A GARCH with Time-Changed Lévy Innovation Model and Its Applications from an Economic Perspective

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The paper constructs a GARCH process with time-changed Lévy innovations from the economic perspective which assumes that the arrival of new information causes the asset return to be stochastic and volatility clustering. The GARCH (1,1) process with generalized hyperbolic innovation is introduced as a general form for the volatility process. The paper uses a special case of the process to discuss the economic meaning behind alternative dynamic behaviors, and then applies it in pricing a European option under the hypothesis that every investor selects the canonic martingale measure.

Introduction

Some researchers have proposed the GARCH process with different innovation distributions from normal law such as, student's *t*-distribution (Bollerslev, 1987), the generalized error distribution (Andersson, 2001), normal inverse Gaussian (Forsberg and Bollerslev, 2002), α-stable distribution (Menn and Rchev, 2005), and generalized secant hyperbolic distribution (Palmitesta and Provasi, 2004). These models are developed for the purpose of fitting empirical facts from market data such as asymmetrically heavy tails with positive excess kurtosis, volatility clustering and the leverage effect.[1]

Some empirical facts contribute to the modeling of asset dynamics. The argument proposed by Geman *et al.* (2001a and 2001b) suggests that financial asset dynamics must have a jump component but need not have a diffusion component. Empirical evidence shows that infinite-activity jumps may perform better than finite-activity compound Poisson jumps. This implies that asset prices actually display many small jumps on a finite time scale. Lévy processes can generate non-normal innovation and accommodate

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[1] The term 'leverage effect' is used to describe the negative correlation between stock returns and their volatilities.

both low and high frequency jumps. Stochastic time changes can produce stochastic volatility, which includes both affine and quadratic volatility dynamics. Therefore, by subordinating the stochastic time changes to Lévy processes, the time-changed Lévy processes simultaneously possess infinite-activity jumps, stochastic volatility, and non-normal distributions.

In this paper, we assume that the noise in the financial market is the sole innovation term of the GARCH process, thus, the return process is constant if there is no noise. This assumption can be observed in a return structure that depends on the risk-free rate and risk premium dynamics (or volatility structure). The economic interpretations are different from previous literature, which assumed only the time-changed return dynamics. As a result of this improvement, we set the innovation process to be a time-changed Lévy process with subordinator T_t. The process represents the impact of the noise on the price, due to the arrival of new information. If the noise arises frequently, then the arrival of information passes quickly, and the price jumps violently. Hence, the asset price or firm value processes vary more. When investors possess these assets and hedge their dynamics, it becomes more likely for the total value to reach the stop loss point of the portfolio (default barrier). The hedge may be in vain and the cost increases. This uncertain condition usually indicates worsening of the economic environment in the real world. On the other hand, a slow arrival time corresponds to steady growth as a result of less noise, which implies a better economic environment.

The Generalized Hyperbolic (GH) process is a pure jump process. We take this process as the noise dynamics, due to its flexibility to fit different market data well, and the GARCH process is adopted as the variance dynamics for volatility clustering in most of the financial data. The return dynamics is assumed to be the GARCH-GH process. This is different from the time-changed Lévy return processes of Carr and Wu (2004). We can apply the GARCH-GH process to derive a European call option pricing model and find the economic meaning behind the return behavior, using different parameter values.

Next, the paper introduces the family of GH distributions with their main properties and establishes the GARCH-type option pricing models with GH innovations. Then, it presents the GARCH process with time-changed Lévy innovation and option pricing models. Finally, it concludes the discussion.

The Model

The Generalized Hyperbolic Distributions

Let the information arrival time, S be a random variable following the Generalized Inverse Gaussian (GIG) distribution and let W be an independent standard normal random variable. We assume that the impact on price due to information arrival is given by the distribution, $\mu S + W_S$, where μ is a constant and W_S is a Brownian motion with mean 0 and variance S; this is called a GH distribution. If S is the calendar time, which has the property of steady increase at constant speed, then the information impact follows a normal distribution with mean μS and variance S, or it can be regarded as a drift-μ Brownian motion subordinated by S. The density function of the GH distribution is as follows (Prause, 1999):

$$p(x;\gamma,\alpha,\beta,\delta,\mu)=a(\gamma,\alpha,\beta,\delta)\left(\delta^2+(x-\mu)^2\right)^{\frac{\gamma}{2}-\frac{1}{4}}K_{\gamma-1/2}\left(\alpha\sqrt{\delta^2+(x-\mu)^2}\right)e^{\beta(x-\mu)}$$

where $a(\gamma,\alpha,\beta,\delta)=\dfrac{\left(\alpha^2-\beta^2\right)^{\gamma/2}}{\sqrt{2\pi}\alpha^{\gamma-1/2}\delta^{\gamma}K_{\gamma}\left(\delta\sqrt{\alpha^2-\beta^2}\right)}$ and K is the modified Bessel function of the second kind. Since GIG distribution is infinitely divisible, GH distributions can be characterized by five parameters: $\gamma, \alpha, \beta, \delta$ and μ. The first three determine the shape of the GH and the last two represent the scale and location, respectively. Both Lévy density and probability density have exponential tails with decay rates $\gamma_+=\alpha-\beta$ on the right side and $\gamma_-=\alpha+\beta$ on the left side. The restrictions on the parameters are as follows: $\mu\in R$ and $\delta\geq 0, |\beta|<\alpha$ if $\gamma>0$; $\delta>0, |\beta|<\alpha$ if $\gamma=0$; $\delta>0, |\beta|\leq\alpha$ if $\gamma<0$. The characteristic function of the GIG distribution has the following form:

$$\Phi(u)=e^{i,\mu u}\left(\frac{\alpha^2-\beta^2}{\alpha^2-(\beta+iu)^2}\right)^{\gamma/2}\frac{K_{\gamma}\left(\delta\sqrt{\alpha^2-(\beta+iu)^2}\right)}{K_{\gamma}\left(\delta\sqrt{\alpha^2-\beta^2}\right)}$$

and the moment generating function can be derived from it as follows:

$$M(u)=e^{\mu u}\left(\frac{\alpha^2-\beta^2}{\alpha^2-(\beta+u)^2}\right)^{\gamma/2}\frac{K_{\gamma}\left(\delta\sqrt{\alpha^2-(\beta+u)^2}\right)}{K_{\gamma}\left(\delta\sqrt{\alpha^2-\beta^2}\right)}$$

There are many well-known probability distribution in the GH family which has a rich structure and a great variety of shapes. Some examples are listed as follows:[2]

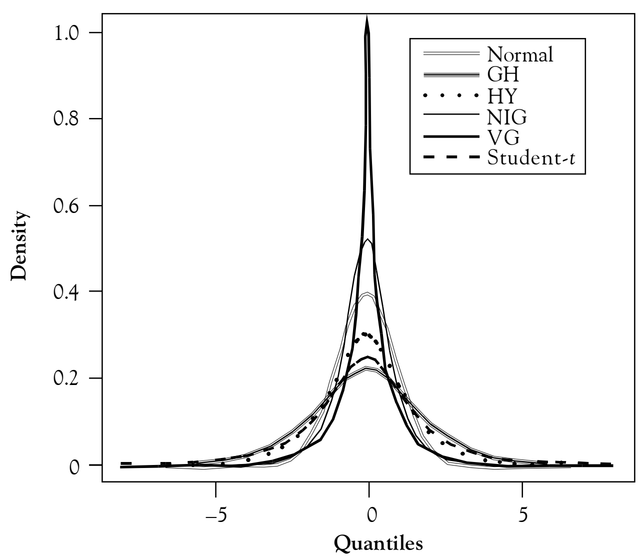
- Normal $\left(\mu+\sigma^2\beta\Delta,\sigma^2\Delta\right)$, when $\alpha, \delta\to\infty, \delta/\alpha\to\sigma^2$ and Δ denotes time period.
- Hyperbolic distribution, when $\gamma=1$.

[2] The last three cases are taken from Blæsild (1999).

- Normal Inverse Gaussian (θ, σ, ν) (NIG), where $\gamma = -0.5$, $\mu = 0$, $\beta = \theta / \sigma^2$, $\alpha = \sqrt{\theta^2/\sigma^4 + 1/(\nu\sigma^2)}$ and $\delta = \sigma / \sqrt{\nu}$.
- Variance Gamma (θ, σ, ν), when $\delta = \mu = 0$, $\alpha = \sqrt{\theta^2/\sigma^4 + 2/(\sigma^2\nu)}$, $\beta = \theta/\sigma^2$ and $\gamma = 1/\nu$.
- Student's t-distribution (ν), when $\gamma = -0.5$, $\alpha = \beta = 0$ and $\delta = \sqrt{\nu}$.
- Cauchy (μ), when $\gamma = -0.5$, $\alpha = \beta = 0$ and $\delta = 1$.
- GIG (γ, τ, ϕ), when $\mu = 0$, $\alpha\delta^2 \to \tau$ and $\alpha - \beta = \phi / 2$.

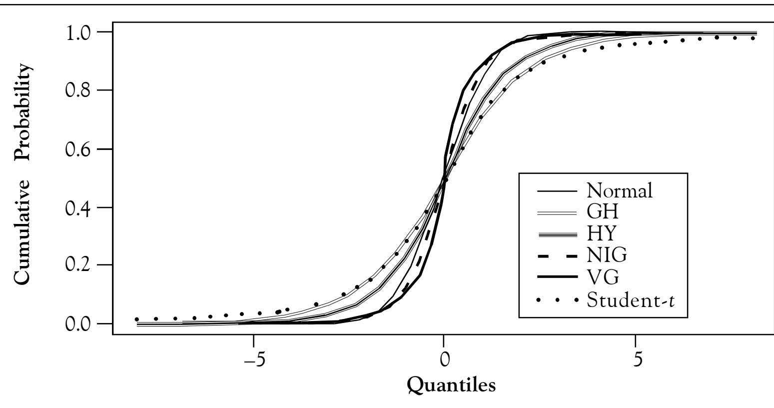
Figures 1 and 2 demonstrate the density and cumulative probability of the GH distribution and its special cases. GH distribution has more flexibility for skewness, kurtosis and tail behavior than most of the other distributions. Thus, we can fit market data well by using this distribution.

Figure 1: Densities of the GH Family



Note: The figure shows a comparison of the densities of the GH family including Normal(0,1), GH(2,1,0,1,0), Hyperbolic=GH(1,1,0,1,0), NIG=GH(–0.5,1,0,1,0), VG=GH(0.5,1,0,0,0), and Student-*t* (4)=GH(–1,0,0,2,0).

Figure 2: Cumulative Probabilities of the GH Family



Note: The figure shows the cumulative probabilities of the GH family, including standard Normal (0,1), GH(2,1,0,1,0), Hyperbolic=GH(1,1,0,1,0), NIG=GH(–0.5,1,0,1,0), VG=GH(0.5,1,0,0,0), and Student-*t* (4)=GH(–1,0,0,2,0).

The GARCH Process with GH Innovation

Some empirical studies, such as Bollerslev (1986) and Akigiray and Booth (1988), suggest that one may adopt GARCH(1,1) as a conditional heteroskedastic model for volatility clustering. Therefore, we nest GH law in the GARCH(1,1) process to combine the two properties of information arrival—stochastic and clustering. We assume that logarithmic asset returns under physical measure P, adhere to the following dynamics:

$$\ln\left(S_t/S_{t-1}\right)=r_t-q_t+\lambda\sigma_t-w\left(\sigma_t;\,\bar{\theta}\right)+\varepsilon_t\left(\sigma_t;\,\bar{\theta}\right),\ t\in N \qquad \ldots(1)$$

$$\varepsilon_t=\sigma_t z_t/\Omega_{t-1}\sim GH\left(\bar{\theta}\right) \qquad \ldots(2)$$

$$\sigma_t^2=\alpha_0+\alpha_1\varepsilon_{t-1}^2+\beta_1\sigma_{t-1}^2,\ t\in N,\ \varepsilon_0=0 \qquad \ldots(3)$$

where S_t denotes the price of the underlying asset at time t. r_t and q_t denote the risk-free rate and dividend yield for the period $[t–1, t]$, respectively. λ can be interpreted as the market price of risk. $\bar{\theta}=(\alpha,\,\beta,\,\gamma,\,\delta,\,\mu)$ stands for the parameter vector of GH. σ_t^2 is the conditional variance of ε_t given $\{\varepsilon_{t-1},\,\varepsilon_{t-2},\,\ldots,\,\varepsilon_0\}$ and $\{z_t\}$ is a sequence of independent and identical random variables with mean 0 and variance 1. $w\left(\sigma_t;\,\bar{\theta}\right)$ is equal to $\log\left(E_{p_t}\left[\exp\left(\varepsilon_t\left(\sigma_t;\,\bar{\theta}\right)\right)\right]\right)$ which neutralizes the expectation of the innovation. The volatility process $(\sigma_t)_{t=1,\,2,\,\ldots,\,T}$ is predictable, and some restrictions on parameters, such as $w > 0$, $\alpha\geq 0$, $\beta\geq 0$ and $\alpha+\beta<1$, will ensure that the unconditional variance of the asset return is positive and bounded (or covariance stationary).

We can further find the conditional expectation,

$$E\left[\bar{S}_t/\bar{S}_{t-1}\,|\,\psi_{t-1}\right]=\exp\left(r_t+\lambda_t\sigma_t\right)$$

where $\bar{S}_t=S_t\,\exp\left(\sum_{k=1}^{t}q_t\right)$ is the stock price including reinvestment of the dividend yield q_t.

The Option Pricing Model Under GARCH-GH Process

The GARCH-GH option pricing model is constructed as follows: consider that the asset dynamics are defined on a filtered probability space $\left(\Omega,\,\psi,\,\left(\psi_t\right)P\right)$. It consists of filtered probability spaces $(\Omega_t,\,\psi_t,\,P_t)_{t\in N}$ on which a sequence $(\varepsilon_t)_{t\in N}$ of independent and identical real random variables are defined.

Definition:

$$\Omega:=\prod_{t\in N}\Omega_t\,,$$

$$\psi_t:=\otimes_{k=1}^{t}\sigma\left(\varepsilon_k\right)\otimes\psi_0\otimes\psi_0\ldots,$$

$$\psi := \sigma\left(\bigcup_{t\in N}\psi_t\right)$$

$$P := \otimes_{t\in N} P_t,$$

where $\psi_0 = \{\phi, \Omega\}$, and ψ_t is the σ-field of all information up to and including time *t*. $\sigma(\varepsilon_k)$ stands for the σ-field generated by ε_k on Ω_k.

Since this process belongs to the class of incomplete models, thus, there is no unique equivalent martingale measure for this model. Duan (1995) and Heston and Nandi (2000) derived a GARCH option pricing model from the Locally Risk-Neutral Valuation Relationship (LRNVR). Rather than restricting investors' preferences, we assume that every investor chooses the canonic martingale measure which is consistent with the assumptions postulated by the LRNVR. Then, a GARCH-GH option pricing model can be introduced as the following theorem.

***Theorem* 1:**

Let a new measure *Q* on ψ_T equivalent to the physical measure *P*, be defined by the Radon-Nikodym derivative, $dQ/dP = Z_T$, where *T* is the time horizon and belongs to natural numbers, and the density process $(Z_t)_{t=1,2,\ldots,T}$ is defined as follows:

$$Z_0 \equiv 1,$$

$$Z_t :\equiv \frac{d\,(P_1 \otimes \ldots \otimes P_{t-1} \otimes Q_t \otimes P_{t+1} \otimes \ldots \otimes P_T)}{dP} \cdot Z_{t-1},\ t = 1, 2, \ldots, T \qquad \ldots(4)$$

and Q_t measure makes the distribution of $\varepsilon_t^* = \varepsilon_t + \sigma_t\lambda$ identical to that of ε_t under P_t measure. Then:

- The discounted asset price process $\{\bar{S}_t\}$ is a martingale with respect to the filtration ψ_t under *Q* measure.
- The conditional one period ahead variances of the return dynamics are unaffected by the change of measure, that is,

  $$Var^Q(\log(S_t/S_{t-1})\,|\,\psi_{t-1}) = Var^P(\log(S_t/S_{t-1})\,|\,\psi_{t-1}),\ a.s.$$

- The return dynamics under *Q* measure can be written as:

  $$\ln(S_t/S_{t-1}) = r - q_t - w\,(\sigma_t; \bar{\theta}) + \varepsilon_t(\sigma_t; \bar{\theta}),\ t = 1, 2, \ldots, T \qquad \ldots(5)$$

  $$\sigma_t^2 = \alpha_0 + \alpha_1\left(\varepsilon_{t-1}^* - \sigma_{t-1}\lambda\right)^2 + \beta_1\sigma_{t-1}^2,\ \varepsilon_0 = 0$$

where $\{\varepsilon_t^*\}$ is a sequence of real random variables on (Ω_t, ψ_t) following $GH(\bar{\theta})$ and $t = 1, 2, \ldots, T$.

Proof of Theorem 1 is provided in the Appendix.

These results enable us to calculate the value of the derivatives of the underlying asset. Without losing generality, we set the dividend yield to be 0 hereafter. The following corollary is introduced to price the vanilla options.

***Corollary* 1:**

The value of a European call option with strike price K and maturity T is given by,

$$C_t^{GARCH-GH} = \exp\left(-\sum_{k=t+1}^{T} r_k\right) E^Q\left[\left(S_T - K\right)^+ \mid \psi_t\right], \qquad \ldots(6)$$

where the underlying asset price S_T at time T is given by

$$S_T = S_t \exp\left(\sum_{k=t+1}^{T}\left(\left(r_k - q_k\right) - w\left(\sigma_k; \bar{\theta}\right) + \varepsilon_k\left(\sigma_k; \bar{\theta}\right)\right)\right) \qquad \ldots(7)$$

Proof:

We choose finite periods $\{t, t+1, t+2, \ldots, T\}$. Then, summing the return dynamics given by Equation 5 over these intervals, we get Equation 7 and the arbitrage free option price at time t can thus be expressed by Equation 6.

All subclasses of the GH family can be nested in the GARCH process. We can apply Theorem 1 and Corollary 1 to price the European option under such GARCH process with Lévy time-changed innovations model, in a manner similar to the above steps.

Special Cases

The GH process incorporates the pure jumps process—VG and NIG—as special cases. The former has finite variation while the latter does not. We consider these two cases as examples. The business time T_t is set to follow an Inverse Gaussian (IG) process with mean t and variance νt, in the first example. The IG describes the distribution of the time that a Brownian motion takes to reach a fixed positive level, if the Gaussian law describes the distribution of the distance at a fixed time in Brownian motion. We assume that the impact caused by the new information is a geometric Brownian motion subordinating T_t. That is, the impact process is a Normal Inverse Gaussian (NIG) and can be expressed as a Brownian motion W with drift θ and volatility σ at time changed by an independent T_t process:

$$X_t^{NIG}(\theta, \sigma, \nu) = \theta T_t^{\nu} + \sigma W\left(T_t^{\nu}\right)$$

Consider another example. If the business time T_t is set to follow a Gamma process with mean t and variance νt, and the impact caused by the new information is assumed to be a geometric Brownian motion subordinating T_t, then, it is known as a Variance Gamma (VG) process and can be expressed as a Brownian motion W with drift θ and volatility σ at time changed by an independent T_t process:

$$X_t^{VG}(\theta, \sigma, \nu) = \theta T_t^{\nu} + \sigma W\left(T_t^{\nu}\right)$$

Alternative innovation distributions of GARCH(1,1) can be constructed by using the procedure described in subsection: The GARCH Process with GH Innovation. We introduce and discuss these two simple special forms of the GARCH-GH process. The first case is the GARCH(1,1) process with $\text{NIG}(0, \sigma, \nu)$ innovations, denoted by the GARCH(1,1)-$\text{NIG}(0, \sigma, \nu)$ in which the parameter $\theta = 0$ means that the expectation of the innovation (noise) equals 0. The second case is the GARCH(1,1) process with $\text{VG}(0, \sigma, \nu)$ innovations, denoted by the GARCH(1,1)-$\text{VG}(0, \sigma, \nu)$. The two cases are further discussed regarding their economic meaning, and they have fewer parameters for easy estimation and quick pricing option. The details are provided in the following subsections.

The GARCH-NIG Option Pricing Model

From Equations 1, 2 and 3, we replace the GH distribution with NIG distribution, and assume that the drift of the NIG innovation is 0, the risk-free interest rate r is constant in every period. Then, the process is called the GARCH(1,1)-$\text{NIG}(0, \sigma, \nu)$ process which under P measure is written as follows:

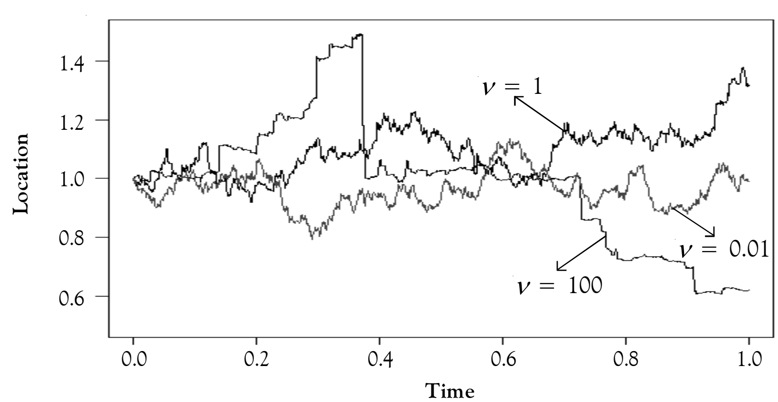
$$\ln \frac{S_t}{S_{t-1}} = r - 0.5 g_t + \lambda \sqrt{h_t} + \varepsilon_t,$$

$$\varepsilon_t \mid F_t \sim NIG\left(0, h_t, \nu\right),$$

$$h_t = w + a\varepsilon_{t-1}^2 + bh_{t-1},\ t = 1, 2, \ldots, T$$

where $g_t = 2\left(1/\nu - \sqrt{1/\nu^2 - h_t/\nu}\right)$, and ε_t is a sequence of independent random variables with mean 0 and variance h_t. Figure 3 displays the dynamic behavior of the GARCH(1,1)-$\text{NIG}(0, \sigma, \nu)$ process with different values of parameter ν, while the other parameters remain fixed. Since ν represents the variance of the subordinator T_t,

Figure 3: The GARCH(1,1)-NIG(0, σ, ν) Process with Different ν Values



Note: The dynamics are modeled as follows: $\ln\left(S_t/S_{t-1}\right) = r - 0.5h_t + \lambda\sqrt{h_t} + \varepsilon_t$, $\varepsilon \mid \Omega_{t-1} \sim NIG\left(0, h_t, \nu\right)$, $h_t = w + \alpha\varepsilon_{t-1}^2 + \beta h_{t-1}$, where $r = 0.01/252$, $\lambda = 0.03$, $\omega = 0.0002$, $\alpha = 0.05$, and $\beta = 0.04$.

as ν increases, T_t becomes increasingly varied. We can see that the jump sizes become larger than those for smaller ν. The larger jumps appear more frequent and in stronger cluster. Moreover, the range of the entire path gets wider. The reason is that the larger ν creates greater volatility h_t in the GARCH structure. These phenomena reflect that the information arrival passes faster and the impact caused by information increases. Volatility clustering, existing popularly in financial markets, can be regarded as the clustering of information arrival. Further in finance, the frequent larger jumps in dynamics of asset prices or firm values always reverse the drift and cause sudden large losses or gains. Hedging is difficult in such situations. Therefore, large ν also means a highly uncertain environment that is flooded with abundance of good or bad news. It usually implies a worse scenario.

For introducing the GARCH(1,1)-NIG$(0, \sigma, \nu)$ option pricing model, we need the following corollary:

***Corollary* 2:**

Under the hypotheses of Theorem 1, the value of a European call option at time t with strike price K and expiring at maturity T is:

$$C_t^{GARCH-NIG} = \exp\left(-r\,(T-t)\right) E_Q\left[\max\left(S_T - K,\, 0\right) | \,\Omega_t\right], \qquad \text{...(8)}$$

where $E_Q[\bullet]$ denotes the expectation under the risk-neutral distribution. The terminal asset price S_T can be written in the following form:

$$S_t \exp\left[r\,(T-t) - 0.5\sum_{i=t+1}^{T} g_i + \sum_{i=t+1}^{T} \varepsilon_i^*\right] \qquad \text{...(9)}$$

where ε_t^* conditional on Ω_{t-1} is a NIG distribution with mean 0 and variance h_t.

Proof:

We replace g_t by $2\left(1/\nu - \sqrt{1/\nu^2 - h_t/\nu}\right)$ in $E^Q\left[\exp\left(r - 0.5\, g_t + \varepsilon_t^*\right) | \,\Omega_{t-1}\right]$, and then $E\left[\ln\left(S_t/S_{t-1}\right)\right]$ is equal to r. Thus, the price process under the risk-neutral probability measure Q is:

$$\ln\left(S_t/S_{t-1}\right) = r - 0.5 g_t + \varepsilon_t^* \qquad \text{...(10)}$$

$$\varepsilon_t^* \,|\, \Omega_{t-1} \sim NIG\left(0, h_t, \nu\right)$$

$$h_t = w + \alpha\left(\varepsilon_{t-1}^* - \lambda\sqrt{h_{t-1}}\right)^2 + \beta h_{t-1},\ t = 1, 2, \ldots, T$$

where $g_t = 2\left(1/\nu - \sqrt{1/\nu^2 - h_t/\nu}\right)$. We choose finite periods $\{t, t+1, t+2, \ldots, T\}$, and summing the return dynamics given by Equation 10 over these intervals, we get Equation 9, and the arbitrage free option price at time t can thus be expressed by Equation 8.

***Proposition* 1:**

The GARCH(1,1) option pricing model of Duan (1995) is a special case when the value of ν approaches 0.

Proof:

Under P measure, the price dynamics are:

$$\ln\left(S_t/S_{t-1}\right) = r - 0.5 g_t + \lambda\sqrt{g_t} + \varepsilon_t,$$

where $\varepsilon_t \mid \Omega_{t-1} \sim NIG\left(0, h_t, \nu\right)$,

$$h_t = w + \alpha\, \varepsilon_{t-1}^2 + \beta h_{t-1} \text{ and } g_t = 2\left(1/\nu - \sqrt{1/\nu^2 - \nu h_t}\right).$$

If ν approaches zero, then the NIG distribution turns into the Normal distribution with mean 0 and variance h_t. That is, $\varepsilon_t \mid \Omega_{t-1} \sim N\left(0, h_t\right)$, the model is Duan's GARCH option pricing model.

The GARCH-VG Option Pricing Model

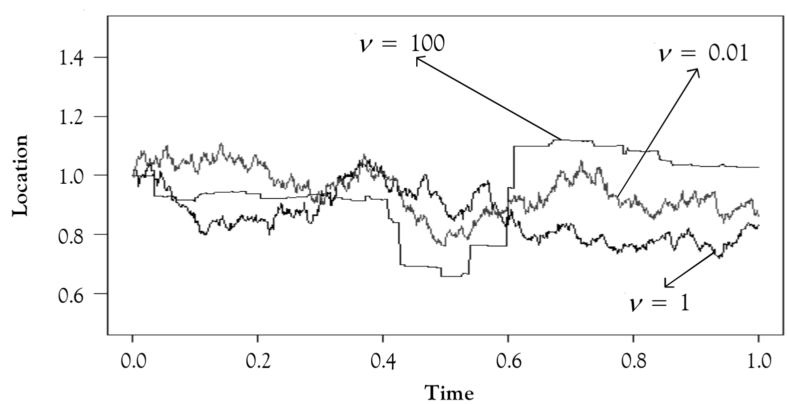
From Equations 1, 2 and 3, we replace the GH distribution with VG distribution, and assume that the drift of the VG innovation is 0, and the risk-free interest rate, r is constant in every period. Then, the process is called the GARCH(1,1)-VG$\left(0, \sigma, \nu\right)$ process, which under physical measure P is written as follows:

$$\ln\left(S_t/S_{t-1}\right) = r - 0.5 g_t + \lambda\sqrt{h_t} + \varepsilon_t$$

$$\varepsilon_t \mid \Omega_t \sim VG\left(0, h_t, \nu\right)$$

$$h_t = w + a\,\varepsilon_{t-1}^2 + b h_{t-1}$$

Figure 4: The GARCH(1,1)-VG(0, σ, ν) Process with Different ν Values



Note: The dynamics are modeled as follows: $\ln\left(S_t/S_{t-1}\right) = r - 0.5 h_t + \lambda\sqrt{h_t} + \varepsilon_t$, $\varepsilon_t \mid \Omega_{t-1} \sim VG\left(0, h_t, \nu\right)$, $h_t = w + \alpha\,\varepsilon_{t-1}^2 + \beta h_{t-1}$, where r = 0.01/252, λ = 0.03, ω = 0.0002, α = 0.05, and β = 0.04.

where $g_t = (-2/h_t)\log(1-h_t\nu/2)$, and ε_t is a sequence of independent random variables with mean 0 and variance h_t. Figure 4 displays the dynamic behavior of GARCH(1,1)-VG$(0,\sigma,\nu)$ with different values of parameter ν, while the other parameters remain unchanged. As discussed in subsection: The GARCH-NIG Option Pricing Model, a larger ν corresponds to worsening of the economic environment.

For introducing the GARCH(1,1)–VG$(0,\sigma,\nu)$ option pricing model, we need the following corollary:

***Corollary* 3:**

Under the hypotheses of Theorem 1, the value of a European call option at time t with strike price K and expiring at maturity T is:

$$C_t^{GARCH-VG} = \exp\left(-r\,(T-t)\right) E_Q\left[\max\left(S_T - K, 0\right) | \,\Omega_t\right]$$

where $E_Q[\bullet]$ denotes the expectation under the risk-neutral distribution and the terminal asset price is, $S_T = S_t \exp\left[r\,(T-t) - 0.5\sum_{i=t+1}^{T} g_i + \sum_{i=t+1}^{T}\varepsilon_i^*\right]$, where ε_t^* conditional on Ω_{t-1} is a VG distribution with mean 0 and variance h_t.

Proof is similar to that of Corollary 2.

***Proposition* 2:**

The GARCH(1,1) option pricing model of Duan (1995) is a special case when the value of ν approaches 0.

Proof is similar to that of Proposition 1.

Conclusion

We consider the dynamic behavior of financial asset prices from an economic perspective which assumes that arrival of new information causes randomness and clustering of volatility of an asset's return. Under this assumption, the GARCH with time-changed Lévy innovations is established. We can describe the rate and the impact of the information arrival via alternative parameters estimated from the market data, and can further explain the economic meaning beyond the model only for fitting the financial data.

A general GARCH(1,1)-GH process has a rich structure that contains well-known GARCH(1,1) processes with alternative innovations. GARCH-GH option pricing models can be developed under the hypothesis that every investor selects the canonic martingale measure. We introduce simpler forms of the option pricing model—GARCH(1,1)-NIG$(0,\sigma,\nu)$ and GARCH(1,1)-VG$(0,\sigma,\nu)$. Forsberg and Bollerslev (2002) show that the former offers very accurate out-of-sample predictions on the ECU/US exchange rates. Further research can be done to compare the empirical performances for different market data. ❖

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Appendix

Proof of Theorem 1 is as follows:

(a) Using Equation 4, the definition of $w\left(\sigma_t; \bar{\theta}\right)$ and the measurability of σ_t with respect to ψ_{t-1}, we obtain

$$E^Q\left[\bar{S}_t \mid F_{t-1}\right] = E^Q\left[S_{t-1}\exp\left(r_t + \lambda\sigma_t - w\left(\sigma_t; \bar{\theta}\right) + \varepsilon_t\left(\sigma_t; \bar{\theta}\right)\right) \mid \psi_{t-1}\right]$$

$$= E^P\left[\frac{Z_t}{Z_{t+1}} S_{t-1}\exp\left(r_t + \lambda\sigma_t - w\left(\sigma_t; \bar{\theta}\right) + \varepsilon_t\left(\sigma_t; \bar{\theta}\right)\right) \mid \psi_{t-1}\right]$$

$$= S_{t-1}\exp\left(r_t + \lambda\sigma_t\right)\cdot\exp\left(-w\left(\sigma_t; \bar{\theta}\right)\right)\cdot E^P\left[\frac{Z_t}{Z_{t-1}}\exp\left(\varepsilon_t\left(\bar{\theta}\right)\right) \mid \psi_t\right]$$

$$= S_{t-1}\exp\left(r_t + \lambda\sigma_t\right)\cdot\frac{1}{E\left[\exp\left(\varepsilon_t\right)\right]}\cdot\exp\left(-\lambda\sigma_t\right)\cdot E\left[\exp\left(\varepsilon_t\right)\right]$$

$$= S_{t-1}\exp\left(r_t\right)$$

(b) Since

$$Var_P\left(\log\left(S_t/S_{t-1}\right) \mid \psi_{t-1}\right) = Var_P\left(\varepsilon_t\left(\bar{\theta}\right) \mid \psi_{t-1}\right) = \sigma_t \quad a.s.,$$

$$Var_Q\left(\log\left(S_t/S_{t-1}\right) \mid \psi_{t-1}\right) = Var_Q\left(\sigma_t\eta_t\left(\bar{\theta}\right) \mid \psi_{t-1}\right) = \sigma_t \quad a.s.,$$

the assertion holds.

(c) Since the distribution of $\varepsilon_t^* = \varepsilon_t + \sigma_t\lambda$ under Q_t measure is the same as that of ε_t under P_t measure, using Equation 1 we can derive,

$$\ln\left(S_t/S_{t-1}\right) = r_t - q_t + \lambda\sigma_t - w\left(\sigma_t; \bar{\theta}\right) + \varepsilon_t\left(\sigma_t; \bar{\theta}\right)$$

$$= r_t - q_t - w\left(\sigma_t; \bar{\theta}\right) + \left(\varepsilon_t\left(\sigma_t; \bar{\theta}\right) + \sigma_t\lambda\right)$$

$$= r_t - q_t - w\left(\sigma_t; \bar{\theta}\right) + \varepsilon_t^*,\ t = 1, 2, \ldots, T,$$

and the variance process,

$$\sigma_t^2 = \alpha_0 + \alpha_1\varepsilon_{t-1}^2 + \beta_1\sigma_{t-1}^2$$

$$= \alpha_0 + \alpha_1\left(\varepsilon_{t-1}^* - \sigma_{t-1}\lambda\right)^2 + \beta_1\sigma_{t-1}^2,\ t = 1, 2, \ldots, T\ \cdot$$

The variance process ε_{t-1} needs to be replaced by $\varepsilon_t^* - \sigma_t\lambda$ in order to ensure the desired result.

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