Insurance: Mathematics and Economics 62 (2015) 257–269

Contents lists available at ScienceDirect

Insurance: Mathematics and Economics

journal homepage: www.elsevier.com/locate/ime

# Comparative ambiguity aversion and downside ambiguity aversion

Yi-Chieh Huang [a], Larry Y. Tzeng [b,c,*], Lin Zhao [d]

[a] Department of Risk Management and Insurance, Feng Chia University, No. 100, Wenhwa Road, Taichung city 40724, Taiwan
[b] Department and Graduate Institute of Finance, National Taiwan University, No. 50, Ln. 144, Sec. 4, Keelung Road, Taipei city 10673, Taiwan
[c] Risk and Insurance Research Center, National Chengchi University, No. 64, Sec. 2, Zhinan Road, Taipei city 11605, Taiwan
[d] The Institute of Systems Science, Chinese Academy of Sciences, No. 55, Zhongguancun East Road, Beijing city 100190, China

ARTICLE INFO

*Article history:*
Received October 2014
Received in revised form
March 2015
Accepted 6 April 2015
Available online 22 April 2015

*JEL classification:*
D80
D81

*Keywords:*
Ambiguity
Downside ambiguity
Comparative ambiguity aversion
Comparative downside ambiguity aversion
Effort

ABSTRACT

This paper first defines an increase in ambiguity and an increase in downside ambiguity. We then provide comparative criteria for ambiguity aversion and downside ambiguity aversion. Different from the finding that the comparative criterion for risk aversion is variant with the measure of the premium to reduce risks, we show that the criteria remain the same, whether the premiums to reduce ambiguity and downside ambiguity are measured by utility or money. Under the criteria, a more ambiguity-averse (downside-ambiguity-averse) individual is shown to spend more effort in reducing ambiguity (downside ambiguity) than a less ambiguity-averse (downside-ambiguity-averse) individual.


## 1. Introduction

Ellsberg (1961) found that individuals tend to prefer a lottery with certain probabilities to a lottery with uncertain probabilities. Such a preference that cannot be explained by the theory of expected utility is known as ambiguity aversion. Since then, researchers have developed various models to characterize this ambiguity-averse preference, such as maxmin expected utility (Gilboa and Schmeidler, 1989), Choquet expected utility (Schmeidler, 1989), $\alpha$-maxmin expected utility (Ghirardato et al., 2004), the smooth model of ambiguity aversion (Klibanoff et al., 2005), and so on.[1] Based on these models, fruitful findings regarding decision making under ambiguity aversion have been discovered.[2] On the other hand, some researchers have devoted to experimental studies and further enriched our understanding of ambiguity preferences.[3]

* Corresponding author at: Department and Graduate Institute of Finance, National Taiwan University, No. 50, Ln. 144, Sec. 4, Keelung Road, Taipei city 10673, Taiwan. Tel.: +886 2 33661082.
*E-mail addresses:* huangyic@fcu.edu.tw (Y.-C. Huang), tzeng@ntu.edu.tw (L.Y. Tzeng), zhaolin@iss.ac.cn (L. Zhao).

[1] Regarding the smooth model of ambiguity aversion (Klibanoff et al., 2005), its intertemporal version which is called recursive smooth ambiguity model (Klibanoff et al., 2009) and its dual representation (Iwaki and Osaki, 2014) have been developed. Furthermore, more general ambiguity models which accommodate the above models and relax their assumptions have also been proposed, such as Maccheroni et al.'s (2006) variational characterization, Siniscalchi's (2009) vector expected utility, Nau's (2011) state-preference theory, Cerreia-Vioglio et al.'s (2011) uncertainty averse representation, and Ghirardato and Siniscalchi's (2012) local and global multiple-prior characterization.

[2] For instance, Gollier (2011) proved that under certain conditions, greater ambiguity aversion decreases the demand for the ambiguous asset but increases the equity premium. Snow (2011) showed that an increase in ambiguity aversion raises both the demand for self-insurance and the demand for self-protection. However, these results do not necessarily hold for more than two states of nature (Alary et al., 2013). Furthermore, Hoy et al. (2014) found that ambiguity aversion can help to explain the fact that few people are willing to receive a free genetic test.

[3] For example, Cabantous (2007) found that insurers charge higher premiums in the presence of ambiguity than in the absence of ambiguity and that the charged premiums are higher when the insurers face with conflict ambiguity than with imprecise ambiguity. Cabantous et al. (2011) also discovered similar results, whereas when considering different types of risks, the result of higher charged premiums for conflict ambiguity than for imprecise ambiguity is reversed under the risks with abundant data such as fire risk. Charness et al. (2013) pointed out that social interactions affect ambiguity attitudes by inducing ambiguity loving and incoherent individuals to behave ambiguity neutrality. Moreover, from the

http://dx.doi.org/10.1016/j.insmatheco.2015.04.002
0167-6687/

In addition to characterizing the preference of ambiguity aversion, researchers have been exploring how to compare the degree of ambiguity aversion across individuals. For example, under the $\alpha$-maxmin expected utility model (Ghirardato et al., 2004), the function $a$, which is the weight on the maxmin expected utility, is used to measure the degree of ambiguity aversion. An individual with a larger value of the function $a$ behaves more pessimistically, indicating a more ambiguity-averse person. Under the smooth ambiguity aversion model (Klibanoff et al., 2005), one is said to be more ambiguity averse than the other when one's ambiguity function is obtained by the other's ambiguity function transformed with a strictly increasing and concave function, given the same utility function and subjective beliefs. Klibanoff et al. (2005) also proposed a measure for ambiguity aversion named the coefficient of ambiguity aversion, which is analogous to the Arrow–Pratt absolute risk aversion index.[4]

Beyond ambiguity aversion, third-order ambiguity attitude has not received much attention in the literature. One notable exception is the study of Baillon (2013). Analogous to risk apportionment (Eeckhoudt and Schlesinger, 2006), Baillon (2013) used the preferences over different options that vary in the probabilities of lotteries to define ambiguity attitudes, which is called ambiguity apportionment. In this approach, ambiguity aversion is defined as the preference of the option with certain probabilities of lotteries and third-order ambiguity attitude, which he called ambiguity prudence, is defined as the preference of the option in which the reduction and variation of the probabilities of lotteries are separated.[5] Furthermore, he showed that if the state space satisfies certain properties, under the smooth model of ambiguity aversion (Klibanoff et al., 2005), ambiguity aversion and ambiguity prudence correspond to an ambiguity function with a negative second derivative and one with a positive third derivative, respectively.[6] However, regarding the intensity of the third-order ambiguity attitude, as far as we know, no paper explores it.

This paper extends this line of the literature by providing comparative criteria for ambiguity aversion and downside ambiguity aversion under the smooth model of ambiguity aversion (Klibanoff et al., 2005). Compared with other models on ambiguity, the smooth ambiguity model is more suitable for our purpose because the preference-based definition of ambiguous events in its framework can disentangle non-constant ambiguity attitude and the ambiguity of an event (Klibanoff et al., 2011). In the literature, stochastic dominance approach has been a prevalent and powerful tool for ordering risks, characterizing risk preferences, and analyzing comparative statics regarding risks under the expected utility model. Given the success of stochastic dominance approach, for ambiguity, we develop similar notions by adopting the smooth ambiguity aversion model (Klibanoff et al., 2005) with a framework like the expected utility model.

Following Rothschild and Stiglitz (1970) and Ekern (1980), we first provide formal definitions of an increase in ambiguity and an increase in downside ambiguity. Then, we propose the conditions under which one is more ambiguity averse or downside ambiguity averse than the other. These conditions are analogous to those proposed by Ross (1981) and Modica and Scarsini (2005), respectively, for risk aversion and downside risk aversion. Furthermore, we find that the conditions of comparative ambiguity aversion and downside ambiguity aversion defined by the monetary premium are the same as those defined by the utility premium. This finding is different from the results on risks in the literature, such as those of Jindapon and Neilson (2007).[7]

As an application of the comparative criteria, we study the problem of spending efforts in reducing ambiguity and downside ambiguity which is related to but different from what Huang (2012) has examined. In the presence of ambiguity, we study the impact of more ambiguity aversion on optimal efforts to reduce ambiguity while Huang (2012) studied its impact on optimal efforts to reduce risks. Moreover, our more ambiguity aversion measure is developed in the spirit of Ross's (1981) comparative risk aversion measure while Huang's (2012) measure is defined as Klibanoff et al.'s (2005) absolute ambiguity aversion measure which was proposed in the spirit of the Arrow–Pratt risk aversion measure. We show that an individual will spend more efforts in reducing ambiguity (downside ambiguity) if and only if he or she is more ambiguity averse (downside ambiguity averse) defined by our criteria. This finding is analogous to the result of Jindapon and Neilson (2007), where efforts are spent to reduce risks.

Our paper contributes to the literature in three ways. First, this paper is, to our best knowledge, the first to offer definitions of an increase in ambiguity and an increase in downside ambiguity for general ambiguous probability distributions. In contrast to the Bernoulli distribution studied by Snow (2011) and Alary et al. (2013), for general probability distributions, the notion of an increase in ambiguity or in downside ambiguity depends crucially on how the unknown parameter in the probability distribution is defined. To use the mean-preserving spread and the mean–variance-preserving spread to describe the increase in ambiguity and the increase in downside ambiguity, respectively, the unknown parameter must be properly specified. Second and moreover, we develop comparative ambiguity aversion and downside ambiguity aversion in the spirit of Ross (1981). This opens up a new way to analyze the intensity of higher-order ambiguity attitudes that complements the apportionment approach adopted by Baillon (2013) when studying higher-order ambiguity attitudes. Third, we clarify that conditions for comparative ambiguity aversion and downside ambiguity aversion are indifferent whether the premiums to reduce ambiguity and downside ambiguity, respectively, are measured by utility or money. This finding identifies a key difference between comparative ambiguity aversion and comparative risk aversion.

The rest of our paper is organized as follows. Section 2 provides the definitions for an increase in ambiguity and an increase in downside ambiguity. Section 3 develops the notion of comparative ambiguity aversion and downside ambiguity aversion, while Section 4 presents an application in spending efforts in reducing ambiguity and downside ambiguity. We conclude the paper in Section 5 and relegate all proofs to the Appendices.

experimental results, Conte and Hey (2013) found that among four types of models (the expected utility model, the smooth ambiguity aversion model, the rank dependent expected utility model, and the $\alpha$-maxmin expected utility model), the smooth ambiguity aversion model has the best explanatory and predictive power for individuals' behavior in the presence of ambiguity.

4 Ahn (2008) proposed a model whose domain is the set of lotteries other than the state space. He proposed a index similar to that of Klibanoff et al. (2005).

5 This preference also represents the preference of the option with a less-spread left tail on the distribution of probabilities of lotteries, which is similar to the definition of downside risk aversion (Menezes et al., 1980). Thus, in this paper, we use the term *downside ambiguity aversion* for third-order ambiguity attitude.

6 In his paper, fourth-order ambiguity attitude (ambiguity temperance) and $N$th-order ambiguity attitude were also defined and the sign of the $N$th derivative of the ambiguity function could be determined by $(-1)^{n+1}$ if the state space satisfies certain properties.

7 In general, conditions of comparative risk aversion defined by the monetary premium lead to Ross's (1981) measure of risk aversion, whereas conditions of comparative risk aversion defined by the utility premium lead to the Arrow–Pratt measure of risk aversion.

## 2. Increases in ambiguity and downside ambiguity

We consider an individual facing uncertain wealth with a distribution density represented by $p(x;\theta)$, where $x$ denotes the wealth value and $\theta$ denotes the parameter on which the distribution density relies. The distribution is ambiguous in the sense that the parameter $\theta$ is not known precisely and taking on values in a space $\Theta$. The ambiguity takes the form of a probability distribution for $\theta$. To describe attitudes toward ambiguity, we adopt the smooth model of ambiguity aversion (Klibanoff et al., 2005). Under this model, the expected utility for a given $\theta$ is

$$U(\theta) = \int p(x;\theta)u(x)dx,$$

where $u$ is the intrinsic vNM utility function. The individual's overall utility under ambiguity is expressed as

$$E_F[\phi(U(\theta))] = \int \phi(U(\theta))\, dF(\theta),$$

where $F : \Theta \to [0, 1]$ denotes the cumulative distribution function of the parameter and $\phi$ is the individual's ambiguity function that is increasing and concave. We make the following assumptions throughout our analysis:

A1. $\Theta = [\underline{\theta}, \overline{\theta}]$;
A2. $U(\theta) : \Theta \mapsto \mathbb{R}$ is fixed, and is continuously differentiable;
A3. $U'(\theta) > 0$ for all $\theta \in \Theta$.

Here, A1 specifies the space $\Theta$ as an interval, in which the boundary $\underline{\theta}$ or $\overline{\theta}$ can be $-\infty$ or $+\infty$.[8] A2 clarifies that we will take the function $U(\theta)$ as given and focus on comparative statics regarding $\phi$. In order to fix $U(\theta)$, we need to fix the intrinsic utility function $u$ and the distribution density $p(x;\theta)$.

A3 is motivated by the fact that the expected utility $U(\theta)$ is monotone in $\theta$ if the change in $\theta$ specifies a change in the wealth distribution in line with the stochastic dominance rule. The monotonicity of $U(\theta)$ is also assumed by the literature on ambiguity such as two recent papers, Gollier (2011) and Snow (2014).[9] In fact, Rothschild and Stiglitz (1970) have proven that an improvement/a deterioration according to the first-order or second-order stochastic dominance always increases/decreases the expected utility, as long as the utility function $u$ is increasing and concave. For most commonly used distributions, the change in the parameter corresponds to an improvement or a deterioration in line with the stochastic dominance rule. As shown in Appendix A, if there is a density function $\boldsymbol{p}(x)$ such that $p(x;\theta) = \boldsymbol{p}(x-\theta)$, then an increase in $\theta$ implies an improvement in line with the first-order stochastic dominance. If there is a density function $\boldsymbol{p}(x)$ and a constant $\mu$ such that $\int_{-\infty}^{+\infty} x\boldsymbol{p}(x)dx = 0$ and $p(x;\theta) = \frac{1}{\theta}\boldsymbol{p}\left(\frac{x-\mu}{\theta}\right)$ ($\theta > 0$), then an increase in $\theta$ implies a deterioration in line with the second-order stochastic dominance.[10] Many distributions can be parameterized in the form $\frac{1}{\sigma}\boldsymbol{p}\left(\frac{x-\mu}{\sigma}\right)$. A natural example is the normal distribution, where $\boldsymbol{p}(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$.

Taking into account the monotonicity of $U(\theta)$, A3 requires that $\theta$ is ordered in the way that a larger $\theta$ produces a higher utility.[11] Assumptions A1–A3 imply that the mapping $U(\theta) : [\underline{\theta}, \overline{\theta}] \mapsto [U(\underline{\theta}), U(\overline{\theta})]$ is one-to-one and onto.

We first define an increase in ambiguity in the spirit of Rothschild and Stiglitz (1970) as follows.

**Definition 1** (*Increase in Ambiguity According to Rothschild and Stiglitz, 1970*)**.** Let $F$ and $G$ be two distributions of $\theta$. We say that $F$ has less ambiguity than $G$ in the sense of Rothschild and Stiglitz (1970), denoted $F \succsim^{RS} G$, if $E_F[\phi(U(\theta))] \geq E_G[\phi(U(\theta))]$ for all $\phi : [U(\underline{\theta}), U(\overline{\theta})] \mapsto \mathbb{R}$ that is twice continuously differentiable and satisfies $\phi' > 0, \phi'' < 0$.

The following lemma provides the condition for an increase in ambiguity according to Rothschild and Stiglitz (1970).

**Lemma 1.** *Let $F$ and $G$ be two distributions of $\theta$. Then, under Assumptions* A1–A3, *$F \succsim^{RS} G$ if and only if*

$$\int_{\underline{\theta}}^{\eta} U'(\theta)[F(\theta) - G(\theta)]d\theta \leq 0, \quad \forall \eta \in [\underline{\theta}, \overline{\theta}]. \tag{1}$$

**Proof.** Please see Appendix B. ■

Following Ekern (1980), we further refine the notion in Rothschild and Stiglitz (1970) by removing the requirement $\phi' > 0$.

**Definition 2** (*Increase in Ambiguity According to Ekern, 1980*)**.** Let $F$ and $G$ be two distributions of $\theta$. We say that $F$ has less ambiguity than $G$ in the sense of Ekern (1980), denoted $F \succsim^{E} G$, if $E_F[\phi(U(\theta))] \geq E_G[\phi(U(\theta))]$ for all $\phi : [U(\underline{\theta}), U(\overline{\theta})] \mapsto \mathbb{R}$ that is twice continuously differentiable and satisfies $\phi'' < 0$.

Lemma 2 gives the conditions for an increase in ambiguity according to Ekern (1980). Since $\phi$ in Definition 2 needs not to be monotone increasing, an additional restriction (see Eq. (2)) is required. The proof is similar to that of Lemma 1 and is thus omitted.

**Lemma 2.** *Let $F$ and $G$ be two distributions of $\theta$. Then, under Assumptions* A1–A3, *$F \succsim^{E} G$ if and only if*

$$\int_{\underline{\theta}}^{\overline{\theta}} U'(\theta)[F(\theta) - G(\theta)]d\theta = 0 \tag{2}$$

*and*

$$\int_{\underline{\theta}}^{\eta} U'(\theta)[F(\theta) - G(\theta)]d\theta \leq 0, \quad \forall \eta \in [\underline{\theta}, \overline{\theta}]. \tag{3}$$

---

8 When $\underline{\theta} = -\infty$ or $\overline{\theta} = +\infty$, we abuse notation a bit to use $[\underline{\theta}, \overline{\theta}]$ to denote $(-\infty, \overline{\theta}]$ or $[\underline{\theta}, +\infty)$.

9 Gollier (2011) demonstrated that the analytically optimal demand for ambiguous asset can be found when (1) the return follows the normal distribution, where $\theta$ denotes its mean; (2) $\theta$ is normally distributed; (3) the individual has a CARA risk preference; and (4) the individual has a CRAA ambiguity preferences. Under the assumptions (1) and (3), $U'(\theta) > 0$ can be found. He further showed the equivalent relationship between an increase in ambiguity aversion and a Monotone-Likelihood-Ratio-order shift on the distribution of the second-order belief by assuming that $U'(\theta) > 0$, where $\theta$ denotes the parameter which determines the distribution of the return. On the other hand, Snow (2014) examined the comparative statics of an increase in ambiguity aversion on the demand for insurance by setting $\theta$ as the loss probability in a two-state model where the assumption of $U'(\theta) < 0$ can be found.

10 With the density $\frac{1}{\sigma}\boldsymbol{p}\left(\frac{x-\mu}{\sigma}\right)$, the mean of the uncertain wealth equals $\mu$, which is independent of $\sigma$; the variance equals $\left(\int_{-\infty}^{+\infty} y^2\boldsymbol{p}(y)dy\right)\sigma^2$, which is proportional to $\sigma^2$. An increase in $\sigma$ in fact creates a mean-preserving increase of risk in the sense of Rothschild and Stiglitz (1970).

11 Otherwise, if $U(\theta)$ is monotone decreasing in $\theta$, one can employ a negative monotonic transformation on $\theta$ such as $\tilde{\theta} = -\theta$ or $\tilde{\theta} = \frac{1}{\theta}$ to obtain $V(\tilde{\theta}) = \int u(x)p\left(x; -\tilde{\theta}\right)dx = U(-\tilde{\theta})$ or $V(\tilde{\theta}) = \int u(x)p\left(x; \frac{1}{\tilde{\theta}}\right)dx = U\left(\frac{1}{\tilde{\theta}}\right)$. Then the expected utility $V(\tilde{\theta})$ is monotone increasing in $\tilde{\theta}$.

Lemmas 1 and 2 demonstrate that the notion of increase in ambiguity depends on $U'(\theta)$. Recall that $\theta$ parameterizes the distribution density $p(x;\theta)$ and different specifications of the parameter give rise to different functional forms of $U$. The term $U'(\theta)$ in (1)–(3) corrects for the effect brought by the parameterization. Eq. (2) can be understood as a generalization of the mean-preservation condition, since when $U'(\theta)$ is constant, it reduces to

$$\int_{\underline{\theta}}^{\overline{\theta}} [F(\theta) - G(\theta)]d\theta = 0.$$

Moreover, when $U'(\theta)$ is constant, Eq. (3) reduces to the well-known second-order stochastic dominance condition

$$\int_{\underline{\theta}}^{\eta} [F(\theta) - G(\theta)]d\theta \le 0, \quad \forall \eta \in [\underline{\theta}, \overline{\theta}].$$

In this case, Eqs. (2) and (3) correspond to the notion that $G$ is a mean-preserving spread of $F$.[12] When $U'(\theta)$ is not constant, we have the following result.

**Proposition 1.** *Under Assumptions* A1–A3, *let* $\tau \in [U(\underline{\theta}), U(\overline{\theta})]$ *be a new parameter. We employ* $\theta = U^{-1}(\tau)$ *to re-parameterize the distribution density* $p(x;\theta)$ *as*

$$\tilde{p}(x;\tau) = p\left(x; U^{-1}(\tau)\right).$$

*Under this new parameterization, the expected utility for a given* $\tau$ *is*

$$V(\tau) = \int u(x)\tilde{p}(x;\tau)dx = \tau,$$

*and the individual's overall utility under ambiguity is expressed as*

$$E_F[\phi(V(\tau))] = \int_{U(\underline{\theta})}^{U(\overline{\theta})} \phi(\tau)dF(\tau).$$

*For distributions of* $\tau$, $F \gtrsim^E G$ *if and only if* $G$ *is a mean-preserving spread of* $F$.

**Proof.** Please see Appendix C. ∎

Proposition 1 shows that when $U'(\theta)$ is not constant, one can re-parameterize the distribution density such that the marginal expected utility is constant with the new parameter. The essence of the re-parameterization is to introduce a new parameter for the wealth distribution such that the expected utility is linear in this new parameter. After the re-parameterization, an increase in ambiguity corresponds exactly to the mean-preserving spread. To illustrate the dependence of $\gtrsim^E$ on the parameterization, let us look at two concrete examples.

**Example 1.** Assume that the uncertain wealth obeys a Bernoulli distribution: There are two states of nature such that $U(\theta) = p(\theta)u(a) + (1 - p(\theta))u(b)$ $(a > b)$. When $p(\theta) = \theta \in [0, 1]$, we have $U(\theta) = \theta[u(a) - u(b)] + u(b)$, which in turn suggests $U'(\theta) = u(a) - u(b) > 0$ is a constant. For $\theta$, $F \gtrsim^E G$ if and only if $G$ is a mean-preserving spread of $F$. This special case was explored by Snow (2011) and Alary et al. (2013).

**Example 2.** Assume that the uncertain wealth obeys a normal distribution $p(x;\mu,\sigma) = \frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{(x-\mu)^2}{2\sigma^2}}$, where $\mu \in \mathbb{R}$ is ambiguous but $\sigma$ is not. Let $u(x) = -e^{-\gamma x}$, $\gamma > 0$. We then have $U(\mu) = -e^{-\gamma(\mu - \frac{\gamma}{2}\sigma^2)}$. In this case, an increase in ambiguity on $\mu$ is not necessarily a mean-preserving spread. However, if we introduce a new parameter $\tau \in (-\infty, 0)$, and re-parameterize the normal distribution of the uncertain wealth as

$$\tilde{p}(x;\tau,\sigma) = p\left(x; -\frac{1}{\gamma}\log(-\tau), \sigma\right) = \frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{\left[x+\frac{1}{\gamma}\log(-\tau)\right]^2}{2\sigma^2}},$$

then we have

$$V(\tau) = \int u(x)\tilde{p}(x;\tau,\sigma)dx = \tau e^{\frac{\gamma^2\sigma^2}{2}},$$

which is linear in $\tau$.[13] For distributions of $\tau$, $F \gtrsim^E G$ if and only if $G$ is a mean-preserving spread of $F$.

In the same spirit as Menezes et al. (1980), we can define an increase in downside ambiguity.

**Definition 3** (*Increase in Downside Ambiguity According to Menezes et al., 1980*). Let $F$ and $G$ be two distributions of $\theta$. We say that $F$ has less downside ambiguity than $G$ in the sense of Menezes et al. (1980), denoted $F \gtrsim^M G$, if $E_F[\phi(U(\theta))] \ge E_G[\phi(U(\theta))]$ for all $\phi$ that is three times continuously differentiable and satisfies $\phi' > 0$, $\phi'' < 0$, $\phi''' > 0$.

The following lemma shows the conditions of an increase in downside ambiguity according to Menezes et al. (1980).

**Lemma 3.** *Let* $F$ *and* $G$ *be two distributions of* $\theta$. *Then, under Assumptions* A1–A3, $F \gtrsim^M G$ *if and only if*

$$\int_{\underline{\theta}}^{\eta} U'(\theta)\,[F(\theta) - G(\theta)]\,d\theta \le 0 \tag{4}$$

*and*

$$\int_{\underline{\theta}}^{\eta} U'(\theta)\left(\int_{\underline{\theta}}^{\theta} U'(\zeta)[F(\zeta) - G(\zeta)]d\zeta\right)d\theta \le 0,$$
$$\forall \eta \in [\underline{\theta}, \overline{\theta}]. \tag{5}$$

**Proof.** Please see Appendix D. ∎

Following Ekern (1980), we further refine the notion in Menezes et al. (1980) by removing the requirement $\phi' > 0$, $\phi'' < 0$.

**Definition 4** (*Increase in Downside Ambiguity According to Ekern, 1980*). Let $F$ and $G$ be two distributions of $\theta$. We say that $F$ has less downside ambiguity than $G$ in the sense of Ekern (1980), denoted $F \gtrsim^E G$, if $E_F[\phi(U(\theta))] \ge E_G[\phi(U(\theta))]$ for all $\phi$ that is three times continuously differentiable and satisfies $\phi''' > 0$.

Lemma 4 gives the conditions for an increase in downside ambiguity according to Ekern (1980). Since $\phi$ in Definition 4 needs not to be monotone increasing and concave, additional restrictions (see Eqs. (6) and (7)) are required. The proof is similar to that of Lemma 3 and is thus omitted.

**Lemma 4.** *Let* $u$ *be given and let* $F$ *and* $G$ *be two distributions of* $\theta$. *Then,* $F \gtrsim^E G$ *if and only if*

$$\int_{\underline{\theta}}^{\overline{\theta}} U'(\theta)[F(\theta) - G(\theta)]d\theta = 0, \tag{6}$$

$$\int_{\underline{\theta}}^{\overline{\theta}} U'(\theta)\left(\int_{\underline{\theta}}^{\theta} U'(\zeta)[F(\zeta) - G(\zeta)]d\zeta\right)d\theta = 0, \tag{7}$$

[12] When $U'(\theta)$ is constant, $U(\theta)$ is linear in $\theta$, and the concavity of overall utility $\phi(U(\theta))$ is attributed entirely to $\phi$. In this case, the concavity of $\phi$ corresponds exactly to the aversion to mean-preserving spreads.

[13] The re-parameterization in this example is defined by $\tau = -e^{-\gamma\mu} \neq U(\mu)$ while the re-parameterization in Proposition 1 is defined by $\tau = U(\mu)$. They are equivalent to each other based on a nonnegative scale change.

and

$$\int_{\underline{\theta}}^{\eta} U'(\theta)\left(\int_{\underline{\theta}}^{\theta} U'(\zeta)[F(\zeta)-G(\zeta)]d\zeta\right)d\theta \le 0,$$
$$\forall \eta \in [\underline{\theta}, \overline{\theta}]. \quad (8)$$

Similar to the characterization of the increase in ambiguity, the notion of increase in downside ambiguity also depends on how the distribution density $p(x;\theta)$ is parameterized, and the term $U'(\theta)$ in (4)–(8) corrects for the effect introduced by the parameterization. Eqs. (6) and (7) can be understood as a generalization of the mean–variance-preservation condition, since when $U'(\theta)$ is constant, Eq. (6) implies that $F$ and $G$ have the same mean, while Eq. (7) implies that $F$ and $G$ have the same variance. Lemma 4 thus reduces to stating that $G$ is a mean–variance-preserving spread of $F$. When $U'(\theta)$ is not constant, we have the following result. The proof is similar to that of Proposition 1 and is thus omitted.

**Proposition 2.** *Under Assumptions* A1–A3, *let the distribution density be re-parameterized by the parameter* $\tau \in [U(\underline{\theta}), U(\overline{\theta})]$ *in the same manner as that specified in Proposition 1. For distributions of* $\tau$, $F \gtrsim^E G$ *if and only if* $G$ *is a mean–variance-preserving spread of* $F$.

The way of the re-parameterization in Proposition 2 is the same as that in Proposition 1. Accordingly, in Example 2, with the same re-parameterization

$$\tilde{p}(x;\tau,\sigma) = p\left(x; -\frac{1}{\gamma}\log(-\tau), \sigma\right) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{\left[x+\frac{1}{\gamma}\log(-\tau)\right]^2}{2\sigma^2}},$$

for distributions of $\tau$, we have $F \gtrsim^E G$ if and only if $G$ is a mean–variance-preserving spread of $F$.

## 3. Comparative ambiguity aversion and downside ambiguity aversion

This section provides comparative conditions for ambiguity aversion and downside ambiguity aversion in the spirit of Ross (1981). For simplicity, an increase in ambiguity and an increase in downside ambiguity are both defined in the sense of Ekern (1980) for all the analyses. Suppose that there are two individuals with ambiguity functions $\phi_1$ and $\phi_2$, respectively. For distributions of $\theta$ such that $F \gtrsim^E G$, define $\pi_i$ as the premium measured in utility the individual is willing to pay to reduce the ambiguity from $G$ to $F$. Mathematically, $\pi_i$ satisfies

$$E_G[\phi_i(U(\theta))] = E_F[\phi_i(U(\theta)-\pi_i)], \quad i=1,2.$$

To ensure that $\pi_i$ is well defined, the domain of $\phi_i$ must be properly extended to cover $[U(\underline{\theta})-\pi_i, U(\overline{\theta})]$, the lower boundary of which depends on $\pi_i$, which further depends on $\phi_i$, $F$ and $G$. To facilitate analysis, we make the following assumption to avoid this issue.

A4. $\lim_{\theta\to\underline{\theta}} U(\theta) = -\infty$ so that $U(\theta) \in (-\infty, U(\overline{\theta})]$.[14]

Under Assumption A4, the domain of the ambiguity functions $\phi_i$ $(i=1,2)$ naturally expands to $(-\infty, U(\overline{\theta})]$, and as a result, $\pi_i$ is well defined for all distributions $F$ and $G$ satisfying $F \gtrsim^E G$. Assumption A4, when combined with A1–A3, implies that the mapping $U(\theta) : [\underline{\theta}, \overline{\theta}] \mapsto (-\infty, U(\overline{\theta})]$ is one-to-one and onto.

The following proposition offers the equivalent characterizations of comparative ambiguity aversion.

[14] Note that $\underline{\theta}$ can be $-\infty$, with which A4 rewrites $\lim_{\theta\to-\infty} U(\theta) = -\infty$.

**Proposition 3** (*Comparative Ambiguity Aversion According to Ross, 1981*)**.** *Let Assumptions* A1–A4 *hold true. For* $\phi_1, \phi_2 : (-\infty, U(\overline{\theta})] \mapsto \mathbb{R}$ *that are twice continuously differentiable and satisfy* $\phi_i' > 0$, $\phi_i'' < 0$ $(i=1,2)$, *the following three statements are equivalent:*

(1) *There exists a constant* $\lambda > 0$ *such that*

$$\frac{\phi_1''(z)}{\phi_2''(z)} \ge \lambda \ge \frac{\phi_1'(w)}{\phi_2'(w)}, \quad \forall z, w \in (-\infty, U(\overline{\theta})].$$

(2) *There exists a twice continuously differentiable function* $h : (-\infty, U(\overline{\theta})] \mapsto \mathbb{R}$ *satisfying* $h' \le 0$, $h'' \le 0$, *and a constant* $\lambda > 0$, *such that* $\phi_1 = \lambda\phi_2 + h$.

(3) $\pi_1 \ge \pi_2$ *for all* $F$ *and* $G$ *such that* $F \gtrsim^E G$.

**Proof.** Please see Appendix E. ■

Proposition 3 offers three equivalent conditions under which individual 1 is more ambiguity averse than individual 2, analogous to Ross's (1981) measure of risk aversion. The first condition is stronger than the measure of ambiguity aversion proposed by Klibanoff et al. (2005). The second condition demonstrates that the more ambiguity-averse ambiguity function $\phi_1$ can be obtained by a linear transformation of $\phi_2$ in which the additive component is decreasing and concave. The third condition clarifies that the more ambiguity-averse individual always pays a higher premium to reduce ambiguity.

We also provide the equivalent characterizations of comparative downside ambiguity aversion as the following proposition, which parallels Proposition 3.

**Proposition 4** (*Comparative Downside Ambiguity Aversion According to Modica and Scarsini, 2005*)**.** *Let Assumptions* A1–A4 *hold true. For* $\phi_1, \phi_2 : (-\infty, U(\overline{\theta})] \mapsto \mathbb{R}$ *that are three times continuously differentiable and satisfy* $\phi_i' > 0$, $\phi_i''' > 0$ $(i=1,2)$, *the following three statements are equivalent:*

(1) *There exists a constant* $\lambda > 0$ *such that*

$$\frac{\phi_1'''(z)}{\phi_2'''(z)} \ge \lambda \ge \frac{\phi_1'(w)}{\phi_2'(w)}, \quad \forall z, w \in (-\infty, U(\overline{\theta})].$$

(2) *There exists a function* $h : (-\infty, U(\overline{\theta})] \mapsto \mathbb{R}$ *that is three times continuously differentiable and satisfy* $h' \le 0$, $h''' \ge 0$, *and a constant* $\lambda > 0$, *such that* $\phi_1 = \lambda\phi_2 + h$.

(3) $\pi_1 \ge \pi_2$ *for all* $F$ *and* $G$ *such that* $F \gtrsim^E G$.

**Proof.** Please see Appendix F. ■

Proposition 4 shows three equivalent conditions under which individual 1 is more downside ambiguity averse than individual 2. These conditions are similar to those for comparative risk aversion proposed by Modica and Scarsini (2005). Note that the first condition implies that $\phi_1'''(z)/\phi_1'(z) \ge \phi_2'''(z)/\phi_2'(z)$ for all $z \in (-\infty, U(\overline{\theta})]$.

So far, we have employed the utility premium to define comparative ambiguity aversion and comparative downside ambiguity aversion. An alternative is to use the monetary premium instead, that is, define $\hat{\pi}_i$ such that

$$\int_{\underline{\theta}}^{\overline{\theta}} \phi_i\left(\int p(x;\theta)u(x)dx\right)dG(\theta)$$
$$= \int_{\underline{\theta}}^{\overline{\theta}} \phi_i\left(\int p(x;\theta)u(x-\hat{\pi}_i)dx\right)dF(\theta), \quad i=1,2.$$

We have the following linkage between the monetary premium and the utility premium.

**Proposition 5.** *Let Assumptions* A1–A4 *hold true. For ambiguity, the following two statements are equivalent:*

(1) $\hat{\pi}_1 \geq \hat{\pi}_2$ *for all* $F$ *and* $G$ *such that* $F \gtrsim^E G$.
(2) $\pi_1 \geq \pi_2$ *for all* $F$ *and* $G$ *such that* $F \gtrsim^E G$.
*For downside ambiguity, the following two statements are equivalent:*
(3) $\hat{\pi}_1 \geq \hat{\pi}_2$ *for all* $F$ *and* $G$ *such that* $F \gtrapprox^E G$.
(4) $\pi_1 \geq \pi_2$ *for all* $F$ *and* $G$ *such that* $F \gtrapprox^E G$.

**Proof.** Please see Appendix G. ■

In fact, if there exists a couple of distributions $F \gtrsim^E G$ ($F \gtrapprox^E G$) satisfying $\pi_1 < \pi_2$, one can easily construct a new couple of distribution $\bar{F} \gtrsim^E \bar{G}$ ($\bar{F} \gtrapprox^E \bar{G}$) such that $\hat{\pi}_1 < \hat{\pi}_2$ and vice versa.

Unlike the findings regarding risk aversion, we find that the conditions of comparative ambiguity aversion and downside ambiguity aversion defined by the monetary premium and by the utility premium yield identical measures of ambiguity aversion and downside ambiguity aversion, as shown in Proposition 5. The rational is that, for risks, the monetary premium affects the argument of the utility function, resulting in Ross's (1981) measure of risk aversion, while the utility premium has no influence on the argument of the utility function, which results in the Arrow–Pratt measure of risk aversion. On the other hand, for ambiguity, since both the monetary premium and the utility premium affect the argument of the ambiguity function, the same measures of ambiguity aversion and downside ambiguity aversion are derived.[15]

## 4. An application

In this section, we illustrate how the comparative conditions developed above can be applied to a comparative static problem that is similar to that of Jindapon and Neilson (2007).[16] Suppose there are two individuals with ambiguity functions $\phi_1$ and $\phi_2$, respectively, and each individual can spend an effort $e \in [0, 1]$ in reducing ambiguity (or downside ambiguity) from $G$ to $eF+(1-e)G$ at a cost of utility $c(e)$. Then, the optimal effort level for individual $i$ is

$$e_i^* = \arg\max_e \int_{\underline{\theta}}^{\overline{\theta}} \phi_i\left(U(\theta) - c(e)\right) d\left[eF(\theta) + (1-e)G(\theta)\right]. \qquad (9)$$

To ensure that $e_i^*$ is well defined, some conditions on $c(e)$ are required and the domain of $\phi_i$ needs to be expanded beyond $[U(\underline{\theta}), U(\overline{\theta})]$. This leads us to make the following three assumptions, where $M$ is a pre-specified positive constant and $c(e)$ is twice continuously differentiable:

A5. $0 \leq c(e) \leq M, c'(e) > 0, c''(e) > 0$ for all $e \in (0, 1)$;
A6. $c'(0) = 0, \lim_{e\to 1} c'(e) = +\infty$;
A7. $\phi_i$ $(i = 1, 2)$ is well defined on $[U(\underline{\theta}) - M, U(\overline{\theta})]$.

A5 shapes $c(e)$ in a general way, where $c'(e) > 0$ and $c''(e) > 0$ capture that both the effort cost and the marginal effort cost is increasing in the effort level. In A6, $c'(0) = 0$ implies that $e = 0$ is not a solution to (9) because the individual's marginal utility is positive when $e = 0$; $\lim_{e\to 1} c'(e) = +\infty$ implies that $e \to 1$ is not a solution either because the individual's marginal utility goes to negative infinity when $e = 1$. Overall, A6 guarantees the existence of an interior solution to (9). A7 guarantees that $\phi_i\left(U(\theta) - c(e)\right)$ is well defined.

The following proposition indicates who will spend more efforts in reducing ambiguity (or downside ambiguity). In the same spirit of Proposition 5, identical results can be obtained if we replace the cost of effort in utility by the cost of effort in money.

**Proposition 6.** *Let Assumptions* A1–A3 *and* A5–A7 *hold true. Consider two ambiguity functions* $\phi_1, \phi_2 : [U(\underline{\theta}) - M, U(\overline{\theta})] \mapsto \mathbb{R}$ *that are three times continuously differentiable and satisfy* $\phi_i' > 0, \phi_i'' < 0, \phi_i''' > 0$, $i = 1, 2$. *Then, the solution to* (9) *is interior and unique. Moreover, for ambiguity, the following two statements are equivalent:*

(1) *Individual* 1 *is more ambiguity averse than individual* 2 *in the sense of Ross (1981): there exists a constant* $\lambda > 0$ *such that*

$$\frac{\phi_1''(z-k)}{\phi_2''(z-k)} \geq \lambda \geq \frac{\phi_1'(w-k)}{\phi_2'(w-k)}, \quad \forall z, w \in [U(\underline{\theta}), U(\overline{\theta})], \quad \forall k \in [0, M].$$

(2) $e_1^* \geq e_2^*$ *for all* $c(e)$ *satisfying Assumptions* A5–A6, *and all* $F$ *and* $G$ *such that* $F \gtrsim^E G$.
*For downside ambiguity, the following two statements are equivalent:*
(3) *Individual* 1 *is more downside ambiguity averse than individual* 2 *in the sense of Modica and Scarsini (2005): there exists a constant* $\lambda > 0$ *such that*

$$\frac{\phi_1'''(z-k)}{\phi_2'''(z-k)} \geq \lambda \geq \frac{\phi_1'(w-k)}{\phi_2'(w-k)}, \quad \forall z, w \in [U(\underline{\theta}), U(\overline{\theta})], \quad \forall k \in [0, M].$$

(4) $e_1^* \geq e_2^*$ *for all* $c(e)$ *satisfying Assumptions* A5 *and* A6, *and all* $F$ *and* $G$ *such that* $F \gtrapprox^E G$.

**Proof.** Please see Appendix H.[17] ■

Jindapon and Neilson (2007) found that when the cost of effort is measured by money, an individual who is more risk averse, as defined by Ross (1981), makes more efforts in reducing risks than a less risk-averse individual. With regard to ambiguity and downside ambiguity, we find a similar result: Proposition 6 shows that the more ambiguity-averse individual, according to Ross (1981) (or the more downside-ambiguity-averse individual, according to Modica and Scarsini, 2005), spends more efforts in reducing ambiguity (or downside ambiguity). When the cost of effort is measured by utility, we obtain the result by using identical comparative conditions, while Jindapon and Neilson (2007) obtained the result by the Arrow–Pratt measure of risk aversion.

15 When the utility premium is defined in the level of the utility under ambiguity, i.e., is deducted outside of the $\phi$ function, the derived measures might be related to Klibanoff et al.'s (2005) absolute ambiguity aversion measure (in the sense of the Arrow–Pratt risk aversion measure) but different from the measures derived by the monetary premium. Thus, in such a setting, we might obtain results analogous to those under risks that the measures depend on how the premium is defined.

16 Li (2009) provided comparative conditions of higher-order Ross risk aversion and discussed the relationship between his results and Jindapon and Neilson (2007)'s results. We adopt an approach close to Li (2009) to develop comparative conditions of second-and-third-order ambiguity attitudes in the sense of Ross (1981). In addition to Jindapon and Neilson (2007), many papers have investigated the comparative static problem of efforts such as Ehrlich and Becker (1972), Dionne and Eeckhoudt (1985), Briys and Schlesinger (1990), Jullien et al. (1999), Kocabiyikoğlu and Popescu (2007), and Chuang et al. (2013).

17 Note that $\frac{\phi_1''(x)}{\phi_2''(x)} \geq \lambda \geq \frac{\phi_1'(y)}{\phi_2'(y)}$, $\forall x, y \in [U(\underline{\theta}) - M, U(\overline{\theta})]$ is sufficient for $\frac{\phi_1''(z-k)}{\phi_2''(z-k)} \geq \lambda \geq \frac{\phi_1'(w-k)}{\phi_2'(w-k)}$, $\forall z, w \in [U(\underline{\theta}), U(\overline{\theta})], \forall k \in [0, M]$. But the converse is not true. Indeed, it is easily seen that the former statement includes the special case $\frac{\phi_1''(U(\underline{\theta})-M)}{\phi_2''(U(\underline{\theta})-M)} \geq \lambda \geq \frac{\phi_1'(U(\overline{\theta}))}{\phi_2'(U(\overline{\theta}))}$ but the latter statement cannot accommodate this case. This remark also applies to the comparison between $\frac{\phi_1'''}{\phi_2'''}$ and $\frac{\phi_1'}{\phi_2'}$.

## 5. Conclusion

This paper first provides general definitions of an increase in ambiguity and an increase in downside ambiguity. Then, we propose comparative criteria for ambiguity aversion and downside ambiguity aversion in the spirit of Ross (1981). We show that these criteria remain the same, whether the premiums to reduce ambiguity and downside ambiguity are defined by utility or money. Finally, we illustrate an application of these criteria by studying who will spend more effort in reducing ambiguity and downside ambiguity.

Since Ross (1981)'s risk aversion measure is widely used in analyzing incentive problems involving risks, the Ross-type ambiguity measure developed in this paper could be used to analyze many other incentive problems involving ambiguity. In addition to the application we proposed in Section 4, one more example is the decision on receiving a genetic test in the presence of ambiguity studied by Hoy et al. (2014). Specifically, one might analyze who is inclined to pay more costs to take a genetic test to reduce ambiguity.

One of the main contribution of this paper is to introduce Ross's (1981) approach to study the intensity of ambiguity attitudes. Since Ross's (1981) approach can be extended to accommodate higher-order risk attitudes (Li, 2009; Denuit and Eeckhoudt, 2010), our paper opens up a new way to study the intensity of higher-order ambiguity attitudes that complements Baillon's (2013) study of ambiguity attitudes by the apportionment approach. Following the approach suggested by Li (2009), the extension of our results to $N$th-order ambiguity attitudes with $N \geq 4$ is straightforward.


## Acknowledgments

The authors appreciate the helps of managing editor Rob Kaas and the valuable comments of two anonymous referees. Larry Y. Tzeng acknowledges financial support from the research grant of Ministry of Science and Technology (MOST) (102-2410-H-002-028-MY3). Lin Zhao acknowledges financial support from the research grant of National Science Foundation of China (NSFC) (No. 71301161).


## Appendices

To simplify the notation, we denote $\mathscr{X}(\eta) = \int_{\underline{\theta}}^{\eta} U'(\theta)[F(\theta) - G(\theta)]d\theta$ and do not mention it again unless there is potential for confusion. We first establish two lemmas that will be used later.

**Lemma A.1.** *Let Assumptions* A1–A3 *hold true. Suppose that* $w_0 \in (U(\underline{\theta}), U(\overline{\theta}))$ *is given and* $\varepsilon > 0$ *is small enough such that* $(w_0 - \varepsilon, w_0 + \varepsilon) \subset [U(\underline{\theta}), U(\overline{\theta})]$. *Let F and G satisfy*

$$\text{Under } F: \theta \equiv U^{-1}(w_0),$$

$$\text{Under } G: \theta = \begin{cases} U^{-1}\left(w_0 - \frac{\varepsilon}{2}\right), & \text{with probability } \frac{1}{2}, \\ U^{-1}\left(w_0 + \frac{\varepsilon}{2}\right), & \text{with probability } \frac{1}{2}. \end{cases}$$

*Then, we have:*

(1) *The support of* $\mathscr{X}(\eta)$ *is* $\left[U^{-1}\left(w_0 - \frac{\varepsilon}{2}\right), U^{-1}\left(w_0 + \frac{\varepsilon}{2}\right)\right] \subset \left(U^{-1}(w_0 - \varepsilon), U^{-1}(w_0 + \varepsilon)\right)$;
(2) $F \gtrsim^E G$.

**Proof of Lemma A.1.** By construction, we have

$$F(\theta) = \begin{cases} 0, & \theta \in [\underline{\theta}, U^{-1}(w_0)), \\ 1, & \theta \in [U^{-1}(w_0), \overline{\theta}], \end{cases}$$

$$G(\theta) = \begin{cases} 0, & \theta \in \left[\underline{\theta}, U^{-1}\left(w_0 - \frac{\varepsilon}{2}\right)\right), \\ \frac{1}{2}, & \theta \in \left[U^{-1}\left(w_0 - \frac{\varepsilon}{2}\right), U^{-1}\left(w_0 + \frac{\varepsilon}{2}\right)\right), \\ 1, & \theta \in \left[U^{-1}\left(w_0 + \frac{\varepsilon}{2}\right), \overline{\theta}\right]. \end{cases}$$

It is easily seen that $F \equiv G$ on $\left[\underline{\theta}, U^{-1}\left(w_0 - \frac{\varepsilon}{2}\right)\right) \cup \left[U^{-1}\left(w_0 + \frac{\varepsilon}{2}\right), \overline{\theta}\right]$,

$$\begin{aligned} \mathscr{X}(\overline{\theta}) &= \int_{\underline{\theta}}^{\overline{\theta}} U'(\theta)[F(\theta) - G(\theta)]d\theta \\ &= \frac{1}{2}\left(-\int_{U^{-1}(w_0 - \frac{\varepsilon}{2})}^{U^{-1}(w_0)} U'(\theta)d\theta + \int_{U^{-1}(w_0)}^{U^{-1}(w_0 + \frac{\varepsilon}{2})} U'(\theta)d\theta\right) \\ &= 0, \end{aligned}$$

and $\mathscr{X}(\theta)$ is supported in $\left[U^{-1}\left(w_0 - \frac{\varepsilon}{2}\right), U^{-1}\left(w_0 + \frac{\varepsilon}{2}\right)\right] \subset (U^{-1}(w_0 - \varepsilon), U^{-1}(w_0 + \varepsilon))$. The fact that $\mathscr{X}(\eta) \leq 0$ for all $\eta \in [\underline{\theta}, \overline{\theta}]$ follows straightforwardly. ■

**Lemma A.2.** *Let Assumptions* A1–A3 *hold true. Suppose that* $z_0, w_0 \in (U(\underline{\theta}), U(\overline{\theta}))$ *are given and* $z_0 < w_0$. *Fix* $\varepsilon > 0$ *small enough such that* $(z_0 - \varepsilon, z_0 + \varepsilon) \subset [U(\underline{\theta}), U(\overline{\theta})]$ *and* $z_0 + \varepsilon < w_0$. *Let* $0 < q < 1$, *F and G satisfy*

$$\text{Under } F: \theta = \begin{cases} U^{-1}(z_0), & \text{with probability } q, \\ U^{-1}(w_0), & \text{with probability } 1 - q, \end{cases}$$

$$\text{Under } G: \theta = \begin{cases} U^{-1}\left(z_0 - \frac{\varepsilon}{2}\right), & \text{with probability } \frac{q}{2}, \\ U^{-1}\left(z_0 + \frac{\varepsilon}{2}\right), & \text{with probability } \frac{q}{2}, \\ U^{-1}(w_0), & \text{with probability } 1 - q. \end{cases}$$

*Then, we have:*

(1) *the support of* $\mathscr{X}(\eta)$ *is* $\left[U^{-1}\left(z_0 - \frac{\varepsilon}{2}\right), U^{-1}\left(z_0 + \frac{\varepsilon}{2}\right)\right] \subset \left(U^{-1}(z_0 - \varepsilon), U^{-1}(z_0 + \varepsilon)\right)$;
(2) $F \gtrsim^E G$.

**Proof of Lemma A.2.** By construction, we have

$$F(\theta) = \begin{cases} 0, & \theta \in [\underline{\theta}, U^{-1}(z_0)), \\ q, & \theta \in [U^{-1}(z_0), U^{-1}(w_0)), \\ 1, & \theta \in [U^{-1}(w_0), \overline{\theta}], \end{cases}$$

$$G(\theta) = \begin{cases} 0, & \theta \in \left[\underline{\theta}, U^{-1}\left(z_0 - \frac{\varepsilon}{2}\right)\right), \\ \frac{q}{2}, & \theta \in \left[U^{-1}\left(z_0 - \frac{\varepsilon}{2}\right), U^{-1}\left(z_0 + \frac{\varepsilon}{2}\right)\right), \\ q, & \theta \in \left[U^{-1}\left(z_0 + \frac{\varepsilon}{2}\right), U^{-1}(w_0)\right), \\ 1, & \theta \in \left[U^{-1}(w_0), \overline{\theta}\right]. \end{cases}$$

It is easily seen that $F \equiv G$ on $\left[\underline{\theta}, U^{-1}\left(z_0 - \frac{\varepsilon}{2}\right)\right) \cup \left[U^{-1}\left(z_0 + \frac{\varepsilon}{2}\right), \overline{\theta}\right]$,

$$\begin{aligned} \mathscr{X}(\overline{\theta}) &= \int_{\underline{\theta}}^{\overline{\theta}} U'(\theta)[F(\theta) - G(\theta)]d\theta \\ &= \frac{q}{2}\left(-\int_{U^{-1}(z_0 - \frac{\varepsilon}{2})}^{U^{-1}(z_0)} U'(\theta)d\theta + \int_{U^{-1}(z_0)}^{U^{-1}(z_0 + \frac{\varepsilon}{2})} U'(\theta)d\theta\right) \\ &= 0, \end{aligned}$$

and $\mathscr{X}(\theta)$ is supported in $\left[U^{-1}\left(z_0-\frac{\varepsilon}{2}\right), U^{-1}\left(z_0+\frac{\varepsilon}{2}\right)\right] \subset (U^{-1}(z_0-\varepsilon), U^{-1}(z_0+\varepsilon))$. The fact that $\mathscr{X}(\eta) \le 0$ for all $\eta \in [\underline{\theta}, \overline{\theta}]$ follows straightforwardly. ■

## Appendix A. Proof of the monotonicity of $U(\theta)$

Assume that the uncertain wealth takes value from $(-\infty, +\infty)$. For the distribution density of the uncertain wealth $p(x;\theta)$, we provide a formal proof for the following two statements on the monotonicity of $U(\theta)$:

(1) If there is a density function $\boldsymbol{p}(x)$ such that $p(x;\theta) = \boldsymbol{p}(x-\theta)$, then an increase in $\theta$ implies an improvement in line with the first-order stochastic dominance. In this case, $U(\theta)$ is increasing in $\theta$ as long as $u$ is increasing.

(2) If there is a density function $\boldsymbol{p}(x)$ and a constant $\mu$ such that $\int_{-\infty}^{+\infty} x\boldsymbol{p}(x)dx = 0$ and $p(x;\theta) = \frac{1}{\theta}\boldsymbol{p}\left(\frac{x-\mu}{\theta}\right)$ $(\theta > 0)$, then an increase in $\theta$ implies a deterioration in line with the second-order stochastic dominance. In this case, $U(\theta)$ is decreasing in $\theta$ as long as $u$ is increasing and concave.

**Proof.** Let $H(x) = \int_{-\infty}^{x} \boldsymbol{p}(t)dt$, $H_\theta(x) = \int_{-\infty}^{x} p(t;\theta)dt$ be the cumulative distribution functions associated with $\boldsymbol{p}(x)$, $p(x;\theta)$, respectively. For statement (1), we have $H_\theta(x) = H(x-\theta)$. For $\theta_1 > \theta_2$, there must be $H_{\theta_1}(x) \le H_{\theta_2}(x)$ for all $x \in (-\infty, +\infty)$. This confirms that the distribution $p(x;\theta_1)$ dominates $p(x;\theta_2)$ in the sense of first-order stochastic dominance. For statement (2), we have $H_\theta(x) = H\left(\frac{x-\mu}{\theta}\right)$. For $\theta_1 > \theta_2 > 0$, there is $H_{\theta_1}(x) \ge H_{\theta_2}(x)$ for $x \le \mu$ and $H_{\theta_1}(x) \le H_{\theta_2}(x)$ for $x \ge \mu$. Then, it follows straightforwardly that

$$\int_{-\infty}^{x} [H_{\theta_2}(t) - H_{\theta_1}(t)]dt \le 0, \quad \forall x \in (-\infty, \mu]. \tag{A.1}$$

Integration by parts yields

$$\int_{-\infty}^{+\infty} [H_{\theta_2}(t) - H_{\theta_1}(t)]dt = -\int_{-\infty}^{+\infty} \left[\frac{t}{\theta_2}\boldsymbol{p}\left(\frac{t-\mu}{\theta_2}\right) - \frac{t}{\theta_1}\boldsymbol{p}\left(\frac{t-\mu}{\theta_1}\right)\right]dt = 0.$$

This equation implies that for $x \in (\mu, +\infty)$, there is

$$\int_{-\infty}^{x} [H_{\theta_2}(t) - H_{\theta_1}(t)]dt = -\int_{x}^{+\infty} [H_{\theta_2}(t) - H_{\theta_1}(t)]dt \le 0. \tag{A.2}$$

Taking (A.1) and (A.2) together, we see that $\int_{-\infty}^{x}[H_{\theta_2}(t) - H_{\theta_1}(t)]dt \le 0$ for all $x \in (-\infty, +\infty)$. This confirms that the distribution $p(x;\theta_2)$ dominates $p(x;\theta_1)$ in the sense of second-order stochastic dominance. ■

## Appendix B. Proof of Lemma 1

Integration by parts yields

$$\begin{aligned} &E_F[\phi(U(\theta))] - E_G[\phi(U(\theta))] \\ &= -\int_{\underline{\theta}}^{\overline{\theta}} \phi'(U(\theta))U'(\theta)[F(\theta) - G(\theta)]d\theta \\ &= -\phi'\left(U(\overline{\theta})\right)\mathscr{X}(\overline{\theta}) + \int_{\underline{\theta}}^{\overline{\theta}} \phi''(U(\theta))U'(\theta)\mathscr{X}(\theta)d\theta. \end{aligned}$$

The sufficiency (the "if" part) then follows straightforwardly. The necessity (the "only if" part) follows from the arguments by contradiction in Rothschild and Stiglitz (1970). ■

## Appendix C. Proof of Proposition 1

Keeping in mind that $U(\theta) = \int u(x)p(x;\theta)dx$, we have

$$\begin{aligned} V(\tau) &= \int u(x)\tilde{p}(x;\tau)dx = \int u(x)p\left(x;U^{-1}(\tau)\right)dx \\ &= U(U^{-1}(\tau)) = \tau. \end{aligned}$$

The remaining part of this proposition is apparent. ■

## Appendix D. Proof of Lemma 3

We employ integration by parts once more to obtain

$$\begin{aligned} &E_F[\phi(U(\theta))] - E_G[\phi(U(\theta))] \\ &= -\phi'\left(U(\overline{\theta})\right)\mathscr{X}(\overline{\theta}) + \phi''\left(U(\overline{\theta})\right)\left(\int_{\underline{\theta}}^{\overline{\theta}} U'(\theta)\mathscr{X}(\theta)d\theta\right) \\ &\quad - \int_{\underline{\theta}}^{\overline{\theta}} \phi'''(U(\theta))U'(\theta)\left(\int_{\underline{\theta}}^{\theta} U'(\eta)\mathscr{X}(\eta)d\eta\right)d\theta. \end{aligned}$$

The sufficiency (the "if" part) then follows straightforwardly. The necessity (the "only if" part) follows from the arguments by contradiction in Menezes et al. (1980). ■

## Appendix E. Proof of Proposition 3

We first prove (1) $\to$ (2). Define $h := \phi_1 - \lambda\phi_2$, in which $\lambda$ is given in (1). Since $\phi_1, \phi_2 : (-\infty, U(\overline{\theta})] \mapsto \mathbb{R}$ are twice continuously differentiable, $h$ is also twice continuously differentiable. When (1) holds true,

$$h' = \phi_1' - \lambda\phi_2' = \phi_2'\left(\frac{\phi_1'}{\phi_2'} - \lambda\right)$$

satisfies $h' \le 0$ due to the facts $\phi_2' > 0$ and $\lambda \ge \frac{\phi_1'}{\phi_2'}$, and

$$h'' = \phi_1'' - \lambda\phi_2'' = \phi_2''\left(\frac{\phi_1''}{\phi_2''} - \lambda\right)$$

satisfies $h'' \le 0$ due to the facts $\phi_2'' < 0$ and $\frac{\phi_1''}{\phi_2''} \ge \lambda$. As a result, $h := \phi_1 - \lambda\phi_2$ satisfies all the properties stated in (2). This proves (1) $\to$ (2).

We then prove (2) $\to$ (3). Given (2), we have

$$\begin{aligned} &E_F[\phi_1(U(\theta) - \pi_1)] \\ &= E_G[\phi_1(U(\theta))] = E_G[\lambda\phi_2(U(\theta)) + h(U(\theta))] \\ &\le E_G[\lambda\phi_2(U(\theta))] + E_F[h(U(\theta))] \\ &= E_F[\lambda\phi_2(U(\theta) - \pi_2) + h(U(\theta))] \\ &\le E_F[\lambda\phi_2(U(\theta) - \pi_2) + h(U(\theta) - \pi_2)] \\ &= E_F[\phi_1(U(\theta) - \pi_2)], \end{aligned}$$

where the first inequality follows from the fact that $F \gtrsim^E G$ and the second inequality follows from the fact that $h' \le 0$. Then $\pi_1 \ge \pi_2$ follows straightforwardly from the fact that $E_F[\phi_1(U(\theta) - \pi_1)] \le E_F[\phi_1(U(\theta) - \pi_2)]$.

To prove (3) $\to$ (1), assume that $F \gtrsim^E G$. Let $\pi_1(t)$ and $\pi_2(t)$ be such that

$$E_{tG+(1-t)F}[\phi_i(U(\theta))] = E_F[\phi_i(U(\theta) - \pi_i(t))], \quad i = 1, 2.$$

Put differently,

$$\begin{aligned} &\int_{\underline{\theta}}^{\overline{\theta}} \phi_i(U(\theta))d[tG(\theta) + (1-t)F(\theta)] \\ &= \int_{\underline{\theta}}^{\overline{\theta}} \phi_i(U(\theta) - \pi_i(t))dF(\theta). \end{aligned}$$

Implicit differentiation and substitution of $\pi_i(0) = 0$ yield

$$\pi_i'(0) = \frac{\int_{\underline{\theta}}^{\overline{\theta}} \phi_i\left(U(\theta)\right) d[F(\theta) - G(\theta)]}{\int_{\underline{\theta}}^{\overline{\theta}} \phi_i'\left(U(\theta)\right) dF(\theta)} = \frac{\int_{\underline{\theta}}^{\overline{\theta}} \phi_i''\left(U(\theta)\right) U'(\theta)\mathscr{X}(\theta)d\theta}{\int_{\underline{\theta}}^{\overline{\theta}} \phi_i'\left(U(\theta)\right) dF(\theta)}, \tag{E.1}$$

where the second equality follows from integration by parts. Assume, for contradiction, that (1) does not hold. Then there exist $z_0, w_0 \in (-\infty, U(\overline{\theta})]$ such that

$$-\frac{\phi_1''(z_0)}{\phi_1'(w_0)} < -\frac{\phi_2''(z_0)}{\phi_2'(w_0)}. \tag{E.2}$$

Because $\phi_i$ is twice continuously differentiable, we can assume $z_0 < U(\overline{\theta})$. Remember that under Assumptions A1–A4, the mapping $U(\theta) : [\underline{\theta}, \overline{\theta}] \mapsto (-\infty, U(\overline{\theta})]$ is one-to-one and onto. Therefore, $U^{-1} : (-\infty, U(\overline{\theta})] \mapsto [\underline{\theta}, \overline{\theta}]$ is well defined. Without loss of generality, we assume $z_0 \le w_0$ and differentiate between two cases.

**Case 1**: $z_0 = w_0$. In this case, by continuity, we extend (E.2) to be

$$-\frac{\phi_1''(z)}{\phi_1'(w_0)} < -\frac{\phi_2''(z)}{\phi_2'(w_0)}, \quad \forall z \in (w_0 - \varepsilon, w_0 + \varepsilon),$$

where $\varepsilon > 0$ is small enough such that $(w_0 - \varepsilon, w_0 + \varepsilon) \subset (-\infty, U(\overline{\theta})]$. Construct $F$ and $G$ in the way specified by Lemma A.1. Then, $\mathscr{X}(\theta)$ is supported in $(U^{-1}(w_0 - \varepsilon), U^{-1}(w_0 + \varepsilon))$ and $\int_{\underline{\theta}}^{\overline{\theta}} \phi_i'\left(U(\theta)\right) dF(\theta) = \phi_i'(w_0)$. By Eq. (E.1), we thus have

$$\pi_1'(0) = \frac{1}{\phi_1'(w_0)} \int_{U^{-1}(w_0-\varepsilon)}^{U^{-1}(w_0+\varepsilon)} \phi_1''\left(U(\theta)\right) U'(\theta)\mathscr{X}(\theta)d\theta < \frac{1}{\phi_2'(w_0)} \int_{U^{-1}(w_0-\varepsilon)}^{U^{-1}(w_0+\varepsilon)} \phi_2''\left(U(\theta)\right) U'(\theta)\mathscr{X}(\theta)d\theta = \pi_2'(0). \tag{E.3}$$

**Case 2**: $z_0 < w_0$. In this case, by continuity, we extend (E.2) to be

$$-\frac{\phi_1''(z)}{q\phi_1'(z_0) + (1-q)\phi_1'(w_0)} < -\frac{\phi_2''(z)}{q\phi_2'(z_0) + (1-q)\phi_2'(w_0)}, \quad \forall z \in (z_0 - \varepsilon, z_0 + \varepsilon),$$

where $\varepsilon > 0$ is small enough such that $(z_0 - \varepsilon, z_0 + \varepsilon) \subset (-\infty, w_0) \subset (-\infty, U(\overline{\theta})]$ and $0 < q \ll 1$.[18] Construct $F$ and $G$ in the way specified by Lemma A.2. Then, $\mathscr{X}(\theta)$ is supported in $(U^{-1}(z_0 - \varepsilon), U^{-1}(z_0 + \varepsilon))$ and

$$\int_{\underline{\theta}}^{\overline{\theta}} \phi_i'\left(U(\theta)\right) dF(\theta) = q\phi_i'(z_0) + (1-q)\phi_i'(w_0), \quad i = 1, 2.$$

By Eq. (E.1), we have

$$\pi_1'(0) = \frac{\int_{U^{-1}(z_0-\varepsilon)}^{U^{-1}(z_0+\varepsilon)} \phi_1''\left(U(\theta)\right) U'(\theta)\mathscr{X}(\theta)d\theta}{q\phi_1'(z_0) + (1-q)\phi_1'(w_0)} < \frac{\int_{U^{-1}(z_0-\varepsilon)}^{U^{-1}(z_0+\varepsilon)} \phi_2''\left(U(\theta)\right) U'(\theta)\mathscr{X}(\theta)d\theta}{q\phi_2'(z_0) + (1-q)\phi_2'(w_0)} = \pi_2'(0). \tag{E.4}$$

To sum up, Eqs. (E.3) and (E.4) demonstrate that in both cases, we have $\pi_1(t) < \pi_2(t)$ for $t$ small enough. This contradicts $\pi_1 \ge \pi_2$ for all $F$ and $G$ such that $F \gtrsim^E G$. ■

[18] Indeed, if we introduce a function $F(z, q) = -\frac{\phi_1''(z)}{q\phi_1'(z_0)+(1-q)\phi_1'(w_0)} + \frac{\phi_2''(z)}{q\phi_2'(z_0)+(1-q)\phi_2'(w_0)}$, then $F(z, q)$ is continuous in both $z$ and $q$, and $F(z_0, 0) < 0$. By continuity, $F(z, q) < 0$ when $z$ is sufficiently close to $z_0$ and $q$ is sufficiently close to zero.

## Appendix F. Proof of Proposition 4

The proof can be performed in the same manner as that for Proposition 3 if we replace the integration by parts

$$\int_{\underline{\theta}}^{\overline{\theta}} \phi_i\left(U(\theta)\right) d[F(\theta) - G(\theta)] = \int_{\underline{\theta}}^{\overline{\theta}} \phi_i''\left(U(\theta)\right) U'(\theta)\mathscr{X}(\theta)d\theta$$

with

$$\int_{\underline{\theta}}^{\overline{\theta}} \phi_i\left(U(\theta)\right) d[F(\theta) - G(\theta)] = -\int_{\underline{\theta}}^{\overline{\theta}} \phi_i'''\left(U(\theta)\right) U'(\theta) \left(\int_{\underline{\theta}}^{\theta} U'(\eta)\mathscr{X}(\eta)d\eta\right) d\theta$$

when $F \gtrapprox^E G$. ■

## Appendix G. Proof of Proposition 5

We prove (1) → (2) by contradiction. Assume that there exists $F \gtrsim^E G$ such that $\pi_1 < \pi_2$. Denote the utility premium and the monetary premium paid for reducing the ambiguity from $tG + (1-t)F$ to $F$ by $\pi_1(t), \pi_2(t)$ and $\hat{\pi}_1(t), \hat{\pi}_2(t)$, respectively. By definition, we have

$$\int_{\underline{\theta}}^{\overline{\theta}} \phi_i\left(U(\theta)\right) d[tG(\theta) + (1-t)F(\theta)] = \int_{\underline{\theta}}^{\overline{\theta}} \phi_i\left(U(\theta) - \pi_i(t)\right) dF(\theta) = \int_{\underline{\theta}}^{\overline{\theta}} \phi_i\left(\int p(x;\theta)u(x - \hat{\pi}_i(t))dx\right) dF(\theta).$$

Implicit differentiation and integration by parts yields

$$\pi_i'(t) = \frac{\int_{\underline{\theta}}^{\overline{\theta}} \phi_i''\left(U(\theta)\right) U'(\theta)\mathscr{X}(\theta)d\theta}{\int_{\underline{\theta}}^{\overline{\theta}} \phi_i'\left(U(\theta) - \pi_i(t)\right) dF(\theta)}, \tag{G.1}$$

$$\hat{\pi}_i'(t) = \frac{\int_{\underline{\theta}}^{\overline{\theta}} \phi_i''\left(U(\theta)\right) U'(\theta)\mathscr{X}(\theta)d\theta}{\int_{\underline{\theta}}^{\overline{\theta}} \phi_i'\left(\int p(x;\theta)u(x - \hat{\pi}_i(t))dx\right) \int p(x;\theta)u'(x - \hat{\pi}_i(t))dxdF(\theta)}. \tag{G.2}$$

Since $\pi_1(0) = \pi_2(0) = 0$ and $\pi_1(1) < \pi_2(1)$, there exists $t_0 \in [0, 1)$ such that $\pi_1(t_0) = \pi_2(t_0) = \pi_0$ and $\pi_1'(t_0) < \pi_2'(t_0)$. By Eq. (G.1), this implies

$$\frac{\int_{\underline{\theta}}^{\overline{\theta}} \phi_1''\left(U(\theta)\right) U'(\theta)\mathscr{X}(\theta)d\theta}{\int_{\underline{\theta}}^{\overline{\theta}} \phi_1'\left(U(\theta) - \pi_0\right) dF(\theta)} < \frac{\int_{\underline{\theta}}^{\overline{\theta}} \phi_2''\left(U(\theta)\right) U'(\theta)\mathscr{X}(\theta)d\theta}{\int_{\underline{\theta}}^{\overline{\theta}} \phi_2'\left(U(\theta) - \pi_0\right) dF(\theta)}. \tag{G.3}$$

We claim that there exist $\theta_a, \theta_b \in [\underline{\theta}, \overline{\theta}]$ such that

$$-\frac{\phi_1''\left(U(\theta_a)\right)}{\phi_1'\left(U(\theta_b) - \pi_0\right)} < -\frac{\phi_2''\left(U(\theta_a)\right)}{\phi_2'\left(U(\theta_b) - \pi_0\right)}.$$

To verify this claim, we define a new two-variable function

$$H_i(\theta_a, \theta_b) = -\frac{\phi_i''\left(U(\theta_a)\right)}{\phi_i'\left(U(\theta_b) - \pi_0\right)}, \quad i = 1, 2$$

and argue by contradiction. If $H_1(\theta_a,\theta_b) \geq H_2(\theta_a,\theta_b)$ for all $(\theta_a,\theta_b) \in [\underline{\theta},\overline{\theta}] \times [\underline{\theta},\overline{\theta}]$, then it follows from the fact $\mathscr{X}(\theta) \leq 0$ that

$$H_1(\theta_a,\theta_b)\mathscr{X}(\theta_a) \leq H_2(\theta_a,\theta_b)\mathscr{X}(\theta_a), \quad \forall(\theta_a,\theta_b) \in [\underline{\theta},\overline{\theta}] \times [\underline{\theta},\overline{\theta}],$$

which integrated over $\theta_a$ implies[19]

$$\int_{\underline{\theta}}^{\overline{\theta}} H_1(\theta_a,\theta_b)\mathscr{X}(\theta_a)d\theta_a \leq \int_{\underline{\theta}}^{\overline{\theta}} H_2(\theta_a,\theta_b)\mathscr{X}(\theta_a)d\theta_a < 0, \quad \forall\theta_b \in [\underline{\theta},\overline{\theta}],$$

which further implies

$$\frac{1}{\int_{\underline{\theta}}^{\overline{\theta}} H_1(\theta_a,\theta_b)\mathscr{X}(\theta_a)d\theta_a} \geq \frac{1}{\int_{\underline{\theta}}^{\overline{\theta}} H_2(\theta_a,\theta_b)\mathscr{X}(\theta_a)d\theta_a}, \quad \forall\theta_b \in [\underline{\theta},\overline{\theta}].$$

Integrating the above over $\theta_b$ yields

$$\int_{\underline{\theta}}^{\overline{\theta}} \left(\frac{1}{\int_{\underline{\theta}}^{\overline{\theta}} H_1(\theta_a,\theta_b)\mathscr{X}(\theta_a)d\theta_a}\right) d\theta_b \geq \int_{\underline{\theta}}^{\overline{\theta}} \left(\frac{1}{\int_{\underline{\theta}}^{\overline{\theta}} H_2(\theta_a,\theta_b)\mathscr{X}(\theta_a)d\theta_a}\right) d\theta_b. \tag{G.4}$$

By definition, we have

$$\int_{\underline{\theta}}^{\overline{\theta}} \left(\frac{1}{\int_{\underline{\theta}}^{\overline{\theta}} H_i(\theta_a,\theta_b)\mathscr{X}(\theta_a)d\theta_a}\right) d\theta_b = -\int_{\underline{\theta}}^{\overline{\theta}} \left(\frac{\phi_i'(U(\theta_b)-\pi_0)}{\int_{\underline{\theta}}^{\overline{\theta}} \phi_i''(U(\theta_a))U'(\theta_a)\mathscr{X}(\theta_a)d\theta_a}\right) d\theta_b = -\frac{\int_{\underline{\theta}}^{\overline{\theta}} \phi_i'(U(\theta)-\pi_0)\,d\theta}{\int_{\underline{\theta}}^{\overline{\theta}} \phi_i''(U(\theta))U'(\theta)\mathscr{X}(\theta)d\theta}. \tag{G.5}$$

Substituting (G.5) into (G.4), we get

$$0 < \frac{\int_{\underline{\theta}}^{\overline{\theta}} \phi_1'(U(\theta)-\pi_0)\,d\theta}{\int_{\underline{\theta}}^{\overline{\theta}} \phi_1''(U(\theta))U'(\theta)\mathscr{X}(\theta)d\theta} \leq \frac{\int_{\underline{\theta}}^{\overline{\theta}} \phi_2'(U(\theta)-\pi_0)\,d\theta}{\int_{\underline{\theta}}^{\overline{\theta}} \phi_2''(U(\theta))U'(\theta)\mathscr{X}(\theta)d\theta},$$

or in other words,

$$\frac{\int_{\underline{\theta}}^{\overline{\theta}} \phi_1''(U(\theta))U'(\theta)\mathscr{X}(\theta)d\theta}{\int_{\underline{\theta}}^{\overline{\theta}} \phi_1'(U(\theta)-\pi_0)\,d\theta} \geq \frac{\int_{\underline{\theta}}^{\overline{\theta}} \phi_2''(U(\theta))U'(\theta)\mathscr{X}(\theta)d\theta}{\int_{\underline{\theta}}^{\overline{\theta}} \phi_2'(U(\theta)-\pi_0)\,d\theta},$$

which is a contradiction to (G.3). Therefore, as long as (G.3) holds true, there must exist some $(\theta_a,\theta_b) \in [\underline{\theta},\overline{\theta}] \times [\underline{\theta},\overline{\theta}]$ such that $H_1(\theta_a,\theta_b) < H_2(\theta_a,\theta_b)$, i.e.,

$$-\frac{\phi_1''(U(\theta_a))}{\phi_1'(U(\theta_b)-\pi_0)} < -\frac{\phi_2''(U(\theta_a))}{\phi_2'(U(\theta_b)-\pi_0)}.$$

Denoting $U(\theta_a)$ and $U(\theta_b)-\pi_0$ by $z_0$ and $w_0$ respectively, then $z_0, w_0 \in (-\infty, U(\overline{\theta})]$ and

$$-\frac{\phi_1''(z_0)}{\phi_1'(w_0)} < -\frac{\phi_2''(z_0)}{\phi_2'(w_0)}.$$

Following the arguments in the proof of Proposition 3, we can construct a pair of distributions $\bar{F}$ and $\bar{G}$ over $\theta$ such that $\bar{F} \gtrsim^E \bar{G}$ and

$$\frac{\int_{\underline{\theta}}^{\overline{\theta}} \phi_1''(U(\theta))U'(\theta)\bar{\mathscr{X}}(\theta)d\theta}{\int_{\underline{\theta}}^{\overline{\theta}} \phi_1'(U(\theta))\left(\int p(x;\theta)u'(x)dx\right)d\bar{F}(\theta)} < \frac{\int_{\underline{\theta}}^{\overline{\theta}} \phi_2''(U(\theta))U'(\theta)\bar{\mathscr{X}}(\theta)d\theta}{\int_{\underline{\theta}}^{\overline{\theta}} \phi_2'(U(\theta))\left(\int p(x;\theta)u'(x)dx\right)d\bar{F}(\theta)},$$

where $\bar{\mathscr{X}}(\eta) = \int_{\underline{\theta}}^{\eta} U'(\theta)[\bar{F}(\theta)-\bar{G}(\theta)]d\theta$. This amounts to saying $\hat{\pi}_1'(0) < \hat{\pi}_2'(0)$ according to Eq. (G.2), which further implies $\hat{\pi}_1(t) < \hat{\pi}_2(t)$ for $t$ small enough. In sum, with a small $t$, we have $\bar{F} \gtrsim^E t\bar{G}+(1-t)\bar{F}$ and the monetary premium paid for reducing the ambiguity from $t\bar{G}+(1-t)\bar{F}$ to $\bar{F}$ satisfies $\hat{\pi}_1(t) < \hat{\pi}_2(t)$, which is a contradiction.
The proof for (2) → (1) is similar.
The proof of (3) ⇔ (4) can be performed in the same manner, with the integration by parts

$$\int_{\underline{\theta}}^{\overline{\theta}} \phi_i(U(\theta))\,d[F(\theta)-G(\theta)] = \int_{\underline{\theta}}^{\overline{\theta}} \phi_i''(U(\theta))U'(\theta)\mathscr{X}(\theta)d\theta$$

replaced by

$$\int_{\underline{\theta}}^{\overline{\theta}} \phi_i(U(\theta))\,d[F(\theta)-G(\theta)] = -\int_{\underline{\theta}}^{\overline{\theta}} \phi_i'''(U(\theta))U'(\theta)\left(\int_{\underline{\theta}}^{\theta} U'(\eta)\mathscr{X}(\eta)d\eta\right)d\theta$$

when $F \gtrsim^E_{\approx} G$. ■

## Appendix H. Proof of Proposition 6

The first-order condition of the objective function is

$$\frac{d}{de}\bigg|_{e=e_i^*} \int_{\underline{\theta}}^{\overline{\theta}} \phi_i(U(\theta)-c(e))d[eF(\theta)+(1-e)G(\theta)] = -c'(e_i^*)\int_{\underline{\theta}}^{\overline{\theta}} \phi_i'\left(U(\theta)-c(e_i^*)\right)d\left[e_i^*F(\theta)+(1-e_i^*)G(\theta)\right] + \int_{\underline{\theta}}^{\overline{\theta}} \phi_i\left(U(\theta)-c(e_i^*)\right)d\left[F(\theta)-G(\theta)\right] = 0.$$

Integrating the second term of the right-hand side of the above equation by parts yields

$$0 = -c'(e_i^*) + \frac{\int_{\underline{\theta}}^{\overline{\theta}} \phi_i''\left(U(\theta)-c(e_i^*)\right)U'(\theta)\mathscr{X}(\theta)d\theta}{\int_{\underline{\theta}}^{\overline{\theta}} \phi_i'\left(U(\theta)-c(e_i^*)\right)d\left[e_i^*F(\theta)+(1-e_i^*)G(\theta)\right]}. \tag{H.1}$$

The second-order derivative of the objective function is

$$\frac{d^2}{de^2}\int_{\underline{\theta}}^{\overline{\theta}} \phi_i(U(\theta)-c(e))d[eF(\theta)+(1-e)G(\theta)] = -c''(e)\int_{\underline{\theta}}^{\overline{\theta}} \phi_i'(U(\theta)-c(e))\,d\left[eF(\theta)+(1-e)G(\theta)\right]$$

[19] Notice that Proposition 5 becomes trivial when $F \equiv G$, and hence the proof focuses on the case $F \neq G$. In this case, if $F \gtrsim^E G$, $\mathscr{X}(\theta) \leq 0$ and in the meanwhile there must exist some $\hat{\theta} \in [\underline{\theta},\overline{\theta}]$ such that $\mathscr{X}(\hat{\theta}) < 0$. This ensures $\int_{\underline{\theta}}^{\overline{\theta}} H_2(\theta_a,\theta_b)\mathscr{X}(\theta_a)d\theta_a < 0$.

$$+ (c'(e))^2 \int_{\underline{\theta}}^{\overline{\theta}} \phi_i''\left(U(\theta) - c(e)\right) d\left[eF(\theta) + (1-e)G(\theta)\right]$$
$$- 2c'(e) \int_{\underline{\theta}}^{\overline{\theta}} \phi_i'''\left(U(\theta) - c(e)\right) U'(\theta)\mathscr{X}(\theta) d\theta < 0,$$

where the last term of the right-hand side of the above equation has been rearranged using integration by parts. The condition $c'(0) = 0$ guarantees that $e = 0$ is not the solution to (9) because the individual's marginal utility is always positive when $e = 0$. The condition $\lim_{e\to 1} c'(e) = +\infty$ guarantees that $e = 1$ is not the solution to (9) because the individual's marginal utility goes to negative infinity when $e \to 1$. These guarantees the existence of an interior solution. The negative sign of the second-order derivative guarantees that the solution to (9) is unique.

To prove (1) $\to$ (2), when individual 1 is more ambiguity averse than individual 2, we have $-\phi_1''(z-k)/\phi_1'(w-k) \geq -\phi_2''(z-k)/\phi_2'(w-k)$ for all $z, w \in [U(\underline{\theta}), U(\overline{\theta})]$ and all $k \in [0, M]$. Accordingly,

$$\frac{\phi_1''\left(U(\theta_a) - c(e_2^*)\right) U'(\theta_a)\mathscr{X}(\theta_a)}{\phi_1'\left(U(\theta_b) - c(e_2^*)\right)} \geq \frac{\phi_2''\left(U(\theta_a) - c(e_2^*)\right) U'(\theta_a)\mathscr{X}(\theta_a)}{\phi_2'\left(U(\theta_b) - c(e_2^*)\right)} \geq 0, \quad \forall \theta_a, \theta_b \in [\underline{\theta}, \overline{\theta}].$$

Integrating the above over $\theta_a$ yields[20]

$$\frac{\int_{\underline{\theta}}^{\overline{\theta}} \phi_1''\left(U(\theta_a) - c(e_2^*)\right) U'(\theta_a)\mathscr{X}(\theta_a) d\theta_a}{\phi_1'\left(U(\theta_b) - c(e_2^*)\right)} \geq \frac{\int_{\underline{\theta}}^{\overline{\theta}} \phi_2''\left(U(\theta_a) - c(e_2^*)\right) U'(\theta_a)\mathscr{X}(\theta_a) d\theta_a}{\phi_2'\left(U(\theta_b) - c(e_2^*)\right)} > 0$$

for all $\theta_b \in [\underline{\theta}, \overline{\theta}]$, or in other words,

$$0 < \frac{\phi_1'\left(U(\theta_b) - c(e_2^*)\right)}{\int_{\underline{\theta}}^{\overline{\theta}} \phi_1''\left(U(\theta_a) - c(e_2^*)\right) U'(\theta_a)\mathscr{X}(\theta_a) d\theta_a} \leq \frac{\phi_2'\left(U(\theta_b) - c(e_2^*)\right)}{\int_{\underline{\theta}}^{\overline{\theta}} \phi_2''\left(U(\theta_a) - c(e_2^*)\right) U'(\theta_a)\mathscr{X}(\theta_a) d\theta_a}$$

for all $\theta_b \in [\underline{\theta}, \overline{\theta}]$. Integrating the above over $\theta_b$ yields

$$0 < \frac{\int_{\underline{\theta}}^{\overline{\theta}} \phi_1'\left(U(\theta_b) - c(e_2^*)\right) d\theta_b}{\int_{\underline{\theta}}^{\overline{\theta}} \phi_1''\left(U(\theta_a) - c(e_2^*)\right) U'(\theta_a)\mathscr{X}(\theta_a) d\theta_a} \leq \frac{\int_{\underline{\theta}}^{\overline{\theta}} \phi_2'\left(U(\theta_b) - c(e_2^*)\right) d\theta_b}{\int_{\underline{\theta}}^{\overline{\theta}} \phi_2''\left(U(\theta_a) - c(e_2^*)\right) U'(\theta_a)\mathscr{X}(\theta_a) d\theta_a},$$

or equivalently,

$$\frac{\int_{\underline{\theta}}^{\overline{\theta}} \phi_1''\left(U(\theta) - c(e_2^*)\right) U'(\theta)\mathscr{X}(\theta) d\theta}{\int_{\underline{\theta}}^{\overline{\theta}} \phi_1'\left(U(\theta) - c(e_2^*)\right) d\theta} \geq \frac{\int_{\underline{\theta}}^{\overline{\theta}} \phi_2''\left(U(\theta) - c(e_2^*)\right) U'(\theta)\mathscr{X}(\theta) d\theta}{\int_{\underline{\theta}}^{\overline{\theta}} \phi_2'\left(U(\theta) - c(e_2^*)\right) d\theta} > 0.$$

We then obtain from (H.1) that

$$-c'(e_2^*) + \frac{\int_{\underline{\theta}}^{\overline{\theta}} \phi_1''\left(U(\theta) - c(e_2^*)\right) U'(\theta)\mathscr{X}(\theta) d\theta}{\int_{\underline{\theta}}^{\overline{\theta}} \phi_1'\left(U(\theta) - c(e_2^*)\right) d\left[e_2^*F(\theta) + (1-e_2^*)G(\theta)\right]}$$
$$\geq -c'(e_2^*) + \frac{\int_{\underline{\theta}}^{\overline{\theta}} \phi_2''\left(U(\theta) - c(e_2^*)\right) U'(\theta)\mathscr{X}(\theta) d\theta}{\int_{\underline{\theta}}^{\overline{\theta}} \phi_2'\left(U(\theta) - c(e_2^*)\right) d\left[e_2^*F(\theta) + (1-e_2^*)G(\theta)\right]}$$
$$= 0,$$

leading to $e_1^* \geq e_2^*$.

To prove (2) $\to$ (1), we argue by contradiction. Suppose that there exist $z_0, w_0 \in [U(\underline{\theta}), U(\overline{\theta})]$ and $k_0 \in [0, M]$ such that

$$-\frac{\phi_1''(z_0 - k_0)}{\phi_1'(w_0 - k_0)} < -\frac{\phi_2''(z_0 - k_0)}{\phi_2'(w_0 - k_0)}. \tag{H.2}$$

Because $\phi_i$ is three times continuously differentiable, we can assume $z_0 \in (U(\underline{\theta}), U(\overline{\theta}))$ and $k_0 \in (0, M)$. We are mainly concerned with the term

$$\mathscr{T}_i(e) = \frac{\int_{\underline{\theta}}^{\overline{\theta}} \phi_i''\left(U(\theta) - k_0\right) U'(\theta)\mathscr{X}(\theta) d\theta}{\int_{\underline{\theta}}^{\overline{\theta}} \phi_i'\left(U(\theta) - k_0\right) d\left[eF(\theta) + (1-e)G(\theta)\right]}, \quad e \in [0, 1]$$

in the following construction. Let

$$c(e) = M\left(1 - \delta\sqrt{1 - e^2}\right), \quad 0 < \delta \leq 1, \tag{H.3}$$

where $\delta$ is to be specified later. Notice that $c(e)$ in (H.3) satisfies Assumptions A5 and A6. Without loss of generality, we assume $z_0 \leq w_0$ and differentiate between two cases.

**Case 1**: $z_0 = w_0$. In this case, by continuity, we extend (H.2) to be

$$-\frac{\phi_1''(w_0 - k_0 - \varepsilon)}{\phi_1'(w_0 - k_0 + \varepsilon)} < -\frac{\phi_2''(w_0 - k_0 + \varepsilon)}{\phi_2'(w_0 - k_0 - \varepsilon)}, \tag{H.4}$$

where $\varepsilon > 0$ is small enough such that $(w_0 - \varepsilon, w_0 + \varepsilon) \subset [U(\underline{\theta}), U(\overline{\theta})]$. Construct $F$ and $G$ in the way specified by Lemma A.1. Then, $\mathscr{X}(\theta)$ is supported in $(U^{-1}(w_0 - \varepsilon), U^{-1}(w_0 + \varepsilon))$ and

$$\mathscr{T}_i(e) = \frac{\int_{U^{-1}(w_0-\varepsilon)}^{U^{-1}(w_0+\varepsilon)} \phi_i''\left(U(\theta) - k_0\right) U'(\theta)\mathscr{X}(\theta) d\theta}{e\phi_i'(w_0 - k_0) + \frac{1-e}{2}\phi_i'\left(w_0 - k_0 + \frac{\varepsilon}{2}\right) + \frac{1-e}{2}\phi_i'\left(w_0 - k_0 - \frac{\varepsilon}{2}\right)}.$$

Due to $\phi_i'' < 0, \phi_i''' > 0$, we have for all $\theta \in (U^{-1}(w_0 - \varepsilon), U^{-1}(w_0 + \varepsilon))$ and all $e \in [0, 1]$ that

$$-\frac{\phi_1''(U(\theta) - k_0)}{e\phi_1'(w_0 - k_0) + \frac{1-e}{2}\phi_1'\left(w_0 - k_0 + \frac{\varepsilon}{2}\right) + \frac{1-e}{2}\phi_1'\left(w_0 - k_0 - \frac{\varepsilon}{2}\right)}$$
$$< -\frac{\phi_1''(w_0 - k_0 - \varepsilon)}{\phi_1'(w_0 - k_0 + \varepsilon)} < -\frac{\phi_2''(w_0 - k_0 + \varepsilon)}{\phi_2'(w_0 - k_0 - \varepsilon)} \quad \text{by (G.4)}$$
$$< -\frac{\phi_2''(U(\theta) - k_0)}{e\phi_2'(w_0 - k_0) + \frac{1-e}{2}\phi_2'\left(w_0 - k_0 + \frac{\varepsilon}{2}\right) + \frac{1-e}{2}\phi_2'\left(w_0 - k_0 - \frac{\varepsilon}{2}\right)},$$

which leads us to

$$\mathscr{T}_1(e) < \mathscr{T}_2(e), \quad \forall e \in [0, 1]. \tag{H.5}$$

We now specify $\delta$ in (H.3) such that when $c(e) = k_0$, there will be $c'(e) = \mathscr{T}_1(e)$. In fact, $c(e) = k_0$ implies

$$e = \sqrt{1 - \frac{1}{\delta^2}\left(1 - \frac{k_0}{M}\right)^2}, \quad 1 - \frac{k_0}{M} \leq \delta \leq 1,$$

$$c'(e) - \mathscr{T}_1(e) = \delta M \sqrt{\left(\frac{\delta}{1 - \frac{k_0}{M}}\right)^2 - 1} - \mathscr{T}_1\left(\sqrt{1 - \frac{1}{\delta^2}\left(1 - \frac{k_0}{M}\right)^2}\right) := f(\delta).$$

[20] Notice that Proposition 6 becomes trivial when $F \equiv G$, and hence the proof focuses on the case $F \neq G$. In this case, if $F \succsim^E G$, $\mathscr{X}(\theta) \leq 0$ and in the meanwhile there must exist some $\hat{\theta} \in [\underline{\theta}, \overline{\theta}]$ such that $\mathscr{X}(\hat{\theta}) < 0$. This ensures $\frac{\int_{\underline{\theta}}^{\overline{\theta}} \phi_2''\left(U(\theta_a) - c(e_2^*)\right) U'(\theta_a)\mathscr{X}(\theta_a) d\theta_a}{\phi_2'\left(U(\theta_b) - c(e_2^*)\right)} > 0$.

$$\mathcal{T}_i(e) = \frac{\int_{U^{-1}(z_0-\varepsilon)}^{U^{-1}(z_0+\varepsilon)} \phi_i''\left(U(\theta)-k_0\right)U'(\theta)\mathcal{X}(\theta)d\theta}{q\left[e\phi_i'(z_0-k_0)+\frac{1-e}{2}\phi_i'\left(z_0-k_0+\frac{\varepsilon}{2}\right)+\frac{1-e}{2}\phi_i'\left(z_0-k_0-\frac{\varepsilon}{2}\right)\right]+(1-q)\phi_i'(w_0-k_0)}.$$

**Box I.**

$$\begin{aligned}
&-\frac{\phi_1''(U(\theta)-k_0)}{q\left[e\phi_1'(z_0-k_0)+\frac{1-e}{2}\phi_1'\left(z_0-k_0+\frac{\varepsilon}{2}\right)+\frac{1-e}{2}\phi_1'\left(z_0-k_0-\frac{\varepsilon}{2}\right)\right]+(1-q)\phi_1'(w_0-k_0)}\\
&< -\frac{\phi_1''(z_0-k_0-\varepsilon)}{q\phi_1'(z_0-k_0+\varepsilon)+(1-q)\phi_1'(w_0-k_0)}\\
&< -\frac{\phi_2''(z_0-k_0+\varepsilon)}{q\phi_2'(z_0-k_0-\varepsilon)+(1-q)\phi_2'(w_0-k_0)} \quad \text{by (H.7)}\\
&< -\frac{\phi_2''(U(\theta)-k_0)}{q\left[e\phi_2'(z_0-k_0)+\frac{1-e}{2}\phi_2'\left(z_0-k_0+\frac{\varepsilon}{2}\right)+\frac{1-e}{2}\phi_2'\left(z_0-k_0-\frac{\varepsilon}{2}\right)\right]+(1-q)\phi_2'(w_0-k_0)},
\end{aligned}$$

**Box II.**

Since $f\left(1-\frac{k_0}{M}\right) = -\mathcal{T}_1(0) < 0$ and for $\varepsilon$ small enough

$$f(1) = M\sqrt{\left(\frac{1}{1-\frac{k_0}{M}}\right)^2 - 1} - \mathcal{T}_1\left(\sqrt{1-\left(1-\frac{k_0}{M}\right)^2}\right) > 0,$$

there must exist a constant $\hat{\delta} \in \left(1-\frac{k_0}{M}, 1\right)$ such that $f(\hat{\delta}) = 0$.[21]

With $\delta = \hat{\delta}$, we have that $\hat{e} = \sqrt{1-\frac{1}{\hat{\delta}^2}\left(1-\frac{k_0}{M}\right)^2}$ satisfies

$$c(\hat{e}) = k_0, \qquad c'(\hat{e}) = \mathcal{T}_1(\hat{e}). \tag{H.6}$$

Thanks to the concavity of the objective function, the $\hat{e}$ satisfying the first-order condition is exactly the optimal effort level. Taking (H.1), (H.5) and (H.6) together, we have

$$\begin{aligned}
0 &= -c'(\hat{e}) + \frac{\int_{\underline{\theta}}^{\overline{\theta}} \phi_1''\left(U(\theta)-c(\hat{e})\right)U'(\theta)\mathcal{X}(\theta)d\theta}{\int_{\underline{\theta}}^{\overline{\theta}} \phi_1'\left(U(\theta)-c(\hat{e})\right)d\left[\hat{e}F(\theta)+(1-\hat{e})G(\theta)\right]}\\
&< -c'(\hat{e}) + \frac{\int_{\underline{\theta}}^{\overline{\theta}} \phi_2''\left(U(\theta)-c(\hat{e})\right)U'(\theta)\mathcal{X}(\theta)d\theta}{\int_{\underline{\theta}}^{\overline{\theta}} \phi_2'\left(U(\theta)-c(\hat{e})\right)d\left[\hat{e}F(\theta)+(1-\hat{e})G(\theta)\right]}.
\end{aligned}$$

The above leads us to $e_1^* = \hat{e}$ and $e_2^* > \hat{e}$, which contradicts (2).

**Case 2**: $z_0 < w_0$. In this case, by continuity, we extend (H.2) to be

$$\begin{aligned}
&-\frac{\phi_1''(z_0-k_0-\varepsilon)}{q\phi_1'(z_0-k_0+\varepsilon)+(1-q)\phi_1'(w_0-k_0)}\\
&< -\frac{\phi_2''(z_0-k_0+\varepsilon)}{q\phi_2'(z_0-k_0-\varepsilon)+(1-q)\phi_2'(w_0-k_0)},
\end{aligned} \tag{H.7}$$

where $\varepsilon > 0$ is small enough such that $(z_0-\varepsilon, z_0+\varepsilon) \subset [U(\underline{\theta}), U(\overline{\theta})]$ and $z_0+\varepsilon < w_0$, and $0 < q \ll 1$.[22]Construct $F$ and $G$ in the way specified by Lemma A.2. Then, $\mathcal{X}(\theta)$ is supported in $(U^{-1}(z_0-\varepsilon), U^{-1}(z_0+\varepsilon))$ and we get the equation in Box I. Due to $\phi_i'' < 0$, $\phi_i''' > 0$, we have for all $\theta \in (U^{-1}(w_0-\varepsilon), U^{-1}(w_0+\varepsilon))$ and all $e \in [0,1]$ the equation given in Box II which leads us to

$$\mathcal{T}_1(e) < \mathcal{T}_2(e), \quad \forall e \in [0,1].$$

The remaining of the proof is exactly the same as that in Case 1. This proves $(2) \to (1)$.

The proof of $(3) \Leftrightarrow (4)$ can be performed in the same manner, with the integration by parts

$$\begin{aligned}
&\int_{\underline{\theta}}^{\overline{\theta}} \phi_i\left(U(\theta)-c(e_i^*)\right)d\left[F(\theta)-G(\theta)\right]\\
&= \int_{\underline{\theta}}^{\overline{\theta}} \phi_i''\left(U(\theta)-c(e_i^*)\right)U'(\theta)\mathcal{X}(\theta)d\theta
\end{aligned}$$

replaced by

$$\begin{aligned}
&\int_{\underline{\theta}}^{\overline{\theta}} \phi_i\left(U(\theta)-c(e_i^*)\right)d\left[F(\theta)-G(\theta)\right]\\
&= \int_{\underline{\theta}}^{\overline{\theta}} \phi_i'''\left(U(\theta)-c(e_i^*)\right)U'(\theta)\left(\int_{\underline{\theta}}^{\theta} U'(\eta)\mathcal{X}(\eta)d\eta\right)d\theta
\end{aligned}$$

when $F \gtrsim^E G$. ■

21 Indeed, for any given $e \in [0,1]$,

$$\lim_{\varepsilon\to 0} \mathcal{T}_i(e) = \lim_{\varepsilon\to 0}\left[\frac{\int_{U^{-1}(w_0-\varepsilon)}^{U^{-1}(w_0+\varepsilon)} \phi_i''(U(\theta)-k_0)U'(\theta)\mathcal{X}(\theta)d\theta}{e\phi_i'(w_0-k_0)+\frac{1-e}{2}\phi_i'\left(w_0-k_0+\frac{\varepsilon}{2}\right)+\frac{1-e}{2}\phi_i'\left(w_0-k_0-\frac{\varepsilon}{2}\right)}\right] = 0.$$

Therefore, we can always fix $\varepsilon$ small enough such that $f(1) > 0$.

22 Indeed, if we introduce a function $F(\varepsilon, q) = -\frac{\phi_1''(z_0-k_0-\varepsilon)}{q\phi_1'(z_0-k_0+\varepsilon)+(1-q)\phi_1'(w_0-k_0)} + \frac{\phi_2''(z_0-k_0+\varepsilon)}{q\phi_2'(z_0-k_0-\varepsilon)+(1-q)\phi_2'(w_0-k_0)}$, then $F(\varepsilon, q)$ is continuous in both $\varepsilon$ and $q$, and $F(0,0) < 0$. By continuity, $F(\varepsilon, q) < 0$ when $\varepsilon$ and $q$ are sufficiently close to zero.

## References


Ahn, D., 2008. Ambiguity without a state space. Rev. Econom. Stud. 75 (1), 3–28.

Alary, D., Gollier, C., Treich, N., 2013. The effect of ambiguity aversion on insurance and self-protection. Econom. J. 123 (573), 1188–1202.

Baillon, A., 2013. Prudence (and more) with respect to ambiguity. Working Paper (http://aurelienbaillon.com/research).

Briys, E., Schlesinger, H., 1990. Risk aversion and the propensities for self-insurance and self-protection. South. Econ. J. 57 (2), 458–467.

Cabantous, L., 2007. Ambiguity aversion in the field of insurance: Insurers' attitude to imprecise and conflicting probability estimates. Theory and Decision 62 (3), 219–240.

Cabantous, L., Hilton, D., Kunreuther, H., Michel-Kerjan, E., 2011. Is imprecise knowledge better than conflicting expertise? Evidence from insurers' decisions in the United States. J. Risk Uncertain. 42 (3), 211–232.

Cerreia-Vioglio, S., Maccheroni, F., Marinacci, M., Montrucchio, L., 2011. Uncertainty averse preferences. J. Econom. Theory 146 (4), 1275–1330.

Charness, G., Karni, E., Levin, D., 2013. Ambiguity attitudes and social interactions: An experimental investigation. J. Risk Uncertain. 46 (1), 1–25.

Chuang, O.-C., Eeckhoudt, L., Huang, R.J., Tzeng, L.Y., 2013. Risky targets and effort. Insurance Math. Econom. 52 (3), 465–468.

Conte, A., Hey, J.D., 2013. Assessing multiple prior models of behaviour under ambiguity. J. Risk Uncertain. 46 (2), 113–132.

Denuit, M.M., Eeckhoudt, L., 2010. Stronger measures of higher-order risk attitudes. J. Econom. Theory 145 (5), 2027–2036.

Dionne, G., Eeckhoudt, L., 1985. Self-insurance, self-protection and increased risk aversion. Econom. Lett. 17 (1–2), 39–42.
Eeckhoudt, L., Schlesinger, H., 2006. Putting risk in its proper place. Amer. Econ. Rev. 96 (1), 280–289.
Ehrlich, I., Becker, G.S., 1972. Market insurance, self-insurance, and self-protection. J. Polit. Econ. 80 (4), 623–648.
Ekern, S., 1980. Increasing Nth degree risk. Econom. Lett. 6 (4), 329–333.
Ellsberg, D., 1961. Risk, ambiguity, and the Savage axioms. Quart. J. Econ. 75 (4), 643–669.
Ghirardato, P., Maccheroni, F., Marinacci, M., 2004. Differentiating ambiguity and ambiguity attitude. J. Econom. Theory 118 (2), 133–173.
Ghirardato, P., Siniscalchi, M., 2012. Ambiguity in the small and in the large. Econometrica 80 (6), 2827–2847.
Gilboa, I., Schmeidler, D., 1989. Maxmin expected utility with non-unique prior. J. Math. Econom. 18 (2), 141–153.
Gollier, C., 2011. Portfolio choices and asset prices: The comparative statics of ambiguity aversion. Rev. Econom. Stud. 78 (4), 1329–1344.
Hoy, M., Peter, R., Richter, A., 2014. Take-up for genetic tests and ambiguity. J. Risk Uncertain. 48 (2), 111–133.
Huang, R.J., 2012. Ambiguity aversion, higher-order risk attitude and optimal effort. Insurance Math. Econom. 50 (3), 338–345.
Iwaki, H., Osaki, Y., 2014. The dual theory of the smooth ambiguity model. Econom. Theory 56 (2), 275–289.
Jindapon, P., Neilson, W.S., 2007. Higher-order generalizations of Arrow–Pratt and Ross risk aversion: A comparative statics approach. J. Econom. Theory 136 (1), 719–728.
Jullien, B., Salanié, B., Salanié, F., 1999. Should more risk-averse agents exert more effort? Geneva Pap. Risk Insur. Theory 24 (1), 19–28.
Klibanoff, P., Marinacci, M., Mukerji, S., 2005. A smooth model of decision making under ambiguity. Econometrica 73 (6), 1849–1892.
Klibanoff, P., Marinacci, M., Mukerji, S., 2009. Recursive smooth ambiguity preferences. J. Econom. Theory 144 (3), 930–976.
Klibanoff, P., Marinacci, M., Mukerji, S., 2011. Definitions of ambiguous events and the smooth ambiguity model. Econom. Theory 48 (2–3), 399–424.
Kocabiyikoğlu, A., Popescu, I., 2007. Managerial motivation dynamics and incentives. Manag. Sci. 53 (5), 834–848.
Li, J., 2009. Comparative higher-degree Ross risk aversion. Insurance Math. Econom. 45 (3), 333–336.
Maccheroni, F., Marinacci, M., Rustichini, A., 2006. Ambiguity aversion, robustness, and the variational representation of preferences. Econometrica 74 (6), 1447–1498.
Menezes, C., Geiss, C., Tressler, J., 1980. Increasing downside risk. Amer. Econ. Rev. 70 (5), 921–932.
Modica, S., Scarsini, M., 2005. A note on comparative downside risk aversion. J. Econom. Theory 122 (2), 267–271.
Nau, R., 2011. Risk, ambiguity, and state-preference theory. Econom. Theory 48 (2–3), 437–467.
Ross, S.A., 1981. Some stronger measures of risk aversion in the small and the large with applications. Econometrica 49 (3), 621–638.
Rothschild, M., Stiglitz, J.E., 1970. Increasing risk: I. A definition. J. Econom. Theory 2 (3), 225–243.
Schmeidler, D., 1989. Subjective probability and expected utility without additivity. Econometrica 57 (3), 571–587.
Siniscalchi, M., 2009. Vector expected utility and attitudes toward variation. Econometrica 77 (3), 801–855.
Snow, A., 2011. Ambiguity aversion and the propensities for self-insurance and self-protection. J. Risk Uncertain. 42 (1), 27–43.
Snow, A., 2014. Ambiguity aversion and wealth effects on demand for insurance. Working Paper.