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# Wave propagation in predator-prey systems 

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#### Abstract

In this paper, we study a class of predator-prey systems of reaction-diffusion type. Specifically, we are interested in the dynamical behaviour for the solution with the initial distribution where the prey species is at the level of the carrying capacity, and the density of the predator species has compact support, or exponentially small tails near $x= \pm \infty$. Numerical evidence suggests that this will lead to the formation of a pair of diverging waves propagating outwards from the initial zone. Motivated by this phenomenon, we establish the existence of a family of travelling waves with the minimum speed. Unlike the previous studies, we do not use the shooting argument to show this. Instead, we apply an iteration process based on Berestycki et al 2005 (Arch. Rational Mech. Anal. 178 57-80) to construct a set of super/ sub-solutions. Since the underlying system does not enjoy the comparison principle, such a set of super/sub-solutions is not based on travelling waves, and in fact the super/sub-solutions depend on each other. With the aid of the set of super/sub-solutions, we can construct the solution of the truncated problem on the finite interval, which, via the limiting argument, can in turn generate the wave solution. There are several advantages to this approach. First, it can remove the technical assumptions on the diffusivities of the species in the existing literature. Second, this approach is of PDE type, and hence it can shed some light on the spreading phenomenon indicated by numerical simulation. In fact, we can compute the spreading speed of the predator species for a class of biologically acceptable initial distributions. Third, this approach might be applied to the study of waves in non-cooperative systems (i.e. a system without a comparison principle).


Keywords: predator-prey system, travelling wave, spreading speed, super/sub-solution
Mathematics Subject Classification: 34A34, 34A35, 34A12, 35K57
(Some figures may appear in colour only in the online journal)

## 1. Introduction

Understanding of biological invasion is a topic of current interest in ecology [37, 38, 41, 42, 47]. In general, invasion of exotic species takes place via propagation of population waves separating the region (the front of the waves) where the exotic species is absent from the region (the wake of the waves) where the exotic species is present at considerable densities. To achieve a deeper understanding of such phenomena, it is believed that the ecology mechanism underlying these waves is the result of the interaction between species and the diffusion movement of species [37, 38, 41]. Hence reaction-diffusion models are commonly employed to understand these waves.

There is a family of travelling waves for a number of ecology models which are in the reaction-diffusion formulations. The existence of such a family of waves in a given model gives rise to an important question: for a given type of ecological event (i.e. for a given initial condition), which member of this family of waves can be developed? This question has been well studied for the reaction-diffusion model which enjoys the comparison principle [8,53]. For instance, the well known Fisher-KPP equation is such an example. The Fisher-KPP equation was employed by Fisher [14] and Kolmogorov, Petrovskii and Piskunov [26] to model population dynamics, and is given by

$$
u_{t}=\mathrm{d} u_{x x}+f(u),
$$

where the function $f:[0, K] \rightarrow \mathbb{R}$ is of class $C^{1}$, and satisfies $f(0)=f(K)=0, f>0$ on $(0, K)$ and the KPP assumption $f(u) \leqslant f^{\prime}(0) u$ for $u \in[0, K]$. Here $K$ can be viewed as a carrying capacity. One of the key properties of the Fisher-KPP equation is that there exists a minimal speed $c_{0}=2 \sqrt{f^{\prime}(0)}$ such that, for each $c \geqslant c_{0}$, the Fisher-KPP equation admits a travelling wave with wave speed $c$ which connects the homogeneous steady states $K$ and 0 . The travelling wave with minimal wave speed $c_{0}$ has a very interesting feature. Specifically, the compactly supported perturbation of the zero state, which can be viewed as the invasion of a small amount of exotic species, will evolve into a pair of diverging waves with the speed equal to the minimal wave speed $c_{0}$. This key feature was first established by Kolmogorov et al [26], and later characterized by Aronson and Weinberger [1, 2]. More refined information about the transition zone of the wave with the minimal speed was given by Uchiyama [50], Lau [28], Bramson [5, 6], and references therein. The travelling wave with speed $c>c_{0}$ can be generated via the evolution of the exponentially decaying perturbation of the zero state, and its speed is determined by the decay rate of the small tails at $\pm \infty$ (see [5, 6, 12, 28, 46, 50]).

Although the wave phenomena in the Fisher-KPP equation have been well understood, almost all realistic models have more than one species involved. Further, many models do not possess the comparison principle, so that the techniques employed in the Fisher-KPP equation cannot be applied to the study of the waves in these models, in particular the computation of the spreading speed of the invasion. Hence, in this paper, we would like to investigate the waves in a class of two-component prey-predator models without the comparison principle. Note that these prey-predator models were employed by Own and Lewis [42] to study the spread of recolonizing lupin plants on Mount St Helens. We will address the following two
questions: (i) the nature of waves in these population systems, and (ii) the way in which waves can be initiated via the invasion of predators into a prey population.

### 1.1. The predator-prey model

More precisely, we consider the predator-prey models, described by nonlinear growth of prey, general predator responses including Holling types I-III, predator mortality, and diffusion of prey and predator. Mathematically, the system under study is described by a nonlinear reaction-diffusion system of two equations which reads

$$
\begin{align*}
u_{t} & =\delta u_{x x}+\varrho(u)-g(u) v, \\
v_{t} & =v_{x x}+\gamma g(u) v-m v, \tag{1.1}
\end{align*}
$$

where $u$ and $v$ are the dimensionless population densities of prey and predator, respectively; $\varrho(u)$ is the growth rate of the prey population; $g(u) v$ describes predation and $g(u)$ is the functional response to the predator population; $m v$ stands for predator mortality; $\gamma$ is the prey consumption efficiency; $\delta$ is the ratio of the diffusivity of the prey to that of the predator; $x$ is the dimensionless distance and $t$ is the dimensionless time. Model (1.1) has been studied for a number of different forms of $\varrho(u)$ [34-36, 38, 40-42]. The prototype of $\varrho(u)$ can be logistic prey growth or bistable nonlinearity in the case of the Allee effect [29, 41].

Our study of system (1.1) is motivated by the following systems.

### 1.1.1. Holling type I functional response.

$$
\begin{align*}
& u_{t}=\delta u_{x x}+\alpha u\left(1-\frac{u}{K}\right)-u v,  \tag{1.2}\\
& v_{t}=v_{x x}+\gamma u v-m v,
\end{align*}
$$

where $\alpha, K, \gamma$ and $m$ are positive constants. Here $\varrho(u)=\alpha u\left(1-\frac{u}{K}\right)$ and $g(u)=u$ is the Holling type I functional response [9, 10].

### 1.1.2. Holling type II functional response.

$$
\begin{align*}
& u_{t}=\delta u_{x x}+\alpha u\left(1-\frac{u}{K}\right)-\frac{u}{1+u} v, \\
& v_{t}=v_{x x}+\gamma \frac{u}{1+u} v-m v, \tag{1.3}
\end{align*}
$$

where $\alpha, K, \gamma$ and $m$ are positive constants. Here $\varrho(u)=\alpha u\left(1-\frac{u}{K}\right)$ and $g(u)=\frac{u}{1+u}$ is the Holling type II functional response [11, 15, 17, 22].

### 1.1.3. Holling type III functional response.

$$
\begin{align*}
& u_{t}=\delta u_{x x}+\alpha u\left(1-\frac{u}{K}\right)-\frac{u^{2}}{1+u^{2}} v, \\
& v_{t}=v_{x x}+\gamma \frac{u^{2}}{1+u^{2}} v-m v, \tag{1.4}
\end{align*}
$$

where $\alpha, K, \gamma$ and $m$ are positive constants. Here $\varrho(u)=\alpha u\left(1-\frac{u}{K}\right)$ and $g(u)=\frac{u^{2}}{1+u^{2}}$ is the Holling type III functional response [30].

Table 1. The values of $(q, p, g, \gamma, \beta)$ for systems (1.2)-(1.6) corresponding to the setting of system (1.7).

| system | $q(u)$ | $p(u)$ | $g(u)$ | $\gamma$ | $\beta$ |
| :--- | :--- | :--- | :--- | :--- | :--- |
| $(1.2)$ | $\alpha u$ | $1-\frac{u}{K}$ | $u$ | $\gamma$ | $\frac{m}{\gamma}$ |
| $(1.3)$ | $\alpha u$ | $1-\frac{u}{K}$ | $\frac{u}{1+u}$ | $\gamma$ | $\frac{m}{\gamma}$ |
| $(1.4)$ | $\alpha u$ | $1-\frac{u}{K}$ | $\frac{u^{2}}{1+u^{2}}$ | $\gamma$ | $\frac{m}{\gamma}$ |
| $(1.5)$ | $\alpha u$ | $1-\frac{u}{K}$ | $1-\mathrm{e}^{-n \mu}$ | $\gamma$ | $\frac{m}{\gamma}$ |
| $(1.6)$ | $\Lambda$ | $1-\frac{\mu}{\Lambda} \cdot u$ | $\vartheta u$ | 1 | $\mu+\varpi$ |

1.1.4. Ivlev type functional response.

$$
\begin{align*}
& u_{t}=\delta u_{x x}+\alpha u\left(1-\frac{u}{K}\right)-\left(1-\mathrm{e}^{-n u}\right) v, \\
& v_{t}=v_{x x}+\gamma\left(1-\mathrm{e}^{-n u}\right) v-m v, \tag{1.5}
\end{align*}
$$

where $\alpha, K, \gamma, m$ and $n$ are positive constants. Here $\varrho(u)=\alpha u\left(1-\frac{u}{K}\right)$ and $g(u)=1-\mathrm{e}^{-n u}$ is the Ivlev type functional response $[3,7,24,25,27,31,33,44,49,51,52,54]$.

### 1.1.5. SIR model.

$$
\begin{align*}
& u_{t}=\delta u_{x x}+\Lambda-\mu u-\vartheta u v, \\
& v_{t}=v_{x x}+\vartheta u v-(\mu+\varpi) v, \tag{1.6}
\end{align*}
$$

where $\Lambda, \mu, \vartheta$, and $\varpi$ are positive constants. Note that $u$ and $v$ stand for the susceptible species and the infective species, respectively. Here $\varrho(u)=\Lambda-\mu u, g(u)=\vartheta u$ [41], and $\gamma=1$ and $m=\mu+\varpi$.

For simplicity of mathematical analysis, we write

$$
\varrho(u):=q(u) p(u) \text { and } m:=\gamma \beta .
$$

Then system (1.1) can be recast into the following system:

$$
\begin{align*}
& u_{t}=\delta u_{x x}+q(u) p(u)-g(u) v,  \tag{1.7a}\\
& v_{t}=v_{x x}+\gamma(g(u)-\beta) v . \tag{1.7b}
\end{align*}
$$

Here $\beta$ is a positive constant, and the hypotheses on $p, q$ and $g$ are given below.

## Hypotheses

(H1) $\quad q \in C^{1}([0, \infty)), q(0) \geqslant 0$, and $q>0$ and $q^{\prime} \geqslant 0$ on $(0, \infty)$.
(H2) $\quad p \in C^{1}([0, \infty)), p(K)=0$ for some positive constant $K$, and $p^{\prime}<0$ on $(0, \infty)$.
(H3) $g \in C^{2}([0, \infty)), g(0)=0, g^{\prime}(0) \geqslant 0, g\left(u_{\beta}\right)=\beta$ for some positive constant $u_{\beta}<K$, and $g^{\prime}>0$ on $(0, \infty)$.
(H4) The function $R(u):=\frac{q(u) p(u)}{g(u)}$ satisfies

$$
R(u) \geqslant R\left(u_{\beta}\right) \text { for } 0<u<u_{\beta} \text { and } R(u) \leqslant R\left(u_{\beta}\right) \text { for } u_{\beta}<u<K .
$$

Now we make comments on hypotheses (H1)-(H4). To begin with, we consider the corresponding kinetic system of system (1.7), which is system (1.7) without diffusion and is given by the following ordinary differential system:

$$
\begin{align*}
& \frac{\mathrm{d} u}{\mathrm{~d} t}=p(u) q(u)-g(u) v, \\
& \frac{\mathrm{~d} v}{\mathrm{~d} t}=\gamma(g(u)-\beta) v . \tag{1.8}
\end{align*}
$$

We first consider the case where the predator is absent. Then system (1.8) is reduced to the scalar equation

$$
\begin{equation*}
\frac{\mathrm{d} u}{\mathrm{~d} t}=p(u) q(u) . \tag{1.9}
\end{equation*}
$$

In view of the assumptions (H1)-(H2), we have that $p(K) q(K)=0, p(u) q(u)>0$ for $u \in(0, K)$ and $p(u) q(u)<0$ for $u>K$. Under this reduced condition, one can verify that $K$ is a stable equilibrium point of equation (1.9). Thus, we can conclude that, in the absence of predator species, the prey species will converge to the environmental carrying capacity state $K$. Second, we note that assumptions (H1)-(H3) imply that $p^{\prime}(K) q(K)<0$ and $g(K)-\beta>0$. Using this reduced condition, one can readily verify that the equilibrium point $(K, 0)$ is a saddle point of system (1.8) with eigenvalues $p^{\prime}(K) q(K)$ and $\gamma(g(K)-\beta)$. On the other hand, assumption (H3) implies that system (1.8) admits a unique coexistence equilibrium point

$$
E_{\beta}:=\left(u_{\beta}, v_{\beta}\right) \text { with } u_{\beta}=g^{-1}(\beta) \text { and } v_{\beta}=R\left(u_{\beta}\right) .
$$

Let $J\left(E_{\beta}\right)$ be the linearized matrix of system (1.8) at $E_{\beta}$. A straightforward computation and $(\mathbf{H} 3)-(\mathbf{H} 4)$ yield that $\operatorname{trace}\left(J\left(E_{\beta}\right)\right)=R^{\prime}\left(u_{\beta}\right) g\left(u_{\beta}\right) \leqslant 0$ and $\operatorname{det}\left(J\left(E_{\beta}\right)\right)=\gamma g\left(u_{\beta}\right) g^{\prime}\left(u_{\beta}\right) v_{\beta}>0$. Hence we can deduce that $E_{\beta}$ is a stable point of system (1.8). Together with the fact that ( $K$, 0 ) is a saddle point of system (1.8), one might expect that there is a wave solution of system (1.7) connecting the equilibria $(K, 0)$ and $E_{\beta}$.

Now we turn to the biologically acceptable initial data of system (1.7). Biological invasion often starts with a local introduction of exotic species. Hence, for the initial distribution of species, the prey is at the level of $K$, which is the carrying capacity for the prey in the absence of predation, and the density of predator species is non-zero only inside a certain bounded region. Specifically, by locally introducing a number of predators into the area which is inhabited by prey at the level of $K$, the initial data of system (1.7) can be formulated as follows:

$$
u(x, 0)=K, \quad v(x, 0)= \begin{cases}\sigma_{0} \vartheta_{0}(x), & |x| \leqslant l_{0}  \tag{1.10}\\ 0, & |x|>l_{0}\end{cases}
$$

where $\vartheta_{0}(\cdot)$ is a non-negative continuous function on $\left[-l_{0}, l_{0}\right]$ with a maximum value of unity and $\vartheta_{0}\left( \pm l_{0}\right)=0 ; \sigma_{0}$ is a positive control parameter and measures the maximum concentration of the predator introduced; $\sigma_{0} \vartheta_{0}(\cdot)$ is the predator density inside the initially invaded patch and $l_{0}$ is the patch diameter for $v[35,36,43]$.

Numerical simulations of system (1.7)-(1.10) with zero flux boundary conditions suggest that under assumptions $(\mathbf{H} \mathbf{1})-(\mathbf{H} 4)$, the corresponding solution evolves into a pair of diverging travelling waves propagating outwards from the initial zone $\left\{|x|<l_{0}\right\}$. Moreover, further numerical evidence indicates that the formation of waves is independent of the initial predator
density parameter $\sigma_{0}$. This dynamical behaviour is very similar to that of the single equation with KPP-type nonlinearity [26]. We note that, in the absence of predator, system (1.7) is reduced to the single KPP equation. Such a similarity of dynamical behaviour between system (1.7) and the KPP equation suggests that there is a family of waves of system (1.7) with minimal wave speed, and that, with the additional assumption that the $v$ component of the initial data decays exponentially as $x \rightarrow \pm \infty$, the corresponding solution of system (1.7) develops into a pair of diverging travelling waves as $t \rightarrow \infty$ moving at the speed determined by the decay rate of the $v$ component of the initial data as $x \rightarrow \pm \infty$. Motivated by these observations, in this paper we would like to analytically study two issues: (i) the existence of a family of waves of system (1.7) with minimal wave speed, and (ii) the evolution of system (1.7) with the initial data imposed by the following constraint:

$$
u(x, 0)=K, \quad v(x, 0): \begin{cases}=\sigma_{0} \vartheta_{0}(x), & |x| \leqslant l_{0}  \tag{1.11}\\ C_{1}^{ \pm} \mathrm{e}^{-\lambda^{ \pm}|x|} \leqslant v(x, 0) \leqslant C_{2}^{ \pm} \mathrm{e}^{-\lambda^{ \pm}|x|}, & \pm x>l_{0}\end{cases}
$$

where $\sigma_{0}$ and $\vartheta_{0}$ are defined as in (1.10), and $\lambda^{ \pm}, C_{1}^{ \pm}$and $C_{2}^{ \pm}$are positive constants. We note that the difference between initial data (1.11) and initial data (1.10) is that there are exponentially decaying tails for the $v$ component of initial data (1.11).

### 1.2. Main results

In this section, we would like to state the main results.
First, we consider the existence of travelling waves of system (1.7). To do this, we first briefly survey the existence of travelling waves of system (1.7). The existence of travelling waves of system (1.7) with specific nonlinearity $(\varrho, g)$ was pioneered by Dunbar [9-11], and further developed by numerous authors. Specifically, the travelling wave of system (1.2) with $\delta \in[0,1]$ was studied by Dunbar [9,10]; that of system (1.3) was established by Dunbar and by Huang et al [11, 22]; that of system (1.4) with $\delta=0$ was investigated by Li and Wu [30]; that of system (1.5) with $\delta \in[0,1]$ was studied by Hsu et al [24]. The existence of waves with a somewhat more generalized class of prey-predator interaction was recently demonstrated in [23, 24, 32]. The numerical computation of travelling waves of system (1.7) with specific nonlinearity $(\varrho, g)$ was given by Owen and Lewis [42]. We remark that previous analytical studies of travelling waves of system (1.7) make a technical assumption on the diffusion coefficient $\delta$. In this paper, we will remove such an assumption on $\delta$.

By a travelling wave solution of system (1.7), we mean a solution of system (1.7) of the form

$$
(u(x, t), v(x, t))=(U(z), V(z)), \quad z=x-c t,
$$

with the boundary conditions $(U, V)(-\infty)=\left(u_{\beta}, v_{\beta}\right)$ and $(U, V)(+\infty)=(K, 0)$. Here the wave speed $c$ is a constant to be determined and the wave profile $(U, V) \in C^{2}(\mathbb{R}) \times C^{2}(\mathbb{R})$ is a pair of non-negative functions. Upon substituting $(u, v)(x, t)=(U, V)(z)$ into (1.7), we are led to the governing system for $(U, V)$ as follows:

$$
\begin{align*}
& \delta U^{\prime \prime}+c U^{\prime}+q(U) p(U)-g(U) V=0,  \tag{1.12a}\\
& V^{\prime \prime}+c V^{\prime}+\gamma(g(U)-\beta) V=0 \tag{1.12b}
\end{align*}
$$

on $\mathbb{R}$, together with the boundary conditions

$$
\begin{equation*}
(U, V)(-\infty)=\left(u_{\beta}, v_{\beta}\right), \quad(U, V)(+\infty)=(K, 0) . \tag{1.13}
\end{equation*}
$$

Here the prime indicates differentiation with respect to $z$. Then the main result on the existence of travelling waves of system (1.7) can be stated in the following theorem.
Theorem 1.1. (existence of travelling waves)
Assume that hypotheses $\mathbf{( H 1 ) - ( \mathbf { H } 4 ) ~ h o l d . ~ T h e n ~ t h e ~ f o l l o w i n g ~ h o l d . ~}$
(I) For each $c<c_{\min }:=2 \sqrt{\gamma[g(K)-\beta]}$, there are no non-negative solutions $(U, V)$ of system (1.12)-(1.13).
(II) For each $c>c_{\text {min }}$, system (1.12)-(1.13) admits a non-negative solution ( $U, V$ ) with the following properties.
(i) $0<U<K$ and $V>0$ over $\mathbb{R}$.
(ii) There exists a $\gamma^{*}>0$ such that there hold
(a) if $\gamma \in\left(0, \gamma^{*}\right)$, then the solution $(U, V)$ approaches $\left(u_{\beta}, v_{\beta}\right)$ monotonically for large $-z$;
(b) if $\gamma>\gamma^{*}$, then the solution $(U, V)$ has exponentially damped oscillations about $\left(u_{\beta}, v_{\beta}\right)$ for large $-z$.
(iii) We have $V(z)=\mathcal{O}\left(\mathrm{e}^{-\lambda z}\right)$ as $z \rightarrow \infty$, where $\lambda$ is given by

$$
\begin{equation*}
\lambda=\lambda(c):=\frac{1}{2} \cdot\left(c-\sqrt{c^{2}-4 \gamma[g(K)-\beta]}\right) . \tag{1.14}
\end{equation*}
$$

We make three comments on the existence of travelling waves of system (1.7). First, the minimal speed $c_{\text {min }}$ of travelling waves of system (1.7) is independent of the ratio $\delta$ of the diffusion rate of $u$ to that of $v$. Second, since we do not make any assumption on the diffusion coefficient $\delta$, theorem 1.1 improves the existence results for travelling waves in the previous literature [9-11, 22-24, 30, 32]. Third, due to the lack of a uniform bound of the $v$ component of travelling wave for $c$ close to $c_{\min }$, we are unable to show the existence of critical waves (i.e. waves with speed $c=c_{\min }$ ). We left this question for our future study.

Next we discuss the asymptotic behaviour of the solution of system (1.7)-(1.11). As we mentioned before, numerical evidence indicates that the solution of system (1.7)-(1.11) develops into a pair of diverging travelling waves as $t \rightarrow \infty$ (see figure 1). To the best of our knowledge, there is no proof for convergence of the solution of system (1.7) with initial data (1.11)/ (1.10) to an expanding wave whose spread, both to the left and to the right, asymptotically takes the form of a travelling wave. Basically, there are two approaches for characterizing the expanding wave phenomenon. The first approach is the delicate construction of a super/subsolution based on travelling wave solutions [13]. However, this approach cannot be applied to system (1.7)-(1.11) because of the lack of a comparison principle for system (1.7) or any equivalent tool which allows a sharp comparison of solutions. The second approach is to look at the evolution of the leading edge, which was first introduced by Aronson and Weinberger [1, 2]. This approach cannot show convergence to an expanding wave in the usual sense. Nevertheless, it can compute the spreading speed of invasion waves. This is the approach which we use to characterize the expanding wave phenomenon of system (1.7)-(1.11).

To state the convergence result, we need additional notations. First, in view of relation (1.14), it follows from a straightforward computation that $\lambda(c)$ is defined for $c \in[2 \sqrt{\gamma[g(K)-\beta]}, \infty)$, and is decreasing in $c \in[2 \sqrt{\gamma[g(K)-\beta]}, \infty)$. Further, we have the limits $\lim _{c \rightarrow(2 \sqrt{\gamma[g(K)-\beta]})} \lambda(c)=\sqrt{\gamma[g(K)-\beta]}$ and $\lim _{c \rightarrow+\infty} \lambda(c)=0$. Taken together, the decay rate $\lambda(c) \in(0, \sqrt{\gamma[g(K)-\beta]}]$ for each admissible wave speed $c \geqslant 2 \sqrt{\gamma[g(K)-\beta]}$. Next, one can verify that relation (1.14) between the wave speed $c$ and the decay rate $\lambda$ is a one-to-one correspondence. Thus, in order to specify the dependence of wave profile ( $U, V$ )




Figure 1. The solution of system (1.2) as a function of the spatial variable $x$ is plotted at $t=0, t=10, t=20$ and $t=30$. The $u$ component of the initial data $\left(u_{0}, v_{0}\right)$ is 1 . The $v$ component of the initial data $\left(u_{0}, v_{0}\right)$ is chosen so that $v_{0}$ is of the hump shape, $v_{0} \sim \mathrm{e}^{0.4417(x-100)}$ for $x$ close to the left end and $v_{0} \sim \mathrm{e}^{-0.4417(x-100)}$ for $x$ close to the right end. The parameter values are $\delta=2, \alpha=1, K=1, \gamma=1.6$ and $m=1$.
and speed $c$ on the decay rate $\lambda=\lambda(c)$ in the statements of theorem 1.2, we denote the wave $(U, V)$ established in theorem 1.1 and the associated wave speed $c$ by $\left(U_{\lambda}, V_{\lambda}\right)$ and $c_{\lambda}$, respectively.

Now we are ready to state the result that the solution of system (1.7)-(1.11) with $\lambda^{ \pm} \in(0, \sqrt{\gamma[g(K)-\beta]})$ develops into a pair of waves propagating with speed $c_{\lambda^{ \pm}}$in the sense of Aronson and Weinberger [1, 2]:

Theorem 1.2. (Evolution of travelling waves)
Assume that hypotheses $(\mathbf{H} \mathbf{1})-(\mathbf{H} 4)$ hold. Suppose that $(u, v)$ is the solution of system (1.7)-(1.11) with $\lambda^{ \pm} \in(0, \sqrt{\gamma[g(K)-\beta]})$. Then we have the following.
(i) There exist non-negative continuous functions $\psi_{\lambda^{+}}^{+}, \phi_{\lambda^{+}}^{+}$and $\varsigma_{\lambda^{+}}^{+}$with $\lim _{x \rightarrow \infty} \psi_{\lambda^{+}}^{+}(x)=K$ and $\lim _{x \rightarrow \infty} \varsigma_{\lambda^{+}}^{+}(x)=\lim _{x \rightarrow \infty} \phi_{\lambda^{+}}^{+}(x)=0$ such that the following hold:
(a) $\psi_{\lambda^{+}}^{+}$vanishes in $\left(-\infty, \zeta_{1}^{+}\right]$and is strictly increasing in $\left[\zeta_{1}^{+}, \infty\right)$ for some constant $\zeta_{1}^{+}>0$, and

$$
K \geqslant u(x, t) \geqslant \psi_{\lambda^{+}}^{+}\left(x-c_{\lambda^{+}} t\right), \quad \forall(x, t) \in \mathbb{R} \times[0, \infty)
$$

(b) $\varsigma_{\lambda^{+}}^{+}$is strictly decreasing in $\mathbb{R}$, $\phi_{\lambda^{+}}^{+}$vanishes in $\left(-\infty, \xi_{1}^{+}\right.$, and is strictly increasing in $\left[\xi_{1}^{+}, \xi_{2}^{+}\right]$and strictly decreasing in $\left[\xi_{2}^{+}, \infty\right)$ for some constants $\xi_{2}^{+}>\xi_{1}^{+}>0$, and

$$
\varsigma_{\lambda^{+}}^{+}\left(x-c_{\lambda^{+}} t\right) \geqslant v(x, t) \geqslant \phi_{\lambda^{+}}^{+}\left(x-c_{\lambda^{+}} t\right), \quad \forall(x, t) \in \mathbb{R} \times[0, \infty) .
$$

(ii) There exist non-negative continuous functions $\psi_{\lambda^{-}}^{-}, \phi_{\lambda^{-}}^{-}$and $\varsigma_{\lambda^{-}}^{-}$with $\lim _{x \rightarrow-\infty} \psi_{\lambda^{-}}^{-}(x)=K$ and $\lim _{x \rightarrow-\infty} \varsigma_{\lambda}^{-}-(x)=\lim _{x \rightarrow-\infty} \phi_{\lambda}^{-}(x)=0$ such that the following hold:
(a) $\psi_{\lambda^{-}}^{-}$vanishes in $\left[\zeta_{1}^{-}, \infty\right]$ and is strictly decreasing in $\left(-\infty, \zeta_{1}^{-}\right]$for some constant $\zeta_{1}^{-}<0$, and

$$
K \geqslant u(x, t) \geqslant \psi_{\lambda}^{-}-\left(x+c_{\lambda}-t\right), \quad \forall(x, t) \in \mathbb{R} \times[0, \infty) ;
$$

(b) $\varsigma_{\lambda^{-}}^{-}$is strictly decreasing in $\mathbb{R}, \phi_{\lambda^{-}}^{-}$vanishes in $\left[\xi_{2}^{-}, \infty\right)$, and is strictly decreasing in $\left[\xi_{1}^{-}, \xi_{2}\right]$ and strictly increasing in $\left(-\infty, \xi_{1}^{-}\right]$for some constants $\xi_{1}^{-}<\xi_{2}^{-}<0$, and

$$
\varsigma_{\lambda}^{-}-\left(x+c_{\lambda}^{-} t\right) \geqslant v(x, t) \geqslant \phi_{\lambda}^{-}\left(x+c_{\lambda}^{-} t\right), \quad \forall(x, t) \in \mathbb{R} \times[0, \infty) .
$$

We make two remarks. First, recall that $\lim _{x \rightarrow \infty} \psi_{\lambda^{+}}^{+}(x)=K$ and $\lim _{x \rightarrow \infty} \varsigma_{\lambda^{+}}^{+}(x)=\lim _{x \rightarrow \infty}$ $\phi_{\lambda^{+}}^{+}(x)=0$. Hence, roughly speaking, assertion (i) of theorem 1.2 states that if one is in the moving coordinate $z=x-c_{\lambda^{+}} t$ with $x \geqslant 0$, then one will see a transition zone connecting a constant steady state $(K, 0)$ and a state sandwiched by $\left(K, \varsigma_{\lambda^{+}}^{+}\right)$and $\left(\psi_{\lambda^{+}}^{+}, \phi_{\lambda^{+}}^{+}\right)$, while assertion (ii) of theorem 1.2 states that if one is in the moving coordinate $z=x+c_{\lambda}-t$ with $x \leqslant 0$, then one will see a transition zone connecting a constant steady state $(K, 0)$ and a state sandwiched by ( $K, \varsigma_{\lambda^{-}}^{-}$) and ( $\psi_{\lambda^{-}}^{-}, \phi_{\lambda^{-}}^{-}$). Taken together, the aforementioned observations suggest that the long time behaviour of the solution $(u, v)$ of the initial value problem (1.7)-(1.11) is a pair of diverging travelling waves whose speed is determined by the decay rate of the initial data as $x \rightarrow \pm \infty$ (see figure 1). From this, we may infer that $(u(x, t), v(x, t)) \rightarrow\left(u_{\beta}, v_{\beta}\right)$ as $t \rightarrow \infty$ for any given $x \in \mathbb{R}$. Second, we note that theorem 1.2 does not address the long time behaviour of the solution $(u, v)$ of initial value problem (1.7)-(1.10). Nevertheless, the conclusion of theorem 1.2 suggests that system (1.7) possesses the wave propagation feature of a KPP-type equation.

Finally, we outline the method for the proof of the main results. To begin with, we note that previous studies on the existence of travelling waves of system (1.7) are based on the shooting arguments [ $9-11,22-24,30,32$. The central point of this approach is the construction of the invariant set, which is nontrivial and strongly depends on the nonlinearity. It is clear that this approach cannot give any information for the evolution of the solution of initial value problem (1.7)-(1.11). In this paper, we will use the PDE approach, instead of shooting arguments, to establish the existence of travelling waves of system (1.7). This approach can also shed some light on the convergence result of the solution of system (1.7)-(1.11). Specifically, the proofs for theorems 1.1 and 1.2 are both based on a pair of coupled super/ sub-solutions of system (1.7). Note that system (1.7) does not enjoy the comparison principle. Hence travelling waves of system (1.7) do not qualify as super/sub-solutions (unless they coincide), and so we cannot bound solutions of system (1.7) componentwise by translates of travelling waves. We note that, due to the lack of a comparison principle for system (1.7), the construction of the sub-solution is based on the super-solution. With the aid of a family of truncated problems whose solutions can be proven to be sandwiched between this pair of coupled super/sub-solutions, one can establish the existence of a family of travelling waves with the minimal speed of system (1.7) through the limiting process. Note that, due to the unbounded property of the $v$ component of the super-solution, one of the key points for the existence of waves is to show that the $v$ component of waves is bounded. We remark that the idea of the framework of the proof for the existence of waves is based on [4]. Next, via the
comparison principle for a single equation, the solution of system (1.7)-(1.11) can be shown to be squeezed between the pair of coupled super/sub-solutions, from which the assertions of theorem 1.2 can be proven.

The remaining parts of this paper are organized as follows. In section 2, we first construct the coupled pairs of super/sub-solutions, and then derive the solution of the truncated problem of system (1.7) on the finite interval $[-l, l]$. Finally, by passing to the limit $l \rightarrow \infty$, we obtain a solution $(U, V)$ of system (1.7) on $\mathbb{R}$ with the condition $(U, V)(\infty)=(K, 0)$, which serves as a candidate for travelling waves of system (1.7). Section 3 is then to establish that the candidate ( $U, V$ ) obtained in section 2 is indeed a travelling wave solution of system (1.7). The two key points of the proof are the estimates of the derivative of $(U, V)$ and the boundedness of the $V$ component. The long time behaviour of system (1.7)-(1.11) is investigated in section 4. In section 5, we give a short discussion and conclusion. Finally, some auxiliary lemmas are given in the appendix.

## 2. Property of waves and construction of a candidate for travelling waves

### 2.1. The minimal speed and decay rate of waves

To begin with, we establish the assertion of theorem 1.1 (I) and the decay rate of the $v$ component of the waves near infinity.
Lemma 2.1. Suppose that hypotheses $(\mathbf{H} \mathbf{1})-(\mathbf{H} 4)$ hold. Then the following statements are valid.
(i) For $c<c_{\text {min }}$, there exist no non-negative solutions of system (1.12)-(1.13).
(ii) For $c>c_{\text {min }}$, if $(U, V)$ is a non-negative solution of system (1.12)-(1.13), then $V(z)=\mathcal{O}\left(\mathrm{e}^{-\lambda z}\right)$ as $z \rightarrow \infty$, where $\lambda$ is given by

$$
\lambda=\frac{1}{2}\left(c \pm \sqrt{c^{2}-4 \gamma[g(K)-\beta]}\right) .
$$

Proof. Linearizing (1.12) around $(K, 0)$ leads to the equations

$$
\begin{align*}
& \delta u^{\prime \prime}+c u^{\prime}+q(K) p^{\prime}(K) u-g(K) v=0,  \tag{2.1a}\\
& v^{\prime \prime}+c v^{\prime}+\gamma(g(K)-\beta) v=0 . \tag{2.1b}
\end{align*}
$$

Note that (2.1b) has two eigenvalues, $-\lambda_{1}$ and $-\lambda_{2}$, where
$\lambda_{1}:=\frac{1}{2}\left(c-\sqrt{c^{2}-4 \gamma[g(K)-\beta]}\right), \quad \lambda_{2}:=\frac{1}{2}\left(c+\sqrt{c^{2}-4 \gamma[g(K)-\beta]}\right)$.

Suppose that $c \leqslant-2 \sqrt{\gamma[g(K)-\beta]}$. Then we have $\lambda_{i}<0, i=1,2$, and so $V(z)$ is unbounded as $z \rightarrow \infty$, which is a contradiction. Therefore, we have $c>-2 \sqrt{\gamma[g(K)-\beta]}$. On the other hand, if $|c|<2 \sqrt{\gamma[g(K)-\beta]}$ holds, then $\lambda_{1}$ and $\lambda_{2}$ form a complex conjugate pair. This would imply that $V(z)$ cannot be of the same sign for $z$ near infinity, a contradiction again. Hence we have $c \geqslant 2 \sqrt{\gamma[g(K)-\beta]}$, which completes the proof of assertion (i). Assertion (ii) follows from the above linearized equation and the definitions of $\lambda_{1}$ and $\lambda_{2}$. The proof of this lemma is thus completed.

### 2.2. Super/sub-solutions

In this section, we will construct a set of a super-solution $\left(U_{\lambda}^{+}, V_{\lambda}^{+}\right)$and a sub-solution $\left(U_{\lambda}^{-}, V_{\lambda}^{-}\right)$. Note that we will not specify the system to which the set of super/sub-solutions ( $U_{\lambda}^{ \pm}, V_{\lambda}^{ \pm}$) is linked. This may be due to the following two observations. First, the set of super/ sub-solutions ( $U_{\lambda}^{ \pm}, V_{\lambda}^{ \pm}$) may apply to system (1.7) with a special class of initial data, not with every class of initial data. (For more detail, we refer the reader to section 4, where we use the set of super/sub-solutions $\left(U_{\lambda}^{ \pm}, V_{\lambda}^{ \pm}\right)$to study the convergence of the solution of system (1.7)-(1.11).) Second, the set of super/sub-solutions ( $U_{\lambda}^{ \pm}, V_{\lambda}^{ \pm}$) also applies to the inhomogeneous linear boundary value problem (see system (2.11) in section 2.3) associated with the truncated problem of system (1.7) whose solutions can generate a candidate for travelling waves via the limiting argument. Based on these two observations, we do not give the specific system to which the set of super/sub-solutions ( $U_{\lambda}^{ \pm}, V_{\lambda}^{ \pm}$) are linked, and hope that it will be clear from the context exactly which system we are discussing at the time.

Next, we make one remark on the construction of the set of super/sub-solutions $\left(U_{\lambda}^{ \pm}, V_{\lambda}^{ \pm}\right)$. The idea of such a construction is motivated by [4]. Specifically, we first construct a supersolution $V_{\lambda}^{+}$for the $V$ component, which is immediately employed to construct a sub-solution $U_{\lambda}^{-}$for the $U$ component. The sub-solution $U_{\lambda}^{-}$is in turn used to generate a sub-solution $V_{\lambda}^{-}$for the $V$ component. The super-solution $U_{\lambda}^{+}$for the $U$ component is always chosen as the constant $K$. Finally, throughout the remainder of this section, we always assume that hypotheses (H1)-(H3) hold and $c>2 \sqrt{\gamma[g(K)-\beta]}$.

Now we will construct super/sub-solutions. For simplicity, we set

$$
P(s):=s^{2}-c s+\gamma[g(K)-\beta] .
$$

Since $c>2 \sqrt{\gamma[g(K)-\beta]}$, the equation $P(s)=0$ has two positive roots, $\lambda$ and $\lambda+d$, where

$$
\lambda=\frac{1}{2}\left(c-\sqrt{c^{2}-4 \gamma[g(K)-\beta]}\right) \text { and } d=\sqrt{c^{2}-4 \gamma[g(K)-\beta]} .
$$

In addition, $P(s)<0$ when $s \in(\lambda, \lambda+d)$.
Lemma 2.2. For a fixed $x_{0} \in \mathbb{R}$, the function $V_{\lambda}^{+}\left(z ; x_{0}\right):=\mathrm{e}^{-\lambda\left(z-x_{0}\right)}$ satisfies the equation

$$
\begin{equation*}
\left(V_{\lambda}^{+}\left(z ; x_{0}\right)\right)^{\prime \prime}+c\left(V_{\lambda}^{+}\left(z ; x_{0}\right)\right)^{\prime}+\gamma[g(K)-\beta] V_{\lambda}^{+}\left(z ; x_{0}\right)=0 \tag{2.2}
\end{equation*}
$$

for all $z \in \mathbb{R}$, where the prime denotes differentiation with respect to $z$.
Proof. Since $P(\lambda)=0$, it follows that

$$
\begin{aligned}
& \left(V_{\lambda}^{+}\left(z ; x_{0}\right)\right)^{\prime \prime}+c\left(V_{\lambda}^{+}\left(z ; x_{0}\right)\right)^{\prime}+\gamma[g(K)-\beta] V_{\lambda}^{+}\left(z ; x_{0}\right)=P(\lambda) V_{\lambda}^{+}\left(z ; x_{0}\right)=0, \forall z \in \mathbb{R} . \\
& \quad \text { Select } 0<\alpha<\min \{c / \delta, \lambda\} \text {. Then } c-\delta \alpha>0 \text { and } \alpha-\lambda<0 \text {. Since } \mathrm{e}^{-\alpha z} \rightarrow 0 \text { and } \mathrm{e}^{(\alpha-\lambda) z} \rightarrow 0 \\
& \text { as } z \rightarrow \infty, \text { there exists } z_{0}>0 \text { such that }
\end{aligned}
$$

$$
\mathrm{e}^{-\alpha z}<K \text { and } \mathrm{e}^{(\alpha-\lambda) z} \leqslant \alpha(c-\delta \alpha) / g(K) \cdot \mathrm{e}^{-\lambda x_{0}}, \forall z \geqslant z_{0},
$$

so

$$
\begin{equation*}
(c-\delta \alpha) \alpha \mathrm{e}^{-\alpha z} \geqslant g(K) \cdot V_{\lambda}^{+}\left(z ; x_{0}\right), \forall z \geqslant z_{0}, \tag{2.3}
\end{equation*}
$$

and

$$
\begin{equation*}
M=M\left(x_{0}\right):=K \mathrm{e}^{\alpha z_{0}}>1 \tag{2.4}
\end{equation*}
$$

In the following, we retain the notation $z_{0}$.
Lemma 2.3. The function $U_{\lambda}^{-}\left(z ; x_{0}\right):=\max \left\{0, K-M \mathrm{e}^{-\alpha z}\right\}$ satisfies the inequality
$\delta\left(U_{\lambda}^{-}\left(z ; x_{0}\right)\right)^{\prime \prime}+c\left(U_{\lambda}^{-}\left(z ; x_{0}\right)\right)^{\prime}+q\left(U_{\lambda}^{-}\left(z ; x_{0}\right)\right) p\left(U_{\lambda}^{-}\left(z ; x_{0}\right)\right)-g\left(U_{\lambda}^{-}\left(z ; x_{0}\right)\right) V_{\lambda}^{+}\left(z ; x_{0}\right) \geqslant 0$
for all $z \neq z_{0}$.
Proof. For $z<z_{0}$, since $U_{\lambda}^{-}\left(z ; x_{0}\right) \equiv 0$ in $\left(-\infty, z_{0}\right)$ and $g(0)=0$, inequality (2.5) follows. For $z>z_{0}, U_{\lambda}^{-}\left(z ; x_{0}\right)=K-M \mathrm{e}^{-\alpha z}$. Noting that $q\left(U_{\lambda}^{-}\left(z ; x_{0}\right)\right) p\left(U_{\lambda}^{-}\left(z ; x_{0}\right)\right) \geqslant 0$, we can use (2.3) and (2.4), and the fact that $g\left(U_{\lambda}^{-}\left(z ; x_{0}\right)\right) \leqslant g(K)$, to deduce that

$$
\begin{aligned}
& \delta\left(U_{\lambda}^{-}\left(z ; x_{0}\right)\right)^{\prime \prime}+c\left(U_{\lambda}^{-}\left(z ; x_{0}\right)\right)^{\prime}+q\left(U_{\lambda}^{-}\left(z ; x_{0}\right)\right) p\left(U_{\lambda}^{-}\left(z ; x_{0}\right)\right) \\
\geqslant & M \alpha(c-\delta \alpha) \mathrm{e}^{-\alpha z} \\
\geqslant & g(K) V_{\lambda}^{+}\left(z ; x_{0}\right) \\
\geqslant & g\left(U_{\lambda}^{-}\left(z ; x_{0}\right)\right) V_{\lambda}^{+}\left(z ; x_{0}\right) .
\end{aligned}
$$

Hence (2.5) holds.
Choose $0<\eta<\min \{\alpha, d\}$. Then $\eta-\alpha<0$ and $P(\lambda+\eta)<0$. For a fixed $x_{1} \in \mathbb{R}$, select

$$
\begin{equation*}
L=L\left(x_{1}, x_{0}\right)>\max \left\{\frac{M}{K} \mathrm{e}^{\lambda x_{1}},-\frac{g^{\prime}(K) M+2 K_{1}}{P(\lambda+\eta)} \gamma \mathrm{e}^{\lambda x_{1}}\right\}, \tag{2.6}
\end{equation*}
$$

where $K_{1}=\max _{0 \leqslant u \leqslant K}\left|g^{\prime \prime}(u)\right| M^{2} / 2$. Set $z_{1}=\left(\ln L-\lambda x_{1}\right) / \eta$. Then $z_{1}>z_{0}>0$ since $z_{0}=\frac{1}{\alpha} \ln \frac{M}{K}$, $L>\frac{M}{K} \mathrm{e}^{\lambda \lambda_{1}}$, and $\eta<\alpha$. In the following, we retain the notation $z_{1}$ and $L$.
Lemma 2.4. The function $V_{\lambda}^{-}\left(z ; x_{1}, x_{0}\right):=\max \left\{0, V_{\lambda}^{+}\left(z ; x_{1}\right)-L \mathrm{e}^{-(\lambda+\eta) z}\right\}$ satisfies the inequality

$$
\begin{equation*}
\left(V_{\lambda}^{-}\left(z ; x_{1}, x_{0}\right)\right)^{\prime \prime}+c\left(V_{\lambda}^{-}\left(z ; x_{1}, x_{0}\right)\right)^{\prime}+\gamma\left[g\left(U_{\lambda}^{-}\left(z ; x_{0}\right)\right)-\beta\right] V_{\lambda}^{-}\left(z ; x_{1}, x_{0}\right) \geqslant 0 \tag{2.7}
\end{equation*}
$$

for all $z \neq z_{1}$.
Proof. For $z<z_{1}$, inequality (2.7) holds immediately since $V_{\lambda}^{-}\left(z ; x_{1}, x_{0}\right) \equiv 0$ in $\left(-\infty, z_{1}\right)$. For $z>z_{1}, V_{\lambda}^{-}\left(z ; x_{1}, x_{0}\right)=V_{\lambda}^{+}\left(z ; x_{1}\right)-L \mathrm{e}^{-(\lambda+\eta) z}$ and $U_{\lambda}^{-}\left(z ; x_{0}\right)=K-M \mathrm{e}^{-\alpha z}$. A simple computation gives that

$$
\begin{aligned}
& \left(V_{\lambda}^{-}\left(z ; x_{1}, x_{0}\right)\right)^{\prime}=\left(V_{\lambda}^{+}\left(z ; x_{1}\right)\right)^{\prime}+(\lambda+\eta) L \mathrm{e}^{-(\lambda+\eta) z} \\
& \left(V_{\lambda}^{-}\left(z ; x_{1}, x_{0}\right)\right)^{\prime \prime}=\left(V_{\lambda}^{+}\left(z ; x_{1}\right)\right)^{\prime \prime}-(\lambda+\eta)^{2} L \mathrm{e}^{-(\lambda+\eta) z}
\end{aligned}
$$

From (H3), we can use Taylor's theorem to deduce that

$$
g\left(U_{\lambda}^{-}\left(z ; x_{0}\right)\right)=g(K)-g^{\prime}(K) M \mathrm{e}^{-\alpha z}+R_{1}(z)
$$

with

$$
\begin{equation*}
\left|R_{1}(z)\right| \leqslant K_{1} \mathrm{e}^{-2 \alpha z} . \tag{2.8}
\end{equation*}
$$

It follows that

$$
\begin{aligned}
& \left(g\left(U_{\lambda}^{-}\left(z ; x_{0}\right)\right)-\beta\right) V_{\lambda}^{-}\left(z ; x_{1}, x_{0}\right) \\
= & \left(g(K)-\beta-g^{\prime}(K) M \mathrm{e}^{-\alpha z}+R_{1}(z)\right)\left(V_{\lambda}^{+}\left(z ; x_{1}\right)-L \mathrm{e}^{-(\lambda+\eta) z}\right) \\
= & (g(K)-\beta) V_{\lambda}^{+}\left(z ; x_{1}\right)-(g(K)-\beta) L \mathrm{e}^{-(\lambda+\eta) z}+R_{2}(z),
\end{aligned}
$$

where
$R_{2}(z)=-g^{\prime}(K) M \mathrm{e}^{\lambda x_{1}} \mathrm{e}^{-(\lambda+\alpha) z}+g^{\prime}(K) M L \mathrm{e}^{-(\lambda+\alpha+\eta) z}+R_{1}(z) \mathrm{e}^{-\lambda\left(z-x_{1}\right)}-L R_{1}(z) \mathrm{e}^{-(\lambda+\eta) z}$.

Since $z>z_{1}>0$ and $L=\mathrm{e}^{\eta z_{1}+\lambda x_{1}}$, it follows that $L \leqslant \mathrm{e}^{\eta z+\lambda x_{1}}$. This, together with (2.8) and the fact that $g^{\prime}(K)>0$ and $\eta-\alpha<0$, gives that

$$
\begin{aligned}
& R_{2}(z) \mathrm{e}^{(\lambda+\eta) z} \\
\geqslant & -g^{\prime}(K) M \mathrm{e}^{\lambda x_{1}} \mathrm{e}^{(\eta-\alpha) z}-K_{1} \mathrm{e}^{\lambda x_{1}} \mathrm{e}^{(\eta-2 \alpha) z}-K_{1} \mathrm{e}^{\lambda x_{1}} \mathrm{e}^{(\eta-2 \alpha) z} \\
\geqslant & -\left(g^{\prime}(K) M+2 K_{1}\right) \mathrm{e}^{\lambda \lambda_{1}} .
\end{aligned}
$$

Together with (2.2) and (2.6), and the definition of $P$, we obtain

$$
\begin{aligned}
& \left(V_{\lambda}^{-}\left(z ; x_{1}, x_{0}\right)\right)^{\prime \prime}+c\left(V_{\lambda}^{-}\left(z ; x_{1}, x_{0}\right)\right)^{\prime}+\gamma\left[g\left(U_{\lambda}^{-}\left(z ; x_{0}\right)\right)-\beta\right] V_{\lambda}^{-}\left(z ; x_{1}, x_{0}\right) \\
\geqslant & \mathrm{e}^{-(\lambda+\eta) z}\left[-P(\lambda+\eta) L-\gamma\left(g^{\prime}(K) M+2 K_{1}\right) \mathrm{e}^{\lambda_{x_{1}}}\right] \\
\geqslant & 0
\end{aligned}
$$

The proof of this lemma is therefore completed.

### 2.3. A truncated problem

In this section, we will use the super/sub solutions $V_{\lambda}^{+}\left(\cdot ; x_{0}\right), U_{\lambda}^{-}\left(\cdot ; x_{0}\right)$, and $V_{\lambda}^{-}\left(\cdot ; x_{1}, x_{0}\right)$ with $x_{0}=x_{1}=0$ established in section 2.2 to construct the solutions of the truncated problem of system (1.12)-(1.13). With the aid of the solution of the truncated problem, we can use the limiting process to obtain a solution $(U, V)$ of system (1.12) satisfying $(U, V)(+\infty)=(K, 0)$ which may serve as a good candidate for travelling wave solutions of system (1.7).

For simplicity, we write $U_{\lambda}^{-}(z ; 0), V_{\lambda}^{-}(z ; 0,0)$, and $V_{\lambda}^{+}(z ; 0)$ as $U^{-}(z), V^{-}(z)$, and $V^{+}(z)$, respectively. Let $l>z_{1}$. We consider the following truncated problem:

$$
\begin{equation*}
\delta U^{\prime \prime}+c U^{\prime}+q(U) p(U)-g(U) V=0 \quad \text { in }(-l, l), \tag{2.9a}
\end{equation*}
$$

$$
\begin{equation*}
V^{\prime \prime}+c V^{\prime}+\gamma[g(U)-\beta] V=0 \quad \text { in }(-l, l), \tag{2.9b}
\end{equation*}
$$

together with the boundary conditions

$$
\begin{equation*}
(U, V)(-l)=\left(U^{-}, V^{-}\right)(-l), \quad(U, V)(l)=\left(U^{-}, V^{-}\right)(l) . \tag{2.10}
\end{equation*}
$$

In the remainder of this section, we will apply the Schauder fixed point theorem to show the existence of solutions of (2.9)-(2.10). For convenience, we set $I_{l}:=[-l, l]$ and $X:=C\left(I_{l}\right) \times C\left(I_{l}\right)$. To begin with, we introduce the set

$$
E:=\left\{(U, V) \in X \mid U^{-} \leqslant U \leqslant U^{+} \equiv K \text { and } V^{-} \leqslant V \leqslant V^{+} \text {in } I_{l}\right\},
$$

which is a closed convex set in the Banach space $X$ equipped with the norm $\left\|\left(f_{1}, f_{2}\right)\right\|_{X}=\left\|f_{1}\right\|_{C(I)}+\left\|f_{2}\right\|_{C(I)}$. Since $U^{-}$and $V^{-}$are non-negative, it follows that $U \geqslant 0$ and $V \geqslant 0$ for any $(U, V) \in E$. Next, we define the mapping $\mathcal{F} E \rightarrow E$ as follows: given $\left(U_{0}, V_{0}\right) \in E$,

$$
\mathcal{F}\left(U_{0}, V_{0}\right):=(U, V),
$$

where $(U, V)$ is the classical solution of the boundary value problem

$$
\begin{align*}
& \delta U^{\prime \prime}+c U^{\prime}+q\left(U_{0}\right) p(U)-g(U) V_{0}=0 \quad \text { in }(-l, l),  \tag{2.11a}\\
& V^{\prime \prime}+c V^{\prime}+\gamma\left[g\left(U_{0}\right) V_{0}-\beta V\right]=0 \quad \text { in }(-l, l)  \tag{2.11b}\\
& (U, V)(-l)=\left(U^{-}, V^{-}\right)(-l), \quad(U, V)(l)=\left(U^{-}, V^{-}\right)(l) . \tag{2.11c}
\end{align*}
$$

Hence by the definition of $\mathcal{F}$, one can see that any fixed point of $\mathcal{F}$ is a classical solution of the problem (2.9)-(2.10). In the remaining part of this section, we will verify that the mapping $\mathcal{F}$ satisfies the condition of the Schauder fixed point theorem.
Lemma 2.5. The mapping $\mathcal{F}$ is well defined; that is, for a given $\left(U_{0}, V_{0}\right) \in E$, there exists a unique solution $(U, V)$ to the boundary value problem (2.11). Moreover, $U^{-} \leqslant U \leqslant U^{+}$and $V^{-} \leqslant V \leqslant V^{+}$in $I_{l}$.

Proof. Since $l>z_{1}>z_{0}>0>-l$, definition of $U^{-}$and $V^{-}$implies that $U^{-}(-l)=0$, $V^{-}(-l)=0,0<U^{-}(l)<K$, and $V^{-}(l)>0$. Noting that system (2.11) is not a coupled system, we can deal with the existence and uniqueness of $U$ and $V$ separately.

First, observing that the equation for $V$ is an inhomogeneous linear equation, the existence and uniqueness of $V$ can be easily obtained by [19, theorem 3.1 of chapter 12]. Moreover, since $V^{\prime \prime}+c V^{\prime}-\gamma \beta V=-\gamma g\left(U_{0}\right) V_{0} \leqslant 0$ on $(-l, l)$ and $V( \pm l)=V^{-}( \pm l) \geqslant 0$, it follows from the maximum principle that $V>0$ over $(-l, l)$.

Now we claim that $V^{-} \leqslant V \leqslant V^{+}$on $I_{l}$. Since $0 \leqslant U^{-} \leqslant U_{0} \leqslant U^{+} \equiv K$, the monotonicity of $g$ implies that $0 \leqslant g\left(U^{-}\right) \leqslant g\left(U_{0}\right) \leqslant g(K)$. Together with the fact that $0 \leqslant V^{-} \leqslant V_{0} \leqslant V^{+}$, we obtain that

$$
g\left(U^{-}\right) V^{-} \leqslant g\left(U_{0}\right) V_{0} \leqslant g(K) V^{+},
$$

so that

$$
\begin{equation*}
V^{\prime \prime}+c V^{\prime}+\gamma\left[g\left(U^{-}\right) V^{-}-\beta V\right] \leqslant 0 \tag{2.12}
\end{equation*}
$$

and

$$
\begin{equation*}
V^{\prime \prime}+c V^{\prime}+\gamma\left[g(K) V^{+}-\beta V\right] \geqslant 0 \tag{2.13}
\end{equation*}
$$

for all $z$ in $(-l, l)$. Now we consider the function $w_{1}=V-V^{-}$. From (2.11c) and the fact $V\left(z_{1}\right)>0$ and $V^{-}\left(z_{1}\right)=0$, we know that $w_{1}\left(z_{1}\right)>0$ and $w_{1}(l)=0$. In addition, (2.7) and (2.12) give that $w_{1}^{\prime \prime}(z)+c w_{1}^{\prime}(z)-\gamma \beta w_{1}(z) \leqslant 0$ for all $z \in\left(z_{1}, l\right)$. Then it follows from the maximum principle that $w_{1} \geqslant 0$ in $\left[z_{1}, l\right]$. This implies that $V^{-} \leqslant V$ in $\left[z_{1}, l\right]$. Together with the fact that $V^{-} \equiv 0 \leqslant V$ in $\left[-l, z_{1}\right]$, we get $V^{-} \leqslant V$ in $I_{l}$. Similarly, we can use (2.13) and the maximum principle to deduce that $V \leqslant V^{+}$in $I_{l}$.

To show the existence of $U$, we first introduce the cut-off function $\phi$ defined by

$$
\phi(\xi):= \begin{cases}0 & \text { for } \xi \leqslant 0 \\ \xi & \text { for } 0<\xi<K, \\ K & \text { for } \xi \geqslant K\end{cases}
$$

and consider the following initial value problem:

$$
\begin{align*}
& \delta U^{\prime \prime}+c U^{\prime}+q\left(U_{0}\right) \tilde{p}(U)-\tilde{g}(U) V_{0}=0,  \tag{2.14a}\\
& U(-l)=\left(U^{-}\right)(-l), \quad U^{\prime}(-l)=m, \tag{2.14b}
\end{align*}
$$

where $\tilde{p}:=p \circ \phi, \tilde{g}:=g \circ \phi$, and $m$ is a constant. For each fixed $m$, noting that the functions $\tilde{p}$ and $\tilde{g}$ satisfy the Lipschitz condition and are bounded over $\mathbb{R}$, and the functions $q\left(U_{0}(\cdot)\right)$ and $V_{0}(\cdot)$ are continuous over $I_{l}$, the initial value problem (2.14) has a unique solution $U(z, m)$ on $I_{l}$. In addition, one can easily deduce from (2.14a) that

$$
\left(\mathrm{e}^{c z / \delta} U^{\prime}(z, m)\right)^{\prime}=-\frac{1}{\delta} \mathrm{e}^{c z / \delta}\left[q\left(U_{0}(z)\right) \tilde{p}(U(z, m))-\tilde{g}(U(z, m)) V_{0}(z)\right],
$$

where the prime denotes differentiation with respect to $z$. Integrating the above equation, we obtain
$U^{\prime}(z, m)=m \mathrm{e}^{-c(l+z) / \delta}-\frac{1}{\delta} \int_{-l}^{z}\left[q\left(U_{0}(\tau)\right) \tilde{p}(U(\tau, m))-\tilde{g}(U(\tau, m)) V_{0}(\tau)\right] \mathrm{e}^{c(\tau-z) / \delta} \mathrm{d} \tau$.

Then an integration of the above equation from $-l$ to $l$ gives
$U(l, m)=\frac{m \delta}{c}\left(1-\mathrm{e}^{-2 c l / \delta}\right)-\frac{1}{\delta} \int_{-l}^{l} \int_{-l}^{z}\left[q\left(U_{0}(\tau)\right) \tilde{p}(U(\tau, m))-\tilde{g}(U(\tau, m)) V_{0}(\tau)\right] \mathrm{e}^{c(\tau-z) / \delta} \mathrm{d} \tau \mathrm{d} z$,
where we have used $U(-l, m)=\left(U^{-}\right)(-l)=0$. Note that the boundedness of $\tilde{p}, \tilde{g}, q\left(U_{0}\right)$, and $V_{0}$ implies that the integral of the above equation is bounded by a constant independent of $m$. Hence, $U(l, m)>U^{-}(l)$ if $m$ is sufficiently large, while $U(l, m)<U^{-}(l)$ if $-m$ is sufficiently large. Since $U(z, m)$ is a continuous function of $m$, there exists $m^{*}>0$ such that $U\left(l, m^{*}\right)=U^{-}(l)$. Then $U(z):=U\left(z, m^{*}\right)$ is a solution of (2.14a) with $U( \pm l)=U^{-}( \pm l)$. Moreover, we have $U>0$ over $(-l, l)$. To see this, recalling that $\tilde{g}(0)=0$, we can rewrite $\tilde{g}(U(z))$ as $\tilde{g}(U(z))=H_{1}(z) U(z)$, where

$$
H_{1}(z):= \begin{cases}\frac{[\tilde{g}(U(z))-\tilde{g}(0)]}{U(z)-0}, & \text { if } U(z) \neq 0, \\ 0, & \text { if } U(z)=0\end{cases}
$$

is a non-negative function since $\tilde{g}$ is non-decreasing. Because $\delta U^{\prime \prime}+c U^{\prime}-H_{1}(z) U V_{0}=$ $\delta U^{\prime \prime}+c U^{\prime}-\tilde{g}(U) V_{0}=-q\left(U_{0}\right) \tilde{p}(U) \leqslant 0$ on $(-l, l)$ and $U( \pm l)=U^{-}( \pm l) \geqslant 0$, it follows from the maximum principle that $U>0$ over $(-l, l)$.

Now we claim that $U^{-} \leqslant U \leqslant U^{+}$in $I_{l}$. To show that $U^{-} \leqslant U$ in $I_{l}$, we consider the function

$$
\begin{aligned}
& H_{2}(z): \\
& \quad= \begin{cases}\frac{\left[\tilde{p}(U(z))-\tilde{p}\left(U^{-}(z)\right)\right] q\left(U^{-}(z)\right)-\left[\tilde{g}(U(z))-\tilde{g}\left(U^{-}(z)\right)\right] V^{+}(z)}{U(z)-U^{-}(z)}, & \text { if } U(z) \neq U^{-}(z), \\
0, & \text { if } U(z)=U^{-}(z),\end{cases}
\end{aligned}
$$

which is non-positive since $q\left(U^{-}(z)\right) \geqslant 0, V^{+}(z) \geqslant 0, \tilde{p}$ is decreasing, and $\tilde{g}$ is increasing. Noticing that $V_{0} \leqslant V^{+}$and $q\left(U_{0}\right) \geqslant q\left(U^{-}\right)$, we deduce from (2.14a) that

$$
\begin{equation*}
\delta U^{\prime \prime}+c U^{\prime}+q\left(U^{-}\right) \tilde{p}(U)-\tilde{g}(U) V^{+} \leqslant 0 \text { in }(-l, l) . \tag{2.16}
\end{equation*}
$$

Then (2.5) and (2.16) imply that the function $w_{2}:=U-U^{-}$satisfies $\delta w_{2}^{\prime \prime}+c w_{2}^{\prime}+H_{2}(z) w_{2} \leqslant 0$ in $\left(z_{0}, l\right)$. In addition, from (2.11c) and the fact that $U\left(z_{0}\right)>0$ and $U^{-}\left(z_{0}\right)=0$, we know that $w_{2}\left(z_{0}\right)>0$ and $w_{2}(l)=0$. Hence the maximum principle asserts that $w_{2} \geqslant 0$ in $\left[z_{0}, l\right]$, which implies that $U^{-} \leqslant U$ in $\left[z_{0}, l\right]$. Together with the fact that $U^{-} \equiv 0 \leqslant U$ in $\left[-l, z_{0}\right]$, we obtain $U^{-} \leqslant U$ in $[-l, l]$. Now we show that $U \leqslant U^{+}$in $I_{l}$. Recall that $U^{+} \equiv K$. Since $\tilde{p}\left(U^{+}\right)=\tilde{p}(K)=0$, it follows that

$$
\delta\left(U^{+}\right)^{\prime \prime}+c\left(U^{+}\right)^{\prime}+q\left(U_{0}\right) \tilde{p}\left(U^{+}\right)=0 \quad \text { in }(-l, l) .
$$

On the other hand, using the fact that $\tilde{g} \geqslant 0$ and $V_{0} \geqslant 0$, we deduce that

$$
\delta U^{\prime \prime}+c U^{\prime}+q\left(U_{0}\right) \tilde{p}(U)=\tilde{g}(U) V_{0} \geqslant 0 \quad \text { in }(-l, l) .
$$

Since $U^{+}( \pm l) \geqslant U^{-}( \pm l)=U( \pm l)$, we can use a similar argument as the proof for $U^{-} \leqslant U$ in $\left[z_{0}, l\right]$ to obtain that $U \leqslant U^{+}$in $I_{l}$.

Finally, since $0 \leqslant U^{-} \leqslant U \leqslant U^{+} \leqslant K$, it follows that $\tilde{p}(U)=p(U)$ and $\tilde{g}(U)=g(U)$, and therefore $U$ is a solution of $(2.11 a)$ with $U( \pm l)=\left(U^{-}\right)( \pm l)$. The uniqueness of $U$ follows from the maximum principle. Hence the proof of this lemma is completed.

Lemma 2.6. $\mathcal{F}$ is a continuous mapping.
Proof. For given $\left(U_{0}, V_{0}\right)$ and $\left(\tilde{U}_{0}, \tilde{V}_{0}\right)$ in $E$, let

$$
\begin{equation*}
(U, V)=\mathcal{F}\left(U_{0}, V_{0}\right) \text { and }(\tilde{U}, \tilde{V})=\mathcal{F}\left(\tilde{U}_{0}, \tilde{V}_{0}\right) \tag{2.17}
\end{equation*}
$$

We first consider the function $w_{1}:=U-\tilde{U}$. It is easy to see that $w_{1}(-l)=w_{1}(l)=0$ and

$$
w_{1}^{\prime \prime}+\frac{c}{\delta} w_{1}^{\prime}+f_{1}(z) w_{1}=h_{1}(z)
$$

where

$$
f_{1}(z)=\frac{1}{\delta}\left[q\left(U_{0}(z)\right) \frac{p(U(z))-p(\tilde{U}(z))}{U(z)-\tilde{U}(z)}-V_{0}(z) \frac{g(U(z))-g(\tilde{U}(z))}{U(z)-\tilde{U}(z)}\right]
$$

and

$$
h_{1}(z)=\frac{1}{\delta}\left[-p(\tilde{U}(z))\left(q\left(U_{0}(z)\right)-q\left(\tilde{U}_{0}(z)\right)\right)+g(\tilde{U}(z))\left(V_{0}(z)-\tilde{V}_{0}(z)\right)\right] .
$$

Note that

$$
-C_{1} \leqslant f_{1} \leqslant 0 \text { and }\left|h_{1}\right| \leqslant C_{2}\left[\left\|U_{0}-\tilde{U}_{0}\right\|_{C\left(I_{I}\right)}+\left\|V_{0}-\tilde{V}_{0}\right\|_{C\left(I_{I}\right)}\right],
$$

where $C_{1}:=1 / \delta\left[q(K)\left\|p^{\prime}\right\|_{C([0, K])}+\mathrm{e}^{\lambda l}\left\|g^{\prime}\right\|_{C([0, K])}\right], C_{2}:=1 / \delta\left[p(0)\left\|q^{\prime}\right\|_{C[0, K]}+g(K)\right]$. Here we have used the fact that $0 \leqslant V_{0} \leqslant V^{+} \leqslant\left\|V^{+}\right\|_{C\left(I_{l}\right)}=\mathrm{e}^{\lambda l}$ and $0 \leqslant \tilde{U} \leqslant U^{+} \equiv K$, and hypotheses (H1)-(H3). Then lemma A. 1 asserts that there exists a positive constant $C_{3}$, depending only on $C_{1}, \delta, c$, and $l$, such that

$$
\left\|w_{1}\right\|_{C\left(I_{l}\right)} \leqslant C_{2} C_{3}\left[\left\|U_{0}-\tilde{U}_{0}\right\|_{C\left(I_{l}\right)}+\left\|V_{0}-\tilde{V}_{0}\right\|_{C\left(I_{l}\right)}\right],
$$

which, together with the definition of $w_{1}$, implies that

$$
\begin{equation*}
\|U-\tilde{U}\|_{C\left(I_{I}\right)} \leqslant C_{2} C_{3}\left[\left\|U_{0}-\tilde{U}_{0}\right\|_{C\left(I_{I}\right)}+\left\|V_{0}-\tilde{V}_{0}\right\|_{C\left(I_{I}\right)}\right] . \tag{2.18}
\end{equation*}
$$

Next, we consider the function $w_{2}=V-\tilde{V}$. One can easily see that $w_{2}$ satisfies $w_{2}(-l)=w_{2}(l)=0$ and

$$
w_{2}^{\prime \prime}+c w_{2}^{\prime}-\gamma \beta w_{2}=h_{2}(z),
$$

where

$$
h_{2}=\gamma\left[g\left(\tilde{U}_{0}\right) \tilde{V}_{0}-g\left(U_{0}\right) V_{0}\right],
$$

which can be rewritten as

$$
\begin{equation*}
h_{2}=\gamma\left[\tilde{V}_{0}\left(g\left(\tilde{U}_{0}\right)-g\left(U_{0}\right)\right)+g\left(U_{0}\right)\left(\tilde{V}_{0}-V_{0}\right)\right] . \tag{2.19}
\end{equation*}
$$

Applying the mean-value theorem, we obtain that

$$
\left|g\left(\tilde{U}_{0}\right)-g\left(U_{0}\right)\right| \leqslant\left\|g^{\prime}\right\|_{C([0, K])}\left|\tilde{U}_{0}-U_{0}\right| .
$$

Together with the fact that $0 \leqslant \tilde{V}_{0} \leqslant\left\|V^{+}\right\|_{C\left(I_{l}\right)}=\mathrm{e}^{\lambda l}$ and $0 \leqslant g\left(U_{0}\right) \leqslant g(K)$ we deduce from (2.19) that

$$
\left|h_{2}\right| \leqslant \gamma\left[\mathrm{e}^{\lambda l}\left\|g^{\prime}\right\|_{C([0, K])}\left\|U_{0}-\tilde{U}_{0}\right\|_{C\left(I_{l}\right)}+g(K)\left\|V_{0}-\tilde{V}_{0}\right\|_{C\left(I_{l}\right)}\right] .
$$

Then lemma A. 1 asserts that there exists a positive constant $C_{4}$, depending only on $\gamma, \beta$, $\left\|g^{\prime}\right\|_{C([0, K]]}, g(K), c, K, \lambda$, and $l$, such that

$$
\left\|w_{2}\right\|_{C\left(I_{l}\right)} \leqslant C_{4}\left(\left\|U_{0}-\tilde{U}_{0}\right\|_{C\left(I_{l}\right)}+\left\|V_{0}-\tilde{V}_{0}\right\|_{C\left(I_{I}\right)}\right),
$$

which, together with the definition of $w_{2}$, implies that

$$
\begin{equation*}
\|V-\tilde{V}\|_{C\left(I_{I}\right)} \leqslant C_{4}\left(\left\|U_{0}-\tilde{U}_{0}\right\|_{C\left(I_{l}\right)}+\left\|V_{0}-\tilde{V}_{0}\right\|_{C\left(I_{l}\right)}\right) . \tag{2.20}
\end{equation*}
$$

Finally, we use (2.17), (2.18), and (2.20), and the definition of the norm $\|\cdot\|_{X}$, to deduce that

$$
\begin{align*}
& \left\|\mathcal{F}\left(U_{0}, V_{0}\right)-\mathcal{F}\left(\tilde{U}_{0}, \tilde{V}_{0}\right)\right\|_{X} \\
= & \|(U, V)-(\tilde{U}, \tilde{V})\|_{X} \\
= & \|U-\tilde{U}\|_{C\left(I_{l}\right)}+\|V-\tilde{V}\|_{C\left(I_{I}\right)} \\
\leqslant & C_{5}\left(\left\|U_{0}-\tilde{U}_{0}\right\|_{C\left(I_{l}\right)}+\left\|V_{0}-\tilde{V}_{0}\right\|_{C\left(I_{l}\right)}\right) \\
= & C_{5}\left\|\left(U_{0}, V_{0}\right)-\left(\tilde{U}_{0}, \tilde{V}_{0}\right)\right\|_{X}, \tag{2.21}
\end{align*}
$$

where $C_{5}=C_{2} C_{3}+C_{4}$. Thus, for a given $\epsilon>0$, we choose $0<\sigma_{1}<\epsilon / C_{5}$. Then, by (2.21), we have

$$
\left\|\mathcal{F}\left(U_{0}, V_{0}\right)-\mathcal{F}\left(\tilde{U}_{0}, \tilde{V}_{0}\right)\right\|_{X}<\epsilon,
$$

for any $\left(U_{0}, V_{0}\right),\left(\tilde{U}_{0}, \tilde{V}_{0}\right) \in E$ such that $\left\|\left(U_{0}, V_{0}\right)-\left(\tilde{U}_{0}, \tilde{V}_{0}\right)\right\|_{X}<\sigma_{1}$. This shows that $\mathcal{F}$ is a continuous mapping. Hence the proof of this lemma is completed.

Lemma 2.7. $\mathcal{F}$ is precompact.
Proof. For a given sequence $\left\{\left(U_{0, n}, V_{0, n}\right)\right\}_{n \in \mathbb{N}}$ in $E$, let $\left(U_{n}, V_{n}\right)=\mathcal{F}\left(U_{0, n}, V_{0, n}\right)$. Then lemma 2.5 gives that $\left(U_{n}, V_{n}\right) \in E$. Since $0 \leqslant U^{-} \leqslant U^{+} \equiv K$ and $0 \leqslant V^{-} \leqslant V^{+} \leqslant \mathrm{e}^{\lambda l}$ in $I_{l}$ and the functions $p, q$, and $g$ are bounded over $[0, K]$, we can easily see from definition of the set $E$ that the sequences

$$
\left\{U_{0, n}\right\},\left\{V_{0, n}\right\},\left\{U_{n}\right\},\left\{V_{n}\right\},\left\{q\left(U_{0, n}\right)\right\},\left\{g\left(U_{0, n}\right)\right\},\left\{g\left(U_{n}\right)\right\} \text { and }\left\{p\left(U_{n}\right)\right\}
$$

are uniformly bounded in $I_{l}$. Then, by lemma A.2, it follows that the sequences

$$
\left\{U_{n}^{\prime}\right\} \text { and }\left\{V_{n}^{\prime}\right\}
$$

are also uniformly bounded in $I_{l}$. Therefore, we can use the Arzelà-Ascoli theorem to obtain a subsequence $\left\{\left(U_{n_{j}}, V_{n_{j}}\right)\right\}$ of $\left\{\left(U_{n}, V_{n}\right)\right\}$ such that

$$
\left(U_{n_{j}}, V_{n_{j}}\right) \rightarrow(U, V),
$$

uniformly in $I_{l}$ as $j \rightarrow \infty$, for some $(U, V) \in E$. Hence the set $\overline{\mathcal{F}(E)}$ is compact in $E$, so $\mathcal{F}$ is precompact.

Finally, with the aid of lemmas 2.5-2.7, we can apply the Schauder fixed point theorem to conclude that $\mathcal{F}$ has a fixed point, which is a non-negative solution of system (2.9)-(2.10). So we have the following existence result for the truncated problem (2.9)-(2.10).
Lemma 2.8. System (2.9)-(2.10) admits a solution $(U, V)$ on $I_{1}$. Moreover,

$$
\begin{equation*}
0 \leqslant U^{-} \leqslant U \leqslant U^{+} \equiv K \text { and } 0 \leqslant V^{-} \leqslant V \leqslant V^{+} \tag{2.22}
\end{equation*}
$$

on $I_{l}$.

### 2.4. The construction of a candidate for travelling waves

In this section, we use the solution $\left(U_{l}, V_{l}\right)$ of the truncated problem (2.9)-(2.10) and the limiting argument to obtain a solution $(U, V)$ of system (1.12) satisfying $(U, V)(+\infty)=(K, 0)$.

Hence if we can show that $(U, V)(-\infty)=\left(u_{\beta}, v_{\beta}\right)$, then $(U, V)$ must be a travelling wave of system (1.7). Thus this observation would suggest that $(U, V)$ is a good candidate for travelling wave solutions of system (1.7). The condition that $(U, V)(-\infty)=\left(u_{\beta}, v_{\beta}\right)$ will be verified in section 3. Now we have the following lemma.

Lemma 2.9. Suppose that hypotheses $\mathbf{( H 1 ) - ( H 3 ) ~ h o l d . ~ I f ~} c>c_{\min }$, then system (1.12) admits a solution $(U, V)$ on $\mathbb{R}$ satisfying $0<U<K$ and $V>0$ over $\mathbb{R}, V(z)=\mathcal{O}\left(\mathrm{e}^{-\lambda z}\right)$ as $z \rightarrow \infty$, where $\lambda$ is given by (1.14), and

$$
(U, V)(+\infty)=(K, 0) \text { and }\left(U^{\prime}, V^{\prime}\right)(+\infty)=(0,0)
$$

Proof. Let $\left\{l_{n}\right\}_{n \in \mathbb{N}}$ be an increasing sequence in $\left(z_{1}, \infty\right)$ such that $l_{n} \rightarrow \infty$ as $n \rightarrow \infty$ and let $\left(U_{n}, V_{n}\right), n \in \mathbb{N}$, be a solution of system (2.9)-(2.10) with $l=l_{n}$. For any fixed $N \in \mathbb{N}$, since the function $V^{+}$is bounded above in $\left[-l_{N}, l_{N}\right]$ and the functions $p, q$ and $g$ are bounded over $[0, K]$, it follows from (2.22) that the sequences

$$
\left\{U_{n}\right\}_{n \geqslant N},\left\{V_{n}\right\}_{n \geqslant N},\left\{q\left(U_{n}\right)\right\}_{n \geqslant N},\left\{p\left(U_{n}\right)\right\}_{n \geqslant N} \text { and }\left\{g\left(U_{n}\right)\right\}_{n \geqslant N}
$$

are uniformly bounded in $\left[-l_{N}, l_{N}\right]$. Then we can use lemma A. 2 in the appendix to infer that the sequences

$$
\left\{U_{n}^{\prime}\right\}_{n \geqslant N} \text { and }\left\{V_{n}^{\prime}\right\}_{n \geqslant N}
$$

are also uniformly bounded in $\left[-l_{N}, l_{N}\right]$. Using (2.9), we can express $U_{n}^{\prime \prime}$ and $V_{n}^{\prime \prime}$ in terms of $U_{n}$, $V_{n}, U_{n}^{\prime}$ and $V_{n}^{\prime}$. Differentiating (2.9), we can use the resulting equations to express $U_{n}^{\prime \prime \prime}$ and $V_{n}^{\prime \prime \prime}$ in terms of $U_{n}, V_{n}, U_{n}^{\prime}, V_{n}^{\prime}, U_{n}^{\prime \prime}$ and $V_{n}^{\prime \prime}$. Consequently, the sequences

$$
\left\{U_{n}^{\prime \prime}\right\}_{n \geqslant N},\left\{V_{n}^{\prime \prime}\right\}_{n \geqslant N},\left\{U_{n}^{\prime \prime \prime}\right\}_{n \geqslant N} \text { and }\left\{V_{n}^{\prime \prime \prime}\right\}_{n \geqslant N}
$$

are uniformly bounded in $\left[-l_{N}, l_{N}\right]$. With the aid of the Arzelà-Ascoli theorem, we can use the diagonal process (see Gilbarg and Trundinger [18]) and the uniform convergence theorem (see Rudin [45]) to obtain a subsequence $\left\{\left(U_{n_{j}}, V_{n_{j}}\right)\right\}$ of $\left\{\left(U_{n}, V_{n}\right)\right\}$ such that

$$
U_{n_{j}} \rightarrow U, U_{n_{j}}^{\prime} \rightarrow U^{\prime}, U_{n_{j}}^{\prime \prime} \rightarrow U^{\prime \prime},
$$

and

$$
V_{n_{j}} \rightarrow V, V_{n_{j}}^{\prime} \rightarrow V^{\prime}, V_{n_{j}}^{\prime \prime} \rightarrow V^{\prime \prime},
$$

uniformly in any compact interval of $\mathbb{R}$ as $n \rightarrow \infty$, for some functions $U$ and $V$ in $C^{2}(\mathbb{R})$. Then it is easy to see that $(U, V)$ is a non-negative solution of system (1.12) and satisfies (2.22) over $\mathbb{R}$. From definitions of $U^{-}$and $V^{+}$, we see that $U^{-}(z) \rightarrow K$ and $V^{+}(z) \rightarrow 0$ as $z \rightarrow \infty$. This, together with (2.22), implies that

$$
\begin{equation*}
(U, V)(+\infty)=(K, 0) \tag{2.23}
\end{equation*}
$$

and $V(z)=\mathcal{O}\left(\mathrm{e}^{-\lambda z}\right)$ as $z \rightarrow \infty$, where $\lambda$ is given by (1.14).
Furthermore, we claim that $0<U<K$ and $V>0$ over $\mathbb{R}$, and

$$
\begin{equation*}
\left(U^{\prime}, V^{\prime}\right)(+\infty)=(0,0) \tag{2.24}
\end{equation*}
$$

For contradiction, we assume that $V\left(\tilde{z}_{1}\right)=0$ for some $\tilde{z}_{1} \in \mathbb{R}$. Then $V^{\prime}\left(\tilde{z}_{1}\right)=0$. Therefore the uniqueness gives $V \equiv 0$, which contradicts the fact that $V \geqslant V^{-}>0$ on $\left(z_{1}, \infty\right)$. Hence $V>0$ over $\mathbb{R}$. To prove $U<K$ over $\mathbb{R}$, we also use a contradictory argument and assume that $U\left(\tilde{z}_{2}\right)=K$ for some $\tilde{z}_{2} \in \mathbb{R}$. In this case, $U^{\prime}\left(\tilde{z}_{2}\right)=0$ and $U^{\prime \prime}\left(\tilde{z}_{2}\right) \leqslant 0$. This contradicts (1.12a) with $z=\tilde{z}_{2}$. Hence $U<K$ over $\mathbb{R}$. Suppose $U\left(\tilde{z}_{3}\right)=0$ for some $\tilde{z}_{3} \in \mathbb{R}$. Then $U^{\prime}\left(\tilde{z}_{3}\right)=0$. Hence the uniqueness of solutions implies that $U \equiv 0$, which contradicts the fact that $U(+\infty)=K$. Hence $U>0$ over $\mathbb{R}$.

To prove (2.24), we use equation (1.12a) to deduce that

$$
\begin{equation*}
U^{\prime}(z)=\mathrm{e}^{-\frac{c}{\delta}(z-s)} U^{\prime}(s)-\frac{1}{\delta} \mathrm{e}^{-\frac{c}{\delta} z} \int_{s}^{z} \mathrm{e}^{\frac{c}{\delta} \tau}(q(U(\tau)) p(U(\tau))-g(U(\tau)) V(\tau)) \mathrm{d} \tau \tag{2.25}
\end{equation*}
$$

By fixing $s$ and letting $z \rightarrow \infty$ in equality (2.25), we immediately deduce that

$$
\begin{aligned}
\limsup _{z \rightarrow \infty}\left|U^{\prime}(z)\right| & \leqslant \frac{1}{\delta} \max _{\tau \geqslant s}|q(U(\tau)) p(U(\tau))-g(U(\tau)) V(\tau)| \limsup _{z \rightarrow \infty} \mathrm{e}^{-\frac{c}{\delta} z} \int_{s}^{z} \mathrm{e}^{\frac{c}{\delta} \tau} \mathrm{~d} \tau \\
& \leqslant \frac{1}{c} \max _{\tau \geqslant s}|q(U(\tau)) p(U(\tau))-g(U(\tau)) V(\tau)|
\end{aligned}
$$

for $s \in \mathbb{R}$. Together with the fact that $q(U(\infty)) p(U(\infty))-g(U(\infty)) V(\infty)=0$, we can deduce that $U^{\prime}(\infty)=0$. Similarly, using equation (1.12b) and arguing as above, we also obtain $V^{\prime}(\infty)=0$.

## 3. Existence of non-critical waves of system (1.7)

In this section, we will establish the assertion of theorem 1.1 (II), which is restated in the following lemma for the convenience of the readers.

Lemma 3.1. Suppose that hypotheses $\mathbf{( H 1 ) - ( \mathbf { H } 4 )}$ hold. Ifc $>c_{\min }$, then system (1.12)-(1.13) admits a non-negative solution $(U, V)$ with the following properties.
(i) $0<U<K$ and $V>0$ over $\mathbb{R}$.
(ii) There exists a $\gamma^{*}>0$ such that there hold
(a) if $\gamma \in\left(0, \gamma^{*}\right)$, then the solution $(U, V)$ approaches $\left(u_{\beta}, \nu_{\beta}\right)$ monotonically for large $-z$;
(b) if $\gamma>\gamma^{*}$, then the solution $(U, V)$ has exponentially damped oscillations about $\left(u_{\beta}, v_{\beta}\right)$ for large $-z$.
(iii) $V(z)=\mathcal{O}\left(\mathrm{e}^{-\lambda z}\right)$ as $z \rightarrow \infty$, where $\lambda$ is given by (1.14).

In the remaining part of this section, we will show lemma 3.1, and therefore we always assume that hypotheses $(\mathbf{H} \mathbf{1})-(\mathbf{H} 4)$ hold and $c>c_{\text {min }}$ throughout this section.

As we mentioned in section 2.4, a good candidate for the solution of system (1.12)-(1.13) is the one given in lemma 2.9 , which will be denoted by $(U, V)$. In view of lemma 2.9 , $(U, V)$ satisfies $(U, V)(\infty)=(K, 0)$ and assertions (i) and (iii) of lemma 3.1. Further, if $(U, V)(-\infty)=\left(u_{\beta}, v_{\beta}\right)$, then we can follow the eigenvalue analysis in $[10,24]$ to show that assertions (ii) of lemma 3.1 hold for ( $U, V$ ), and hence we omit the proof of assertions (ii) of lemma 3.1 here. Hence, in order to complete the proof of lemma 3.1, it remains to verify that ( $U, V$ ) satisfies

$$
\begin{equation*}
(U, V)(-\infty)=\left(u_{\beta}, v_{\beta}\right) \tag{3.1}
\end{equation*}
$$

To show equality (3.1), we set

$$
\begin{equation*}
\hat{\mathbf{u}}(z)=U(-z) \text { and } \hat{\mathbf{v}}(z)=V(-z) \tag{3.2}
\end{equation*}
$$

Hence (3.1) is equivalent to the equality

$$
\begin{equation*}
(\hat{\mathbf{u}}, \hat{\mathbf{v}})(\infty)=\left(u_{\beta}, v_{\beta}\right) . \tag{3.3}
\end{equation*}
$$

Throughout the remainder of this section, we will keep the notation $(\hat{\mathbf{u}}, \hat{\mathbf{v}})$. One can verify that the governing equation for $(\mathbf{u}, \mathbf{v})=(\hat{\mathbf{u}}, \hat{\mathbf{v}})$ is given by

$$
\begin{align*}
& \delta \mathbf{u}^{\prime \prime}-c \mathbf{u}^{\prime}=g(\mathbf{u}) \mathbf{v}-q(\mathbf{u}) p(\mathbf{u}),  \tag{3.4a}\\
& \mathbf{v}^{\prime \prime}-c \mathbf{v}^{\prime}=\gamma[\beta-g(\mathbf{u})] \mathbf{v}, \tag{3.4b}
\end{align*}
$$

where the prime denotes the differentiation with respect to $z$. Further, the definition of ( $\hat{\mathbf{u}}, \hat{\mathbf{v}}$ ) and lemma 2.9 gives that $(\hat{\mathbf{u}}, \hat{\mathbf{v}})$ is a solution of (3.4) on $\mathbb{R}$ satisfying

$$
\begin{equation*}
0<\hat{\mathbf{u}}<K \text { and } \hat{\mathbf{v}}>0 \tag{3.5}
\end{equation*}
$$

over $\mathbb{R}$, and

$$
(\hat{\mathbf{u}}, \hat{\mathbf{v}})(-\infty)=(K, 0) \text { and }\left(\hat{\mathbf{u}}^{\prime}, \hat{\mathbf{v}}^{\prime}\right)(-\infty)=(0,0)
$$

In order to show equality (3.3) (i.e. $(\hat{U}, \hat{V})(-\infty)=\left(u_{\beta}, v_{\beta}\right)$ ), we first rewrite (3.4) as a system of first-order ODEs:

$$
\begin{align*}
\mathbf{u}^{\prime} & =\mathbf{w}  \tag{3.6a}\\
\delta \mathbf{w}^{\prime} & =c \mathbf{w}+g(\mathbf{u}) \mathbf{v}-q(\mathbf{u}) p(\mathbf{u})  \tag{3.6b}\\
\mathbf{v}^{\prime} & =\mathbf{y}  \tag{3.6c}\\
\mathbf{y}^{\prime} & =c \mathbf{y}+\gamma[\beta-g(\mathbf{u})] \mathbf{v} . \tag{3.6d}
\end{align*}
$$

Next, following the idea of [10], we define the Lyapunov function $\mathcal{L}: \mathbb{R}^{4} \rightarrow \mathbb{R}$ by

$$
\begin{align*}
\mathcal{L}(\mathrm{u}, \mathrm{w}, \mathrm{v}, \mathrm{y}) & :=-\gamma\left(\delta \mathrm{w}-c \mathrm{u}-\delta \beta \frac{\mathrm{w}}{g(\mathrm{u})}+c \beta \int_{u_{\beta}}^{\mathrm{u}} \frac{1}{g(\xi)} \mathrm{d} \xi\right)-\left(\mathrm{y}-c \mathrm{v}-v_{\beta} \frac{\mathrm{y}}{\mathrm{v}}+c v_{\beta} \ln \frac{\mathrm{v}}{v_{\beta}}\right) \\
& =\mathcal{L}_{1}(\mathrm{u}, \mathbf{w}, \mathrm{v}, \mathrm{y})+\mathcal{L}_{2}(\mathrm{u}, \mathbf{w}, \mathrm{v}, \mathrm{y}) . \tag{3.7}
\end{align*}
$$

A simple computation gives that the orbital derivative of $\mathcal{L}$ along the solution $\chi(z):=(\hat{\mathbf{u}}(z), \hat{\mathbf{w}}(z), \hat{\mathbf{v}}(z), \hat{\mathbf{y}}(z))$, where $\hat{\mathbf{w}}:=\hat{\mathbf{u}}^{\prime}(z)$ and $\hat{\mathbf{y}}:=\hat{\mathbf{v}}^{\prime}(z)$, of system (3.6), is

$$
\begin{aligned}
& \frac{\mathrm{d}}{\mathrm{~d} z} \mathcal{L}(\chi(z)) \\
= & \nabla \mathcal{L}(\chi(z)) \cdot \chi^{\prime}(z) \\
= & -\delta \beta \hat{\mathbf{w}}(z)^{2} \frac{g^{\prime}(\hat{\mathbf{u}}(z))}{g(\hat{\mathbf{u}}(z))^{2}}-v_{\beta} \frac{\hat{\mathbf{y}}(z)^{2}}{\hat{\mathbf{v}}(z)^{2}}+\gamma\left[g\left(u_{\beta}\right)-g(\hat{\mathbf{u}}(z))\right]\left[R\left(u_{\beta}\right)-R(\hat{\mathbf{u}}(z))\right],
\end{aligned}
$$

which, together with hypotheses (H3)-(H4), yields

$$
\frac{\mathrm{d}}{\mathrm{~d} z} \mathcal{L}(\chi(z)) \leqslant 0 .
$$

Hence

$$
\begin{equation*}
\mathcal{L}(\chi(z)) \leqslant \mathcal{L}(\chi(0)), \forall z \geqslant 0 . \tag{3.8}
\end{equation*}
$$

The strategy for the remaining part of the proof of equality (3.3) follows [10]. In fact, we will construct an open bounded set $\mathcal{D}$ such that the solution $\chi(z)=(\hat{\mathbf{u}}(z), \hat{\mathbf{w}}(z), \hat{\mathbf{v}}(z), \hat{\mathbf{y}}(z))$ of system (3.6) is positively invariant in $\mathcal{D}$ for all $z \geqslant 0$, and that the function $\mathcal{L}$ is continuous and bounded below in $\mathcal{D}$. Then, together with LaSalle's invariance principle, the assertion $(\hat{\mathbf{u}}, \hat{\mathbf{v}})(\infty)=\left(u_{\beta}, v_{\beta}\right)$ can be established, which would complete the proof of theorem 1.1.

The two key points for the construction of the open bounded set $\mathcal{D}$ are a priori estimates of the derivative of ( $\hat{\mathbf{u}}, \hat{\mathbf{v}}$ ) and the boundedness of $\hat{\mathbf{v}}$, which will be given in sections 3.1, 3.3 and 3.2, respectively. We remark that, unlike the previous studies [9-11, 22-24, 30, 32], we will not employ the shooting argument or the phase-space argument to construct the open bounded set $\mathcal{D}$. We also note that the arguments for some of the estimates in the remainder of this section are motivated by [10, 24].

### 3.1. Estimates of the derivative of $\hat{\mathbf{v}}$

To construct the open bounded set $\mathcal{D}$, we first derive the estimate for the derivative of ( $\hat{\mathbf{u}}, \hat{\mathbf{v}}$ ). Recall that ( $\hat{\mathbf{u}}, \hat{\mathbf{v}}$ ) is defined by (3.2).

Lemma 3.2. For each $z \in \mathbb{R}$, the following inequalities hold:

$$
\begin{align*}
& \hat{\mathbf{v}}^{\prime}(z) \leqslant \frac{c}{2} \hat{\mathbf{v}}(z),  \tag{3.9}\\
& \hat{\mathbf{v}}^{\prime}(z) \geqslant-\frac{\gamma \beta}{c} \hat{\mathbf{v}}(z)  \tag{3.10}\\
& \hat{\mathbf{u}}^{\prime}(z) \leqslant \frac{1}{c} \max _{\mathrm{u} \in[0, K]} q(\mathbf{u}) p(\mathbf{u}) . \tag{3.11}
\end{align*}
$$

Proof. We first establish inequality (3.9). To do this, we consider the function $\psi_{1}(z):=\hat{\mathbf{v}}^{\prime}(z)-c / 2 \cdot \hat{\mathbf{v}}(z)$. Hence, in order to establish (3.9), it suffices to show that $\psi_{1}(z) \leqslant 0$ for all $z \in \mathbb{R}$. For contradiction, we assume that $\psi_{1}\left(\eta_{0}\right)>0$ for some $\eta_{0} \in \mathbb{R}$. Using $c>c_{\text {min }}=2 \sqrt{\gamma[g(K)-\beta]}$ and the fact that $\hat{\mathbf{v}}>0$ and $\hat{\mathbf{u}} \in(0, K)$ on $\mathbb{R}$, we deduce from (3.4b) that

$$
\begin{aligned}
\psi_{1}^{\prime}-\frac{c}{2} \psi_{1} & =\hat{\mathbf{v}}^{\prime \prime}-c \hat{\mathbf{v}}^{\prime}+\frac{c^{2}}{4} \hat{\mathbf{v}} \\
& =\gamma[\beta-g(\hat{\mathbf{u}})] \hat{\mathbf{v}}+\frac{c^{2}}{4} \hat{\mathbf{v}} \\
& >\gamma[\beta-g(K)] \hat{\mathbf{v}}+\frac{c^{2}}{4} \hat{\mathbf{v}}>0
\end{aligned}
$$

and hence that $\mathrm{e}^{-c / 2 z} \psi_{1}(z)$ is strictly increasing in $z$. Together with $\psi_{1}\left(\eta_{0}\right)>0$, we obtain $\psi_{1}(z)>0$ for all $z>\eta_{0}$. Hence we deduce that $\hat{\mathbf{v}}(z) \geqslant \hat{\mathbf{v}}\left(\eta_{0}\right) \mathrm{e}^{c\left(z-\eta_{0}\right) / 2}$ for all $z>\eta_{0}$. On the other
hand, from lemma 2.8 and the definition of $V^{+}$, we have $\hat{\mathbf{v}}(z) \leqslant \mathrm{e}^{\lambda z}$ for $z \in \mathbb{R}$. Since $\lambda<c / 2$, we obtain a contradiction. Hence (3.9) holds.

Next, we show (3.10). To this end, we consider the function

$$
\Phi(z):=c \hat{\mathbf{v}}^{\prime}(z)+\gamma \beta \hat{\mathbf{v}}(z) .
$$

It suffices to show that $\Phi(z) \geqslant 0$ for all $z \in \mathbb{R}$. Note that

$$
\Phi(z)=\hat{\mathbf{v}}(z)\left(c \frac{\hat{\mathbf{v}}^{\prime}(z)}{\hat{\mathbf{v}}(z)}+\gamma \beta\right)>0
$$

for all sufficiently large $-z$ since

$$
\lim _{z \rightarrow-\infty} \hat{\mathbf{v}}^{\prime}(z) / \hat{\mathbf{v}}(z)=\lambda>0 .
$$

For contradiction, we assume that there exists $\hat{z}_{1} \in \mathbb{R}$ such that $\Phi\left(\hat{z}_{1}\right)=0$ and $\Phi^{\prime}\left(\hat{z}_{1}\right) \leqslant 0$. Then there are two possibilities: either

$$
\begin{equation*}
\Phi(z) \leqslant 0, \forall z \geqslant \hat{z}_{1} \tag{3.12}
\end{equation*}
$$

or

$$
\begin{equation*}
\Phi\left(\hat{z}_{2}\right)=0 \text { and } \Phi^{\prime}\left(\hat{z}_{2}\right) \geqslant 0 \tag{3.13}
\end{equation*}
$$

for some $\hat{z}_{2} \geqslant \hat{z}_{1}$. For the first case, (3.12) yields

$$
c \hat{\mathbf{v}}^{\prime}(z) \leqslant-\gamma \beta \hat{\mathbf{v}}(z), \forall z \geqslant \hat{z}_{1} .
$$

Together with the fact that $g(\hat{\mathbf{u}}) \hat{\mathbf{v}}>0$, we deduce from (3.4b) that

$$
\hat{\mathbf{v}}^{\prime \prime}=c \hat{\mathbf{v}}^{\prime}+\gamma[\beta-g(\hat{\mathbf{u}})] \hat{\mathbf{v}} \leqslant-\gamma g(\hat{\mathbf{u}}) \hat{\mathbf{v}}(z)<0, \forall z \geqslant \hat{\mathrm{z}}_{1},
$$

which implies that $\hat{\mathbf{v}}^{\prime}$ is decreasing in $\left[\hat{z}_{1}, \infty\right)$. Hence $\hat{\mathbf{v}}^{\prime}(z) \leqslant \hat{\mathbf{v}}^{\prime}\left(\hat{z}_{1}\right)=-\frac{\gamma \beta}{c} \hat{\mathbf{v}}\left(\hat{z}_{1}\right)<0$, which implies that $\hat{\mathbf{v}}(z)$ is negative for large $z$. This is a contradiction to the fact that $\hat{\mathbf{v}}$ is positive on $\mathbb{R}$. For the second case, (3.13) yields that

$$
\begin{equation*}
\hat{\mathbf{v}}^{\prime}\left(\hat{z}_{2}\right)=-\frac{\gamma \beta}{c} \hat{\mathbf{v}}\left(\hat{z}_{2}\right) \tag{3.14}
\end{equation*}
$$

and

$$
\begin{equation*}
\hat{\mathbf{v}}^{\prime \prime}\left(\hat{z}_{2}\right) \geqslant-\frac{\gamma \beta}{c} \hat{\mathbf{v}}^{\prime}\left(\hat{z}_{2}\right) . \tag{3.15}
\end{equation*}
$$

Then we deduce from (3.4b) that

$$
\begin{aligned}
0= & \hat{\mathbf{v}}^{\prime \prime}\left(\hat{z}_{2}\right)-c \hat{\mathbf{v}}^{\prime}\left(\hat{z}_{2}\right)-\gamma \beta \hat{\mathbf{v}}\left(\hat{z}_{2}\right)+\gamma g\left(\hat{\mathbf{u}}\left(\hat{z}_{2}\right)\right) \hat{\mathbf{v}}\left(\hat{z}_{2}\right) \\
\geqslant & -\frac{\gamma \beta}{c} \hat{\mathbf{v}}^{\prime}\left(\hat{z}_{2}\right)+\gamma \beta \hat{\mathbf{v}}\left(\hat{z}_{2}\right)-\gamma \beta \hat{\mathbf{v}}\left(\hat{z}_{2}\right) \\
& \quad(\text { by }(3.14) \text { and (3.15), and the fact that } g(\hat{\mathbf{u}}) \hat{\mathbf{v}}>0) \\
= & \left(\frac{\gamma \beta}{c}\right)^{2} \hat{\mathbf{v}}\left(\hat{z}_{2}\right) \quad(\text { by }(3.14)) \\
> & 0,
\end{aligned}
$$

a contradiction again. Hence (3.10) is proven.

Finally，we establish（3．11）．To this end，we first claim that $\hat{\mathbf{u}}^{\prime}(z) \leqslant 1 / c \max _{\mathbf{u} \in[0, K]} q(\mathbf{u}) p(\mathbf{u})$ at any local maximum point $z$ of $\hat{\mathbf{u}}^{\prime}$ ．Let $\eta_{0}$ be an arbitrary local maximum point of $\hat{\mathbf{u}}^{\prime}$ ．Then we have $\hat{\mathbf{u}}^{\prime \prime}\left(\eta_{0}\right)=0$ ，which，together with（3．4a），gives that $-c \hat{\mathbf{u}}^{\prime}\left(\eta_{0}\right)=g\left(\hat{\mathbf{u}}\left(\eta_{0}\right)\right) \hat{\mathbf{v}}\left(\eta_{0}\right)-$ $q\left(\hat{\mathbf{u}}\left(\eta_{0}\right)\right) p\left(\hat{\mathbf{u}}\left(\eta_{0}\right)\right) \geqslant-\max _{\mathrm{u} \in[0, K]} q(\mathbf{U}) p(\mathbf{U})$ ，and hence that $\hat{\mathbf{u}}^{\prime}\left(\eta_{0}\right) \leqslant 1 / c \max _{u \in[0, K]} q(u) p(u)$ ．Hence the assertion of this claim is established．Now we note that $\hat{\mathbf{u}}^{\prime}(-\infty)=0<1 / c \max _{u \in[0, K]} q(u) p(u)$ ． Hence，if（3．11）does not hold，then there exists $\hat{\eta}_{0}$ such that $\hat{\mathbf{u}}^{\prime}\left(\hat{\eta}_{0}\right)>1 / c \max _{\mathrm{u} \in[0, K]} q(\mathbf{u}) p(\mathrm{u})$ ． Together with the claim and the continuity of $\hat{\mathbf{u}}^{\prime}$ ，we have $\hat{\mathbf{u}}^{\prime}(z)>1 / c \cdot \max _{\mathbf{u} \in[0, K]} q(\mathbf{u}) p(\mathbf{u})>0$ for all $z \geqslant \hat{\eta}_{0}$ ，which contradicts the boundedness of $\hat{\mathbf{u}}$ ．Hence（3．11）is shown．The proof of this lemma is therefore completed．

## 3．2．Boundedness of $\hat{\mathrm{v}}$

In this section，we show that $\hat{\mathbf{v}}$ is bounded over $\mathbb{R}$ ．Note that $\lim _{z \rightarrow-\infty} \hat{\mathbf{v}}(z)=0$ ．For con－ tradiction，we assume that $\lim \sup _{z \rightarrow \infty} \hat{\mathbf{v}}(z)=\infty$ ．Then there are two possibilities：either
 these two possibilities．

3．2．1．The case where $\lim _{z \rightarrow \infty} \hat{\mathbf{v}}(z)=\infty$ ．In this section，we will exclude the possibility that $\lim _{z \rightarrow \infty} \hat{\mathbf{v}}(z)=\infty$ ．Specifically，we state it in the following lemma．

Lemma 3．3．The solution $(\hat{\mathbf{u}}, \hat{\mathbf{v}})$ cannot satisfy $\lim _{z \rightarrow \infty} \hat{\mathbf{v}}(z)=\infty$ ．
Proof．For contradiction，we assume that $\lim _{z \rightarrow \infty} \hat{\mathbf{v}}(z)=\infty$ ．Consider the function $\Psi:=\gamma\left[\delta \hat{\mathbf{u}}^{\prime}-c \hat{\mathbf{u}}\right]+\hat{\mathbf{v}}^{\prime}-c \hat{\mathbf{v}}$ ．Using $(3.4 a)-(3.4 b), \lim _{z \rightarrow \infty} \hat{\mathbf{v}}(z)=\infty$ ，and the boundedness of $q(\hat{\mathbf{u}}) p(\hat{\mathbf{u}})$ ，we find that

$$
\begin{equation*}
\Psi^{\prime}(z)=\gamma[\beta \hat{\mathbf{v}}-q(\hat{\mathbf{u}}) p(\hat{\mathbf{u}})] \rightarrow \infty \text { as } z \rightarrow \infty, \tag{3.16}
\end{equation*}
$$

which implies that $\lim _{z \rightarrow \infty} \Psi(z)=\infty$ ．Together with（3．9），we deduce that $\delta \hat{\mathbf{u}}^{\prime}-c \hat{\mathbf{u}} \rightarrow \infty$ as $z \rightarrow \infty$ ， a contradiction to the boundedness of $\hat{\mathbf{u}}$ ．This completes the proof of this lemma．

3．2．2．The case where $\lim \inf _{z \rightarrow \infty} \hat{\mathbf{v}}(z)<\lim _{\sup _{z \rightarrow \infty}} \hat{\mathbf{v}}(z)=\infty$ ．In this section，we will exclude the possibility that $\lim _{\inf }^{z \rightarrow \infty} ⿵ 冂 人 一 \mathbf{v}(z)<\lim \sup _{z \rightarrow \infty} \hat{\mathbf{v}}(z)=\infty$ ．This case is more dif－ ficult．For contradiction，we assume that $\lim \inf _{z \rightarrow \infty} \hat{\mathbf{v}}(z)<\lim \sup _{z \rightarrow \infty} \hat{\mathbf{v}}(z)=\infty$ ．Then it fol－ lows that $\hat{\mathbf{v}}(z)$ oscillates infinitely many times as $z \rightarrow \infty$ ．To obtain a contradiction，we need several auxiliary lemmas．

Lemma 3．4．$\hat{\mathbf{u}}(z)$ oscillates infinitely many times as $z \rightarrow \infty$ ．
Proof．Recall that $\hat{\mathbf{v}}(z)$ oscillates infinitely many times as $z \rightarrow \infty$ ．Moreover，for the point $\tilde{z}_{1}$ where $\hat{\mathbf{v}}^{\prime}(z)$ takes its local maximum，with the use of $(3.4 b)$ ，we have $\hat{\mathbf{v}}^{\prime}\left(\tilde{z}_{1}\right)>0$ and $g\left(\hat{\mathbf{u}}\left(\tilde{z}_{1}\right)\right)>\beta$ ， while for the point $\tilde{z}_{2}$ where $\hat{\mathbf{v}}^{\prime}(z)$ takes its local minimum，we have $\hat{\mathbf{v}}^{\prime}\left(\tilde{z}_{2}\right)<0$ and $g\left(\hat{\mathbf{u}}\left(\tilde{z}_{2}\right)\right)<\beta$ ． Since $g(\mathbf{u})$ is monotonically increasing in $0<\mathbf{u}<K$ and $g\left(u_{\beta}\right)=\beta$ ，it follows that $\hat{\mathbf{u}}\left(\tilde{z_{1}}\right)>u_{\beta}$ and $\hat{\mathbf{u}}\left(\tilde{z}_{2}\right)<u_{\beta}$ ．With this observation，we discover that $\hat{\mathbf{u}}(z)$ also oscillates infinitely many times as $z \rightarrow \infty$ ，thereby completing the proof of this lemma．

Lemma 3．5．$\quad \hat{\mathbf{u}}$ has a positive lower bound over $\mathbb{R}$ ．

Proof. Recall that $\hat{\mathbf{u}}(-\infty)=K>0, \hat{\mathbf{u}}(z)>0$ for all $z \in \mathbb{R}$, and that $\hat{\mathbf{u}}$ oscillates infinitely many times as $z \rightarrow \infty$. Therefore, if the conclusion is false, then there exists a sequence of positive numbers $\left\{z_{n}\right\} \rightarrow \infty$ such that $\hat{\mathbf{u}}$ has a local minimum at $z_{n}$ and $\hat{\mathbf{u}}\left(z_{n}\right) \rightarrow 0$ as $n \rightarrow \infty$.

Using (3.9) and (3.10), we get that

$$
\mathcal{L}_{2}(\chi(z)) \geqslant-\left(-\frac{c}{2} \hat{\mathbf{v}}(z)+\frac{\gamma \beta v_{\beta}}{c}+c v_{\beta} \ln \frac{\hat{\mathbf{v}}(z)}{v_{\beta}}\right):=\psi_{3}(\hat{\mathbf{v}}(z))
$$

Since $\psi_{3}(0+)=\infty$ and $\psi_{3}(\infty)=\infty$, the function $\psi_{3}$ is bounded below in $(0, \infty)$. Hence the above inequality implies that $\mathcal{L}_{2}(\chi(z))$ is bounded below for $z \geqslant 0$. In addition, since $\hat{\mathbf{u}}$ attains a local minimum at each point $z=z_{n}$, it follows that $\hat{\mathbf{w}}\left(z_{n}\right)=\hat{\mathbf{u}}^{\prime}\left(z_{n}\right)=0$, and so

$$
\mathcal{L}_{1}\left(\chi\left(z_{n}\right)\right)=\gamma\left(c \hat{\mathbf{u}}\left(z_{n}\right)-c \beta \int_{u_{\beta}}^{\hat{\mathbf{u}}\left(z_{n}\right)} \frac{1}{g(\xi)} \mathrm{d} \xi\right) \rightarrow \infty \text { as } n \rightarrow \infty,
$$

where we have used $\hat{\mathbf{u}}\left(z_{n}\right) \rightarrow 0$ as $n \rightarrow \infty$, and $g(u)=\mathrm{O}\left(u^{1+\rho}\right)$ for some $\rho \geqslant 0$. Taken together, we conclude that $\mathcal{L}\left(\chi\left(z_{n}\right)\right) \rightarrow \infty$ as $n \rightarrow \infty$. This is a contradiction to the fact that $\mathcal{L}(\chi(z))$ is decreasing in $z$. Hence this completes the proof of this lemma.

Lemma 3.6. There exists an $M_{1} \geqslant v_{\beta}$ such that, for $\hat{\mathbf{v}}(z) \geqslant M_{1}$ with $z \geqslant 0$, we have $\hat{\mathbf{u}}^{\prime}(z)<0$. In the following, we retain the notation $M_{1}$.

Proof. First, since $c \mathrm{v} / 2-c v_{\beta} \ln \mathrm{v} \rightarrow \infty$ as $\mathrm{v} \rightarrow \infty$, there exists a large $M_{1} \geqslant v_{\beta}$ such that, for $\mathrm{v} \geqslant M_{1}$, we have
$\left(\frac{c}{2} \mathrm{v}-c v_{\beta} \ln \mathrm{v}\right)+c v_{\beta} \ln v_{\beta}>\frac{\gamma \delta}{c}\left(\max _{\mathrm{u} \in[0, K]} q(\mathbf{u}) p(\mathbf{u})\right)+\gamma c \beta \int_{u_{\beta}}^{K} \frac{1}{g(\xi)} \mathrm{d} \xi+2|\mathcal{L}(\chi(0))|$.

Next, using (3.9) and (3.11), we infer that for all $z$ with $\hat{\mathbf{w}}(z) \geqslant 0$ there holds

$$
\begin{equation*}
\mathcal{L}_{1}(\hat{\mathbf{u}}, \hat{\mathbf{w}}, \hat{\mathbf{v}}, \hat{\mathbf{y}})(z) \geqslant-\frac{\gamma \delta}{c} \cdot\left(\max _{\mathrm{u} \in[0, K]} q(\mathbf{u}) p(\mathbf{u})\right)-\gamma c \beta \int_{u_{\beta}}^{K} \frac{1}{g(\xi)} \mathrm{d} \xi, \tag{3.18}
\end{equation*}
$$

and for all $z$ with $\hat{\mathbf{y}}(z) \geqslant 0$ there holds

$$
\begin{equation*}
\mathcal{L}_{2}(\hat{\mathbf{u}}, \hat{\mathbf{w}}, \hat{\mathbf{v}}, \hat{\mathbf{y}})(z) \geqslant\left(\frac{c}{2} \hat{\mathbf{v}}(z)-c v_{\beta} \ln \hat{\mathbf{v}}(z)\right)+c v_{\beta} \ln v_{\beta} . \tag{3.19}
\end{equation*}
$$

Now, for $z$ with $\hat{\mathbf{v}}(z) \geqslant v_{\beta}$ and $\hat{\mathbf{y}}(z)<0$, we estimate $\mathcal{L}_{2}(\hat{\mathbf{u}}, \hat{\mathbf{w}}, \hat{\mathbf{v}}, \hat{\mathbf{y}})$ as follows:

$$
\begin{align*}
\mathcal{L}_{2}(\hat{\mathbf{u}}, \hat{\mathbf{w}}, \hat{\mathbf{v}}, \hat{\mathbf{y}})(z) & =-\hat{\mathbf{y}}(z)+c \hat{\mathbf{v}}(z)+v_{\beta} \frac{\hat{\mathbf{y}}(z)}{\hat{\mathbf{v}}(z)}-c v_{\beta} \ln \frac{\hat{\mathbf{v}}(z)}{v_{\beta}} \\
& =-\left(1-\frac{v_{\beta}}{\hat{\mathbf{v}}(z)}\right) \hat{\mathbf{y}}+\left(c \hat{\mathbf{v}}(z)-c v_{\beta} \ln \hat{\mathbf{v}}(z)\right)+c v_{\beta} \ln v_{\beta} \\
& \geqslant\left(c \hat{\mathbf{v}}(z)-c v_{\beta} \ln \hat{\mathbf{v}}(z)\right)+c v_{\beta} \ln v_{\beta} . \tag{3.20}
\end{align*}
$$

We are now ready to establish the assertion of this lemma. For contradiction, suppose that there exists a $\hat{z}_{1} \geqslant 0$ such that $\hat{\mathbf{v}}\left(\hat{z}_{1}\right) \geqslant M_{1}$ and $\hat{\mathbf{w}}\left(\hat{z}_{1}\right)=\hat{\mathbf{u}}^{\prime}\left(\hat{z}_{1}\right) \geqslant 0$. Then with the aid of (3.18)(3.20), it follows from the choice of $M_{1}$ that

$$
\mathcal{L}\left(\chi\left(\hat{z}_{1}\right)\right)=\mathcal{L}(\hat{\mathbf{u}}, \hat{\mathbf{w}}, \hat{\mathbf{v}}, \hat{\mathbf{y}})\left(\hat{z}_{1}\right)>2|\mathcal{L}(\chi(0))|,
$$

which contradicts (3.8). The proof of this lemma is thus completed.
Lemma 3.7. Suppose that $\hat{\mathbf{v}}\left(\hat{z}_{0}\right) \geqslant M_{1}$ and $\hat{\mathbf{v}}^{\prime}\left(\hat{z}_{0}\right)=0$ for some $\hat{z}_{0} \in \mathbb{R}$. Then $\hat{\mathbf{v}}\left(\hat{z}_{0}\right)$ cannot be a local minimum.

Proof. Suppose not. Then we have $\hat{\mathbf{v}}^{\prime \prime}\left(\hat{z}_{0}\right) \geqslant 0$, which, together with $(3.4 b)$, yields $g\left(\hat{\mathbf{u}}\left(\hat{z}_{0}\right)\right) \leqslant \beta$. On the other hand, since $\hat{\mathbf{v}}(z)$ oscillates for large $z$, we can find a $\hat{z}_{1}>\hat{z}_{0}$ such that $\hat{\mathbf{v}}^{\prime}(z) \geqslant 0$ for $z \in\left(\hat{z}_{0}, \hat{z}_{1}\right)$ and $\hat{\mathbf{v}}(z)$ takes its local maximal value at $z=\hat{z}_{1}$. Hence we have $\hat{\mathbf{v}}^{\prime \prime}\left(\hat{z}_{1}\right) \leqslant 0$, which, together with (3.4b) again, yields $g\left(\hat{\mathbf{u}}\left(\hat{z}_{1}\right)\right) \geqslant \beta$. In view of the mean-value theorem and the fact that $g(\mathbf{u})$ is increasing in $0<\mathbf{u}<K$, there exists a $\hat{z}_{2} \in\left(\hat{z}_{0}, \hat{z}_{1}\right)$ such that $\hat{\mathbf{u}}^{\prime}\left(\hat{z}_{2}\right) \geqslant 0$. However, since $\hat{\mathbf{v}}\left(\hat{z}_{2}\right)>M_{1}$, we have $\hat{\mathbf{u}}^{\prime}\left(\hat{z}_{2}\right)<0$ by lemma 3.6. This is a contradiction, thus completing the proof of this lemma.

Lemma 3.8. There exist positive constants $k_{1}$ and $M_{2}>M_{1}$ such that for $z \geqslant 0$ with $\hat{\mathbf{v}}(z) \geqslant M_{2}$, we have $\hat{\mathbf{v}}(z) \leqslant-k_{1} \hat{\mathbf{u}}^{\prime}(z)$. In the following, we retain the notations $k_{1}$ and $M_{2}$.
Proof. To begin with, we set up some notations. Set $u^{b}=\inf \{\hat{\mathbf{u}}(z): z \in \mathbb{R}\}$. According to lemma 3.5, we have $u^{b}>0$, which, together with the monotonicity of $g$, yields $g(\hat{\mathbf{u}}(z)) \geqslant g\left(u^{b}\right)>0$ for all $z \in \mathbb{R}$. Set

$$
k_{1}:=\frac{4 \gamma \delta \beta}{g\left(u^{b}\right) c} .
$$

Since

$$
\frac{c}{4} \mathrm{v}-c v_{\beta} \ln \frac{\mathrm{v}}{v_{\beta}} \rightarrow \infty \text { as } \mathrm{v} \rightarrow \infty
$$

there exists a large $M_{2}>M_{1}$ such that

$$
\begin{equation*}
\frac{c}{4} \hat{\mathbf{v}}(z)-c v_{\beta} \ln \frac{\hat{\mathbf{v}}(z)}{v_{\beta}} \geqslant \gamma c \beta \int_{u_{\beta}}^{K} \frac{1}{g(\xi)} \mathrm{d} \xi+2|\mathcal{L}(\chi(0))|, \tag{3.21}
\end{equation*}
$$

for all $z$ with $\hat{\mathbf{v}}(z) \geqslant M_{2}$. Now we set $\mathcal{Z}:=\left\{z \geqslant 0: \hat{\mathbf{v}}(z) \geqslant M_{2}\right\}$.
Next we estimate $\mathcal{L}_{1}(\chi(z))$ for $z \in \mathcal{Z}$. From lemma 3.6, we have $\hat{\mathbf{w}}(z)=\hat{\mathbf{u}}^{\prime}(z)<0$ for $z \in \mathcal{Z}$. Together with the fact that $g(\hat{\mathbf{u}}(z)) \geqslant g\left(u^{b}\right)$ for $z \in \mathbb{R}$, he following inequality holds for $z \in \mathcal{Z}$,

$$
\begin{align*}
\mathcal{L}_{1}(\chi(z)) & =-\gamma \delta \hat{\mathbf{w}}(z)+\gamma c \hat{\mathbf{u}}(z)+\gamma \delta \beta \frac{\hat{\mathbf{w}}(z)}{g(\hat{\mathbf{u}}(z))}-\gamma c \beta \int_{u_{\beta}}^{\hat{\mathbf{u}}(z)} \frac{1}{g(\xi)} \mathrm{d} \xi, \\
& \geqslant \frac{\gamma \delta \beta}{g\left(u^{b}\right)} \cdot \hat{\mathbf{w}}(z)-\gamma c \beta \int_{u_{\beta}}^{K} \frac{1}{g(\xi)} \mathrm{d} \xi . \tag{3.22}
\end{align*}
$$

Now we turn to estimate $\mathcal{L}_{2}(\chi(z))$ for $z \in \mathcal{Z}$. Indeed, since $v_{\beta} / \hat{\mathbf{v}}(z)<1$ for $z \in \mathcal{Z}$, we can use (3.9) to deduce that, for $z \in \mathcal{Z}$, it holds that

$$
\begin{align*}
\mathcal{L}_{2}(\chi(z)) & =-\hat{\mathbf{y}}(z)+c \hat{\mathbf{v}}(z)+v_{\beta} \frac{\hat{\mathbf{y}}(z)}{\hat{\mathbf{v}}(z)}-c v_{\beta} \ln \frac{\hat{\mathbf{v}}(z)}{v_{\beta}} \\
& =\left(-\left(1-\frac{v_{\beta}}{\hat{\mathbf{v}}(z)}\right) \hat{\mathbf{y}}(z)+\frac{c}{2} \hat{\mathbf{v}}(z)\right)+\frac{c}{4} \hat{\mathbf{v}}(z)+\left(\frac{c}{4} \hat{\mathbf{v}}(z)-c v_{\beta} \ln \frac{\hat{\mathbf{v}}(z)}{v_{\beta}}\right) \\
& \geqslant \frac{c}{4} \hat{\mathbf{v}}(z)+\left(\frac{c}{4} \hat{\mathbf{v}}(z)-c v_{\beta} \ln \frac{\hat{\mathbf{v}}(z)}{v_{\beta}}\right) \quad(\text { using (3.9)) } \\
& \geqslant \frac{c}{4} \hat{\mathbf{v}}(z)+\left(\gamma c \beta \int_{u_{\beta}}^{K} \frac{1}{g(\xi)} \mathrm{d} \xi+2|\mathcal{L}(\chi(0))|\right) \quad(\text { using (3.21)). } \tag{3.23}
\end{align*}
$$

In view of (3.22)-(3.23) and the definition of $k_{1}$, we have that, for $z \in \mathcal{Z}$,

$$
\begin{align*}
\mathcal{L}(\chi(z)) & =\mathcal{L}_{1}(\chi(z))+\mathcal{L}_{2}(\chi(z)) \\
& \geqslant \frac{c}{4} \cdot\left(\hat{\mathbf{v}}(z)+k_{1} \hat{\mathbf{w}}(z)\right)+2|\mathcal{L}(\chi(0))| . \tag{3.24}
\end{align*}
$$

Now we are ready to show the assertion of this lemma. For contradiction, we assume that there exists a $\hat{z}_{0} \in \mathcal{Z}$ such that $\hat{\mathbf{v}}\left(\hat{z}_{0}\right)>-k_{1} \hat{\mathbf{u}}^{\prime}\left(\hat{z}_{0}\right)$. Then (3.24) immediately gives $\mathcal{L}\left(\chi\left(\hat{z}_{0}\right)\right)>2|\mathcal{L}(\chi(0))|$, which contradicts (3.8). Hence the proof is completed.
 Specifically, we state it in the following lemma.

Lemma 3.9. The solution $(\hat{\mathbf{u}}, \hat{\mathbf{v}})$ cannot satisfy the inequality $\lim _{\inf }^{z \rightarrow \infty} ⿵ ⺆ \hat{\mathbf{v}}(z)<\lim \sup _{z \rightarrow \infty}$ $\hat{\mathbf{v}}(z)=\infty$.
 view of lemmas 3.6 and 3.7 , we can choose positive numbers $\hat{z}_{0}$ and $\hat{z}_{1}$ such that $\hat{\mathbf{v}}\left(\hat{z}_{0}\right)=M_{2}$, $\hat{\mathbf{v}}^{\prime}(z) \geqslant 0$ for $z \in\left[\hat{z}_{0}, \hat{z}_{1}\right), \hat{\mathbf{v}}^{\prime}\left(\hat{z}_{1}\right)=0, \hat{\mathbf{u}}^{\prime}(z)<0$ for $z \in\left[\hat{z}_{0}, \hat{z}_{1}\right]$, and

$$
\begin{equation*}
c \hat{\mathbf{v}}\left(\hat{z}_{1}\right)>c M_{2}+\gamma k_{1} \int_{0}^{K}|g(\hat{\mathbf{u}})-\beta| \mathrm{d} \hat{\mathbf{u}} . \tag{3.25}
\end{equation*}
$$

## Hence $\left[\hat{z}_{0}, \hat{z}_{1}\right] \subset \mathcal{Z}:=\left\{z \geqslant 0: \hat{\mathbf{v}}(z) \geqslant M_{2}\right\}$.

Integrating (3.4b) from $\hat{z}_{0}$ to $\hat{z}_{1}$ and rearranging the resulting equation, we have

$$
c\left(\hat{\mathbf{v}}\left(\hat{z}_{1}\right)-M_{2}\right)+\hat{\mathbf{v}}^{\prime}\left(\hat{z}_{0}\right)=\int_{\hat{z}_{0}}^{\hat{z}_{1}} \gamma(g(\hat{\mathbf{u}}(\xi))-\beta) \hat{\mathbf{v}}(\xi) \mathrm{d} \xi .
$$

Recall that $\hat{\mathbf{v}}\left(\hat{z}_{1}\right)>M_{2}$ and $\hat{\mathbf{v}}^{\prime}\left(\hat{z}_{0}\right)>0$. Together with lemma 3.8, it follows from the above equation that

$$
\begin{align*}
c\left(\hat{\mathbf{v}}\left(\hat{z}_{1}\right)-M_{2}\right)+\hat{\mathbf{v}}^{\prime}\left(\hat{z}_{0}\right) & \leqslant \gamma \int_{\hat{z}_{0}}^{\hat{z}_{1}}|g(\hat{\mathbf{u}}(\xi))-\beta| \hat{\mathbf{v}}(\xi) \mathrm{d} \xi \\
& \leqslant-\gamma k_{1} \int_{\hat{z}_{0}}^{\hat{z}_{1}}|g(\hat{\mathbf{u}}(\xi))-\beta| \hat{\mathbf{u}}^{\prime}(\xi) \mathrm{d} \xi \\
& =\gamma k_{1} \int_{\hat{\mathbf{u}}\left(\hat{z_{1}}\right)}^{\hat{\mathbf{z}}\left(\hat{z}_{0}\right)}|g(\hat{\mathbf{u}})-\beta| \mathrm{d} \hat{\mathbf{u}} \\
& \leqslant \gamma k_{1} \int_{0}^{K}|g(\hat{\mathbf{u}})-\beta| \mathrm{d} \hat{\mathbf{u}} . \tag{3.26}
\end{align*}
$$

Since $\hat{\mathbf{v}}^{\prime}\left(\hat{z}_{0}\right) \geqslant 0$, (3.26) implies that

$$
c\left(\hat{\mathbf{v}}\left(\hat{z}_{1}\right)-M_{2}\right) \leqslant \gamma k_{1} \int_{0}^{K}|(g(\hat{\mathbf{u}})-\beta)| \mathrm{d} \hat{\mathbf{u}},
$$

which contradicts (3.25). Hence the proof of this lemma is completed.

### 3.3. Estimate of the derivative of $\hat{\mathbf{u}}$

In this section, we give the estimate for the derivative of $\hat{\mathbf{u}}$, which, together with (3.9) and (3.10), in turn give the construction of the open bounded set $\mathcal{D}$. To do this, we recall from section 3.2 that $\hat{\mathbf{v}}$ is bounded over $\mathbb{R}$, and hence that there exists a positive constant $B$ such that $\hat{\mathbf{v}}(z)<B$ for all $z \in \mathbb{R}$. In the following, we retain the notation $B$. Now the estimate for the derivative of $\hat{\mathbf{u}}$ is given in the following lemma.

Lemma 3.10. There exist positive constants $L_{i}, i=1,2$, such that

$$
\begin{equation*}
-L_{1} g(\hat{\mathbf{u}}(z))<\hat{\mathbf{u}}^{\prime}(z)<L_{2} g(\hat{\mathbf{u}}(z)) \tag{3.27}
\end{equation*}
$$

for all $z \geqslant 0$. In the following, we retain the notation $L_{i}, i=1,2$.
Proof. (1) We show that $-L_{1} g(\hat{\mathbf{u}}(z))<\hat{\mathbf{u}}^{\prime}(z)$ for all $z \geqslant 0$, if $L_{1}$ is a sufficiently large constant such that $-L_{1} g(\hat{\mathbf{u}}(0))<\hat{\mathbf{u}}^{\prime}(0)$ and $L_{1} \geqslant 2 B / c$.

Let

$$
\Phi_{1}(z):=\hat{\mathbf{u}}^{\prime}(z)+L_{1} g(\hat{\mathbf{u}}(z)) .
$$

It suffices to show that $\Phi_{1}(z)>0$ for all $z \geqslant 0$. Note that $\Phi_{1}(0)>0$. For contradiction, we assume that there exists $\hat{z}_{1}>0$ such that $\Phi_{1}\left(\hat{z}_{1}\right)=0$ and $\Phi_{1}^{\prime}\left(\hat{z}_{1}\right) \leqslant 0$. Then there are two possibilities: either

$$
\begin{equation*}
\Phi_{1}(z) \leqslant 0, \forall z \geqslant \hat{z}_{1} \tag{3.28}
\end{equation*}
$$

or

$$
\begin{equation*}
\Phi_{1}\left(\hat{z}_{2}\right)=0 \text { and } \Phi_{1}^{\prime}\left(\hat{z}_{2}\right) \geqslant 0, \tag{3.29}
\end{equation*}
$$

for some $\hat{z}_{2} \geqslant \hat{z}_{1}$. For the first case, (3.28) gives

$$
c \hat{\mathbf{u}}^{\prime}(z) \leqslant-2 B g(\hat{\mathbf{u}}(z)), \forall z \geqslant \hat{z}_{1} .
$$

Together with the fact that $0 \leqslant \hat{\mathbf{v}} \leqslant B$ and $q(\hat{\mathbf{u}}) p(\hat{\mathbf{u}})>0$, we deduce from (3.4a) that

$$
\delta \hat{\mathbf{u}}^{\prime \prime}=c \hat{\mathbf{u}}^{\prime}+g(\hat{\mathbf{u}}) \hat{\mathbf{v}}-q(\hat{\mathbf{u}}) p(\hat{\mathbf{u}}) \leqslant-B g(\hat{\mathbf{u}})<0, \forall z \geqslant \hat{z}_{1},
$$

which implies that $\hat{\mathbf{u}}^{\prime}$ is decreasing in $\left[\hat{z}_{1}, \infty\right)$. Hence $\hat{\mathbf{u}}^{\prime}(z) \leqslant \hat{\mathbf{u}}^{\prime}\left(\hat{z}_{1}\right)=-L_{1} g\left(\hat{\mathbf{u}}\left(\hat{z}_{1}\right)\right)<0$ for all $z \geqslant \hat{z}_{1}$, which contradicts the boundedness of $\hat{\mathbf{u}}$. For the second case, (3.29) yields that

$$
\begin{equation*}
\hat{\mathbf{u}}^{\prime}\left(\hat{z}_{2}\right)=-L_{1} g\left(\hat{\mathbf{u}}\left(\hat{z}_{2}\right)\right) \tag{3.30}
\end{equation*}
$$

and

$$
\begin{equation*}
\hat{\mathbf{u}}^{\prime \prime}\left(\hat{z}_{2}\right) \geqslant-L_{1} g^{\prime}\left(\hat{\mathbf{u}}\left(\hat{z}_{2}\right)\right) \hat{\mathbf{u}}^{\prime}\left(\hat{z}_{2}\right) . \tag{3.31}
\end{equation*}
$$

Using (3.4a), we deduce that

$$
\begin{aligned}
0= & \delta \hat{\mathbf{u}}^{\prime \prime}\left(\hat{z}_{2}\right)-c \hat{\mathbf{u}}^{\prime}\left(\hat{z}_{2}\right)+q\left(\hat{\mathbf{u}}\left(\hat{z}_{2}\right)\right) p\left(\hat{\mathbf{u}}\left(\hat{z}_{2}\right)\right)-g\left(\hat{\mathbf{u}}\left(\hat{z}_{2}\right)\right) \hat{\mathbf{v}}\left(\hat{z}_{2}\right) \\
\geqslant & -\delta L_{1} g^{\prime}\left(\hat{\mathbf{u}}\left(\hat{z}_{2}\right)\right) \hat{\mathbf{u}}^{\prime}\left(\hat{z}_{2}\right)+c L_{1} g\left(\hat{\mathbf{u}}\left(\hat{z}_{2}\right)\right)-g\left(\hat{\mathbf{u}}\left(\hat{z}_{2}\right)\right) B \\
& (\text { by }(3.30) \text { and }(3.31), \text { and the fact that } q(\hat{\mathbf{u}}) p(\hat{\mathbf{u}})>0, g(\hat{\mathbf{u}})>0 \text { and } 0<\hat{\mathbf{v}} \leqslant B) \\
\geqslant & \delta L_{1}^{2} g^{\prime}\left(\hat{\mathbf{u}}\left(\hat{z}_{2}\right)\right) g\left(\hat{\mathbf{u}}\left(\hat{z}_{2}\right)\right)+g\left(\hat{\mathbf{u}}\left(\hat{z}_{2}\right)\right) B \quad\left(\text { by }(3.30) \text { and definition of } L_{1}\right) \\
> & 0, \quad\left(\text { use the fact that } g(\hat{\mathbf{u}})>0 \text { and } g^{\prime}(\hat{\mathbf{u}})>0\right),
\end{aligned}
$$

a contradiction again.
(2) We show that there exists a positive constant $L_{2}$ such that

$$
\begin{equation*}
\hat{\mathbf{u}}^{\prime}(z)<L_{2} g(\hat{\mathbf{u}}(z)), \forall z \geqslant 0 . \tag{3.32}
\end{equation*}
$$

To this end, we observe

$$
\begin{align*}
& -\gamma c \beta \int_{u_{s}}^{u} \frac{1}{g(\xi)} \mathrm{d} \xi \geqslant-\gamma c \beta \int_{u_{s}}^{K} \frac{1}{g(\xi)} \mathrm{d} \xi \text { for } u \in\left[u_{s}, K\right] \\
& -\gamma c \beta \int_{u_{s}}^{u} \frac{1}{g(\xi)} \mathrm{d} \xi=\gamma c \beta \int_{u}^{u_{s}} \frac{1}{g(\xi)} \mathrm{d} \xi \geqslant 0 \text { for } u \in\left[0, u_{s}\right] . \tag{3.33}
\end{align*}
$$

Since $\hat{\mathbf{v}}$ is bounded, one can easily use (3.9) and (3.10) to deduce that $\mathcal{L}_{2}(\chi(\cdot))$ is bounded below on $[0, \infty)$. This, together with (3.8), implies that $\mathcal{L}_{1}(\chi(\cdot))$ is bounded above on $[0, \infty)$. Recall that

$$
\mathcal{L}_{1}(\chi(z))=-\gamma\left(\delta \hat{\mathbf{u}}^{\prime}(z)-c \hat{\mathbf{u}}(z)-\delta \beta \frac{\hat{\mathbf{u}}^{\prime}(z)}{g(\hat{\mathbf{u}}(z))}+c \beta \int_{u_{s}}^{\hat{\mathbf{u}}(z)} \frac{1}{g(\xi)} \mathrm{d} \xi\right) .
$$

Then, using (3.11) and (3.33), the upper boundedness of $\mathcal{L}_{1}(\chi(\cdot))$, and the fact that $0<\hat{\mathbf{u}}<K$ on $\mathbb{R}$, we infer that $\hat{\mathbf{u}}^{\prime}(z) / g(\hat{\mathbf{u}}(z))$ is bounded above over $z \geqslant 0$. Hence there exists a positive constant $L_{2}$ such that (3.32) holds. The proof of this lemma is thus completed.

### 3.4. Construction of the set $\mathcal{D}$ and the proof of lemma 3.1

Now we are in a position to construct the open bounded set $\mathcal{D}$ of system (3.6). Indeed, set
$\mathcal{D}:=\left\{(\mathrm{u}, \mathrm{w}, \mathrm{v}, \mathrm{y}) \mid 0<\mathrm{u}<K, 0<\mathrm{v}<B,-L_{1} g(\mathrm{u})<\mathrm{w}<L_{2} g(\mathrm{u}),-\frac{2 \gamma \beta}{c} \mathrm{v}<\mathrm{y}<c \mathrm{v}\right\}$.
Then (3.5), the boundedness of $\hat{\mathbf{v}}$ (see section 3.2), lemma 3.10, and (3.9)-(3.10) assert that the solution $\chi(z)=(\hat{\mathbf{u}}(z), \hat{\mathbf{w}}(z), \hat{\mathbf{v}}(z), \hat{\mathbf{y}}(z))$ with $(\hat{\mathbf{w}}, \hat{\mathbf{y}})=\left(\hat{\mathbf{u}}^{\prime}, \hat{\mathbf{v}}^{\prime}\right)$ of system (3.6) is positively invariant in $\mathcal{D}$ for all $z \geqslant 0$.

Recall the Lyapunov function $\mathcal{L}$ defined by (3.7). In view of (3.8), the orbital derivative of $\mathcal{L}$ along $\chi(z)$ is non-positive. One can easily see that $\mathcal{L}$ is continuous, and, by lemma 3.10, (3.9)-(3.10), and the boundedness of $\hat{\mathbf{u}}$ and $\hat{\mathbf{v}}$, that $\mathcal{L}$ is bounded below on $\mathcal{D}$. Taken together, it follows from LaSalle's invariance principle that $\chi(z) \rightarrow\left(u_{\beta}, 0, v_{\beta}, 0\right)$ as $z \rightarrow \infty$, and so
$(U, V)(-\infty)=(\hat{\mathbf{u}}, \hat{\mathbf{v}})(\infty)=\left(u_{\beta}, v_{\beta}\right)$. This completes the proof of lemma 3.1, and hence the proof of theorem 1.1.

## 4. Evolution of disturbance

In this section, we will establish the assertion of theorem 1.2. The idea of the proof is to construct wave-like functions propagating with the speed $c_{\lambda}$.

### 4.1. Global existence and uniqueness of solutions

First, we establish the global existence and uniqueness of solutions of system (1.7) with the initial condition (1.11). For this, we need to obtain an a priori bound for solutions.

Lemma 4.1. Suppose that hypotheses $\mathbf{( H 1 ) - ( H 4 ) ~ h o l d . ~ L e t ~}(u, v)$ be the solution of system (1.7) on $\mathbb{R} \times[0, T]$ with bounded initial data $\left(u_{0}, \nu_{0}\right)$ satisfying

$$
0 \leqslant u_{0} \leqslant K, v_{0} \geqslant 0 .
$$

Then there exist positive constants $\hat{C}_{0}$ and $\omega$, independent of $T$, such that

$$
0 \leqslant u(x, t) \leqslant K \text { and } 0 \leqslant v(x, t) \leqslant \hat{C}_{0} \mathrm{e}^{\omega t} \text { for all } x \in \mathbb{R} \text { and } t \in[0, T] .
$$

Proof. First, following the argument of [39, p. 270], we have that

$$
0 \leqslant u(x, t) \leqslant K \text { and } v(x, t) \geqslant 0 \text { for all } x \in \mathbb{R} \text { and } t \in[0, T]
$$

To deduce the upper bound for $v$, we use the above inequality for $(u, v)$ and equation (1.7b) to obtain that $v(x, t)$ is a sub-solution of the equation

$$
\begin{equation*}
v_{t}=v_{x x}+\gamma[g(K)-\beta] v, \tag{4.1}
\end{equation*}
$$

for $(x, t) \in \mathbb{R} \times(0, T]$. On the other hand, it is easy to check that $\bar{v}(x, t):=\hat{C}_{0} \mathrm{e}^{\omega t}$ is a supersolution of (4.1) with $\bar{v}(x, 0) \geqslant v_{0}(x)$, provided that $\hat{C}_{0}$ and $\omega$ are sufficiently large constants that $\hat{C}_{0} \geqslant\left\|v_{0}\right\|_{\infty}$ and $\omega \geqslant \gamma[g(K)-\beta]$. Then it follows from the maximum principle that $v(x, t) \leqslant \bar{v}(x, t)$ for $(x, t) \in \mathbb{R} \times(0, T]$. This completes the proof of this lemma.

With the help of lemma 4.1, we can apply the standard argument (see [39, p. 271]) and use [48, theorem 14.4] and [48, lemma 14.3] to obtain the global existence, uniqueness, and regularity of solutions of system (1.7) with the initial condition (1.11).

### 4.2. Comparison lemmas

In this section, we show that if the $v$ component of initial data (1.11) is squeezed between $V_{\lambda}^{-}\left(\cdot ; x_{1}, x_{0}\right)$ and $V_{\lambda}^{+}\left(\cdot ; x_{0}\right)$ for some $x_{0}$ and $x_{1} \in \mathbb{R}$, then the solution of system (1.7) with initial data (1.11) is squeezed between $\left(U_{\lambda}^{-}\left(x-c_{\lambda} t ; x_{0}\right), V_{\lambda}^{-}\left(x-c_{\lambda} t ; x_{1}, x_{0}\right)\right)$ and $\left(K, V_{\lambda}^{+}\left(x-c_{\lambda} t ; x_{0}\right)\right)$ for all $t>0$.

Lemma 4.2. Suppose that hypotheses $\mathbf{( H 1 ) - ( \mathbf { H } )}$ hold. Let $(u, v)$ be the solution of system (1.7) on $\mathbb{R} \times[0, \infty)$ with bounded initial data $\left(u_{0}, v_{0}\right)$ satisfying

$$
\begin{equation*}
u_{0}(x)=K \text { and } V_{\lambda}^{-}\left(x ; x_{1}, x_{0}\right) \leqslant v_{0}(x) \leqslant V_{\lambda}^{+}\left(x ; x_{0}\right) \tag{4.2}
\end{equation*}
$$

for all $x \in \mathbb{R}$, and for some $x_{0}, x_{1} \in \mathbb{R}$ and $\lambda \in(0, \sqrt{\gamma[g(K)-\beta]})$. Then we have

$$
\begin{align*}
& U_{\lambda}^{-}\left(x-c_{\lambda} t ; x_{0}\right) \leqslant u(x, t) \leqslant K  \tag{4.3a}\\
& V_{\lambda}^{-}\left(x-c_{\lambda} t ; x_{1}, x_{0}\right) \leqslant v(x, t) \leqslant V_{\lambda}^{+}\left(x-c_{\lambda} t ; x_{0}\right) \tag{4.3b}
\end{align*}
$$

for all $(x, t) \in \mathbb{R} \times[0, \infty)$.
Proof. To begin with, we show $v(x, t) \leqslant V_{\lambda}^{+}\left(x-c_{\lambda} t ; x_{0}\right)$ for all $(x, t) \in \mathbb{R} \times[0, \infty)$. For this, lemma 4.1 implies that $0 \leqslant u(x, t) \leqslant K$ and $v(x, t) \geqslant 0$ for all $(x, t) \in \mathbb{R} \times[0, \infty)$. In view of equation (1.7b), it follows that $v(x, t)$ is a sub-solution of the equation

$$
\begin{equation*}
v_{t}=v_{x x}+\gamma[g(K)-\beta] v . \tag{4.4}
\end{equation*}
$$

Note that $V_{\lambda}^{+}\left(x-c_{\lambda} t ; x_{0}\right)$ is a solution of equation (4.4) by lemma 2.2. Now, with the use of the fact that $v_{0}(\cdot) \leqslant V_{\lambda}^{+}\left(\cdot ; x_{0}\right)$ on $\mathbb{R}$, we can employ the comparison principle to equation (4.4) to obtain that $v(x, t) \leqslant V_{\lambda}^{+}\left(x-c_{\lambda} t ; x_{0}\right)$ for all $(x, t) \in \mathbb{R} \times[0, \infty)$.

Next we establish $u(x, t) \geqslant U_{\lambda}^{-}\left(x-c_{\lambda} t ; x_{0}\right)$ for all $(x, t) \in \mathbb{R} \times[0, \infty)$. To do this, we recall the definition of $z_{0}$ given in section 2.2. For each $t \geqslant 0$, we set $x_{t}:=c_{\lambda} t+z_{0}$. Then for $(x, t)$ with $x \leqslant x_{t}$ and $t \geqslant 0$, since $U_{\lambda}^{-}\left(x-c_{\lambda} t ; x_{0}\right)=0$, it is obvious that $u(x, t) \geqslant U_{\lambda}^{-}\left(x-c_{\lambda} t ; x_{0}\right)$. Next we consider the region $\Omega_{1}:=\left\{(x, t): x \geqslant x_{t}, t \geqslant 0\right\}$. Using equation (1.7a) and the right-handside inequality of (4.3a), we know that $u(x, t)$ is a super-solution of the equation

$$
\begin{equation*}
u_{t}=\delta u_{x x}+q(u) p(u)-g(u) V_{\lambda}^{+}\left(x-c_{\lambda} t ; x_{0}\right) \tag{4.5}
\end{equation*}
$$

in $\Omega_{1}$. Note that $U_{\lambda}^{-}\left(x-c_{\lambda} t ; x_{0}\right)$ is a sub-solution of equation (4.5) in $\Omega_{1}$ by lemma 2.3. Thus, together with the fact that $u_{0}(\cdot) \geqslant U_{\lambda}^{-}\left(\cdot ; x_{0}\right)$ on $\mathbb{R}$ and $u(x, t) \geqslant U_{\lambda}^{-}\left(x-c_{\lambda} t ; x_{0}\right)$ for $(x, t)=$ $\left(x_{t}, t\right)$ and $t \geqslant 0$, we can employ the comparison principle to equation (4.5) to deduce that $u(x, t) \geqslant U_{\lambda}^{-}\left(x-c_{\lambda} t ; x_{0}\right)$ for all $(x, t) \in \Omega_{1}$. Hence we conclude that $u(x, t) \geqslant U_{\lambda}^{-}\left(x-c_{\lambda} t ; x_{0}\right)$ for all $(x, t) \in \mathbb{R} \times[0, \infty)$.

Finally we prove $v(x, t) \geqslant V_{\lambda}^{-}\left(x-c_{\lambda} t ; x_{1}, x_{0}\right)$ for all $(x, t) \in \mathbb{R} \times[0, \infty)$. To begin with, we recall the definition of $z_{1}$ given in section 2.2 , and set $\tilde{x}_{t}:=c_{\lambda} t+z_{1}$ for each $t \geqslant 0$. Then, in view of the observation that $V_{\lambda}^{-}\left(x-c_{\lambda} t ; x_{1}, x_{0}\right)=0$ for $(x, t)$ with $x \leqslant \tilde{x}_{t}$ and $t \geqslant 0$, we have that $v(x, t) \geqslant V_{\lambda}^{-}\left(x-c_{\lambda} t ; x_{1}, x_{0}\right)$ for $x \leqslant \tilde{x}_{t}$ and $t \geqslant 0$. Next we consider the region $\Omega_{2}:=\left\{(x, t): x \geqslant \tilde{x}_{t}, t \geqslant 0\right\}$. With the use of the left-hand-side inequality of equation (4.3) and equation (1.7b), it follows that $v(x, t)$ is a super-solution of the equation

$$
\begin{equation*}
v_{t}=v_{x x}+\gamma\left[g\left(U_{\lambda}^{-}\left(x-c_{\lambda} t ; x_{0}\right)\right)-\beta\right] v \tag{4.6}
\end{equation*}
$$

in $\Omega_{2}$. Note that $V_{\lambda}^{-}\left(x-c_{\lambda} t ; x_{1}, x_{0}\right)$ is a sub-solution of equation (4.6) in $\Omega_{2}$ by lemma 2.4, and that $v_{0}(\cdot) \geqslant V_{\lambda}^{-}\left(\cdot ; x_{1}, x_{0}\right)$ on $\mathbb{R}$ and $v(x, t) \geqslant V_{\lambda}^{-}\left(x-c_{\lambda} t ; x_{1}, x_{0}\right)$ for $(x, t)=\left(\tilde{x}_{t}, t\right)$ and $t \geqslant 0$. Therefore, the comparison principle yields that $v(x, t) \geqslant V_{\lambda}^{-}\left(x-c_{\lambda} t ; x_{1}, x_{0}\right)$ for $x \geqslant \tilde{x}_{t}$ and $t \geqslant 0$. Hence we have that $v(x, t) \geqslant V_{\lambda}^{-}\left(x-c_{\lambda} t ; x_{1}, x_{0}\right)$ for all $(x, t) \in \mathbb{R} \times[0, \infty)$, thereby completing the proof of this lemma.

### 4.3. The proof of theorem 1.2

With the aid of lemma 4.2, we are ready to establish theorem 1.2. We will only establish assertions (i) of theorem 1.2, since the other assertions can be shown similarly. Now, in view of
condition (1.11), sufficiently large numbers $x_{0}$ and $\left|x_{1}\right|$ with $x_{1}<0$ can be chosen in such a way that condition (4.2) in lemma 4.2 with $\lambda:=\lambda^{+}$holds. Then we can employ lemma 4.2 and the definitions of $U_{\lambda^{+}}^{-}$and $V_{\lambda^{+}}^{ \pm}$to deduce that

$$
\begin{align*}
& \max \left\{0, K-M \mathrm{e}^{-\alpha\left(x-c_{\lambda}+t\right)}\right\} \leqslant u(x, t) \leqslant K \\
& \max \left\{0, \mathrm{e}^{-\lambda^{+}\left(x-c_{\lambda} t-x_{1}\right)}-L \mathrm{e}^{-\left(\lambda^{+}+\eta\right)\left(x-c_{\lambda}+t\right)}\right\} \leqslant v(x, t) \leqslant \mathrm{e}^{-\lambda^{+}\left(x-c_{\lambda} t-x_{0}\right)} \tag{4.7}
\end{align*}
$$

for all $(x, t) \in \mathbb{R} \times[0, \infty)$, where $M=M\left(x_{0}\right), L=L\left(x_{1}, x_{0}\right)$ and $\eta$ are positive constants defined in section 2.2. Set $\psi_{\lambda^{+}}^{+}(x):=\max \left\{0, K-M \mathrm{e}^{-\alpha x}\right\}, \phi_{\lambda^{+}}^{+}(x):=\max \left\{0, \mathrm{e}^{-\lambda^{+}\left(x-x_{1}\right)}-L \mathrm{e}^{-\left(\lambda^{+}+\eta\right) x}\right\}$ and $\varsigma_{\lambda^{+}}^{+}(x):=\mathrm{e}^{-\lambda^{+}\left(x-x_{0}\right)}$. Then inequality (4.7) immediately implies assertion (i) of theorem 1.2 , thereby completing the proof of theorem 1.2.

## 5. Conclusion and discussion

In this paper, we have established a family of travelling waves with minimum wave speed for a class of predator-prey models. We have also shown that, for the initial distribution where the prey species is at the level of the carrying capacity $K$, and the predator species has positive, compactly supported perturbation of the zero state with exponentially small tails, the corresponding solution will evolve into a pair of diverging waves whose speeds are determined by the tail behaviour of the initial data.

Recently, Holzer and Scheel [20, 21] investigated a class of two-component coupled Fisher-KPP equations where one species decouples from the other species. They showed that the evolution of positive, compactly supported perturbations of the unstable homogeneous steady state can give rise to a pair of diverging waves propagating with different speeds for different species. This phenomenon is called anomalous spreading. Their theory suggests that anomalous spreading arises due to poles of the pointwise Green's function of the linearized system around the unstable homogeneous steady state. Specifically, the linearized system associated with the systems of Holzer and Scheel in the moving coordinate frame $\xi=x-s t$ with $s>0$ reads

$$
\begin{align*}
& u_{t}=\mathrm{d} u_{\xi \xi}+s u_{\xi}+\alpha u+\beta v, \\
& v_{t}=v_{\xi \xi}+s v_{\xi}+v . \tag{5.1}
\end{align*}
$$

On the other hand, the linearization of the unstable homogeneous steady state $(K, 0)$ for system (1.7) leads to the following linear system:

$$
\begin{align*}
& u_{t}=\mathrm{d} u_{\xi \xi}+s u_{\xi}+q(K) p^{\prime}(K) u-g(K) v, \\
& v_{t}=v_{\xi \xi}+s v_{\xi}+\gamma(g(K)-\beta) v . \tag{5.2}
\end{align*}
$$

According to assumptions (H1)-(H3), we have

$$
q(K) p^{\prime}(K)<0, g(K)>0, \text { and } \gamma(g(K)-\beta)>0 .
$$

Since Holzer and Scheel in [20,21] made an assumption that the parameters $\alpha$ and $\beta$ in (5.1) are positive, their result cannot be applied to model (1.7). Hence anomalous spreading may not occur in system (1.7) under assumptions (H1)-(H3).

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## Appendix

In this appendix, we collect some a priori estimates in [16] for solutions of the inhomogeneous linear equation

$$
\begin{equation*}
w^{\prime \prime}(z)+A w^{\prime}(z)+f(z) w(z)=h(z) . \tag{A.1}
\end{equation*}
$$

Lemma A.1. (Lemma 3.2 of [16])
Let $A$ be a positive constant and let $f$ and $h$ be continuous functions on [a, b]. Suppose that $w \in C([a, b]) \cap C^{2}((a, b))$ satisfies differential equation (2.17) in $(a, b)$ and $w(a)=w(b)=0$. If

$$
-C_{1} \leqslant f \leqslant 0 \text { and }|h| \leqslant C_{2} \text { on }[a, b],
$$

for some constants $C_{1}, C_{2}$, then there exists a positive constant $C_{3}$, depending only on $A, C_{1}$, and the length of the interval $[a, b]$, such that

$$
\|w\|_{C([a, b])} \leqslant C_{2} C_{3} .
$$

Lemma A.2. (Lemma 3.3 of [16])
Let $A, f$, and $h$ be as in lemma A.1. Suppose that $w \in C([a, b]) \cap C^{2}((a, b))$ satisfies (2.17) in $(a, b)$. If $\|w\|_{C([a, b])} \leqslant C_{0}$ for some constant $C_{0}$, then there exists a positive constant $C_{4}$, depending only on $A, C_{0}, C_{1}, C_{2}$, and the length of the interval $[a, b]$, such that

$$
\left\|w^{\prime}\right\|_{C([a, b])} \leqslant C_{4} .
$$

## References

[1] Aronson D G and Weinberger H F 1975 Nonlinear Diffusion in Population Genetics, Combustion, and Nerve Pulse Propagation (Partial Differential Equations and Related Topics, Lecture Notes in Mathematics vol 446) ed J A Goldstein (Berlin: Springer) pp 5-49
[2] Aronson D G and Weinberger H F 1978 Multidimensional nonlinear diffusion arising in population genetics Adv. Math. 30 33-76
[3] Baek H K, Kim S D and Kim P 2009 Permanence and stability of an Ivlev-type predator-prey system with impulsive control strategies Math. Comput. Modelling 50 1385-93
[4] Berestycki H, Hamel F, Kiselev A and Ryzhik L 2005 Quenching and propagation in KPP reactiondiffusion equations with a heat loss Arch. Ration. Mech. Anal. 178 57-80
[5] Bramson M 1978 Maximal displacement of branching Brownian motion Commun. Pure Appl. Math. 31 531-81
[6] Bramson M 1983 Convergence of solutions of the Kolmogorov equation to travelling waves Mem. Am. Math. Soc. 44285
[7] Cheng K S, Hsu S B and Lin S S 1981 Some results on global stability of a predator-prey model J. Math. Biol. 12 115-26
[8] Diekmann O 1979 Run for your life: a note on the asymptotic speed of propagation of an epidemic J. Differ. Equ. 33 58-73
[9] Dunbar S R 1983 Travelling wave solutions of diffusive Lotka-Volterra equations J. Math. Biol. 17 11-32
[10] Dunbar S R 1984 Traveling wave solutions of diffusive Lotka-Volterra equations: a heteroclinic connection in $\mathbb{R}^{4}$ Trans. Am. Math. Soc. 286 557-94
[11] Dunbar S R 1986 Traveling waves in diffusive predator-prey equations: periodic orbits and point-to-periodic heteroclinic orbits SIAM J. Appl. Math. 46 1057-78
[12] Ebert U and van Saarloos W 2000 Front propagation into unstable states: universal algebraic convergence towards uniformly translating pulled fronts Physics D 146 1-99
[13] Fife P C and McLeod J B 1977 The approach of solutions of non-linear diffusion equations to traveling front solutions Arch. Ration. Mech. Anal. 65 335-61
[14] Fisher R A 1937 The wave of advance of advantageous genes Ann. Eugenics 7 353-69
[15] Freedman H I 1980 Deterministic Mathematical Models in Population Ecology (New York: Dekker)
[16] Fu S-C 2014 The existence of traveling wave fronts for a reaction-diffusion system modelling the acidic nitrate-ferroin reaction Q. Appl. Math. 72 649-64
[17] Gardner R A 1984 Existence of traveling wave solutions of predator-prey systems via the connection index SIAM J. Appl. Math. 44 56-79
[18] Gilbarg D and Trundinger N S 1983 Elliptic Partial Differential Equations of Second Order (New York: Springer)
[19] Hartman P 1982 Ordinary Differential Equations (Basel: Birkhäuser)
[20] Holzer M 2014 Anomalous spreading in a system of coupled Fisher-KPP equations Physica D 270 1-10
[21] Holzer M and Scheel A 2014 Accelerated fronts in a two stage invasion process SIAM J. Math. Anal. 46 397-427
[22] Huang J, Lu G and Ruan S 2003 Existence of traveling wave solutions in a diffusive predator-prey model J. Math. Biol. 46 132-52
[23] Huang W 2012 Traveling wave solutions for a class of predator-prey systems J. Dyn. Differ. Equ. 24 633-44
[24] Hsu C-H, Yang C-R, Yang T-H and Yang T-S 2012 Existence of traveling wave solutions for diffusive predator-prey type systems J. Differ. Equ. 252 3040-75
[25] Ivlev V S 1961 Experimental Ecology of the Feeding of Fishes (New Haven, CT: Yale University)
[26] Kolmogorov A N, Petrovsky I G and Piscounov N S 1937 A study of the diffusion equation with increase in the amount of substance, and its application to a biological problem Bull. Mosc. Univ. Math. Mech. 1-26
[27] Kooij R E and Zegeling A 1996 A predator prey model with Ivlev's functional response J. Math. Anal. Appl. 198 473-89
[28] Lau K-S 1985 On the nonlinear diffusion equation of Kolmogorov, Petrovsky, and Piscounov J. Differ. Equ. 59 44-70
[29] Lewis M A and Kareiva P 1993 Allee dynamics and the spread of invading organisms Theor. Popul. Biol. 43 141-58
[30] Li W and Wu S 2008 Traveling waves in a diffusive predator-prey model with Holling type-III functional response Chaos Solitons Fractals 37 476-86
[31] Ling L and Wang W 2009 Dynamics of a Ivlev-type predator-prey system with constant rate harvesting Chaos Solitons Fractals 41 2139-53
[32] Lin X, Wu C and Weng P 2011 Traveling wave solutions for a predator-prey system with sigmoidal response function J. Dynam. Differ. Equ. 23 903-21
[33] May R 1972 Limit cycles in predator-prey communities Science 177 900-2
[34] May R 1974 Model Ecosystems (Princeton, NJ: Princeton University)
[35] Morozov A, Petrovskii S and Li B-L 2006 Spatiotemporal complexity of patchy invasion in a predator-prey system with the Allee effect J. Theor. Biol. 238 18-35
[36] Morozov A and Petrovskii S 2009 Excitable population dynamics, biological control failure, and spatiotemporal pattern formation in a model ecosystem Bull. Math. Biol. 71 863-87
[37] Murray J D 2004 Mathematical Biology. I: An Introduction (Berlin: Springer)
[38] Murray J D 2004 Mathematical Biology. II: Spatial Models and Biomedical Applications (Berlin: Springer)
[39] Needham D J and Merkin J H 1991 The development of travelling waves in a simple isothermal chemical system with general orders of autocatalysis and decay Phil. Trans. R. Soc. A 337 261-74
[40] Nisbet R M and Gurney W S C 1982 Modeling Fluctuating Populations (New York: Wiley)
[41] Okubo A and Levin S A 2001 Diffusion and Ecological Problems, Modern Prespectives (Berlin: Springer)
[42] Owen M R and Lewis M A 2001 How predation can slow, stop or reverse a prey invasion Bull. Math. Biol. 63 655-84
[43] Petrovskii S V, Malchow H, Hilker F M and Venturino E 2005 Patterns of patchy spread in deterministic and stochastic models of biological invasion and biological control Biol. Invasions 7 771-93
[44] Rosenzweig M L 1971 Paradox of enrichment: destabilization of exploitation ecosystems in ecological time Science 171 385-7
[45] Rudin W 1976 Principles of Mathematical Analysis (New York: McGraw-Hill)
[46] van Saarloos W 2003 Front propagation into unstable states Phys. Rep. 386 29-222
[47] Shigesada N and Kawasaki K 1997 Biological Invasions: Theory and Practice (Oxford: Oxford University Press)
[48] Smoller J 1983 Shock Waves and Reaction-Diffusion Equations (Berlin: Springer)
[49] Sugie J 1998 Two-parameter bifurcation in a predator-prey system of Ivlev type J. Math. Anal. Appl. 217 349-71
[50] Uchiyama K 1978 The behavior of solutions of some nonlinear diffusion equations for large time J. Math. Kyoto Univ. 18 453-508
[51] Wang J, Shi J and Wei J 2011 Predator-prey system with strong Allee effect in prey J. Math. Biol. 62 291-331
[52] Wang H and Wang W 2008 The dynamical complexity of a Ivlev-type prey-predator system with impulsive effect Chaos Solitons Fractals 38 1168-76
[53] Weinberger H F 1982 Long time behavior of a class of biological models SIAM J. Math. Anal. 13 353-96
[54] Xiang Z and Song X 2009 The dynamical behaviors of a food chain model with impulsive effect and Ivlev functional response Chaos Solitons Fractals 39 2282-93

