International Journal of Intelligent Technologies and Applied Statistics
Vol.7, No.1 (2014) pp.19–26, DOI: 10.6148/IJITAS.2014.0701.02

A Fast Numerical Method for Solving Large Sparse Linear Systems

Hsuan-Ku Liu[1*], Tien-Tsai Chang[2] and Kai-Yi Zeng[3]

[1]*Department of Mathematics and Information Education, National Taipei University of Education, Taipei, Taiwan*
[2]*Department of Mathematical Sciences, National Chengchi University, Taipei, Taiwan*
[3]*Department of Mathematics and Information Education, National Taipei University of Education, Taipei, Taiwan*

ABSTRACT

This paper applies the extrapolated homotopy method to solve large sparse linear systems. An optimal extrapolation parameter obtained in this paper makes the approximate solution convergent rapidly. Numerical results demonstrate that the homotopy method embedded with the optimal extrapolation parameter is more accurate than the direct and iterative numerical methods.

Keywords: Homotopy method; Large sparse linear systems; Richardson method

1. Introduction

Consider a linear system as the form:

$$Ax = b \tag{1}$$

Where A is a large sparse matrix and x and v are two vectors. The direct methods are spent a lot of computational time in solving a large system. For example, Liu [2] demonstrated that the computation times for the Gauss-Jordan elimination and the LU decomposition exceed 10^4 for a 1,000 × 1,000 linear system. Many numerical methods, such as iterative methods [3] and homotopy methods [1] and [2], are developed to find a solution for a large sparse linear system numerically. Keramati [1] first adapted to construct a homotopy solving a linear system in which the splitting matrix is given as an identity. In contrast to the iterative method, we

* Corresponding author: HKLiu.nccu@gmail.com

call this method as homotopy method. Liu [2] adapted the Richardson method, the Jacobi method, and the Gauss-Seidel method to select the splitting matrix.

In this paper, an extrapolation parameter is embedded in the HM. The major objective is to study the optimum extrapolation parameters such that the homotopy series converges rapidly for the large sparse system and the large sparse block system. When the matrices are symmetric positive definite, the optimum extrapolation parameters are obtained for the large sparse linear systems and for the large sparse block linear systems, respectively. When A is a symmetric positive definite matrix with spectrum radius close to 0.5, the error for the extrapolated homotopy method is about 10^{-15} (30 iterations), while the error is about 10^{-15} (30 iterations) for the iteration method and the homotopy method (see [1] for the iteration method). When A is a large sparse block matrix with spectrum radius close to 1, the error for the extrapolated homotopy method is about 10^{-10} (30 iterations), while the errors for the homotopy method is about 10^{-1} (30 iterations), respectively. This implies that the extrapolated homotopy method is more efficient than the traditional homotopy methods and the iteration methods for large sparse systems, especially for the large sparse block linear systems.

This paper is organized as follows. In Section 2, we introduce the basic concept in the extrapolated homotopy method. In Section 2.1 and 2.2, we provide an optimum extrapolation parameter of large sparse systems and large sparse block systems, respectively. In Section 3, we display two numerical examples on the large sparse system and the large sparse block system, respectively. The concise conclusion is provided in Section 4.

2. Basic concept in the extrapolated homotopy method

The extrapolated homotopy method defines a homotopy $H(u, p)$ by $H(u,0) = \frac{1}{\alpha}u - b$ and $H(u, 1) = Au - b$ and a convex homotopy as follows:

$$H(u,p) = (1-p)(\frac{1}{\alpha}u - b) + p(Au - b) = 0 \tag{2}$$

For the freedom to select an extrapolation parameter α. Regarding p as an expanding parameter yields:

$$u = u_0 + u_1 p + u_2 p^2 + \cdots \tag{3}$$

Which gives an approximate solution for (1) as:

$$v = \lim_{p \to 1}(u_0 + u_1 p + u_2 p^2 + \cdots)$$

Substituting (3) into (2) and equating the terms with the identical power of p, we obtain:

$$\begin{aligned} p^0 &: \quad u_0 = \alpha b \\ p^i &: \quad u_i = (I - \alpha A)u_{i-1} = (I - \alpha A)^i u_0, i = 1, 2, 3, \ldots, \end{aligned}$$

Where I is the identity matrix. Therefore, the approximate solution for (1) is obtained as:

$$\lim_{p \to 1} \sum_{i=0}^{\infty} (I - \alpha A)^i u_0 \tag{4}$$

With $(I - \alpha A)^0 = I$. According to (4), we do not compute the homotopy series $\sum_{i=0}^{\infty}(I - \alpha A)^i$ again as the change is only b in the linear system. Hence, adapting the homotopy method to solve a sequence of linear systems is better than adapting the iterative methods.

Let $\rho(S) = \max_{\lambda \in \sigma(S)} |\lambda|$ be the spectrum radius, where $\sigma(S)$ denotes the set of all eigenvalues of S and is called the spectrum of S. Following theorems are proposed to verify the convergence of the series u. Since the series $\sum_{k=0}^{\infty} S^k$ converges if and only if $\rho(S) < 1$ [3], we obtain the following results (See Liu [2] for the details).

Theorem 1

The series:

$$u = \sum_{i=0}^{\infty} (I - \alpha A)^i (\alpha b)$$

Converges if and only if $\rho(I - \alpha A) < 1$.

Moreover, the following corollary holds for $\rho(I - \alpha A) \le \| I - \alpha A \|$ for any matrix norm.

Corollary 2

The series:

$$u = \sum_{i=0}^{\infty} (I - \alpha A)^i (\alpha b)$$

Converges if $\| I - \alpha A \| < 1$.

Therfore, the extrapolation parameter α is selected such that one of the following two conditions holds: (i) $\rho(I - \alpha A) < 1$; and (ii) $\| I - \alpha A \| < 1$.

2.1 Optimization the extrapolation parameter for large sparse linear systems

Let $m(A)$ and $M(A)$ denote the algebraically smallest and largest eigenvalues of matrix A, respectively. A real matrix A is called positive definite if $x^T Ax > 0$ for all nonzero real vector x in R^n, where x^T denotes the transpose of the vector x. In this section, we derive an optimal parameter for the extrapolated homotopy method, when the matrix A is symmetric positive definite.

Theorem 3

When A is symmetric positive definite, the optimum extrapolation parameter is:

$$\alpha_{\text{opt}} = \frac{2}{M(A) + m(A)}$$

Proof

Since A is symmetric positive definite, there is an normal matrix Q such that $A = Q^T \Lambda Q$, where $\Lambda = \text{diag}\ (\lambda_1, \lambda_2, \ldots, \lambda_n)$, λ_i is the eigenvalue of A. Hence, we have $I - \alpha A = I - \alpha Q^T \Lambda Q = Q^T (I - \Lambda)Q$. This implies that:

$$\rho(Q^T (I - \Lambda)Q) = \rho(I - \Lambda) = \max(|1 - \max(\alpha\lambda_i)|, |1 - \min(\alpha\lambda_i)|).$$

To minimize $\rho(Q^T (I - \Lambda)Q)$, we get:

$$\alpha_{\text{opt}} = \frac{2}{M(A) + m(A)}$$

2.2 Optimization the extrapolation parameter for large block linear systems

Consider the block linear system problem $Ax = b$, where:

$$A = \begin{bmatrix} B & C \\ C & B \end{bmatrix},\ x = \begin{bmatrix} x \\ \bar{x} \end{bmatrix},\ b = \begin{bmatrix} b \\ \bar{b} \end{bmatrix}$$

The homotopy series in (3) is displayed as:

$$v = \sum_{i=0}^{\infty} \begin{bmatrix} I - \alpha B & \alpha C \\ \alpha C & I - \alpha B \end{bmatrix}^i u_0$$

Theorem 4

When $B + C$ and $B - C$ are symmetric positive definite, the optimum extrapolation parameter is obtained as:

$$\alpha_{\text{opt}} = \frac{2}{a + d}$$

Where $a = \max\,(M(B + C), m(B - C))$ and $d = \min\,(m(B + C), m(B - C))$.

To prove this theorem, we first consider the following lemma.

Lemma 5

Let a, b, c, and d be four real numbers with $a \geq b \geq c \geq d$. The optimum solution ω for the min-max problem:

$$w\,\text{minmax}\{|1 - ax|, |1 - bx|, |1 - cx|, |1 - dx|\} \tag{5}$$

is $x = \frac{2}{a+d}$*; meanwhile the optimum value is* $\frac{a-d}{a+d}$.

Proof

The function $|1 - wx|$ is:

$$|1 - wx| = \begin{cases} 1 - wx, & \text{if } x < 1/w \\ wx - 1, & \text{if } x \geq 1/w \end{cases}$$

For all $w = a, b, c, d$. The relation of functions are:

$$1 - ax < 1 - bx < 1 - cx < 1 - dx \tag{6}$$

And

$$dx - 1 < cx - 1 < bx - 1 < ax - 1 \tag{7}$$

For all $x > 0$. The intersection of $1 - dx$ and $ax - 1$ is appeared at $x = \frac{2}{a+d}$. As $x < \frac{2}{a+d}$, we have $1 - dx \geq ax - 1$; hence $1 - dx > ax - 1 > bx - 1 > cx - 1$ by (6) and $1 - dx > 1 - cx > 1 - bx > 1 - ax$ by (7). As $x > \frac{2}{a+d}$, we have $ax - 1 \geq 1 - dx$; hence $ax - 1 > bx - 1 > cx - 1 > dx - 1$ for all $w = b, c, d$ by (7) and $ax - 1 > 1 - dx > 1 - cx > 1 - bx$ by (6). This implies that the optimal solution of (5) is $x = \frac{2}{a+d}$.

Proof of Theorem 4

Since $\begin{bmatrix} B & C \\ C & B \end{bmatrix} = \begin{bmatrix} \frac{I_n}{\sqrt{2}} & -\frac{I_n}{\sqrt{2}} \\ \frac{I_n}{\sqrt{2}} & \frac{I_n}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} B+C & 0 \\ 0 & B-C \end{bmatrix} \begin{bmatrix} \frac{I_n}{\sqrt{2}} & \frac{I_n}{\sqrt{2}} \\ -\frac{I_n}{\sqrt{2}} & \frac{I_n}{\sqrt{2}} \end{bmatrix}$ and $\begin{bmatrix} \frac{I_n}{\sqrt{2}} & \frac{I_n}{\sqrt{2}} \\ -\frac{I_n}{\sqrt{2}} & \frac{I_n}{\sqrt{2}} \end{bmatrix}$ is an orthogonal matrix, we have:

$$I_{2n} - \alpha \begin{bmatrix} B & C \\ C & B \end{bmatrix} = \begin{bmatrix} \frac{I_n}{\sqrt{2}} & -\frac{I_n}{\sqrt{2}} \\ \frac{I_n}{\sqrt{2}} & \frac{I_n}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} I_n - \alpha(B+C) & 0 \\ 0 & I_n - \alpha(B-C) \end{bmatrix} \begin{bmatrix} \frac{I_n}{\sqrt{2}} & \frac{I_n}{\sqrt{2}} \\ -\frac{I_n}{\sqrt{2}} & \frac{I_n}{\sqrt{2}} \end{bmatrix}$$

This implies that:

$$\rho\left(I - \alpha \begin{bmatrix} B & C \\ C & B \end{bmatrix} \right) = \rho\left(\begin{bmatrix} (I_n - \alpha\Lambda^{(B+C)}) & 0 \\ 0 & (I_n - \alpha\Lambda^{(B-C)}) \end{bmatrix} \right)$$
$$= \max\{|1 - \alpha M(B+C)|, |1 - \alpha M(B-C)|, |1 - \alpha m(B+C)|, |1 - \alpha m(B-C)|\}$$

Where $\Lambda^{(B+C)}$ and $\Lambda^{(B-C)}$ are eigenvalues of $B + C$ and $B - C$, respectively.

Hence the optimum extrapolation parameter is the solution of the min-max problem:

$$\min_{\alpha} \max(|1 - \alpha M(B+C)|, |1 - \alpha M(B-C)|, |1 - \alpha m(B+C)|, |1 - \alpha m(B-C)|).$$

By Lemma 5, we get the optimum extrapolation parameter:

$$w_{opt} = \frac{2}{a+d}$$

Where $a = \max\,(M(B + C), M(B - C))$ and $d = \min\,(m(B + C), m(B - C))$.

3. Comparison results

In this section, comparison results are obtained by applying the homotopy method and extrapolated homotopy method on finding an approximate solution of linear systems $Ax = b$. Here the large sparse matrix A is non-block in Example 1 and is block in Example 2. Since the real solution is not known, errors are computed by the maximum absolute difference between Ax and b.

Example 1

The homotopy method and the extrapolated homotopy method are applied to find a numerical solution of the system:

$$Ax = b$$

Where A is a 1,000 × 1,000 matrix with $\rho(I - A) = 0.4997$ and b is a 1,000 × 1 vector. The errors and computation times for the homotopy method and the extrapolated homotopy methods are displayed in Table 1.

After 10 iterations and 30 iterations, the errors for the extrapolated homotopy method are $4.61e-06$ and $1.89e-15$, respectively, while the errors for the homotopy method are $3.18e-04$ and $2.26e-10$, respectively. This implies that extrapolated homotopy methods converge rapidly than the homotopy method when the large sparse system has a small spectrum radius.

Table 1. The HM and the extrapolated HM for the large sparse systems.

Method		HM			Extrapolated HM	
Iterations	10	30	50	10	30	50
MAD	3.18e-04	2.26e-10	1.77e-15	4.61e-06	1.89e-15	1.89e-15
Time (Sec.)	0.011	0.0309	0.0519	0.0145	0.0352	0.0761

Example 2

The homotopy method and the extrapolated homotopy method are applied to find a numerical solution of the block linear system:

$$Ax = b$$

Where $A = \begin{bmatrix} B & C \\ C & B \end{bmatrix}$ is a 1,000 × 1,000 matrix with $\rho(I - A) = 0.9986$ and b is a 1,000 × 1 vector. The errors and computation times for the homotopy method and the extrapolated homotopy methods are displayed in Table 2.

After 10 iterations, 20 iterations and 30 iterations, the errors for the extrapolated homotopy method are $4.22e-06$, $4.64e-11$ and $2.11e-15$, respectively, while the errors for the homotopy method are 0.5391, 0.4370 and 0.3439, respectively. As we can see, the extrapolated homotopy methods converge rapidly then the traditional homotopy methods for large block systems with small spectrum radius.

Table 2. The HM and extrapolated HM for the large sparse block linear system.

Method	HM			Extrapolated HM		
Iterations	10	20	30	10	20	30
MAD	0.5391	0.4370	0.3439	4.22e-06	4.64e-11	2.11e-15
Time (Sec.)	0.0242	0.0351	0.0463	0.0122	0.0233	0.0334

4. Conclusions

This paper studies the numerical solution of a large sparse linear system. The extrapolated homotopy method is developed for solving a large sparse linear systems and a large sparse block linear systems; meanwhile the optimum extrapolation parameters are obtained for both two systems. Observing the numerical results, we find that the extrapolated homotopy method converges rapidly than the traditional homotopy methods and the iteration methods. In future studies, the extrapolated homotopy Jacobi methods as well as their optimum extrapolation parameters would be developed for solving the large sparse linear systems.

Referecne

[1] B. Keramati (2009). An approach to the solution of linear system of equations by He's homotopy perturbation method, *Chaos, Solitons & Fractals*, 41, 152-156.

[2] H.-K. Liu (2011). Application of homotopy perturbation methods for solving systems of linear equations, *Applied Mathematics and Computation*, 217, 5259-5264.

[3] Y. Saad (2003). Iterative methods for sparse linear systems, In *Society for Industrial and Applied Mathematics*, 2nd ed, Philadelphia, PA.