the period $(T_0, \tau]$ and

$$I(T_{0j}) = \begin{cases} 1, & K_{0j}^L \leq L(T_{0j}, T_{0j}) \leq K_{0j}^U \\ 0 & \text{otherwise} \end{cases} \tag{8}$$

for $j = 1, 2, \ldots n_0$. The coupon at date T_{i+1}, for $i = 1, \ldots, n-1$, is defined to be

$$C(T_{i+1}, T_{i+1}) = \left[\sum_{j=1}^{N_i} (L(T_i, T_i) + \Delta_{ij}) \times I(T_{ij}) \right] / D_i \tag{9}$$

where

$$I(T_{ij}) = \begin{cases} 1, & K_{ij}^{L} \le L(T_{ij}, T_{ij}) \le K_{ij}^{U}, \\ 0 & \text{otherwise,} \end{cases} \tag{10}$$

for $j = 1, 2, \ldots, N_i$. The principal, assumed to be 1, is paid back at date T_n.

The pre-specified range ($[K_{ij}^{L}, K_{ij}^{U}]$) for each observed date can vary daily or across different compounding periods. As given in Nunes (2004), a more general case that allows the range to vary daily is derived. In addition, the pre-specified spread (Δ_{ij}) is allowed to vary daily, which reflects the different compensation arising from selling the digital options with different ranges. As the spreads in each compounding period are set to a fixed level, the pricing formula degenerates to the case considered in Nunes (2004), but in the multifactor LMM framework.

Before deriving the FRN pricing formula, the four preliminary financial products, namely delayed digital options, delayed range digital options, delayed interest-or-nothing digital options, and delayed interest-or-nothing range digital options are priced first. Then, an FRN is priced using the fact that an FRN is a linear combination of these four products.

Delayed Digital Options (DO)

An interest rate delayed digital call (put) option (DC (DP)) pays one currency unit at maturity T_{i+1} if the reference interest rate $L(T_{ij}, T_{ij})$ that matured previously at time T_{ij} with the compounding period $[T_{ij}, T_{ij}^{*}]$ lies above (below) the strike rate K_{ij}. The final payoff of this option at time T_{i+1} is precisely given as follows[5]:

$$DO(T_{i+1}, T_{i+1}; T_{ij}; K_{ij}) = 1_{\{\theta L(T_{ij}, T_{ij}) > \theta K_{ij}\}}$$

where $L(T_{ij}, T_{ij})$ is a matured LIBOR rate for the period $[T_{ij}, T_{ij}^{*}]$, θ set to 1 stands for a digital call and -1 for a digital put.

Theorem 1: The value of the DO at time τ is given as follows:

$$DO(\tau, T_{i+1}; T_{ij}; K_{ij}) = P(\tau, T_{i+1})\, N(\theta d(T_{ij})) \tag{11}$$

with

$$d(T_{ij}) = \frac{\ln\left(\frac{L(\tau, T_{ij})}{K_{ij}}\right) + \rho(\tau, T_{ij}; T_{i+1}) - \frac{1}{2}V(\tau, T_{ij})}{\sqrt{V(\tau, T_{ij})}} \tag{12}$$

[5] $1_{\{\cdot\}}$ is an indicator function, defined as follows:

$$l_{\{A\}}(\omega) = \begin{cases} 1 & \text{if } \omega \in A \\ 0 & \text{otherwise} \end{cases}$$

where

$$\rho(\tau, T_{ij}; T_{i+1}) = \int_{\tau}^{T_{ij}} \gamma(t, T_{ij}) \cdot (\overline{\sigma}^{\tau}(t, T_{ij}^{*}) - (\overline{\sigma}^{\tau}(t, T_{i+1}))dt \quad (13)$$

$$V(\tau, T_{ij}) = \int_{\tau}^{T_{ij}} \|\gamma(t, T_{ij})\|^2 dt. \quad (14)$$

Proof: The proof is given in Appendix A.

Delayed Digital Range Options

A delayed range digital option (DRO) is similar to a DO except that the payment occurs as the reference rate lies inside a pre-specified range $[K_{ij}^{L}, K_{ij}^{U}]$. The final payoff of a general DRO at time T_{i+1} is defined as follows:

$$DRO(T_{i+1}, T_{i+1}; T_{ij}; K_{ij}) = 1_{\{K^{L}{}_{ij} \leq L(T_{ij}, T_{ij}) \leq K^{U}{}_{ij}\}}. \quad (15)$$

Based on the property in probability measure theory, the DRO payoff can be expressed in terms of two DC payoffs. It means that (15) can be rewritten as follows:

$$DRO(T_{i+1}, T_{i+1}; T_{ij}; K_{ij}) = DC(T_{i+1}, T_{i+1}; T_{ij}; K_{ij}^{L}) - DC(T_{i+1}, T_{i+1}; T_{ij}; K_{ij}^{U}).$$

Theorem 2: The time τ value of the DRO is equal to

$$DRO(\tau, T_{i+1}; T_{ij}; K_{ij}) = DC(\tau, T_{i+1}; T_{ij}; K_{ij}^{L}) - DC(\tau, T_{i+1}; T_{ij}; K_{ij}^{U}). \quad (16)$$

Remark 1: For a DO, if the maturity date T_{ij} of its reference rate equals T_{i+1}, which is also the maturity date of the DO, then the DO becomes an ordinary digital option without delaying its payoff. Similarly, as $T_{ij} = T_{i+1}$, a DRO also becomes an ordinary digital range option.

Delayed Interest-or-Nothing Digital Options (DIO)

A delayed interest-or-nothing digital call (put) option (DIC (DIP)) pays a floating interest payment $L(T_i,T_i)$ at maturity date T_{i+1} if the reference interest rate $L(T_{ij},T_{ij})$ is above (below) a pre-specified strike rate K_{ij}. The contract can be stated formally by specifying its final payoff at time T_{i+1} as follows:

$$DIO(T_{i+1}, T_{i+1}; T_{ij}; K_{ij}) = L(T_i, T_i) 1_{\{\theta L(T_{ij}, T_{ij}) > \theta K_{ij}\}}$$

$\theta = 1$ stands for a digital call option and -1 for a digital put option.

Theorem3: The time τ value of the DIO is given as follows:

$$DIO(\tau, T_{i+1}; T_{ij}; K_{ij}) = P(\tau, T_{i+1})L(\tau, T_i)\exp(\rho(\tau, T_i; T_{i+1}))N(\theta e(T_{ij})) \quad (17)$$

with

$$e(T_{ij}) = \frac{\ln(\frac{L(\tau, T_{ij})}{K_{ij}}) + \eta(\tau, T_i; T_{ij}; T_{i+1}) - \frac{1}{2}V(\tau, T_{ij})}{\sqrt{V(\tau, T_{ij})}}$$

where

$$\eta(\tau, T_i; T_{ij}; T_{i+1}) = \int_{\tau}^{T_i} \gamma(t, T_{ij)}.(\overline{\sigma}^{\tau}(t, T^*_{ij}) - \overline{\sigma}^{\tau}(t, T_{i+1}) + \gamma(t, T_i))dt \quad (18)$$

$\rho(\cdot, T_{ij}; T_{i+1})$ and $V(\tau, T_{ij})$ are defined in Equations (13) and (14).

Proof: The proof is given in Appendix B.

Delayed Interest-or-Nothing Range Digital Options

A delayed interest-or-nothing range digital option (DIRO) pays a floating interest payment at maturity T_{i+1} if the reference interest rate $L(T_{ij}, T_{ij})$ lies within a pre-specified range $[K^L_{ij}, K^U_{ij}]$ The final payoff of the DIRO is defined as follows:

$$DIRO(T_{i+1}, T_{i+1}; T_{ij}; K_{ij}) = L(T_{ij}, T_{ij})1_{\{K^L_{ij} \leq L(T_{ij}, T_{ij}) \leq K^U_{ij}\}}.$$

Similar to DROs, DIROs can also be expressed in terms of two DICs, i.e.

$$DIRO(T_{i+1}, T_{i+1}; T_{ij}; K_{ij}) = DIC(T_{i+1}, T_{i+1}; T_{ij}; K^L_{ij}) - DIC(T_{i+1}, T_{i+1}; T_{ij}; K^U_{ij}).$$

Thus, the pricing formula of the DIRO can be expressed in terms of the pricing formulas of DICs, and the result is presented in the following theorem.

Theorem 4: The time τ value of the DIRO is equal to

$$DIRO(\tau, T_{i+1}; T_{ij}; K_{ij}) = DIC(\tau, T_{i+1}; T_{ij}; K^L_{ij}) - DIC(\tau, T_{i+1}; T_{ij}; K^U_{ij}). \quad (19)$$

Remark 2: If the maturity date T_{ij} of its reference rate equals T_{i+1}, which is also the maturity date of the DIO and DIRO, then the DIO and DIRO become, respectively, an ordinary interest-or-nothing digital option and an ordinary interest-or-nothing range option without delaying its payoff.

Range Notes

As seen in Definition 1, an FRN is a linear combination of DROs, DIROs, and a ZCB. With the above preliminary theorems, the pricing formula of the FRN can be derived and is given in the following theorem.

Theorem 5: For the FRN as defined in Definition 1, the time τ value of its first coupon is equal to:

$$C(\tau, T_1) = \left[V(T_0, \tau)P(\tau, T_1) + \sum_{j=1}^{n_0}(L(T_0, T_0) + \Delta_{0j})DRO(\tau, T_1; T_{0j}; K_{0j})\right]/D_0.$$

The time τ value of other coupons is given as follows:

$$C(\tau, T_{i+1}) = \sum_{j=1}^{N_i}\frac{1}{D_i}[DIRO(\tau, T_{i+1}; T_{ij}; K_{ij}) + \Delta_{ij} \times DRO(\tau, T_{i+1}; T_{ij}; K_{ij})]$$

for $i = 1, 2, \ldots, n - 1$. The time τ value of the principal, assumed to be 1, is equal to $P(\tau, T_n)$.

Thus, the time τ value of the FRN is equal to:

$$FRN = \sum_{i=1}^{n} C(\tau, T_i) + P(\tau, T_n). \tag{20}$$

Proof: According to Definition 1, an FRN is a linear combination of DROs, DIROs, and a ZCB. Thus, the pricing formula of the FRN can be easily obtained by employing Theorems 1–4.

Similar to the pricing formula as given in Nunes (2004), the formula is derived in a multifactor LMM framework that has an advantage of enhancing the model calibration to the interest rate covariance matrix observed in the market. Because the LMM prices caps and floors consistently with Black's formula widely used in the market, the implied volatility quoted in the market is consistent with the model volatility, which makes the calibration procedure of the LMM to be easier than other interest rate models.

In addition, since the underlying interest rates of FRNs are usually the LIBOR rates, modeling interest rate behavior based on LIBOR rates rather than other abstract interest rates, such as instantaneous spot or forward rates, can avoid a complicated transformation from abstract interest rates to the LIBOR rates, which makes our pricing formula more intuitively clearer. Therefore, all parameters in the pricing model (20) can be easily calibrated from market data and the formula is more tractable and feasible for practical implementation.

Unlike the Gaussian HJM term-structure model, the LMM has another advantage that it can avoid the pricing error arisen from negative rates with positive probability. Rogers (1996) indicated that the Gaussian HJM model may cause some pricing error in many cases. However, since the LMM has a log-normal volatility structure and the underlying LIBOR rates are positive, the pricing error can be avoided, and thereby making the pricing more accurate.

CONCLUSIONS

This study has derived an approximate pricing formula of floating range notes within the context of the multifactor LMM. As compared with the previous researches within the HJM-type term-structure model, our pricing formulas derived under the LMM have some advantages: easy and flexible to calibrate the model parameters, more intuitively clearer, and avoiding the pricing error resulting from negative rates. Therefore, the pricing formulas within a multifactor LMM are more suitable for practical implementation.

APPENDIX A: PROOF OF THEOREM 1

The DO under the forward measure $Q^{T_{i+1}}$ is priced as follows:

$$\begin{aligned} DO(\tau, T_{i+1}; T_{ij}; K_{ij}) &= P(\tau, T_{i+1}) E^{Q^{T_{i+1}}}(1_{\{\theta L(T_{ij}, T_{ij}) > \theta K_{ij}\}} | F_\tau) \\ &= P(\tau, T_{i+1}) Q^{T_{i+1}}(\theta L(T_{ij}, T_{ij}) > \theta K_{ij} | F_\tau). \end{aligned} \tag{A.1}$$

Applying Propositions 1 and 2, $L(T_{ij}, T_{ij})$ under the measure $Q^{T_{i+1}}$ is given by

$$L(T_{ij},T_{ij}) = L(\tau,T_{ij}) \exp\left(\rho(\tau,T_{ij};T_{i+1}) - \frac{1}{2} V(\tau,T_{ij}) + Z_{ij} \right) \tag{A.2}$$

where

$$Z_{ij} = \int_{\tau}^{T_{ij}} \gamma(t, T_{ij}) dW(t) \sim N(0, V(\tau, T_{ij}))$$

and $\rho(\tau, T_{ij}; T_{i+1})$ and $V(\tau, T_{ij})$ are defined, respectively, in (13) and (14).

As $\theta = 1$ and taking (A.2) into (A.1), the pricing formula of the DC is defined as follows:

$$
\begin{aligned}
&DC(\tau, T_{i+1}; T_{ij}; K_{ij})\\
&\quad = P(\tau, T_{i+1})Q^{T_{i+1}}\left(L(\tau, T_{ij})\exp\left(\rho(\tau, T_{ij}; T_{i+1}) - \frac{1}{2}V(\tau, T_{ij}) + Z_{ij}\right) > K_{ij}|F_\tau\right)\\
&\quad = P(\tau, T_{i+1})Q^{T_{i+1}}\left(\rho(\tau, T_{ij}; T_{i+1}) - \frac{1}{2}V(\tau, T_{ij}) + Z_{ij} > \ln\frac{K_{ij}}{L(\tau,T_{ij})}|F_\tau\right)\\
&\quad = P(\tau, T_{i+1})Q^{T_{i+1}}\left(\frac{-Z_{ij}}{\sqrt{V(\tau,T_{ij})}} \le \frac{\ln\frac{L(\tau,T_{ij})}{K_{ij}} + \rho(\tau,T_{ij}; T_{i+1}) - \frac{1}{2}V(\tau,T_{ij})}{\sqrt{V(\tau,T_{ij})}}|F_\tau\right)\\
&\quad = P(\tau, T_{i+1})N(d(T_{ij}))
\end{aligned}
$$

where $d(T_{ij})$ is defined in (12).

Similarly, as $\theta = 1$, the pricing formula of the DP is defined as follows:

$$
\begin{aligned}
&DP(\tau, T_{i+1}; T_{ij}; K_{ij})\\
&\quad = P(\tau, T_{i+1})Q^{T_{i+1}}\left(L(\tau, T_{ij})\exp\left(\rho(\tau, T_{ij}; T_{i+1}) - \frac{1}{2}V(\tau, T_{ij}) + Z_{ij}\right) < K_{ij}|F_\tau\right)\\
&\quad = P(\tau, T_{i+1})Q^{T_{i+1}}\left(\rho(\tau, T_{ij}; T_{i+1}) - \frac{1}{2}V(\tau, T_{ij}) + Z_{ij} < \ln\frac{K_{ij}}{L(\tau,T_{ij})}|F_\tau\right)\\
&\quad = P(\tau, T_{i+1})Q^{T_{i+1}}\left(\frac{Z_{ij}}{\sqrt{V(\tau,T_{ij})}} \le \frac{\ln\frac{L(\tau,T_{ij})}{K_{ij}} + \rho(\tau,T_{ij};T_{i+1}) - \frac{1}{2}V(\tau,T_{ij})}{\sqrt{V(\tau,T_{ij})}}|F_\tau\right)\\
&\quad = P(\tau, T_{i+1})N(-d(T_{ij})).
\end{aligned}
$$

APPENDIX B: PROOF OF THEOREM 3

The DIO under the forward measure $Q^{T_{i+1}}$ is priced as follows:

$$DIO(\tau, T_{i+1}; T_{ij}; K_{ij}) = P(\tau, T_{i+1})E^{Q^{T_{i+1}}}(L(T_i, T_i)1_{\{\theta L(T_{ij},T_{ij}) > \theta K_{ij}\}}|F_\tau).$$

Applying Equation (A.2), $DIO(\tau, T_{i+1}; T_{ij}; K_{ij})$ can be rewritten as

$$P(\tau, T_{i+1})L(\tau, T_i)\exp(\rho(\tau, T_i; T_{i+1}))E^{Q^{T_{i+1}}}\left(\frac{dR^{T_{i+1}}}{dQ^{T_{i+1}}}1_{\{\theta L(T_{ij},T_{ij}) > \theta K_{ij}\}}|F_\tau\right).$$

where

$$\frac{dR^{T_{i+1}}}{dQ^{T_{i+1}}} = \exp\left(\int_\tau^{T_i}\gamma(t, T_i)dW(t) - \frac{1}{2}\int_\tau^{T_i}\|\gamma(t, T_i)\|^2 dt\right)$$

denotes Radon-Nikodým derivative that defines a new equivalent measure $R^{T_{i+1}}$ on the same measurable space (Ω,F). Thus, one obtains

$$DIO(\tau, T_{i+1}; T_{ij}; K_{ij}) = P(\tau, T_{i+1})L(\tau, T_i)\exp(\rho(\tau, T_i; T_{i+1}))R^{T_{i+1}}(\theta L(T_{ij}, T_{ij}) > \theta K_{ij}|F_\tau).$$

According to Girsanov's Theorem, $L(T_{ij}, T_{ij})$ is derived under the measure $R^{T_{i+1}}$ as follows:

$$L(T_{ij}, T_{ij}) = L(\tau,T_{ij})\exp\left(\eta(\tau, T_i; T_{ij}; T_{i+1}) - \frac{1}{2}V(\tau, T_{ij}) + Z_{ij}\right)$$

where $V(\tau, T_{ij})$, Z_{ij}, and $\eta(\tau, T_i; T_{ij}; T_{i+1})$ are defined, respectively, in (14), (A.3), and (18).

Similar to the deriving process of DC in Appendix A, the pricing formula of the DIC can be derived by setting $\theta = 1$ and by substituting (B.2) into (B.1). In the same way, the pricing formula of the DIP can be derived by setting $\theta = -1$.

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