Valuation of Interest Rate Spread Options in a Multifactor LIBOR Market Model

TING-PIN WU AND SON-NAN CHEN

TING-PIN WU
is an assistant professor in the Department of Statistics, National Taipei University in Taiwan.
tpwu@mail.ntpu.edu.tw

SON-NAN CHEN
is a professor in the Department of Banking and Finance, National Chengchi University in Taiwan.
slchen@nccu.edu.tw

Within the multifactor LIBOR market model, the authors examine three types of interest rate spread options: LIBOR vs. LIBOR, LIBOR vs. swap rate, and swap rate vs. swap rate. These financial products are widely traded in the marketplace or are embedded in structured notes, such as CMS range accruals and steepeners. In the first case, the authors show that the drift has an impact on the pricing which differs from the results of previous research. The authors also present a new approach to approximating the distribution of a forward swap rate under the LIBOR market model and then employ it to price CMS spread options. The numerical examples show that the approximate pricing formulas are robustly accurate as compared with Monte Carlo simulation using recent two-year data.

Interest rate spread options (IRSOs), also known as interest rate difference options, are contracts written on the difference between two interest rates. The subject interest rates may be short rates, intermediate rates, or long rates. For example, a 6-month LIBOR rate contains information regarding short-term interest rates while a 10-year or 20-year constant maturity swap (CMS) rate describes the overall level of the yield curve. Thus, IRSOs function as instruments for practitioners to view relative changes in the different ranges of the yield curve.

The emergence of IRSOs have two main motivations. The first is to control the risk due to changes in the shape of the yield curve. Interest rates have become more volatile during the past decades and thus the issue of managing the risk of interest rates has attracted greater attention from financial institutions and market participants. In order to hedge interest rate risk, various types of financial instruments have been implemented, such as interest rate caps, floors, collars, swaps, and swaptions.

Though basis swaps can be used to hedge exposure in the spread of two yields for financial institutions whose assets and liabilities are dependent on different floating reference rates, as structured instruments they may be inefficient for a specific purpose. In contrast, IRSOs can be tailored to hedge risk that depends on whether the spread of two interest rates is above or below a specified level, or within or outside a specified range on a specific future date. IRSOs can also be used as ancillary instruments for basis swaps. For example, an end-user can use a basis swap to capitalize on anticipated yield curve movements while purchasing IRSOs to eliminate downside risk.

Second, IRSOs can be used to enhance profits from a change in the spread between two interest rates or to lock in a current spread. For example, in October 2005 the spread between the 2-year and 10-year U.S. dollar CMS rate was around 80 basis points, which is significantly narrow. (By comparison, the spread reached 110 basis points in January 2005 and 160 basis points in January 2004.) Due to the flatness of the yield curve, investors were able to generate attractive returns as the yield curve steepened.

The most popular CMS spread option products are steepeners and range accruals. Steepeners pay a high coupon the first year, and the investor is subsequently paid a coupon based on the spread between two CMS rates (for example, 2-year and 10-year CMS rates), multiplied by a specified leverage ratio. Range accruals pay a high coupon if (for example) the 10-year CMS rate minus the 2-year CMS rate remains within a pre-specified range or is above (below) a certain barrier for every day of the coupon period. According to the data given in Sawyer [2005], a trading volume of $30 billion of CMS spread options was recorded in 2005, and it has since increased.

There have been several studies on the pricing of IRSOs. Longstaff [1990] examined an IRSO on a yield spread based on an extended version of the Cox, Ingersoll, and Ross [1985] interest rate model. Both Fu [1996] and Miyazaki and Yoshida [1998] derived pricing formulas for yield-spread options within the Gaussian Heath, Jarrow, and Morton (1992, HJM) framework. Fu [1996] introduced a two-factor HJM model for pricing IRSOs, which permits imperfect correlations for interest rates of different maturities.

Fu [1996] further priced an IRSO on the difference between two LIBOR rates with different compounding periods. His pricing formula showed that the drift terms would not affect the option price. According to the theory of the LIBOR market model (LMM), however LIBOR rates with different expiration dates will not be martingales under the same probability measure; thus the drift terms actually have an impact on the option price.

The main purpose of this article is to provide closed-form pricing formulas for IRSOs within the multifactor LMM framework. Three types of IRSOs are considered: an IRSO on the difference between two LIBOR rates with different compounding periods, an IRSO on the difference between a LIBOR rate and a swap rate, and an IRSO on the difference between two swap rates with different tenors. The pricing formula of the first-type IRSO is shown to have a drift effect which is different from the result in Fu [1996]. The last two types of IRSOs are usually embedded in steepeners and range accruals. It is well-known that the LMM and the swap market model (SMM) are not compatible and thus the distribution of swap rates within the LMM framework is unknown. In this article, we also present a new approach to approximate the distribution of swap rates in the multifactor LMM. The resulting pricing formulas are shown to be sufficiently accurate via Monte Carlo simulation.

An additional contribution is to be able to speed up computation. For example, CMS range accruals have been widely traded in financial market and its component elements are CMS spread options. In practice, the prices of CMS range accruals are computed based on Monte Carlo simulations. For financial institutions issuing hundreds of CMS range accruals, it is too inefficient to provide daily price quotations to customers. This problem can be solved with the accurate closed-form pricing formulas presented in this article.

The LMM was developed by Musiela and Rutkowski [1997], Miltersen, Sandmann, and Sondermann [1997], and Brace, Gatarek, and Musiela [1997, BGM]. It has been widely used by market practitioners because its underlying interest rate is a LIBOR rate, which is market-observable. The cap and floor pricing formulas within the LMM framework are in fact the Black formula which has been consistent with market practice and also makes the calibration procedure easier. The underlying distribution of the LIBOR rate in the LMM is a lognormal distribution rather than a Gaussian distribution, which avoids pricing errors due to the negative rates with positive probabilities.[1] In addition, most popular and actively traded interest rate products can be priced within the LMM framework so that interest rate risks can be managed consistently and efficiently.

THE LIBOR MARKET MODEL AND SOME RELATED TECHNIQUES

In this section, we briefly review the LIBOR market model, a drift-adjustment technique in different measure, and a lognormalization technique for LIBOR rates under different measure. We also introduce a new lognormalization approach for swap rates under the LMM.

Review of the LIBOR Market Model

Assume that trading takes place continuously over an interval $[0, T]$, $0 < T < \infty$. The uncertainty is described by the filtered spot martingale probability space $(\Omega, F, Q, \{\mathcal{F}_t\}_{t\in[0,T]})$ and an m-dimensional independent standard Brownian motion $W(t) = (W_1(t), W_2(t), \ldots, W_m(t))$ is defined on the probability space. The flow of information accruing to all the agents in the economy is represented by the filtration $\{\mathcal{F}_t\}_{t\in[0,\mathcal{T}]}$ which satisfies the usual hypotheses.[2] Note that Q denotes the spot martingale probability measure. The notations are as follows:

$P(t,T)$ = the time t price of a zero-coupon bond paying one dollar at time T

$L^\delta(t,T)$ = the forward LIBOR rate contracted at time t and applied to the period $[T, T+\delta]$ with $0 \le t \le T \le T+\delta \le \mathcal{T}$

Q^T = the forward martingale measure with respect to the numéraire $P(\cdot, T)$.

The relationship between $L^\delta(t, T)$ and $P(t, T)$ can be expressed as follows:

$$L^\delta(t,T) = (P(t,T) - P(t,T+\delta))/\delta P(t,T+\delta) \quad (1)$$

BGM [1997] modeled interest rate behavior in terms of the forward LIBOR rates based on the arbitrage-free conditions in HJM [1992]. We briefly specify their results.

Proposition 1 The LIBOR Rate Dynamics under the Measure Q

The dynamics of the LIBOR rate $L^\delta(t, T)$ under the spot martingale measure Q is given as follows:

$$\frac{dL^\delta(t,T)}{L^\delta(t,T)} = \gamma^\delta(t,T)\cdot\sigma_p(t,T+\delta)dt + \gamma^\delta(t,T)\cdot dW(t) \quad (2)$$

where $0 \le t \le T \le \mathcal{T}$, $\sigma_p(t,\cdot)$ is defined as follows:

$$\sigma_p(t,T) = \begin{cases} \sum_{j=1}^{[\delta^{-1}(T-t)]} \dfrac{\delta L^\delta(t,T-j\delta)}{1+\delta L^\delta(t,T-j\delta)}\gamma^\delta(t,T-j\delta) & t\in[0,T-\delta] \ \& \ T-\delta>0 \\ 0 & \text{otherwise} \end{cases} \quad (3)$$

where δ is some designated length of time,[3] $[\delta^{-1}(T-t)]$ denotes the greatest integer that is less than $\delta^{-1}(T-t)$ and the deterministic function $\gamma: R_+^2 \to R^m$ is bounded and piecewise continuous.

According to the derivation process in BGM [1997], $\{\sigma_p(t, T)\}_{t\in[0,T]}$ stands for the volatility process of the bond price $P(t, T)$. Notice that Equation (2) is a stochastic process of a LIBOR rate under the spot martingale measure Q. It is sometimes desirable to know the processes under other martingale measures. The following proposition specifies the general rule under which the LIBOR rate dynamics are changed following a change of the underlying measure. This rule is useful for deriving the pricing formulas of interest rate derivatives.

Proposition 2 The Drift Adjustment Technique in Different Measures

The dynamics of a forward LIBOR rate $L^\delta(t, T)$ under an arbitrary forward martingale measure Q^S is given as follows:

$$\frac{dL^\delta(t,T)}{L^\delta(t,T)} = \gamma^\delta(t,T)\cdot(\sigma_p(t,T+\delta) - \sigma_p(t,S))dt + \gamma^\delta(t,T)\cdot dW(t) \quad (4)$$

where $0 \le t \le \min(S, T)$.[4]

A Lognormalization Technique for the LMM

According to the definition of the bond volatility process in Equation (3), $\{\sigma_p(t,\cdot)\}_{t\in[0,\cdot]}$ in (2) and (4) is stochastic rather than deterministic. The stochastic differential Equations (2) and (4) are not solvable, and the distribution of $L^\delta(T, T)$ is unknown. However, given a fixed initial time, assumed to be 0, we can approximate $\sigma_p(t, T)$ by $\bar{\sigma}_p^0(t,T)$, which is defined by

$$\bar{\sigma}_p^0(t,T) = \begin{cases} \sum_{j=1}^{[\delta^{-1}(T-t)]} \dfrac{\delta L^\delta(0,T-j\delta)}{1+\delta L^\delta(0,T-j\delta)}\gamma^\delta(t,T-j\delta), & t\in[0,T-\delta] \ \& \ T-\delta>0 \\ 0 & \text{otherwise} \end{cases} \quad (5)$$

where $0 \le t \le T \le \mathcal{T}$. The calendar time of the process $\{L^\delta(t, T-j\delta)\}_{t\in[0,T-j\delta]}$ in (5) is frozen at its initial time 0 and thus the process $\{\bar{\sigma}_p^0(t,T)\}_{t\in[0,T]}$ becomes deterministic. By substituting $\bar{\sigma}_p^0(t,T+\delta)$ for $\sigma_p(t, T+\delta)$ in the drift terms of (2) and (4), their drift and volatility terms will be deterministic, so we can solve it and find the approximate distribution of $L^\delta(T, T)$ to be lognormally distributed.

This technique is the so-called Wiener chaos order 0 approximation, which was first used by BGM [1997] for pricing interest rate swaptions. It was developed further in Brace, Dun, and Barton [1998] and formalized by Brace and Womersley [2000]. The approximation also appears in Schlögl [2002]. This approximation has been shown to be very accurate. We present the result in the following proposition and its proof in Appendix A.

Proposition 3 The Lognormalized LIBOR Market Model

The dynamics of a lognormalized forward LIBOR rate $L^\delta(t,T)$ under an arbitrary forward martingale measure Q^S is given by

$$\frac{dL^\delta(t,T)}{L^\delta(t,T)} = \Delta_0^\delta(t,T;S)dt + \gamma^\delta(t,T)\cdot dW(t) \tag{6}$$

where

$$\Delta_0^\delta(t,T;S) = \gamma^\delta(t,T)\cdot(\bar{\sigma}_P^0(t,T+\delta) - \bar{\sigma}_P^0(t,S)) \tag{7}$$

and $0 \le t \le \min(S, T)$.

An Approximate Distribution of a Swap Rate in the LMM Framework

Besides the LMM, the SMM as shown by Jamshidian [1997] provides the swaption prices with the Black swaption formula, which has become the widely accepted standard pricing formula by the swaption market. It is well-known, however, that the LMM and the SMM are not compatible in that a swap rate and a LIBOR rate cannot be lognormally distributed under the same measure. Therefore, choosing either of the two models as a pricing foundation is problematic. Brace, Dun, and Barton [1998] have suggested that the LMM should be adopted as the central model due to its mathematical tractability, and we follow their suggestion.

In this way, the first problem we encounter is how to find the approximate distribution of swap rates under the LMM framework. It is known that a swap rate is roughly a weighted average of LIBOR rates. Moreover, LIBOR rates under the LMM framework are approximately lognormally distributed. Therefore, the distribution of a swap rate is roughly a weighted average of lognormal distributions. In this subsection, we present a new approach to find the approximate distribution of a swap rate under the LMM framework.

Consider a swap rate, observed at t and with reset dates $\{T_0^\delta, T_1^\delta, ..., T_{n-1}^\delta\}$ and payment dates $\{T_0^\delta, T_1^\delta, ..., T_n^\delta\}$, defined as follows:

$$S_n^\delta(t,T) = \sum_{i=0}^{n-1} w_i^\delta(t) L^\delta(t, T_i^\delta)^5 \tag{8}$$

where

$$w_i^\delta(t) = \frac{P(t,T_{i+1}^\delta)}{\sum_{j=0}^{n-1} P(t,T_{j+1}^\delta)} \tag{9}$$

Brigo and Mercurio [2001] indicate that empirical studies have shown the variability of the w_i^δ to be small compared to the variability of the forward LIBOR rates.[6] Therefore, we can freeze the value of the processes $w_i^\delta(t)$ to its initial values $w_i^\delta(0)$ and obtain

$$S_n^\delta(t,T) \cong \sum_{i=0}^{n-1} w_i^\delta(0) L^\delta(t, T_i^\delta) \tag{10}$$

Note that $S_n^\delta(T, T)$ is a weighted average of lognormally distributed variables and its distribution is unknown. Although $S_n^\delta(T, T)$ is not a lognormal distribution, it can be well-approximated by a lognormal distribution with the correct first two moments.[7] The accuracy of this technique has been examined by Mitchell [1968]. Furthermore, many areas of science have verified the high accuracy of the lognormal approximation for the sum of lognormal random variables (e.g., Aitchison and Brown [1957], Crow and Shimizu [1988], Levy [1992], Limpert, Stahel, and Abbt [2001], and Borovkova, Permana, and Weide [2007]) (In addition, we will provide detailed empirical results in the subsection Calibration Procedure and Numerical Study to show the robust accuracy of the resulting pricing formulas based on moment matching approximation.)

Based on these studies, we assume that $\ln S_n^\delta(T, T)$ has a normal distribution with mean m and variance ν^2. The moment generating function for $\ln S_n^\delta(T, T)$ is given by

$$M_{\ln S}(h) = E[S_n^\delta(T,T)^h] = \exp\left(mh + \frac{1}{2}\nu^2 h^2\right) \tag{11}$$

Taking $h = 1$ and $h = 2$ in (11), we obtain the following two conditions to solve for m and ν^2:[8]

$$m = 2\ln E[S_n^\delta(T,T)] - \frac{1}{2}\ln E[S_n^\delta(T,T)^2] \tag{12}$$

$$\nu^2 = \ln E[S_n^\delta(T,T)^2] - 2\ln E[S_n^\delta(T,T)] \tag{13}$$

VALUATION OF INTEREST RATE SPREAD OPTIONS

In this section, we price three types of IRSOs within the LIBOR market model framework: an option on the difference between two LIBOR rates with different compounding periods, an option between a swap rate and a LIBOR rate, and an option between two swap rates.

Valuation of First-Type IRSOs (FIRSOs)

A FIRSO is an option on the difference between two LIBOR rates with different compounding periods. Its final payoff is given as follows:

$$C_1(T) = (L^{\delta}(T,T) - L^{\eta}(T,T))^{+} \tag{14}$$

where $L^{\delta}(T, T)$ and $L^{\eta}(T, T)$ denote the T-matured LIBOR rates with compounding period δ and η, respectively. T is the option's maturity date and $(a)^{+} = \mathrm{Max}(a, 0)$.

For managing assets and liabilities, FIRSOs are usually used to enhance the interest return of assets or reduce the interest cost from liabilities in a more direct way. If a financial manager desires to manage the risk of an interest rate spread via a long-period basis swap, he may use FIRSOs as ancillary instruments to eliminate the downside risks of particular payments. In addition, if investors have accurate views of the spread between LIBOR rates at some specific time, they can take profits by employing a corresponding FIRSO.

The pricing formula of FIRSOs is presented in the following theorem. Its proof is given in Appendix B.

Theorem 1 *The pricing formula of FIRSOs with the final payoff specified in* (14) *is given as follows:*

$$C_1(0) = P(0,T)\left\{ L^{\delta}(0,T)\exp\left(\int_0^T \Delta_0^{\delta}(u,T;T)du\right)N(d_{11}) - L^{\eta}(0,T)\exp\left(\int_0^T \Delta_0^{\eta}(u,T;T)du\right)N(d_{12})\right\} \tag{15}$$

where $\Delta_0^{\eta}(u,T;T)$ *is defined similarly to* $\Delta_0^{\delta}(u,T;T)$ *as given in* (7), *and*

$$d_{11} = \frac{\ln\dfrac{L^{\delta}(0,T)}{L^{\eta}(0,T)} + \int_0^T (\Delta_0^{\delta}(u,T;T) - \Delta_0^{\eta}(u,T;T))du + \frac{1}{2}V_1^2}{V_1}$$

$$d_{12} = d_{11} - V_1$$

$$V_1^2 = \int_0^T \| \gamma^{\delta}(u,T) - \gamma^{\eta}(u,T)\|^2\, du$$

The pricing formula (15) somewhat resembles the Margrabe [1978] formula in the LMM framework. In contrast with the pricing formula as given in Fu [1996], Equation (15) indicates that drift terms of LIBOR rates do affect the IRSO price. In addition, all the parameters appearing in (15) can be extracted easily from market data, making the pricing formula more tractable and feasible for practitioners.

Valuation of Second-Type IRSOs (SIRSOs)

A SIRSO is an option on the difference between a LIBOR rate and a swap rate. The final payoff is given as follows:

$$C_2(T) = (S_n^{\eta}(T,T) - L^{\delta}(T,T))^{+} \tag{16}$$

where $S_n^{\eta}(t, T)$ is an n-year forward swap rate with a year fraction η observed at time t.

SIRSOs enable practitioners to benefit from taking a view on the spread between a short-term rate and a long-term rate. SIRSOs are usually embedded in the range accruals, which are popular in the structured note market. SIRSOs can also be employed as ancillary instruments for a CMS to remove the downside risks of particular payments.

The pricing formula of SIRSOs is presented in the following theorem. Its proof is given in Appendix C.

Theorem 2 *The pricing formula of SIRSOs with the final payoff specified in* (16) *is given as follows:*

$$C_2(0) = P(0,T)\left(\exp\left(M_{2,\eta} + \frac{1}{2}v_{2,\eta}^2\right)N(d_{2,1}) - \exp\left(M_{2,\delta} + \frac{1}{2}v_{2,\delta}^2\right)N(d_{2,2})\right) \tag{17}$$

where

$$d_{2,1} = \frac{M_{2,\eta} + \frac{1}{2}v_{2,\eta}^2 - M_{2,\delta} - \frac{1}{2}v_{2,\delta}^2 + \frac{1}{2}V_2^2}{V_2}$$

$$d_{2,2} = d_{2,1} - V_2$$

$$M_{2,\delta} = \ln L^{\delta}(0,T) + \int_0^T \left(\Delta_0^{\delta}(u,T;T) - \frac{1}{2} \| \gamma^{\delta}(u,T) \|^2 \right) du$$

$$v_{2,\delta}^2 = \int_0^T \| \gamma^{\delta}(u,T) \|^2 \, du$$

$$M_{2,\eta} = 2\ln \mathrm{E}^{Q^T}[S_n^{\eta}(T,T)] - \frac{1}{2}\ln \mathrm{E}^{Q^T}[S_n^{\eta}(T,T)^2]$$

$$v_{2,\eta}^2 = \ln \mathrm{E}^{Q^T}[S_n^{\eta}(T,T)^2] - 2\ln \mathrm{E}^{Q^T}[S_n^{\eta}(T,T)]$$

$$V_2^2 = v_{2,\delta}^2 + v_{2,\eta}^2 - 2\sum_{i=0}^{n-1} w_i^{\eta}(0) \int_0^T \gamma^{\delta}(u,T) \cdot \gamma^{\eta}(u,T_i^{\eta}) du$$

and $\mathrm{E}^{Q^T}[S_n^{\eta}(T,T)]$ *and* $\mathrm{E}^{Q^T}[S_n^{\eta}(T,T)^2]$ *are defined, respectively, in (C3) and (C4).*

The pricing Formula (17) also somewhat resembles the formula in Margrabe [1978] in the LMM framework, where end-users are more familiar with employing it. As with Equation (15), the parameters in (17) can be extracted easily from market-quoted prices.

Valuation of Third-Type IRSOs (TIRSOs)

A TIRSO is an option on the difference between two CMS rates with the final payoff as follows:

$$C_3(T) = (S_n^{\delta}(T,T) - S_m^{\eta}(T,T))^+ \tag{18}$$

TIRSOs are usually embedded in the range accrual, which is very popular in the structured note market. TIRSOs are traded by investors who wish to take a position on future relative changes in different parts of the yield curve. TIRSOs can also be used as ancillary instruments for a two-way constant maturity swap. As with Equation (15), the parameters in the pricing Formula (19) can be extracted easily from market-quoted prices. Consequently, the pricing formula that follows is workable in practice.

The pricing formula of TIRSOs is presented in the following theorem. Its proof is given in Appendix D.

Theorem 3 *The pricing formula of TIRSOs with the final payoff specified in* (18) *is given as follows:*

$$C_3(0) = P(0,T)\left(\exp\left(M_{3,\delta} + \frac{1}{2}v_{3,\delta}^2 \right) N(d_{3,1}) - \exp\left(M_{3,\eta} + \frac{1}{2}v_{3,\eta}^2 \right) N(d_{3,2}) \right) \tag{19}$$

where

$$d_{3,1} = \frac{M_{3,\delta} + \frac{1}{2}v_{3,\delta}^2 - M_{3,\eta} - \frac{1}{2}v_{3,\eta}^2 + \frac{1}{2}V_3^2}{V_3}$$

$$d_{3,2} = d_{2,1} - V_3$$

$$M_{3,\delta} = 2\ln \mathrm{E}^{Q^T}[S_n^{\delta}(T,T)] - \frac{1}{2}\ln \mathrm{E}^{Q^T}[S_n^{\delta}(T,T)^2]$$

$$v_{3,\delta}^2 = \ln \mathrm{E}^{Q^T}[S_n^{\delta}(T,T)^2] - 2\ln \mathrm{E}^{Q^T}[S_m^{\delta}(T,T)]$$

$$M_{3,\eta} = 2\ln \mathrm{E}^{Q^T}[S_n^{\eta}(T,T)] - \frac{1}{2}\ln \mathrm{E}^{Q^T}[S_m^{\eta}(T,T)^2]$$

$$v_{3,\eta}^2 = \ln \mathrm{E}^{Q^T}[S_m^{\eta}(T,T)^2] - 2\ln \mathrm{E}^{Q^T}[S_m^{\eta}(T,T)]$$

$$V_3^2 = v_{3,\delta}^2 + v_{3,\eta}^2 - 2\sum_{i=0}^{n-1}\sum_{j=0}^{m-1} w_i^{\delta}(0) w_j^{\eta}(0) \int_0^T \gamma^{\delta}(u,T_j^{\delta}) \cdot \gamma^{\eta}(u,T_j^{\eta}) \, du$$

and $\mathrm{E}^{Q^T}[S_n^{\delta}(T,T)]$, $\mathrm{E}^{Q^T}[S_n^{\delta}(T,T)^2]$, $\mathrm{E}^{Q^T}[S_n^{\eta}(T,T)]$, *and* $\mathrm{E}^{Q^T}[S_m^{\delta}(T,T)^2]$ *are, respectively, defined in (D2), (D3), (D4), and (D5).*

The three types of IRSOs, which are usually embedded in structured notes, are widely traded in the financial market. In practice, these structured notes are priced based on Monte Carlo simulations. For financial institutions issuing hundreds of structured notes, however, Monte Carlo simulations are too time consuming to compute the daily prices of issued structured notes for investors each day. The next section examines the closed-form pricing formulas in Theorems 3.1~3.3 and shows them to be sufficiently and robustly accurate by comparing the Monte Carlo simulation. The formulas consequently provide a time-reducing approach to price structured notes and make the management of price quotations easier.

CALIBRATION PROCEDURE AND NUMERICAL STUDY

This section first presents the calibration procedure and then provides some numerical examinations for the approximate formulas derived in the previous section.

Calibration Procedure

The cap and floor pricing formulas within the LMM framework are in fact the Black formula, which has been extensively used in market practice. Thus, the model volatilities can be extracted directly from quoted implied volatilities for cap prices. However, the correlation matrix of the forward LIBOR rates cannot be extracted from the quotations of cap prices since the standard pricing formula of caplets involves only a single forward LIBOR rate. In practice, two approaches are employed to calibrate correlations between LIBOR rates. The first is presented by Rebonato [1999], who applies an historical correlation matrix to engage in calibration, and the second is based on the price quotations of swaptions.[9] Both approaches are tractable and widely used in the marketplace.

In this article, we adopt the Rebonato [1999] approach to engage in a simultaneous calibration of the LMM to the volatilities and correlation matrix of the forward LIBOR rates.[10] We assume that there are n forward LIBOR rates in the m-factor framework. The steps to calibrate the parameters are given as follows.

First, we assume that each forward LIBOR rate, $L(\cdot, t_i)$, has a constant instantaneous volatility, namely, for $i = 1, ..., n$, $\gamma(\cdot, t_i) = v_i$. The setting is as presented in Exhibit 1.[11] Thus, if the market-quoted volatility for t_1-year cap is ξ_1, then $v_1 = \xi_1$. Next, for $i = 2, ..., n$ if the t_i-year cap is ξ_i, then $v_i = \xi_i^2 t_i^2 - \xi_{i-1}^2 t_{i-1}^2$.

Second, we use the historical data of the forward LIBOR rates to derive a market correlation matrix Σ. Σ is an n-rank ($m \leq n$), positive-definite, and a symmetric matrix, and can be written as

$$\Sigma = H\Gamma H$$

where H is a real orthogonal matrix and Γ is a diagonal matrix. Let $A \equiv H\Gamma^{1/2}$ and, thus, $\Sigma = AA'$. In this way, we can find an m-rank ($m \leq n$) matrix B so that $\Sigma^B = BB'$ is an approximate correlation matrix for Σ.

The advantage of finding B is that we may replace the n-dimensional original Brownian motion $dW(t)$ with $BdZ(t)$ where $dZ(t)$ is an m-dimensional Brownian motion. In other words, we change the market correlation structure

$$dW(t)dW(t)' = \Sigma dt$$

to an approximate correlation structure

$$BdZ(t)(BdZ(t))' = BdZ(t)dZ(t)'B' = BB'dt = \Sigma^B dt$$

The remaining problem is how to find a suitable matrix B. Rebonato [1999] proposed a method described as follows. Assume that the ik-th element of B for $i = 1, 2, ..., n$ is specified as follows:

EXHIBIT 1
Instantaneous Volatilities of $L(t, \cdot)$

Instant.Total Vol.	Time $t \in (t_0, t_1)$	(t_1, t_2)	(t_2, t_3)	...	(t_{n-2}, t_{n-1})
Fwd Rate: $L(t, t_1)$	v_1	Dead	Dead	...	Dead
$L(t, t_2)$	v_2	v_2	Dead	...	Dead
⋮	...	...	...	...	...
$L(t, t_n)$	v_n	v_n	v_n	...	v_n

$$b_{i,k} = \begin{cases} \cos\theta_{i,k} \Pi_{j=1}^{k-1} \sin\theta_{i,j} & \text{if } k = 1, 2, \ldots, m-1 \\ \Pi_{j=1}^{k-1} \sin\theta_{i,j} & \text{if } k = m \end{cases}$$

Thus, $\Sigma^B = BB'$ is a function of $\Theta = \{\theta_{i,k}\}_{i=1,\ldots,n;k=1,\ldots,m-1}$. We can obtain an optimal solution $\hat{\Theta}$ by solving the following optimization problem.

$$\min_{\Theta} \sum_{i,j=1}^{n} \left|\Sigma_{i,j}^{B} - \Sigma_{i,j}\right|^2 \qquad (20)$$

where $\Sigma_{i,j}$ is the ij-th element of Σ, and $\Sigma^B_{i,j}$ is the ij-th element of Σ^B, specifically defined as follows:

$$\Sigma_{i,j}^{B} = \sum_{k=1}^{m} b_{i,k} b_{j,k}$$

By substituting $\hat{\Theta}$ into B, we obtain an optimal matrix $\hat{B}$ such that $\hat{\Sigma}^B (= \hat{B}\hat{B}')$ is an approximate correlation matrix for Σ.

Third, we use $\hat{B}$ to distribute the instantaneous total volatility, v_i, to each Brownian motion without changing the amount of the instantaneous total volatility.[12] That is,

$$v_i(\hat{b}_{i-k+1,1}, \hat{b}_{i-k+1,2}, \ldots, \hat{b}_{i-k+1,m}) = (\gamma_1(t,t_i), \gamma_2(t,t_i), \ldots, \gamma_m(t,t_i))$$

where $t_{k-1} \le t < t_k$, $k = 1, \ldots, i$ and $i = 1, 2, \ldots, n$.

This procedure is a general calibration method without a constraint on choosing the number of factors, m. The number of random shocks, m, may depend on the maturity range of interest rates involved in the considered financial product.[13] For example, we may use a three-factor model, i.e., $m = 3$, to capture the shift and twist of the entire yield curve. The first two random shocks can be interpreted, respectively, as the short-term and long-term factors causing the shift of different maturity ranges on the yield curve. The correlation between the short-term and long-term interest rates is specified by the third random shock. According to this feature, the numerical examples in the following section are based on a three-factor model.

Numerical Study

This subsection provides some practical examples that examine the accuracy of the approximate pricing formulas presented in Theorems 1–3 by comparing the results with Monte Carlo simulations. The first case is an IRSO on the difference between a 6m-LIBOR rate and a 3m-LIBOR rate; the second case is between a 7-year swap rate and a

EXHIBIT 2

The Numerical Examples of Theorem 1

Date	Time to Maturity	Theorem 1	MC	SE
2007/1/1	1	5.7551×10^{-5}	5.7561×10^{-5}	1.1548×10^{-6}
	3	3.5085×10^{-4}	3.5076×10^{-4}	3.6374×10^{-6}
	5	5.1147×10^{-4}	5.1247×10^{-4}	6.2260×10^{-6}
2007/7/2	1	2.4179×10^{-4}	2.4178×10^{-4}	1.8705×10^{-6}
	3	5.2946×10^{-4}	5.2946×10^{-4}	4.0401×10^{-6}
	5	5.9586×10^{-4}	5.9593×10^{-4}	6.3528×10^{-6}
2008/1/1	1	2.0770×10^{-5}	2.0776×10^{-5}	9.4075×10^{-7}
	3	7.5396×10^{-4}	7.5390×10^{-4}	8.3932×10^{-6}
	5	9.3120×10^{-4}	9.3408×10^{-4}	1.2973×10^{-5}
2008/7/1	1	1.3033×10^{-3}	1.3033×10^{-3}	9.1545×10^{-6}
	3	1.0637×10^{-3}	1.0635×10^{-3}	1.2236×10^{-5}
	5	9.9555×10^{-4}	9.9987×10^{-4}	1.5991×10^{-5}

Notes: The 1-, 3-, and 5-year IRSOs on the difference between a 6m-LIBOR rate and a 3m-LIBOR rate are semiannually priced based on the market data over the past two years. The market data are listed in Appendix E. The notional value is assumed to be \$1. The simulation is based on 10,000 paths. MC stands for the result of the Monte Carlo simulation, and SE stands for the standard error.

EXHIBIT 3
The Numerical Examples of Theorem 2

Date	Time to Maturity	Theorem 2	MC	SE
2007/1/1	1	1.6847×10^{-3}	1.6846×10^{-3}	2.4359×10^{-5}
	3	3.4602×10^{-3}	3.4600×10^{-3}	4.4089×10^{-5}
	5	3.6115×10^{-3}	3.6117×10^{-3}	4.6739×10^{-5}
2007/7/2	1	3.4947×10^{-3}	3.4947×10^{-3}	3.2942×10^{-5}
	3	3.7791×10^{-3}	3.7792×10^{-3}	4.3340×10^{-5}
	5	3.6164×10^{-3}	3.6165×10^{-3}	4.4806×10^{-5}
2008/1/1	1	9.9053×10^{-3}	9.9053×10^{-3}	4.8197×10^{-5}
	3	8.2671×10^{-3}	8.2670×10^{-3}	7.6276×10^{-5}
	5	6.7070×10^{-3}	6.7088×10^{-3}	7.5803×10^{-5}
2008/7/1	1	1.0943×10^{-2}	1.0944×10^{-2}	5.4960×10^{-5}
	3	7.6029×10^{-3}	7.6026×10^{-3}	7.8739×10^{-5}
	5	6.6668×10^{-3}	6.6658×10^{-3}	7.7912×10^{-5}

Notes: The 1-, 3-, and 5-year IRSOs on the difference between a 7-year swap rate and a 6m-LIBOR rate are semiannually priced based on the market data over the past two years. The year fraction of the 7-year swap rate is a half year. The market data are listed in Appendix E. The notional value is assumed to be \$1. The simulation is based on 10,000 paths. MC stands for the result of the Monte Carlo simulation, and SE stands for the standard error.

6m-LIBOR rate; the third case is between a 10-year swap rate and a 2-year swap rate. The swap rates are reset semiannually.

For each case, we consider three times to maturity, namely 1, 3, and 5 years. To examine the accuracy and robustness of the presented formulas for different market scenarios, IRSOs are priced semiannually for the recent 2-year market data. Appendix E lists the market data used for calibration and simulations. The notional principal for each example is assumed to be \$1 and simulations are based on 10,000 paths.

The results, listed in Exhibits 2–4, show that the approximate formulas are sufficiently accurate by comparison to Monte Carlo simulations and are robust as four market parameters over the past two years are employed. Even for the more volatile market data (on date 2008/7/1), the approximate formulas are still shown to be quite acceptable.

In practice IRSOs, such as CMS spread options and CMS range accruals, are priced based on Monte Carlo simulation, which is too time consuming to provide customers daily price quotations, especially for a financial company issuing hundreds of IRSO-type structured notes. Based on the empirical examinations given in Exhibits 2–4, the pricing results of the presented formulas are robustly close to the results from simulations. This ensures that the approximate formulas are good substitutes for simulations and solve the problem of excessive time consumption that simulations present. Accordingly, the presented formulas can enhance customer service by providing customers with quick daily price quotations. With these advantages, the presented formulas are worth recommending for practical implementation.

CONCLUSIONS

We have developed the pricing formulas of three types of IRSOs within the LMM framework: an option on the difference between two LIBOR rates with different compounding periods, an option on the difference between a swap rate and a LIBOR rate, and an option on the difference between two swap rates. These financial products are widely traded in the marketplace or are embedded in structured notes, such as CMS range accruals and steepeners.

IRSOs can be used as ancillary instruments for interest rate swaps to enhance profit from a change in the spread between two interest rates or to lock in a current spread. They can also be employed to control risks due to changes in the shape of the yield curve.

EXHIBIT 4
The Numerical Examples of Theorem 3

Date	Time to Maturity	Theorem 3	MC	SE
2007/1/1	1	2.8030×10^{-3}	2.8030×10^{-3}	2.9121×10^{-5}
	3	3.6149×10^{-3}	3.6148×10^{-3}	4.2738×10^{-5}
	5	3.4973×10^{-3}	3.4974×10^{-3}	4.1149×10^{-5}
2007/7/2	1	3.7419×10^{-3}	3.7418×10^{-3}	3.1073×10^{-5}
	3	3.6444×10^{-3}	3.6445×10^{-3}	4.1017×10^{-5}
	5	3.1434×10^{-3}	3.1438×10^{-3}	3.7546×10^{-5}
2008/1/1	1	1.0096×10^{-3}	1.0097×10^{-3}	5.1243×10^{-5}
	3	7.2788×10^{-3}	7.2889×10^{-3}	6.9623×10^{-5}
	5	5.6175×10^{-3}	5.6472×10^{-3}	6.1297×10^{-5}
2008/7/1	1	7.3652×10^{-3}	7.3698×10^{-3}	5.6085×10^{-5}
	3	6.6315×10^{-3}	6.6521×10^{-3}	7.0309×10^{-5}
	5	5.3660×10^{-3}	5.4040×10^{-3}	6.0976×10^{-5}

Notes: The 1-, 3-, and 5-year IRSOs on the difference between a 10-year swap rate and a 2-year swap rate are semiannually priced based on the market data over the past two years. The market data are listed in Appendix E. The notional value is assumed to be \$1. The simulation is based on 10,000 paths. MC stands for the result of the Monte Carlo simulation, and SE stands for the standard error.

The approximate formulas have been found to be robustly accurate as compared with the benchmark result from Monte Carlo simulation using the most recent 2-year data. The approximate formulas save time and make daily quotations faster. This is especially important for financial companies that have issued a large number of IRSO-type structured notes. Therefore, the pricing models examined in this article are worth recommending to market practitioners.

APPENDIX A

A Simple Proof for Proposition 2.2

It is well known that under the spot martingale measure Q, the price dynamics of a money market account $M(\cdot)$ and $P(\cdot, T)$ for $T \in [0, \mathcal{T}]$ are given as follows:

$$\frac{dM(t)}{M(t)} = r(t)dt \tag{A1}$$

$$\frac{dP(t,T)}{P(t,T)} = r(t)dt - \sigma_p(t,T) \cdot dw(t) \tag{A2}$$

where $W(t)$ is a Brownian motion on Q, $r(t)$ is an instantaneous short rate at time t and $t \in [0, T]$. Moreover, according to (2), the LIBOR rate dynamics are given as follows:

$$\frac{dL^{\delta}(t,T)}{L^{\delta}(t,T)} = \gamma^{\delta}(t,T) \cdot \sigma_p(t,T+\delta)dt + \gamma^{\delta}(t,T) \cdot dW(t) \tag{A3}$$

Now, we want to find the forward martingale measure Q^S with respect to a numéraire $P(\cdot, S)$ with $S \in [0, T]$. According to the martingale pricing theory, under Q^S any dynamics of an asset price expressed in terms of $P(\cdot, S)$ must be a martingale. This is a clue to find the measure Q^S and the dynamics of asset prices under Q^S.

Thus, all assets are denominated in terms of $P(\cdot, S)$ as follows:

$$F^M(t;S) = \frac{M(t)}{P(t,S)} \tag{A4}$$

$$F(t,T;S) = \frac{P(t,T)}{P(t,S)} \tag{A5}$$

Based on Itô's Lemma, the dynamics of $F^M(t; S)$ and $F(t, T; S)$ are derived as follows:

$$\frac{dF(t,T;S)}{F(t,T;S)} = \sigma_p(t,S) \cdot (\sigma_p(t,S) - \sigma_p(t,T))dt + (\sigma_p(t,S) - \sigma_p(t,T)) \cdot dW(t) \tag{A6}$$

$$\frac{dF^M(t;S)}{F^M(t;S)} = \|\sigma_p(t,S)\|^2 dt + \sigma_p(t,S) \cdot dW(t) \tag{A7}$$

By observing (A6) and (A7), we can find a Randon-Nikodým derivative as follows:

$$\frac{dQ^S}{dQ} = \exp\left(-\frac{1}{2} \int_0^{\min\{S,T\}} \| \sigma_P(u,S) \|^2 \, du - \int_0^{\min\{S,T\}} \sigma_P(u,S) \cdot dW(u) \right) \quad \text{(A8)}$$

to define the target forward measure Q^S. $W^{Q^S}(t)$ is a standard Brownian motion under Q^S defined as follows:

$$W^{Q^S}(t) = W(t) + \int_0^{\min\{S,T\}} \sigma_P(u,S) du \quad \text{(A9)}$$

By taking (A9) into (A3), we could obtain the dynamics of $L^\delta(t,T)$ under Q^S as given in (4).

APPENDIX B

The Proof of Theorem 1

By applying Proposition 2.3, under the measure Q^T, the dynamics of $\{L^*(t,T)\}$ for $* = \delta$ and η is given as follows:

$$\frac{dL^*(t,T)}{L^*(t,T)} = \Delta_0^*(t,T;T)dt + \gamma^*(t,T) \cdot dW(t)$$

Therefore,

$$L^*(T,T) = L^*(0,T) \exp\left(\int_0^T \left(\Delta_0^*(u,T;T) - \frac{1}{2} \| \gamma^*(u,T) \|^2 \right) du + \int_0^T \gamma^*(u,T) \cdot dW(u) \right) \quad \text{(B1)}$$

$$\ln L^*(T,T) = \ln L^*(0,T) + \int_0^T \left(\Delta_0^*(u,T;T) - \frac{1}{2} \| \gamma^*(u,T) \|^2 \right) du + \int_0^T \gamma^*(u,T) \cdot dW(u) \quad \text{(B2)}$$

$$\ln L^*(T,T) \sim N(M_{1,*}, v_{1,*}^2) \quad \text{(B3)}$$

where

$$M_{1,*} = \ln L^*(0,T) + \int_0^T \left(\Delta_0^*(u,T;T) - \frac{1}{2} \| \gamma^*(u,T) \|^2 \right) du$$

$$v_{1,*}^2 = \int_0^T \| \gamma^*(u,T) \|^2 \, du$$

Moreover, based on (B2),

$$V_1^2 = Var(\ln L^\delta(T,T) - \ln L^\eta(T,T)) = \int_0^T \| \gamma^\delta(u,T) - \gamma^\eta(u,T)^2 du$$

We present a useful lemma for the following deriving process.

Lemma B1 *If* $X \sim N(\mu_X, \sigma_X^2)$ *and* $Y \sim N(\mu_Y, \sigma_Y^2)$, *then we have*

$$\mathrm{E}\left[\left(\exp\left(X - \frac{1}{2}\sigma_X^2 \right) - \exp\left(Y - \frac{1}{2}\sigma_Y^2 \right) \right)^+ \right] = \exp(\mu_X) N(\eta) - \exp(\mu_Y) N(\eta - \zeta)$$

where

$$\eta = \frac{\mu_X - \mu_Y + \frac{1}{2}\zeta^2}{\zeta}, \quad \text{and } \zeta^2 = Var(X - Y)$$

Based on the martingale valuation method, the pricing formula of the FIRSO at its initial time 0 can be obtained by calculating the following expectation:

$$C_1(0) = P(0,T) \mathrm{E}^{Q^T} [(L^\delta(T,T) - L^\eta(T,T))^+]$$

By employing (B3) and Lemma B1, the pricing formula of the FIRSO can be derived as shown in Theorem 1.

APPENDIX C

The Proof of Theorem 2

The pricing formula of SIRSOs can be obtained by deriving the following equation:

$$C_2(0) = P(0,T) \mathrm{E}^{Q^T} [(S_n^\eta(T,T) - L^\delta(T,T))^+] \quad \text{(C1)}$$

According to (B3), we have

$$\ln L^\delta(T,T) \sim N(M_{2,\delta}, v_{2,\delta}^2) \quad \text{(C2)}$$

where $M_{2,\delta} = M_{1,\delta}$ and $v_{2,\delta}^2 = v_{1,\delta}^2$.

By employing the approximate method in the subsection titled An Approximate Distribution…, $S_n^\eta(T,T)$ is assumed to be a lognormal distribution with the correct first two moments given as follows:

$$E^{Q^T}[S_n^\eta(T,T)] = \sum_{i=0}^{n-1} w_i^\eta(0) E^{Q^T}[L^\eta(T,T_i^\eta)]$$
$$= \sum_{i=0}^{n-1} w_i^\eta(0) L^\eta(0,T_i^\eta) \exp\left(\int_0^T \Delta_0^\eta(u,T_i^\eta;T)du\right) \quad \text{(C3)}$$

$$E^{Q^T}[S_n^\eta(T,T)^2] = E^{Q^T}\left[\left(\sum_{i=0}^{n-1} w_i^\eta(0) L^\eta(T,T_i^\eta)\right)^2\right]$$
$$= \sum_{i=0}^{n-1} w_i^\eta(0)^2 E^{Q^T}[L^\eta(T,T_i^\eta)^2]$$
$$+ 2\sum_{i=0}^{n-2}\sum_{j=i+1}^{n-1} w_i^\eta(0) w_j^\eta(0) E^{Q^T}[L^\eta(T,T_i^\eta) L^\eta(T,T_j^\eta)] \quad \text{(C4)}$$

where

$$E^{Q^T}\left[L^\eta(T,T_i^\eta)^2\right]$$
$$= L^\eta(0,T_i^\eta)^2 \exp\left(\int_0^T 2\Delta_0^\eta(u,T_i^\eta;T) + \gamma^\eta(u,T_i^\eta)^2 du\right)$$
$$E^{Q^T}\left[L^\eta(T,T_i^\eta) L^\eta(T,T_j^\eta)\right]$$
$$= L^\eta(0,T_i^\eta) L^\eta(0,T_j^\eta) \exp\left(\int_0^T \xi_0^\eta(u,T_i^\eta,T_j^\eta;T)du\right)$$
$$\xi_0^\eta(u,T_i^\eta,T_j^\eta;T)$$
$$= \Delta_0^\eta(u,T_i^\eta;T) + \Delta_0^\eta(u,T_j^\eta;T) + \gamma^\eta(u,T_i^\eta) \cdot \gamma^\eta(u,T_j^\eta)$$

By applying (12) and (13), the mean and variance of $\ln S_n^\eta(T,T)$ can be computed, respectively, as follows:

$$M_{2,\eta} = 2\ln E^{Q^T}[S_n^\eta(T,T)] - \frac{1}{2}\ln E^{Q^T}[S_n^\eta(T,T)^2] \quad \text{(C5)}$$

$$v_{2,\eta}^2 = \ln E^{Q^T}[S_n^\eta(T,T)^2] - 2\ln E^{Q^T}[S_n^\eta(T,T)] \quad \text{(C6)}$$

For computing (C1), we have to further derive

$$V_2^2 = \text{Var}\,(\ln S_n^\eta(T,T) - \ln L^\delta(T,T))$$
$$= v_{2,\delta}^2 + v_{2,\eta}^2 - 2\text{Cov}\,(\ln L^\delta(T,T), \ln S_n^\eta(T,T)) \quad \text{(C7)}$$

While the covariance part in (C7) is not analytically solvable, we may approximately derive it by replacing $S_n^\eta(T,T)$ with $G_n^\eta(T,T)$, where

$$G_n^\eta(T,T) = \prod_{i=0}^{n-1} L^\eta(T,T_i^\eta)^{w_i^\eta(0)}$$

It means that we use the geometric weighted average to replace the arithmetic weighted average which is also used in Vorst [1992]. Therefore,

$$\text{Cov}\,(\ln L^\delta(T,T), \ln S_n^\eta(T,T))$$
$$= E^{Q^T}\left[\ln L^\delta(T,T) \ln S_n^\eta(T,T)\right]$$
$$- E^{Q^T}\left[\ln L^\delta(T,T)\right] E^{Q^T}\left[\ln S_n^\eta(T,T)\right]$$
$$\cong E^{Q^T}\left[\ln L^\delta(T,T) \ln G_n^\eta(T,T)\right]$$
$$- E^{Q^T}\left[\ln L^\delta(T,T)\right] E^{Q^T}\left[\ln G_n^\eta(T,T)\right]$$
$$= \sum_{i=0}^{n-1} w_i^\eta(0) \text{Cov}\,(\ln L^\delta(T,T), \ln L^\eta(T,T_i^\eta))$$
$$= \sum_{i=0}^{n-1} w_i^\eta(0) \int_0^T \gamma^\delta(u,T) \cdot \gamma^\eta(u,T_i^\eta) du \quad \text{(C8)}$$

By observing (C8), the covariance between $\ln L^\delta(T,T)$ and $\ln S_n^\eta(T,T)$ can be approximately viewed as the weighted covariances between $\ln L^\delta(T,T)$ and $\ln L^\eta(T,T_i^\eta)$ which is involved in $\ln S_n^\eta(T,T)$.

With the aforementioned knowledge, the pricing formula of the SIRSO can be priced via Lemma B1.

Appendix D

The Proof of Theorem 3

The pricing formula of SIRSOs can be obtained by deriving the following equation:

$$C_3(0) = P(0,T) E^{Q^T}[(S_n^\delta(T,T) - S_m^\eta(T,T))^+] \quad \text{(D1)}$$

According to the approximate method in the subsection titled An Approximate Distribution ..., the distributions of $S_n^\delta(T,T)$ and $S_m^\eta(T,T)$ are assumed to be lognormal distributions with the correct first two moments, i.e.,

$$\ln S_n^\delta(T,T) \sim N(M_{3,\delta}, v_{3,\delta}^2)$$
$$\ln S_m^\eta(T,T) \sim N(M_{3,\eta}, v_{3,\eta}^2)$$

where

$$M_{3,\delta} = 2\ln E^{Q^T}[S_n^\delta(T,T)] - \frac{1}{2}\ln E^{Q^T}[S_n^\delta(T,T)^2]$$

$$v_{3,\delta}^2 = \ln E^{Q^T}[S_n^\delta(T,T)^2] - 2\ln E^{Q^T}[S_n^\delta(T,T)]$$

$$M_{3,\eta} = 2\ln E^{Q^T}[S_m^\eta(T,T)] - \frac{1}{2}\ln E^{Q^T}[S_m^\eta(T,T)^2]$$

$$v_{3,\delta}^2 = \ln E^{Q^T}[S_m^\eta(T,T)^2] - 2\ln E^{Q^T}[S_m^\eta(T,T)]$$

Moreover, $E^{Q^T}[S_n^\delta(T,T)]$, $E^{Q^T}[S_n^\delta(T,T)^2]$, $E^{Q^T}[S_m^\eta(T,T)]$, and $E^{Q^T}[S_m^\eta(T,T)]$ are derived as follows:

$$E^{Q^T}[S_n^\delta(T,T)] = \sum_{i=0}^{n-1} w_i^\delta(0) E^{Q^T}[L^\delta(T,T_i^\delta)] = \sum_{i=0}^{n-1} w_i^\delta(0) L^\delta(0,T_i^\delta)\exp\left(\int_0^T \Delta_0^\delta(u,T_i^\delta;T)du\right) \quad \text{(D2)}$$

$$\begin{aligned} E^{Q^T}[S_n^\delta(T,T)^2] &= E^{Q^T}\left[\left(\sum_{i=0}^{n-1} w_i^\delta(0) L^\delta(T,T_i^\delta)\right)^2\right] \\ &= \sum_{i=0}^{n-1} w_i^\delta(0)^2 E^{Q^T}\left[L^\delta(T,T_i^\delta)^2\right] \\ &\quad + 2\sum_{i=0}^{n-2}\sum_{j=i+1}^{n-1} w_i^\delta(0) w_j^\delta(0) E^{Q^T} \\ &\quad \times \left[L^\delta(T,T_i^\delta) L^\delta(T,T_j^\delta)\right] \end{aligned} \quad \text{(D3)}$$

where

$$E^{Q^T}\left[L^\delta(T,T_i^\delta)^2\right] = L^\delta(0,T_i^\delta)^2 \times \exp\left(\int_0^T 2\Delta_0^\delta(u,T_i^\delta;T) + \gamma^\delta(u,T_i^\delta)^2 du\right)$$

$$E^{Q^T}\left[L^\delta(T,T_i^\delta) L^\delta(T,T_j^\delta)\right] = L^\delta(0,T_i^\delta) L^\delta(0,T_j^\delta) \times \exp\left(\int_0^T \xi_0^\delta(u,T_i^\delta,T_j^\delta;T)du\right)$$

$$\xi_0^\delta(u,T_i^\delta,T_j^\delta;T) = \Delta_0^\delta(u,T_i^\delta;T) + \Delta_0^\delta(u,T_j^\delta;T) + \gamma^\delta(u,T_i^\delta)\cdot\gamma^\delta(u,T_j^\delta)$$

and

$$E^{Q^T}[S_m^\eta(T,T)] = \sum_{i=0}^{m-1} w_i^\eta(0) E^{Q^T}[L^\eta(T,T_i^\eta)] = \sum_{i=0}^{m-1} w_i^\eta(0) L^\eta(0,T_i^\eta)\exp\left(\int_0^T \Delta_0^\eta(u,T_i^\eta;T)du\right) \quad \text{(D4)}$$

$$\begin{aligned} E^{Q^T}[S_m^\eta(T,T)^2] &= E^{Q^T}\left[\left(\sum_{i=0}^{m-1} w_i^\eta(0) L^\eta(T,T_i^\eta)\right)^2\right] \\ &= \sum_{i=0}^{m-1} w_i^\eta(0)^2 E^{Q^T}\left[L^\eta(T,T_i^\eta)^2\right] \\ &\quad + 2\sum_{i=0}^{m-2}\sum_{j=i+1}^{m-1} w_i^\eta(0) w_j^\eta(0) E^{Q^T} \\ &\quad \times \left[L^\eta(T,T_i^\eta) L^\eta(T,T_j^\eta)\right] \end{aligned} \quad \text{(D5)}$$

where

$$E^{Q^T}\left[L^\eta(T,T_i^\eta)^2\right] = L^\eta(0,T_i^\eta)^2 \times \exp\left(\int_0^T 2\Delta_0^\eta(u,T_i^\eta;T) + \gamma^\eta(u,T_i^\eta)^2 du\right)$$

$$E^{Q^T}\left[L^\eta(T,T_i^\eta) L^\eta(T,T_j^\eta)\right] = L^\eta(0,T_i^\eta) L^\eta(0,T_j^\eta) \times \exp\left(\int_0^T \xi_0^\eta(u,T_i^\eta,T_j^\eta;T)du\right)$$

$$\xi_0^\eta(u,T_i^\eta,T_j^\eta;T) = \Delta_0^\eta(u,T_i^\eta;T) + \Delta_0^\eta(u,T_j^\eta;T) + \gamma^\eta(u,T_i^\eta)\cdot\gamma^\eta(u,T_j^\eta)$$

For computing (D1), we further derive

$$\begin{aligned} V_3^2 &= \text{Var}(\ln S_n^\delta(T,T) - \ln S_m^\eta(T,T)) \\ &= v_{3,\delta}^2 + v_{3,\eta}^2 - 2\text{Cov}(\ln S_n^\delta(T,T), \ln S_m^\eta(T,T) \end{aligned} \quad \text{(D6)}$$

As indicated in Appendix C, the covariance part in (D6) may be approximately derived as follows:

$$\begin{aligned} &\text{Cov}(\ln S_n^\delta(T,T), \ln S_m^\eta(T,T)) \\ &= \mathrm{E}^{Q^T}\left[\ln S_n^\delta(T,T)\ln S_m^\eta(T,T)\right] - \mathrm{E}^{Q^T}\left[\ln S_n^\delta(T,T)\right] E^{Q^T}\left[\ln S_m^\eta(T,T)\right] \\ &\cong \mathrm{E}^{Q^T}\left[\ln G_n^\delta(T,T)\ln G_m^\eta(T,T)\right] - \mathrm{E}^{Q^T}\left[\ln G_n^\delta(T,T)\right] E^{Q^T}\left[\ln G_m^\eta(T,T)\right] \end{aligned}$$

where

$$G_n^\delta(T,T) = \prod_{i=0}^{n-1} L^\delta(T,T_i^\delta)^{w_i^\delta(0)}$$

$$G_m^\eta(T,T) = \prod_{i=0}^{m-1} L^\eta(T,T_i^\eta)^{w_i^\eta(0)}$$

$$\begin{aligned} \text{Cov}(\ln S_n^\delta(T,T), \ln S_m^\eta(T,T)) &\cong \sum_{i=0}^{n-1}\sum_{j=0}^{m-1} w_i^\delta(0) w_j^\eta(0) \times \text{Cov}(\ln L^\delta(T,T_i^\delta), \ln L^\eta(T,T_j^\eta)) \\ &= \sum_{i=0}^{n-1}\sum_{j=0}^{m-1} w_i^\delta(0) w_j^\eta(0) \times \int_0^T \gamma^\delta(u,T_j^\delta)\cdot\gamma^\eta(u,T_j^\eta)du. \end{aligned} \quad \text{(D7)}$$

By observing Equation (D.7), the covariance between $\ln S_n^\delta(T,T)$ and $\ln S_m^\delta(T,T)$ can be approximately considered as the weighted covariances between $\ln L^\delta(T,T_i^\delta)$ and

EXHIBIT 5

The Three-factor B Matrix

LIBOR	Factor 1	Factor 2	Factor 3	LIBOR	Factor 1	Factor 2	Factor 3
L (·, 0~1)	0.9108	–0.353	0.2139	L (·, 8~9)	0.8522	0.5066	–0.1311
L (·, 1~2)	0.9384	–0.2647	0.2222	L (·, 9~10)	0.6721	0.7063	–0.2226
L (·, 2~3)	0.9702	–0.1666	0.1761	L (·, 10~11)	0.7757	0.4961	–0.3901
L (·, 3~4)	0.9935	–0.0259	0.1111	L (·, 11~12)	0.5431	0.6495	–0.5321
L (·, 4~5)	0.9951	0.0974	0.0164	L (·, 12~13)	0.8414	0.4082	–0.3541
L (·, 5~6)	0.9648	0.2614	–0.0293	L (·, 13~14)	0.7141	0.5198	–0.4689
L (·, 6~7)	0.9381	0.3403	–0.0648	L (·, 14~15)	0.5383	0.6184	–0.5726
L (·, 7~8)	0.8967	0.4256	–0.1219				

Notes: The forward LIBOR rates within each year are assumed to be perfectly correlated, and consequently the correlation matrix of all relevant LIBOR rates, namely Σ, could be reduced to a 15×15 matrix and calculated from the historical forward rate data during 2007/1/1–2008/7/1. Matrix B is computed based on Σ by employing the Rebonato [1999] approach and $BB' \approx \Sigma$.

$\ln L^{\eta}(T, T_i^{\eta})$ which are, respectively, involved in $\ln S_n^{\delta}(T, T)$ and $\ln S_m^{\eta}(T, T)$.

With the aforementioned knowledge, the pricing formula of the TIRSO can be priced via Lemma B1.

APPENDIX E

The Market Data

The data on 3- and 6-month inital LIBOR rates and cap volatilities used for the numerical study can be obtained from Datastream. Exhibit 5 shows the matrix B, which is calibrated from the historical correlation matrix of LIBOR rates based on the Rebonato approach.

ENDNOTES

[1]As examined in Rogers [1996], the Gaussian term structure model has an important theoretical limitation: The rate can attain negative values with positive probability, which may cause pricing errors in many cases.

[2]The filtration $\{\mathcal{F}_t\}_{t \in [0,T]}$ is right continuous, and $\mathcal{F}_0$ contains all the Q-null sets of $\mathcal{F}$.

[3]For ease of computation in Equation (3), we may fix δ (for example, $\delta = 0.25$).

[4]We employ $W(t)$ to denote an independent m-dimensional standard Brownian motion under an arbitrary measure without causing any confusion.

[5]$0 \le t \le T$, $T = T_0^{\delta}$ and $\delta = T_i^{\delta} - T_{i-1}^{\delta}$, $i = 1, 2, \ldots, n$.

[6]See also Brace and Womersley [2000] for the proof of the low variability.

[7]As indicated by the empirical studies in Brigo and Mercurio [2001], forward swap rates obtained from lognormal forward LIBOR rates are not far from being lognormal under the relevant measure.

[8]$\mathrm{E}[S_n^{\delta}(t,T)]$ and $\mathrm{E}[S_n^{\delta}(t,T)^2]$ are computable.

[9]For this approach in detail, please refer to Brigo and Mercurio [2001].

[10]Since caps and swaptions are actively traded financial instruments, a price inconsistency between the two products is almost impossible. Thus, the calibration based only on cap data is not unreasonable. In addition, even if end-users adopt the other calibration approach, our pricing formulas remain workable.

[11]For other assumptions of volatility structures, please refer to Brigo and Mercurio [2001].

[12]Note the Euclidean norm of each row vector of B is 1.

[13]For more details regarding the performance of one- and multi-factor models, please refer to Driessen, Klaassen, and Melenberg [2003] and Rebonato [1999].

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