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On the valuation of reverse mortgage insurance

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This article presents a closed-form formula for calculating the loan-to-value (LTV) ratio in an adjusted-rate reverse mortgage (RM) with a lump sum payment. Previous literatures consider the pricing of RM in a constant interest rate assumption and price it on fixed-rate loans. This paper successfully considers the dynamic of interest rate and the adjustable-rate RM simultaneously. This paper also considers the housing price shock into the valuation model. Assuming that house prices follow a jump diffusion process with a stochastic interest rate and that the loan interest rate is adjusted instantaneously according to the short rate, we demonstrate that the LTV ratio is independent of the term structure of interest rates. This argument holds even when housing prices follow a general process: an exponential Lévy process. In addition, the HECM (Home Equity Conversion Mortgage) program may be not sustainable, especially for a higher level of housing price volatility. Finally, when the loan interest rate is adjusted periodically according to the LIBOR rate, our finding reveals that the LTV ratio is insensitive to the parameters characterizing the CIR model.

Keyword: reverse mortgage; option pricing; jump diffusion process; exponential Lévy process

1. Introduction

Demographic aging offers one of the most serious challenges that developed and developing countries face. The elderly dependent ratio – defined as the ratio of the number of senior dependents (over 65 years of age) to the total working-age population (aged 15–65 years) – keeps rising in most countries, which means that the overall economy faces a greater burden to support an aging population. Governments and industries seek to decrease their financial burden by deferring the retirement age, reducing the received benefits in a defined benefit pension plan, and/or transferring the defined benefit pension scheme to a defined contribution pension scheme. Eventually, it becomes difficult for pension and social security systems to provide sufficient benefits. Therefore, reverse mortgage (RM) products might provide a great alternative solution.

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For many seniors, the equity of their residence is their greatest asset, yet it is unavailable to them unless they take a home equity loan or sell the equity. For example, in the US, the American Housing Survey shows that more than 12.5 million elderly have no mortgage debt, and the median value of mortgaged-free homes is US $127,959. A RM involves borrowing money by using the equity in their homes as collateral. Therefore, it enables the seniors to finance their retirement years without selling the property and having to pay back the money until the borrower dies, sell the property, or move out permanently. At the mortgage's due date, the loan gets repaid with accumulated interest through the sale of the property. Moreover, the lender can only receive the minimum of the entire debt or the net value of the property, which prevents the borrower from owing more than the value of the house. This nonrecourse clause makes the RM difficult to value.

There are several payment options for RMs: lump sum payment, line of credit, term, tenure, modified term (a combination of line of credit with term), and modified tenure (a combination of line of credit with tenure payment). In Home Equity Conversion Mortgage (HECM) scheme, the adjustable rate option is available with all kinds of payment options, whereas the fixed rate option is only available for the lump sum payment. Prior to 2009, nearly all HECMs carried adjustable interest rates. The lump sum RMs, including the fixed-rate and the adjustable-rate option, have dominated the market (nearly 70% market share) since mid-2009[1]. An alternative to a RM is the sale and leaseback financing, i.e. the homeowner sells the home and then leases it from the purchaser. Although the sale and leaseback transaction is prevailing in business, it is not popular in home equity conversion.

The main purpose of this study is to develop a framework to model the embedded risks and value the non-recourse provision of the RM products under an HECM scheme. As pointed out by Phillips & Gwin (1992), loan providers of RMs are mainly involved with longevity risk, housing price risk, and stochastic interest rate risk. The existing literature on risk modeling of the HECM program has three flaws. First, some use static mortality tables and thus neglect the longevity risk (see Weinrobe, 1988; Szymanoski, 1994; Tse, 1995; Zhai, 2000). We employ a generalized Lee–Carter model to model mortality risks and construct dynamic mortality tables. Second, previous HECM models assume home prices are driven by a geometric Brownian motion (e.g. Szymanoski, 1994; Ma et al., 2007; Wang et al., 2007). However, most of the housing price returns exhibit negative skewness and positive excess kurtosis. In addition, extensive literatures suggest that external shocks might lead to a string of house price changes. These external shocks include money supply shocks (Lastrapes, 2002), redevelopment (Lee et al., 2005), and foreign property shocks (Wilson et al., 2007). The housing price jump risk is getting more attention to the mortgage-pricing research after the subprime crisis (Chen, Chang, et al., 2010). Therefore, we use exponential Lévy processes to model house price returns. Finally, as pointed by Chen, Cox, et al.

[1] For more detail, please refer to 'RMs–Report to Congress' published by the Consumer Financial Protection Bureau on 28 June 2012.

(2010), HECM loans almost exclusively opt for adjustable interest rates in practice[2], yet most literatures choose a constant interest rate assumption (e.g. Chinloy & Megbolugbe, 1994; Szymanoski, 1994; Chen, Cox, et al., 2010; Li et al., 2010). This paper incorporates the CIR (Cox et al., 1985) stochastic model of the short interest rate in pricing adjustable-rate RMs.

This paper builds a modeling and pricing framework for adjustable-rate RM products with a lump sum payment. The complexity of valuation problems comes from the fact that the adjustable-rate RM products are involved with multiple risks. Therefore, most literatures (such as Chen, Cox, et al., 2010) use Monte Carlo simulation to numerically calculate the values of RM products. The first contribution of this paper is to directly derive a closed-form formula for the adjustable-rate RM insurance with longevity risk, housing price risk, and stochastic interest rate risk. In addition, the loan-to-value (LTV) ratio is determined, under which the present value of premium income is equal to the cost of RM insurance. The second contribution of this paper is to prove that the level of LTV ratio of an adjusted-rate RM is independent with the interest rate parameters if the RM rates are adjusted instantaneously based on the short rates. In other words, when the loan interest rates are adjusted instantaneously according to short rates, the valuation of adjusted-rate RMs is only connected with the housing price risk, longevity risk, and interest rate spread on the loan. We also assess the risk profile of RM insurance in empirical and numerical analyses. Finally, for the adjustable-rate RMs, the loan interest rate is adjusted monthly or annually according to the LIBOR rate in practice. The third contribution of this paper is to demonstrate that when the loan interest rate is adjusted yearly based on LIBOR rate the impact of parameters characterizing the CIR models on the LTV ratio is also trivial.

The remainder of this article is organized as follows: in the next section, we outline the valuation framework, including the identification of the RM contract. After we present the valuation formula in the Section 3, we extend the housing price model to a general Lévy process in the Section 4. We present the numerical results and sensitivity analysis and then offer several conclusions.

2. The model

In this section, we first describe the contract structures of RMs and the LTV ratios, which provide the basis for our valuation. In addition, as Phillips and Gwin (1992) point out, the crucial impact factors of RMs are longevity risk, housing price risk, and stochastic interest rate risk. We then model the dynamics of the spot interest rates, the house prices, and the mortality rates sequentially.

[2] In FHA's (Federal Housing Administration) HECM program, borrower can choose an adjustable interest rate or a fixed-rate RM. If one chooses an adjustable interest rate, one may choose to have the interest rate adjust monthly or annually. Lenders may not adjust annually adjusted HECMs by more than two percentage points per year and not by more than five total percentage points over the life of the loan. FHA does not require interest rate caps on monthly adjusted HECMs. (For more details, please refer to http://portal.hud.gov/hudportal/HUD?src=/program_offices/housing/sfh/hecm/hecmabou).

2.1. RM contract and LTV ratio

We investigate RM with a lump sum payment, analogous to the US HECM program (Szymanoski, 1994). The initial property value, denoted $H(0)$, enables us to determine the lump sum payment. The borrower receives a lump sum payment, $BAL(0)$, and does nothing else, as long as the house is his or her principal residence. We assume that the loan becomes due and payable only at the borrower's death. At the terminal date of the loan, the outstanding balance is payable, and the remainder of the value belongs to any heirs of the borrower if the home value is greater than the outstanding balance. However, the lender must accept the home value if it is less than the loan amount. To cover this contingent loss, the lender charges an initial premium, or $100\pi_0$ percent of the initial house value, at the inception of the contract, as well as an annual assessment of $100\pi_m$ percent of the outstanding balance for the life of the loan. These charges are not paid by cash but rather accrue to the outstanding balance of the loan.

For the HECM demonstration, the premium structure is a combination of a one-time upfront charge and an annual assessment (π_m) proportional to the outstanding balance for the life of the loan. Because the sequential premium π_m of the RM insurance gets charged over the life of the loan and accrued to the outstanding balance with the interest rate, the size of π_m not only affects the present value of the future insurance premium but also influences the probability that a contingent loss occurs, and its severity.

Let the loan interest rate be equal to the short rate plus an interest rate spread π_r. The death of the borrower can happen at the end of each year for our purposes. $BAL(t)$, the outstanding balance at time t, is determined by the outstanding balance at time $t-1$ plus the premium charge with interest accrued; that is,

$$BAL(t_1) = (BAL(t_0) + \pi_0 H(t_0)) \exp\left\{ \int_{t_0}^{t_1} (r(s) + \pi_r) ds \right\}, \tag{1}$$

$$BAL(t_{j+1}) = BAL(t_j)(1 + \pi_m) \exp\left\{ \int_{t_j}^{t_{j+1}} (r(s) + \pi_r) ds \right\}, \quad j = 1, 2, \ldots, \tag{2}$$

where $r(t)$ is the instantaneous rate of interest (short rate) at time t; π_0 is the initial premium rate, equal to a percentage of the house value, and π_m is the sequential annual premium rate, which equals a percentage of the outstanding balance. In turn, we can reduce Equations (1) and (2) to

$$BAL(t_{j+1}) = BAL(t_0)\pi(j+1) \exp\left\{ \int_{t_0}^{t_{j+1}} (r(s) + \pi_r) ds \right\}, \quad j = 0, 1, 2, \ldots, \tag{3}$$

where

$$\pi(j+1) = \left(1 + \pi_0 \frac{H(t_0)}{BAL(t_0)}\right)(1+\pi_m)^j \qquad \text{for} \quad j = 0, 1, 2, \ldots . \tag{4}$$

Through the pricing process, it is more convenient to set the valuation date t_0 to 0. Thus, at the valuation date $t_0(=0)$, the money market account is defined by,

$$B(t) = \exp\left(\int_0^t r(u)du\right). \tag{5}$$

According to Equation (3), we have,

$$\frac{BAL(t_{j+1})}{B(t_{j+1})} = BAL(0)\pi(j+1)e^{\pi_r t_{j+1}}, \quad j = 0, 1, 2, \ldots . \tag{6}$$

Under risk-neutral probability measure Q, the value of the RM insurance, which is also the present value of the expected losses from future claims, equals the expectation of discounted future cash flows under Q. Let x_0 be the age of the borrower at time t_0 and ω be the final age at which all people perish; then, the value of the RM insurance $V(0)$ is of the form,

$$\begin{aligned} V(0) &= \sum_{j=1}^{\omega-x_0} E_Q\left[\frac{\left({}_{t_{j-1}}p_{x_0,t_0} - {}_{t_j}p_{x_0,t_0}\right)\left(BAL(t_j) - H(t_j)\right)^+}{B(t_j)}\right] \\ &= \sum_{j=1}^{\omega-x_0} E_Q\left[\left({}_{t_{j-1}}p_{x_0,t_0} - {}_{t_j}p_{x_0,t_0}\right)\left(BAL(0)\pi(j)e^{\pi_r t_j} - DH(t_j)\right)^+\right], \end{aligned} \tag{7}$$

where $[A]^+ = \max\{A, 0\}$; $DH(t) \equiv H(t)/B(t)$ is the discounted housing price; and ${}_{t_n}p_{x_0,t_0}$ is the probability that the cohort aged x_0 in year t_0 will survive till age x_0+n and satisfies ${}_{t_0}p_{x_0,t_0} = 1$. In views of Equation (7), the RM insurance, contingent on the value of the property, is a put option underlying the property. When the option is in the money, the borrower receives nothing from the sale of the house but retains the right to reside there by keeping the equity. The borrower would not sell the property in this case. For simplicity, ignoring the possibilities that the borrower sells the property or moves out permanently, which increases the early terminated probability of the loan, we consider only the borrower's death as a cause of the loan becoming due and payable when pricing a RM insurance.

In contrast, the present value of the premium charges, denoted by $R(0)$, is,

$$R(0) = \pi_0 H(0) + E_Q\left[\sum_{j=1}^{\omega-x_0} \frac{{}_{t_j}p_{x_0,t_0} BAL(t_j)\pi_m}{B(t_j)}\right] = \pi_0 H(0) + E_Q\left[\sum_{j=1}^{\omega-x_0} {}_{t_j}p_{x_0,t_0} BAL(0)\pi(j)e^{\pi_r t_j}\pi_m\right]. \tag{8}$$

The initial advance $BAL(0)$ can be determined by setting the value of the RM insurance equal to the present value of the premium charges, namely,

$$V(0) = R(0). \tag{9}$$

Note that in views of Equations (7) and (8), both $V(0)$ and $R(0)$ are mainly determined by the discounted housing price and survival probabilities, but are irrelevant to the interest rates, which implies that the LTV ratio is independent of interest rate parameters when the discounted housing price and survival probabilities are independent of interest rate parameters. Employing the CIR model, jump diffusion model, and Lee Carter model to capture the dynamics of the spot interest rates, the housing prices, and the mortality rates, the goal of this paper is to derive the LTV ratio in closed form to prove this argument.

2.2. Interest rate process

In the paper, we use the CIR model to capture the interest rate dynamic:

$$dr(t) = \alpha(\beta_r - r(t))dt + \sigma_r\sqrt{r(t)}dW_r(t), \tag{10}$$

where α is the speed of reversion; β_r is the long-run short interest rate; σ_r is the instantaneous volatility; and $W_r(t)$ is a standard Brownian motion under risk neutral probability measure Q. Consequently, the time t price of a zero coupon bond maturing at time T, $P(t, T)$, under measure Q is as follows:

$$P(t,T) = A(t,T)e^{-B(t,T)r(t)}, \tag{11}$$

where

$$A(t,T) = \left[\frac{2\gamma e^{(\gamma+\alpha)(T-t)/2}}{(\gamma+\alpha)(e^{\gamma(T-t)}-1)+2\gamma}\right]^{\frac{2\alpha\beta_r}{\sigma_r^2}}, \tag{12}$$

$$B(t,T) = \frac{2(e^{\gamma(T-t)}-1)}{(\gamma+\alpha)(e^{\gamma(T-t)}-1)+2\gamma} \quad \text{and} \quad \gamma = \sqrt{\alpha^2 + 2\sigma_r^2}. \tag{13}$$

2.3. House price model

We treat the house price H as a jump diffusion process. Equivalently, the behavior of the changes in housing prices can be divided into two parts: (1) continuous diffusion, which is responsible for the usual housing price movement and described by a traditional Brownian motion, and (2) discontinuous jumps, which correspond to the arrival of new information that is important to the housing market. Under the risk-neutral measure Q, the house price process is governed by,

$$\frac{dH(t)}{H(t)} = (r(t) - \delta(t))dt + \hat{\sigma}_H(t) \cdot dW(t) + d(J(t) - \lambda\beta t), \tag{14}$$

where $\delta(t)$ is a maintenance yield (or rental rate) for the house; $W(t) = [W_r(t), W_H(t)]'$ is a two-dimensional standard Brownian motion under measure Q; $\hat{\sigma}_H(t)$ is the volatility vector of the housing price and satisfies $\hat{\sigma}_H(t) = \sigma_H(t)\left[\rho_{Hr}, \sqrt{1-\rho_{Hr}^2}\right]'$, that is, $|\hat{\sigma}_H(t)| = \sigma_H(t)$; $|\cdot|$ denotes the Euclidean norm in R^2; and ρ_{Hr} is the correlation coefficient between the interest rate process and the house price process. Moreover, $J(t)$, a compound Poisson process, is defined by $J(t) = \sum_{i=1}^{N(t)} (Y_i - 1)$, where $\{N(t);\ t \geq 0\}$ is a Poisson process with intensity λ, and the jump sizes $\{Y_1, Y_2, \ldots\}$ are independent and identically distributed random variables with mean $E(Y_i) = \beta + 1$. The random variables $\{Y_1, Y_2, \ldots\}$ are assumed to be independent of $\{W(t);\ t \geq 0\}$ and $\{N(t);\ t \geq 0\}$.

Note that $J(t) - \lambda\beta t$ is a Q-martingale, so the jump process does not change the drift term on average but only affects the volatility of housing returns.

2.4. *Mortality model*

When treating mortality in this study, we use the Lee–Carter model (Lee & Carter, 1992), which has gained widespread acceptance and has proved effective for estimating mortality. Lee (2000) discusses several recent extensions, applications, and methodological improvements to the Lee–Carter model. However, mortality forecasting still is based on historical mortality data, which presents anticipative processes under P-measures in the arbitrage pricing theory. The recent Wang (2000) transformation provides an elegant and tractable method to deal with this issue. Denuit et al. (2007) employ the Lee–Carter model to price a risky coupon survivor bond; they apply the Wang transform to determine the market price of bearing the mortality risk. Following the approach proposed by Denuit et al. (2007), we apply the Lee–Carter model, accompanied by the Wang transform, to price RM insurance.

2.4.1. Lee–Carter model

We use the Lee–Carter model to assess the mortality-related risks. Let $m_{x,t}$ be the central death rate for age x at time t. The Lee–Carter model assumes:

$$\ln(m_{x,t}) = \alpha_x + \beta_x k_t + e_{x,t}, \tag{15}$$

where α_x presents the average specific pattern of mortality over age group x, β_x describes the pattern of deviations from the age x profile when the parameter k varies, k_t explains the change of mortality over time t, and $e_{x,t}$ describes the error term, which is expected to be white noise with zero mean and a relatively small variance (Lee, 2000).

We use the singular value decomposition approximation (Lee & Carter, 1992) to fit the solutions of the parameters. By stipulating that $\sum_t k_t = 0$ and $\sum_x \beta_x = 1$, we assume $\hat{\alpha}_x$ is simply the average value over time of $\ln(m_{x,t})$, and $\hat{k}_t$ is the sum over various ages of $\ln(m_{xt}) - \hat{\alpha}_x$. For each age group x, we can obtain $\hat{\beta}_x$ by regressing $\ln(m_{xt}) - \hat{\alpha}_x$ on $\hat{k}_t$ without a constant term.

Following Lee & Carter (1992), we forecast future values of k_t with:

$$k_t = k_{t-1} + z + \varepsilon_t, \tag{16}$$

where z is the drift parameter, and ε_t is a sequence of independent and identically normal distributions with mean 0 and variance σ^2. We assume that the values $k_1, \ldots, k_{t_0}$ are known but that k_{t_j} are unknown and must be forecast, where $t_j = t_0 + j$, for any natural number j. By virtue of Equation (16), we have,

$$k_{t_j} = k_{t_0} + jz + \sum_{i=1}^{j} \varepsilon_{t_i}, \tag{17}$$

Moreover, conditional on t_0, k_{t_j} is normal distributed with mean $k_{t_0} + jz$ and variance $j\sigma^2$.

2.4.2. Wang risk transform

Wang (2000) proposes a transformation for pricing contingent claims that can be traded or not. The Wang transform is based on the idea that the annuity market price takes into account the uncertainty in the mortality table, as well as the uncertainty in the lifetime of an annuitant once the table is given. Because contracts contingent on mortality rates usually are not traded on financial markets, Wang's transformation helps value mortality-linked securities (Lin & Cox, 2005; Dowd et al., 2006; Liao et al., 2007; Denuit et al., 2007). Consequently, we also use Wang transform to consider the market price of mortality risk. Following Denuit et al. (2007), we define the probability that the cohort aged x_0 in year t_0 will survive to age $x_0 + n$ as

$$_{t_n}p_{x_0,t_0} = \exp\left(- \sum_{j=0}^{n-1} m_{x_0+j,t_0+j} \right) = \exp\left(- \sum_{j=0}^{n-1} \exp\left(\alpha_{x_0+j} + \beta_{x_0+j} k_{t_0+j}\right) \right). \tag{18}$$

Let F_n be the cumulative distribution function of $_{t_n}p_{x_0,t_0}$; that is,

$$F_n(u) = \Pr\left[_{t_n}p_{x_0,t_0} \leq u \right], \quad 0 \leq u \leq 1. \tag{19}$$

Based on the Wang transform with a transformation parameter τ, under the risk-neutral measure Q, the probability of the cohort aged x_0 in year t_0 surviving to age $x_0 + n$ is given by

$$Pr_Q\left[_{t_n}p_{x_0,t_0} \leq u \right] = \Phi(\Phi^{-1}(F_n(u)) + \tau), \tag{20}$$

where Φ is the distribution function of the standard Normal distribution, and the parameter τ measures the market price of mortality risk. Therefore, according to the risk-neutral measure Q, the expected probability of the cohort aged x_0 in year t_0 surviving to age $x_0 + n$, $S(t_n)$, takes the form:

$$S(t_n) = E_Q\left[_{t_n}p_{x_0,t_0} \right] = \int_0^1 \left(1 - \Phi(\Phi^{-1}(F_n(u)) + \tau)\right) du. \tag{21}$$

For a more detailed demonstration of this application, see Denuit et al. (2007).

3. LTV ratio in closed form

We determine the lump sum payment $BAL(0)$[3] when the present value of the insurance premiums covers the present value of expected losses from future claims. According to Equations (10), (14), and (15), we derive the closed-form solutions of the value of the RM insurance $V(0)$ and the present value of the premium charges $R(0)$ in Proposition 1.

Proposition 1: Assume that the mortality process in Equation (15) and the financial asset prices (i.e. interest rate process in Equation (10) and house price process in Equation (14)) are independent. Let Y_i be lognormal distributed with $E[\ln Y_i] = \theta$ and $\mathrm{Var}[\ln Y_i] = \sigma_Y^2$. The present value of the premium charges $R(0)$ and the value of the RM insurance $V(0)$ are

$$R(0) = \pi_0 H(0) + \sum_{j=1}^{\omega - x_0} S(t_j) BAL(0)\pi(j)\pi_m e^{\pi_r t_j}, \tag{22}$$

$$V(0) = \sum_{j=1}^{\omega - x_0} \left[S(t_{j-1}) - S(t_j)\right] C(t_j), \tag{23}$$

respectively, where

$$\begin{aligned} C(t_j) = {} & BAL(0)\pi(j) e^{\pi_r t_j} \sum_{m=0}^{\infty} \frac{e^{-\lambda t_j}(\lambda t_j)^m}{m!} \Phi(-d_{2j}(m)) \\ & - H(0) e^{-\int_0^{t_j} \delta(s) ds} \sum_{m=0}^{\infty} \frac{e^{-\lambda_Q t_j}(\lambda_Q t_j)^m}{m!} \Phi(-d_{1j}(m)), \end{aligned} \tag{24}$$

where $\lambda_Q = \lambda(\beta + 1)$;

$$d_{1j}(m) = \frac{\ln\left(\frac{H(0)}{BAL(0)\pi(j)}\right) + r_{mj} + \frac{1}{2}\sigma_{mj}^2}{\sigma_{mj}}; \quad d_{2j}(m) = d_{1j}(m) - \sigma_{mj} \tag{25}$$

$$r_{mj} = m\theta + \frac{1}{2}\sigma_Y^2 m - \int_0^{t_j} (\delta(s) + \lambda\beta + \pi_r) ds; \quad \text{and } \sigma_{mj}^2 = \int_0^{t_j} \sigma_H(s)^2 ds + \sigma_Y^2 m \tag{26}$$

where Φ is the standard normal cumulative distribution function.

Proof: See Appendix 1.

An interesting and meaningful result is that neither the value of the RM insurance nor the present value of the premium charges correlates with the interest rate process. Thus, the LTV, equal to the ratio of the lump sum payment to the house price at the valuation date

[3] In this study, the valuation date is t_0. We use 0 instead of t_0 sometimes for simplicity without ambiguousity.

($BAL(0)/H(0)$), is independent of the parameters of the interest rate model – a definitely important and significant result of our study. There are three assumptions contributing to this argument. First, the loan interest rate equals the short rate plus an interest rate spread which is independent of the level of interest rate. Second, the rental rate is independent of the level of interest rate. Third, the mortality rate process is independent of financial asset processes. Consequently, the discounted housing price and survival probabilities are independent of interest rate parameters which in turn lead to the independence between the LTV ratio and the term structure of interest rates. In the next section, we show that the argument still holds even when the house price process follows a more general stochastic process: the exponential Lévy process.

4. Pricing RMs with exponential Lévy processes

For pricing RM insurance contracts, in the literature most of the pricing papers assume that the housing price process follows traditional geometric Brownian motion. However, US housing prices in the past two decades have varied significantly in response to government policy changes and catastrophic events. Chen, Chang, et al. (2010) demonstrate empirically that the national average new home prices for single-family mortgages jumped notably over the period 1986–2008. On 14 occasions, the monthly housing price changed more than 10% per month. The highest monthly housing price returns reached 20.85%, in June 1992, whereas the lowest monthly housing price returns fell to −22.76% in November 2007. Geometric Brownian motion only reflects the normal events but not the abnormal jumps. As a result, it is necessary to develop a suitable framework for a housing price process that includes jump risks.

In this section, we explore Lévy processes as a way to model the housing price process with jump risks. Roughly speaking, a Lévy process is a continuous time stochastic process with stationary independent increments, analogous to i.i.d. innovations in a discrete setting. Two important examples of Lévy processes include Brownian motion (the only purely continuous Lévy process) and the compound Poisson process underlying the jump diffusion model. In this section, the housing price dynamics $\{H(t);\ t \geq 0\}$ can be modeled as exponential Lévy processes, including Brownian motion for normal innovations and pure jump Lévy process for non-normal innovations as follows:

$$H(t) = H(0)B(t)\exp\left(A(t) + \int_0^t \hat{\sigma}_H(s) \cdot dW(s) + L(t)\right), \tag{27}$$

where $A(t)$, a process of finite variation, is often determined by no-arbitrage or equilibrium pricing relations and thus depends on the specification of Brownian motion and pure jump Lévy process; and $\{L(t);\ t \geq 0\}$, independent from the Brownian motion $\{W(t);\ t \geq 0\}$, is a pure jump Lévy process, such that the sample paths of L are right-continuous with left limits, and $L(u) - L(t)$ is independent of $L(t)$ and distributed as $L(u - t)$. The definitions of $B(t)$, $\hat{\sigma}_H(s)$, and $W(s)$ are consistent with previous definitions.

For any $\omega \in R$, the following Lévy-Khinchin representation (Bertoin, 1996) states that a characteristic function $\phi(\omega)$ of a pure jump Lévy process $\{L(t);\, t \geq 0\}$ can be expressed as,

$$\phi_t(\omega) \equiv E_Q(\exp(i\omega L(t))) = \exp(-t\psi(\omega)), \tag{28}$$

where the characteristic exponent $\psi(\omega)$ is given by,

$$\psi(\omega) = \int_{R-\{0\}} \left(1 - e^{i\omega x} + i\omega x 1_{|x|<1}\right) v(dx), \tag{29}$$

and the Lévy process is uniquely specified by the Lévy measure v, which satisfies the following conditions:

$$\int_{-\infty}^{\infty} 1_{|x|\geq 1}\, v(dx) < \infty \quad \text{and} \quad \int_{-\infty}^{\infty} x^2 1_{|x|<1} v(dx) < \infty \tag{30}$$

where the symbol 1_D denotes an indicator function of D. The first condition means that Lévy processes L has finite number of large jumps (large jumps are defined as jumps with absolute values greater than 1) and the second condition states Lévy measure must be square-integrable around the origin.

Under the risk-neutral measure Q, the discounted housing price process must satisfy the following expression:

$$E_Q\left(\frac{H(t)}{B(t)}\right) = H(0)\exp\left(-\int_0^t \delta(s)ds\right), \tag{31}$$

where $\delta(t)$ is the maintenance yield (or rental rate) for the house. By virtue of Equation (27), we have,

$$\begin{aligned} E_Q\left(\frac{H(t)}{B(t)}\right) &= H(0)E_Q\left(\exp\left(A(t) + \int_0^t \hat{\sigma}_H(s)\cdot dW(s) + L(t)\right)\right) \\ &= H(0)\exp\left(A(t) + \frac{1}{2}\int_0^t \sigma_H^2(s)ds - t\psi(-i)\right) \end{aligned} \tag{32}$$

or equivalently, $A(t)$ can be expressed as follows:

$$A(t) = t\psi(-i) - \int_0^t \left(\delta(s) + \frac{1}{2}\sigma_H^2(s)\right)ds. \tag{33}$$

Therefore, by virtue of Equation (33), the housing price process under the risk-neutral measure Q becomes,

$$\ln H(t) = \ln H(0) + \int_0^t \left(r(s) - \delta(s) - \frac{1}{2}\sigma_H^2(s) \right) ds + t\psi(-i) + \int_0^t \hat{\sigma}_H(s) \cdot dW(s) + L(t). \tag{34}$$

For a pure jump Lévy process, it can display either finite or infinite activity. In the former case, the aggregate jump arrival rate is finite, whereas in the latter case, an infinite number of jumps can occur over any finite time interval. We list the Lévy measures and the corresponding characteristic exponents of some useful pure jump Lévy processes in Table 1. Since a pure jump Lévy process L is uniquely characterized by its Lévy measure, according to Table 1, we can reduce the housing price process in Equation (34) to that in Equation (14) by setting,

$$v(dx) = \frac{\lambda}{\sqrt{2\pi\sigma_Y^2}} \exp\left(\frac{-(x-\theta)^2}{2\sigma_Y^2} \right) dx. \tag{35}$$

Consequently, different selections of Lévy measures can contribute to different patterns of housing price dynamics.

When the housing price follows the exponential Lévy process, the value of the RM insurance $V(0)$ is given by Proposition 2.

Proposition 2: Assume that the mortality process in Equation (15) and the financial asset prices (i.e. interest rate process in Equation (10) and house price process in Equation (14)) are independent. The present value of the premium charges $R(0)$ is given by Proposition 1 and the closed-form solution of the RM insurance is provided as follows:

$$V(0) = \sum_{j=1}^{\omega - x_0} \left[S(t_{j-1}) - S(t_j) \right] C_L(t_j), \tag{36}$$

Table 1. Lévy measure and characteristic exponents for pure jump Lévy processes.

Lévy components	Lévy measures	Characteristic exponents
Finite activity pure jump Lévy components		
Merton (1976)	$\lambda \frac{1}{\sqrt{2\pi\sigma_Y^2}} \exp\left(-\frac{(x-\theta)^2}{2\sigma_Y^2}\right)$	$\lambda\left(1 - e^{i\omega\theta - \frac{1}{2}\sigma_Y^2\omega^2}\right)$
Kou (2002)	$\lambda \frac{1}{2\eta} \exp\left(-\frac{\lvert x-k\rvert}{\eta}\right)$	$\lambda\left(1 - e^{i\omega k}\frac{1-\eta^2}{1+\omega^2\eta^2}\right)$
Eraker et al. (2003)	$\lambda \frac{1}{\eta} \exp\left(-\frac{x}{\eta}\right)$	$\lambda\left(1 - \frac{1}{1-i\omega\eta}\right)$
Infinite activity pure jump Lévy components		
NIG, Barndorff-Nielsen (1998)	$e^{\beta x}\frac{\delta\alpha}{\pi\lvert x\rvert}K_1(\alpha\lvert x\rvert)$	$-\delta\left[\sqrt{\alpha^2-\beta^2} - \sqrt{\alpha^2-(\beta+i\omega)^2}\right]$
Generalized hyperbolic, Eberlein et al. (1998)	$\frac{e^{\beta x}}{\lvert x\rvert}\left[\int_0^\infty \frac{e^{-\sqrt{2y+\alpha^2}\lvert x\rvert}}{\pi^2 y(J_{\lvert\lambda\rvert}^2(\delta\sqrt{2y})+Y_{\lvert\lambda\rvert}^2(\delta\sqrt{2y}))}dy + 1_{\lambda\geq 0}\lambda e^{-\alpha\lvert x\rvert}\right]$	$-\ln\left[\frac{\sqrt{\alpha^2-\beta^2}}{\sqrt{\alpha^2-(\beta+i\omega)^2}}\right]^\lambda \left[\frac{\kappa_\lambda\left(\delta\sqrt{\alpha^2-(\beta+i\omega)^2}\right)}{\kappa_\lambda\left(\delta\sqrt{\alpha^2-\beta^2}\right)}\right]$
CGMY, Carr et al. (2002)	$\begin{cases} Ce^{-G\lvert x\rvert}\lvert x\rvert^{-Y-1}, & x<0, \\ Ce^{-M\lvert x\rvert}\lvert x\rvert^{-Y-1}, & x>0 \end{cases}$	$C\Gamma(-Y)[M^Y-(M-i\omega)^Y+G-(G+i\omega)^Y]$
VG, Madan et al. (1998)	$\frac{\exp(Ax-B\lvert x\rvert)}{\kappa\lvert x\rvert}$ $\left(A=\frac{\alpha}{v^2}, B=\sqrt{\frac{2v^2}{\kappa}+\alpha^2}/v^2\right)$	$\frac{1}{\kappa}\ln\left(1-i\omega\alpha\kappa+\frac{1}{2}v^2\omega^2\kappa\right)$
LS, Carr and Wu (2003)	$c\lvert x\rvert^{-\alpha-1}, \quad x<0$	$-c\Gamma(-\alpha)(i\omega)^\alpha$

where

$$C_L(t_j) = K_j - H(0)e^{-\int_0^{t_j} \delta(s)\,ds} + C(t_j, k_j), \tag{37}$$

$$C(t_j, k_j) = \frac{e^{-\alpha k_j}}{2\pi} \int_{-\infty}^{\infty} e^{-i\omega k_j} \frac{\phi_{h(t_j)}(\omega - (\alpha+1)i)}{(\alpha + i\omega)(\alpha + 1 + i\omega)} d\omega, \tag{38}$$

$$\phi_{h(t)}(\omega) = \exp\left(i\omega[\ln H(0) + A(t)] - \frac{\omega^2}{2} \int_0^t \sigma_H^2(s)ds - t\,\psi(\omega) \right), \tag{39}$$

$$K_j = \exp(k_j) = BAL(0)\pi(j)e^{\pi_r t_j}. \tag{40}$$

Proof: See Appendix 2.

Similar to the case that the housing price index follows a jump diffusion process, neither the value of the RM insurance nor the present value of the premium charges is related to the interest rate process. Thus, even when the housing price process follows a general exponential Lévy process, the parameters of stochastic interest rate model are also irrelevant to the determination of LTV.

5. Numerical results

In this section, assuming that the housing price process follows a jump diffusion process for ease of the analysis, we first present the numerical results for the fair LTV ratio. We also study the sustainability of the HECM program. Finally, a numerical study is carried out to validate that the LTV ratio is insensitive to the parameters of the CIR model even when a yearly adjustable-rate RM is considered.

5.1. Fair LTV ratio

In this subsection, we first present the numerical results for a representative base case, and then depict the sensitivity of the LTV ratios and the option values by varying the level of several parameters (i.e. borrower's age, annual premium rate π_m, and interest rate spread π_r). The issuer of the RM insurance can understand the characteristics of this product by observing the sensitivity analysis. Table 2 lists the parameters of the base case. In addition, we use US male mortality data from the Human Mortality Database[4] to fit the Lee–Carter model,

[4] See http://www.mortality.org/.

Table 2. Parameters of the base case[a]

	Parameter	Value
x_0	Borrower's age	70
ω	Final age all lives are assumed to end	110
π_0	Initial premium rate	0.02
π_m	Sequential annual premium rate	0.005
π_r	Interest rate spread	0.015
$\delta(t)$	Maintenance yield (assumed to be constant)	0.02
σ_H	Volatility coefficient for housing price	0.0739
λ	Frequency of jumps	8.1676
θ	Mean of ln Y_i	−0.0021
σ_Y	Standard deviation of ln Y_i	0.0344
τ	Market price of risk of mortality	–0.5

[a]The assumption of π_0 and π_m are in line with the prevailing HECM program before 4 October 2010 (Chen, Cox, et al., 2010). Parameters of housing price model are borrowed from Lee et al. (2012), which employ monthly observations, from January 1973 to December 2010, of the national average prices of previously occupied homes for conventional single-family mortgages in the United States as a proxy for housing prices.

with observations from 1970 to 2005. The market price of mortality risk, denoted by τ, is assumed to be –0.5[5] by referring to Denuit et al. (2007), who measure τ on the basis of Belgian data and it ranges from −0.44 to −0.49 across various discount rates.

For this base case with the parameters listed in Table 2, the LTV is 32.973%, which means that the lump sum payment is 32.973 when house price is 100 at the inception of the loan. The option value, or the value of the RM insurance, is 5.065. Based on Proposition 1, the hedging ratios, delta ($\partial V(0)/\partial H(0)$) and gamma ($\partial delta/\partial H(0)$), can be straightforwardly calculated as −0.0997 and 0.0024, respectively.

We next examine the sensitivity of the LTV ratios and option values by varying the level of parameter values, including the borrower's age, the annual premium rate, and the interest rate spread. The issuers are capable of adjusting the initial premium rate (π_0), the annual premium rate (π_m), and the interest rate spread (π_r). In addition, they can also flexibly determine the LTV ratio together with the option value, to suit the corresponding risks and market competition.

Figure 1 depicts a three-dimensional plot of LTV ratios for different level of ages and annual premium rates. From Figure 1, it can be seen that the higher the level of annual premium rate, the higher the LTV ratio. This is consistent with our intuition. The probability that the value of the house being smaller than the outstanding balance becomes larger as the greater of the lump sum payment; and this corresponds to a larger premium charge. Figure 1 also reveals that the elder the borrower, the higher the LTV ratio. For its economic implication, the present value of the house is the sum of the present values of future rental incomes, which consists of the rental income during the period that the borrower is alive and the rental income during the period that the borrower is dead. According to the RM mechanism,

[5] Denuit et al. (2007) measure the market price of mortality risk for a 65-year-old individual based on the Belgian data. We apply the approach proposed by Denuit et al. (2007) to introduce a way to consider the market price of mortality risk. Since the predicting of the market price of mortality risk is not our purpose, it is set exogenously and is assumed to be constant at various ages.

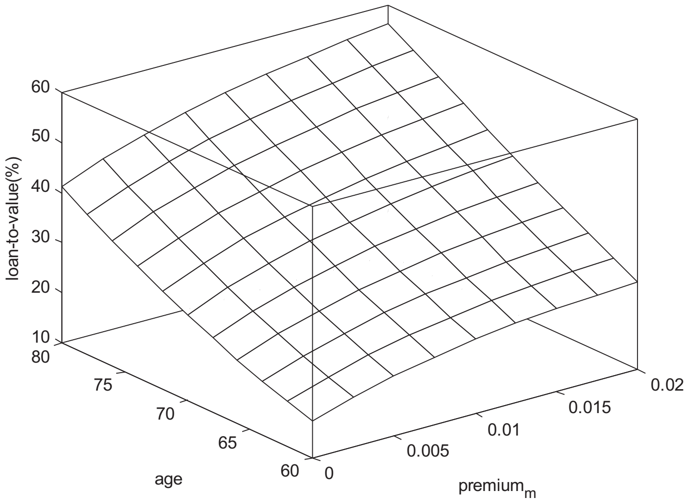

Figure 1. Three-dimensional plot of LTV ratios for different ages and π_m.

the borrower uses the rental income after his or her death in exchange for the lump sum payment at the inception. An older borrower can borrow more money because his or her expected death comes sooner and the present value of the rental income after death is greater.

Figure 2 depicts the relationship among the option values, ages, and annual premium rates. Since the annual premium rate represents the premium charge for covering the contingent loss when the value of the house property is less than the outstanding balance, we can

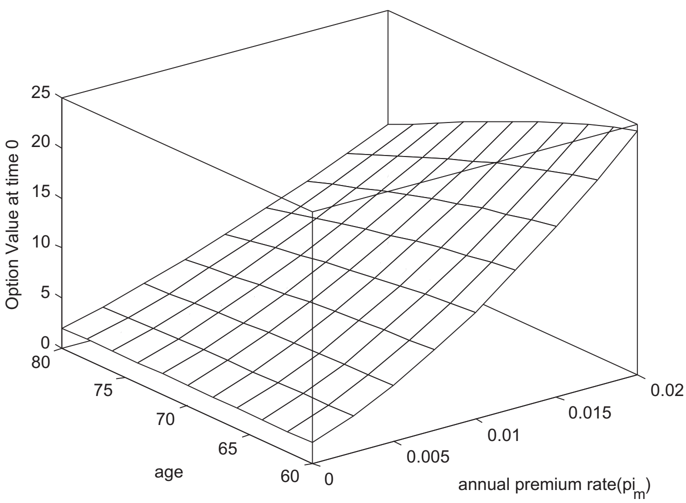

Figure 2. Three-dimensional plot of the option values for different ages and π_m.

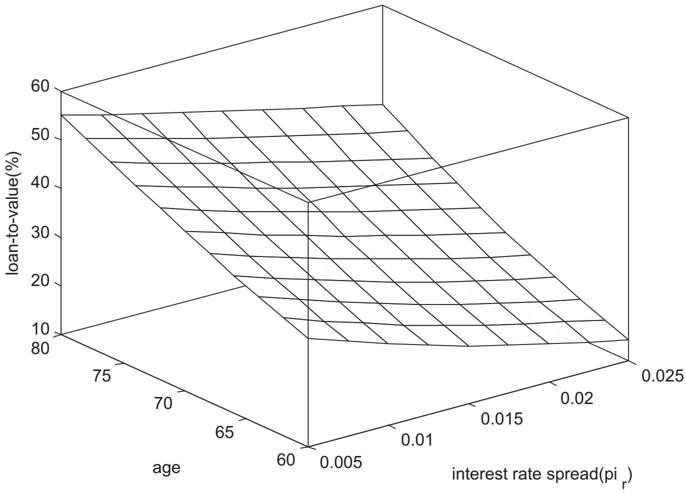

Figure 3. Three-dimensional plot of LTV ratios for different ages and π_r.

see that, in spite of the borrower's age, the option value is an increasing function of the annual premium rate. In addition, as the borrower's age decreases, the option value increases even as the LTV ratio falls since the longer the expected maturity date the higher the option value.

In Figures 3 and 4, we depict the LTV ratios as well as the option values in terms of different ages and interest rate spreads. As shown in Figure 3, the LTV ratio is negatively related to the interest rate spread. Besides, it increases with the borrower's age, which is consistent with our previous finding. Figure 4 indicates that the option value is negatively related to interest rate spread. In sum, Figures 1–4 reveal that, for a borrower aged x_0, the issuer can increase the LTV ratio by charging a higher annual premium rate and/or charging a lower interest rate spread. However, for a specific π_m (or π_r), the greater LTV corresponds to a greater option value for an age-specific borrower.

With Table 3, we also consider the sensitivity of the LTV ratios by varying the level of the market price of risk. Table 3 reveals that the higher the level of risk parameter (in absolute value), i.e. the higher the level of survival probability as well as the level of mortality improvements, the lower the LTV ratio. However, compared with the volatility of housing returns, the impact of the market price of risk is insignificant.

5.2. Sustainability of the HECM Program

In this subsection, following Chen, Cox, et al. (2010), we look at the sustainability of the HECM program in the United States. Chen, Cox, et al. (2010) consider the sustainability ratio of the present value of premium charges ($R(0)$) to the present value of RM insurance ($V(0)$) and conclude that the HECM program is sustainable. Following their approach, we examine the sustainability of the HECM program.

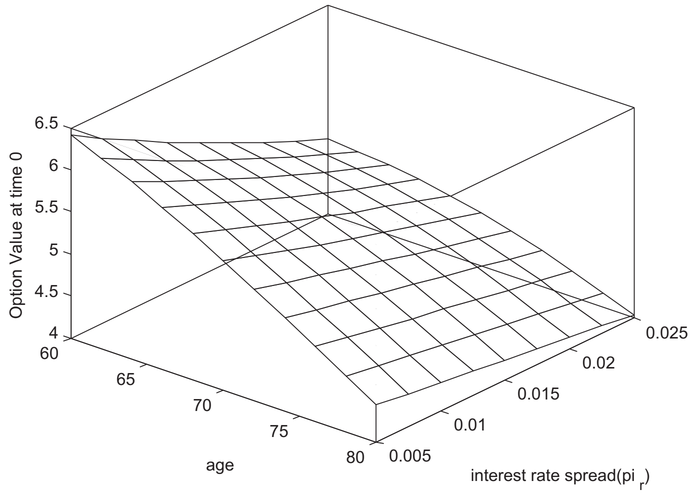

Figure 4. Three-dimensional plot of the option values for different ages and π_r.

Table 3. LTV ratios for different mortality risk premiums.

	LTV ratio (%)
LC model with $\tau = 0$	33.323
LC model with $\tau = -0.5$ (Base Case)	32.973
LC model with $\tau = -1$	32.634

Table 4. Values of RM insurance and sustainability ratios.

σ_H	0.025	0.05	0.0739	0.10	0.125
$V(0)$	70,339	72,635	76,052	80,901	86,341
$R(0)/V(0)$	0.326	0.316	0.301	0.283	0.265
λ	0	4	8.1676	12	16
$V(0)$	63,914	70,211	76,052	80,893	85,512
$R(0)/V(0)$	0.359	0.326	0.301	0.283	0.268
θ	−0.04	−0.03	−0.02	−0.01	−0.0021
$V(0)$	88,089	83,045	79,157	76,747	76,052
$R(0)/V(0)$	0.260	0.276	0.290	0.299	0.301
σ_Y	0	0.02	0.0344	0.04	0.06
$V(0)$	63,964	68,364	76,052	79,701	94,468
$R(0)/V(0)$	0.358	0.335	0.301	0.288	0.243

Following Chen, Cox, et al. (2010), the initial house value and the initial loan advance are assumed to be \$300,000 and \$187,057, respectively. The present value of premium charges then equals \$22,922 over all cases. Table 4 lists the present value of RM insurance

$V(0)$ and the sustainability ratio under a range of housing price parameters[6]. The higher the level of $R(0)/V(0)$, the higher is the possibility for the HECM program to be sustainable. The values of RM insurance together with the sustainability ratio, are irrelevant to the term structure of interest rates. However, they are sensitive to the volatility of the housing price. Accordingly, we can see that from Table 4, the increases in σ_H, λ, θ (in absolute value), and σ_Y, which in turn lead to an increase in the volatility of the housing price, cause an increase in the value of the RM insurance. In addition, the present value of premium charges collected by FHA (Federal Housing Administration) is less than the present value of the total expected claim losses for each case; that is, the HECM program is not sustainable according to our base case. Obviously, it is different from the main finding of Chen, Cox, et al. (2010) that HECM program is sustainable. By using the simulation approach, we find that the volatility of the housing price is nearly 1.5% of Chen, Cox, et al. (2010) but 12.3% in our base case. After the subprime crisis, the annual premium for the HECM standard in the RM program insured by the FHA has been raised from 0.5 to 1.25% on 4 October 2010. The FHA asserts that the previous premium rate, 0.5%, was underestimated. Consequently, this evidence reveals that our findings are consistent with the FHA's perspective.

5.3. *Adjustable-rate RMs: A realistic case*

In light of our analysis, we consider the loan interest rate adjusted according to the short rate plus an interest rate spread, demonstrating that the interest rate risk (short rate) is irrelevant to the fair valuation of RMs. However, in HECM program, the loan interest rate for adjustable-rate products is adjusted according to the one-month CMT (Constant Maturity Treasury) rate or LIBOR rate, not the short rate. Moreover, it is adjusted monthly or annually, depending on the choice of the borrower. Consequently, this paper provides a more realistic case for adjustable-rate RMs, the loan interest rate of which is adjusted monthly or annually according to the LIBOR rate.

Let $L(t_j, t_k)$ be the prevailing discrete forward rate at time t_j over the time interval $[t_k, t_k + \Delta t]$ and $\Delta t = t_{k+1} - t_k$. As a result, $L(t_k, t_k)$ is the spot LIBOR rate at time t_k and is given by $L(t_k, t_k) = (1/\Delta t)[1/P(t_k, t_{k+1}) - 1]$, where $P(t_k, t_{k+1})$ is the time t_k price of a zero coupon bond that pays one dollar at time t_{k+1}. We assume that the loan interest rate is adjusted yearly according to the spot LIBOR rate. More specifically, instead of Equations (1) and (2), the outstanding balance at time t is of the form:

$$BAL(t_1) = (BAL(0) + \pi_0 H(0))[1 + L(0,0)\Delta t]e^{\pi_r \Delta t}, \tag{41}$$

$$BAL(t_{j+1}) = BAL(t_j)(1 + \pi_m)\left[1 + L(t_j, t_j)\Delta t\right]e^{\pi_r \Delta t}, \quad j = 1, 2, \ldots, \tag{42}$$

[6] $V(0)$ and $R(0)$ are NRP (non-recourse provision) and MIP (mortgage insurance premiums) defined in Chen, Cox, et al. (2010), respectively.

Table 5. Parameters of the CIR model.

Parameter	$r(0)$	α	β_r	σ_r	ρ_{Hr}
Value	0.03	0.2	0.05	0.02	0.25

Note: ρ_{Hr} is the correlation coefficient between the interest rate and house price.

Table 6. The sensitivity of the LTV ratios to interest rate parameters (Unit: %).

	0	0.03	0.06	0.09	0.12
LTV	32.759	32.803	32.842	32.877	32.906
α	0.01	0.1	0.2	0.3	0.4
LTV	32.822	32.803	32.803	32.806	32.809
β_r	0.01	0.03	0.05	0.07	0.09
LTV	32.847	32.830	32.803	32.764	32.712
σ_r	0.01	0.02	0.03	0.04	0.05
LTV	32.832	32.803	32.772	32.739	32.705
ρ_{Hr}	−1	−0.5	0	0.5	1
LTV	33.048	32.948	32.851	32.755	32.662

Similarly, Equations (41) and (42) can be reduced to,

$$\begin{aligned} BAL(t_j) &= (BAL(0) + \pi_0 H(0))(1+\pi_m)^{j-1} \prod_{k=0}^{j-1} [1 + L(t_k, t_k)\Delta t] e^{\pi_r \Delta t} \\ &= BAL(0)\pi(j) \prod_{k=0}^{j-1} [1 + L(t_k, t_k)\Delta t] e^{\pi_r \Delta t}. \end{aligned} \tag{43}$$

Substituting $BAL(t_j)$ defined in Equation (43) into Equations (7) and (8), we can compute the present value of the RM insurance ($V(0)$) and the present value of the premium charges ($R(0)$), and in turn determine the LTV ratio through Monte-Carlo simulation.[7]

For the base case with the parameters listed in Table 2 together with the parameters of the CIR model in Table 5, the LTV ratio is 32.803% when the loan interest rate is adjusted yearly according to LIBOR rate, which is very close to 32.973% when the loan interest rate is adjusted instantaneously according to short rate. Table 6 shows the sensitivity of the LTV ratios by varying the level of interest rate parameters. We can see that the LTV ratios are insensitive to the level of interest rate parameters. Consequently, for adjustable-rate RMs, the LTV ratio is less dependent on the interest rate parameters.

6. Conclusion

Most borrowers of HECM loans choose adjustable interest rates, yet most literatures choose a constant interest rate assumption. This paper successfully considers the dynamic of interest

[7] Each simulated result is based on 10,000 simulation paths. The time partition (dt) is one month.

rate and the adjustable-rate RM simultaneously. We also successfully consider the housing price shock into the valuation model. Both of them make a further progress on the literature of RMs.

In this study, when the RM rate is adjusted instantaneously based on the short rate, we present a closed-form formula for calculating the LTV in a lump sum payment adjustable-rate RM. We assume that the house price model follows a general Lévy process and that the interest rate model follows the CIR model. With the assumption that the RM rate is the short rate plus an interest rate spread, we demonstrate that the LTV ratio is independent of the term structure of interest rate. This argument differs from preceding literature and provides a new perspective on the design of RMs. We also provide a more realistic case with the RM rate adjusted periodically based on the LIBOR rate. We reach a similar conclusion that the LTV ratio is insensitive to the parameters of the CIR model even when a yearly adjustable-rate RM is considered.

We provide a sensitivity analysis of the value of RMs against the annual premium rate and the interest rate spread. We find that the issuer can increase the LTV by charging a higher annual premium rate and/or charging a lower interest rate spread. However, for an age-specific borrower, a greater LTV represents a greater option value. The issuer, therefore, should understand the influence of these variables to determine the appropriate features of its products. In addition, we also demonstrate that the HECM program may not be sustainable, especially for a higher level of housing price volatility. Consequently, after the subprime crisis, the FHA has adjusted the previous premium rate from 0.5 to 1.25% on 4 October 2010.

Due to mortality improvement a wide range of mortality models have been proposed and discussed (e.g. Lee and Carter, 1992; Brouhns et al., 2002; Renshaw and Haberman, 2003; Cairns et al., 2006; Koissi et al., 2006; Melnikov and Romaniuk, 2006; Li and Chan, 2007; Cairns et al., 2009; Biffis et al., 2010; Yang et al., 2010; Hainaut, 2012). More sophisticated mortality models might help companies hedge against longevity risk. Consequently, for further research, it will be an interesting topic to discuss the impact of mortality model risk on pricing lump sum RM contracts.

In this study, we do not consider a capped property value, which limits the amount of value that can be used to determine cash advances and the amount of the insurance. We also only consider a lump sum payment, which indicates that we ignore annuity payment and line of credit RMs. These issues also provide some topics of interest for further research.

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References

Barndorff-Nielsen, O. E. (1998). Processes of normal inverse Gaussian type. *Finance and Stochastics* **2**, 41–68.
Bertoin, J. (1996). *Levy processes*. Cambridge Tracts in Mathematics 121. Cambridge: Cambridge University Press.

Biffis, E., Denuit, M. & Devolder, P. (2010). Stochastic mortality under measure changes. *Scandinavian Actuarial Journal* **4**, 284–311.

Brouhns, N., Denuit, M. & Vermunt, J. K. (2002). A Poisson log-bilinear regression approach to the construction of projected life tables. *Insurance: Mathematics and Economics* **31**, 373–393.

Cairns, A. J. G., Blake, D. & Dowd, K. (2006). A two-factor model for stochastic mortality with parameter uncertainty: theory and calibration. *Journal of Risk and Insurance* **73**, 687–718.

Cairns, A. J. G., Blake, D., Dowd, K., Coughlan, G. D., Epstein, D., Ong, A. & Balevich, I. (2009). A quantitative comparison of stochastic mortality models using data from England and Wales and the United States. *North American Actuarial Journal* **13**, 1–35.

Carr, P. & Madan, D. B. (1999). Option valuation using the fast fourier transform. *Journal of Computational Finance* **2** (4), 61–73.

Carr, P., Geman, H., Madan, D. B. & Yor, M. (2002). The fine structure of asset returns: an empirical investigation. *Journal of Business* **75**, 305–332.

Carr, P. & Wu, L. (2003). The finite moment log stable process and option pricing. *Journal of Finance* **58**, 753–777.

Chen, H., Cox, S. H. & Wang, S. S. (2010). Is the home equity conversion mortgage in the United States sustainable – evidence from pricing mortgage insurance premiums and non-recourse provisions using the conditional Esscher transform. *Insurance: Mathematics and Economics* **46**, 371–384.

Chen, M. C., Chang, C. C., Lin, S. K. & Shyu, S. D. (2010). Estimation of housing price jump risks and their impact on the valuation of mortgage insurance contacts. *Journal of Risk and Insurance* **77** (2), 399–422.

Chinloy, P. & Megbolugbe, I. F. (1994). Reverse mortgages: contracting and crossover risk. *Journal of the American Real Estate and Urban Economics Association* **22** (2), 367–386.

Cox, J. C., Ingersoll, J. E., Jr & Ross, S. A. (1985). A theory of the term structure of interest rate. *Econometrica* **53** (2), 385–407.

Denuit, M., Devolder, P. & Goderniaux, A. C. (2007). Securitization of longevity risk: pricing survivor bonds with Wang transform in the Lee–Carter framework. *The Journal of Risk and Insurance* **74** (1), 87–113.

Dowd, K., Blake, D., Cairns, A. J. G. & Dawson, P. (2006). Survivor swaps. *Journal of Risk and Insurance* **73** (1), 1–17.

Eberlein, E., Keller, U. & Prause, K. (1998). New insights into smile, mispricing, and value at risk: the hyperbolic model. *Journal of Business* **71** (3), 371–405.

Eraker, B., Johannes, M. & Poison, N. (2003). The impact of jumps in equity index volatility and returns. *Journal of Finance* **58** (3), 1269–1300.

Hainaut, D. (2012). Multidimensional Lee–Carter model with switching mortality processes. *Insurance: Mathematics and Economics* **50**, 236–246.

Koissi, M. C., Shapiro, A. F. & Högnäs, G. (2006). Evaluating and extending the Lee–Carter model for mortality forecasting: bootstrap confidence interval. *Insurance: Mathematics and Economics* **38**, 1–20.

Kou, S. G. (2002). A jump-diffusion model for option pricing. *Management Science* **48**, 1086–1101.

Lastrapes, W. D. (2002). The real price of housing and money supply shocks: time series evidence and theoretical simulations. *Journal of Housing Economics* **11**, 40–74.

Lee, B. S., Chung, E. C. & Kim, Y. H. (2005). Dwelling age, redevelopment, and housing prices: the case of apartment complexes in Seoul. *Journal of Real Estate Finance and Economics* **30** (1), 55–80.

Lee, R. D. (2000). The Lee–Carter method for forecasting mortality, with various extensions and applications. *North American Actuarial Journal* **4**, 80–91.

Lee, R. D. & Carter, L. R. (1992). Modeling and forecasting U.S. mortality. *Journal of the American Statistical Association* **87**, 659–675.

Lee, Y. T., Wang, C. W. & Huang, H. C. (2012). On the valuation of reverse mortgages with regular tenure payments. *Insurance: Mathematics and Economics* **51** (2), 430–441.

Li, S. H., and W. S. Chan. (2007). The Lee–Carter model for forecasting mortality, revisited. *North American Actuarial Journal* **11** (1), 68–89.

Li, J. S. H., Hardy, M. R. & Tan, K. S. (2010). On pricing and hedging the no-negative-equity guarantee in equity release mechanisms. *The Journal of Risk and Insurance* **77** (2), 499–522.

Liao, H. H., Yang, S. S. & Huang, I. H. (2007). *The design of securitization for longevity risk: pricing under stochastic mortality model with tranche technique*. Taipei: Third International Longevity Risk and Capital Market Solutions Symposium.

Lin, Y. J. & Cox, S. H. (2005). Securitization of mortality risks in life annuities. *Journal of Risk and Insurance* **72** (2), 227–252.

Ma, S., Kim, G. & Lew, K. (2007). Estimating reverse mortgage insurer's risk using stochastic models. Paper Presented at the Asia-Pacific Risk and Insurance Association 2007 Annual Meeting, Taipei.

Madan, D. B., Carr, P. P. & Chang, E. C. (1998). The variance gamma process and option pricing. *European Finance Review* **2**, 79–105.

Melnikov, A. & Romaniuk, Y. (2006). Evaluating the performance of Gompertz, Makeham and Lee–Carter mortality models for risk management with unit-linked contracts. *Insurance: Mathematics and Economics* **39**, 310–329.

Merton, R. C. (1976). Option pricing when underlying stock returns are discontinuous. *Journal of Financial Economics* **3**, 125–144.
Phillips, W. A. & Gwin, S. B. (1992). Reverse mortgage. *Transactions of the Society of Actuaries* **44**, 289–323.
Renshaw, A. E. & Haberman, S. (2003). Lee–Carter mortality forecasting with age-specific enhancement. *Insurance: Mathematics and Economics* **33**, 255–272.
Szymanoski, E. J. (1994). Risk and the home equity conversion mortgage. *Real Estate Economics* **22** (2), 347–366.
Tse, Y. K. (1995). Modelling reverse mortgages. *Asia Pacific Journal of Management* **12** (2), 79–95.
Wang, S. S. (2000). A class of distortion operators for pricing financial and insurance risks. *The Journal of Risk and Insurance* **67** (1), 15–36.
Wang, L., Valdez, E. A. & Piggott, J. (2007). Securitization of longevity risk in reverse mortgages. Working paper, Available at SSRN: http://ssrn.com/abstract=1087549.
Weinrobe, M. (1988). An insurance plan to guarantee reverse mortgage. *Journal of Risk and Insurance* **55** (4), 644–659.
Wilson, P. J., Stevenson, S. & Zurbruegg, R. (2007). Foreign property shocks and the impact on domestic securitized real estate markets: an unobserved components approach. *The Journal of Real Estate Finance and Economics* **34**, 407–424.
Yang, S. S., Yue, J. C. & Huang, H. C. (2010). Modelling longevity risk using principle component. *Insurance: Mathematics and Economics* **46**, 254–270.
Zhai, D. H. (2000). Reverse mortgage securitizations: understanding and gauging the risks. Structure Finance, Moody's Investors Service Special Report.

Appendix 1. Proof of Proposition 1.

By virtue of Equations (6) and (21), the present value of the premium charges $R(0)$ is,

$$R(0) = \pi_0 H(0) + E_Q\left[\sum_{j=1}^{\omega-x_0} \frac{{}_{t_j}p_{x_0,t_0} BAL(t_j)\pi_m}{B(t_j)}\right] = \pi_0 H(0) + \sum_{j=1}^{\omega-x_0} E_Q\left[{}_{t_j}p_{x_0,t_0}\right] E_Q\left[\frac{BAL(t_j)\pi_m}{B(t_j)}\right]$$
$$= \pi_0 H(0) + \sum_{j=1}^{\omega-x_0} S(t_j) BAL(0)\pi(j)\pi_m e^{\pi_r t_j},$$

where we use the assumption that the mortality process and the financial asset price processes are independent.

For the present value of the RM insurance $V(0)$, it can be expressed as follows:

$$V(0) = \sum_{j=1}^{\omega-x_0} E_Q\left[\frac{\left[{}_{t_{j-1}}p_{x_0,t_0} - {}_{t_j}p_{x_0,t_0}\right]\left[BAL(t_j) - H(t_j)\right]^+}{B(t_j)}\right] = \sum_{j=1}^{\omega-x_0} (S(t_{j-1}) - S(t_j))C(t_j),$$

where

$$C(t_j) = E_Q\left[\frac{BAL(t_j)1_D}{B(t_j)}\right] - E_Q\left[\frac{H(t_j)1_D}{B(t_j)}\right] = A_1 - A_2,$$

and $D = \{BAL(t_j) \geq H(t_j)\}$.

To derive A_1, applying Equation (6), we have,

$$A_1 = E_Q[BAL(0)\pi(j)e^{\pi_r t_j}1_D] = BAL(0)\pi(j)e^{\pi_r t_j} Pr_Q(BAL(t_j) \geq H(t_j)).$$

Using Ito's Lemma, we obtain the expression for housing price as follows:

$$H(t_j) = H(0)B(t_j)\exp\left\{-\int_0^{t_j}(\lambda\beta + \delta(s) + \frac{1}{2}\sigma_H^2(s))ds + \int_0^{t_j}\hat{\sigma}_H(s)\cdot dW(s) + \sum_{n=1}^{N(t_j)}\ln Y_n\right\} \tag{A.1}$$

Therefore, using Equation (A.1), we have,

$$\begin{aligned} &Pr_Q\big(BAL(t_j) \geq H(t_j)\big) \\ &= Pr_Q\left\{BAL(0)\pi(j)e^{\pi_r t_j} \geq H(0)\exp\left(-\int_0^{t_j}\left(\delta(s) + \lambda\beta + \frac{1}{2}\sigma_H^2(s)\right)ds + \int_0^{t_j}\hat{\sigma}_H(s)\cdot dW(s) + \sum_{n=1}^{N(t_j)}\ln Y_n\right)\right\} \\ &= Pr_Q\left\{\int_0^{t_j}\hat{\sigma}_H(s)\cdot dW(s) + \sum_{n=1}^{N(t_j)}\ln Y_n \leq -x_j\right\} \end{aligned} \tag{A.2}$$

where

$$x_j = \ln\left(\frac{H(0)}{BAL(0)\pi(j)}\right) - \int_0^{t_j}\left(\pi_r + \delta(s) + \lambda\beta + \frac{1}{2}\sigma_H^2(s)\right)ds.$$

Let $\ln Y_n = \theta + \sigma_Y Z_n \sim N(\theta, \sigma_Y^2)$, where Z_n is a standard normal random variable and is independent of Z_m, for $n \neq m$. Therefore, Equation (A.2) can be rewritten as follows:

$$\begin{aligned} Pr_Q\big(BAL(t_j) \geq H(t_j)\big) &= \sum_{m=0}^{\infty}\frac{e^{-\lambda t_j}(\lambda t_j)^m}{m!}Pr_Q\left\{\int_0^{t_j}\hat{\sigma}_H(s)\cdot dW(s) + \sum_{n=1}^{m}(\theta + \sigma_Y Z_n) \leq -x_j\right\} \\ &= \sum_{m=0}^{\infty}\frac{e^{-\lambda t_j}(\lambda t_j)^m}{m!}Pr_Q\left\{\frac{\int_0^{t_j}\hat{\sigma}_H(s)\cdot dW(s) + \sigma_Y\sum_{n=1}^{m}Z_n}{\sqrt{\int_0^{t_j}\sigma_H^2(s)\,ds + \sigma_Y^2 m}} \leq \frac{-(x_j + m\theta)}{\sqrt{\int_0^{t_j}\sigma_H^2(s)\,ds + \sigma_Y^2 m}}\right\} \\ &= \sum_{m=0}^{\infty}\frac{e^{-\lambda t_j}(\lambda t_j)^m}{m!}\Phi\{-d_{2j}(m)\}. \end{aligned}$$

To derive A_2, applying Equation (A.1), we have,

$$\begin{aligned} A_2 &= E_Q\left[H(0)\exp\left(-\int_0^{t_j}(\delta(s) + \lambda\beta)ds\right)\xi_{t_j}^R\left(\prod_{n=1}^{N(t_j)}Y_n\right)1_D\right] \\ &= H(0)\exp\left(-\int_0^{t_j}(\delta(s) + \lambda\beta)ds\right)E_R\left[\left(\prod_{n=1}^{N(t_j)}Y_n\right)1_D\right], \end{aligned}$$

where

$$\left.\frac{dR}{dQ}\right|_{F_T} = \xi_T^R = exp\left\{\int_0^T\hat{\sigma}_H(s)\cdot dW(s) - \frac{1}{2}\int_0^T\sigma_H^2(s)ds\right\}.$$

By Girsanov's theorem, the process $W_R(t)$, defined by,

$$dW_R(t) = dW(t) - \hat{\sigma}_H(t)dt,$$

is a standard Brownian motion under probability measure R. Under the probability measure R, the housing price process satisfies,

$$\frac{dH(t)}{H(t)} = (r(t) - \delta(t) + \sigma_H^2(t))dt + \hat{\sigma}_H(t) \cdot dW_R(t) + d(J(t) - \lambda\beta t).$$

Similarly, using Ito's Lemma, we obtain,

$$H(t_j) = H(0)B(t_j)\exp\left\{ \int_0^{t_j} \left(-\delta(s) - \lambda\beta + \tfrac{1}{2}\sigma_H^2(s)\right)ds + \int_0^{t_j} \hat{\sigma}_H(s) \cdot dW_R(s) + \sum_{n=1}^{N(t_j)} \ln Y_n \right\}.$$

Therefore, A_2 is of the form:

$$\begin{aligned}
A_2 &= H(0)\exp\left(-\int_0^{t_j} (\delta(s) + \lambda\beta)ds\right) E_R\left[\left(\prod_{n=1}^{N(t_j)} \exp(\theta + \sigma_Y Z_n)\right) 1_D\right] \\
&= H(0)\exp\left(-\int_0^{t_j} (\delta(s) + \lambda\beta)ds\right) \sum_{m=0}^{\infty} \frac{e^{-\lambda t_j}(\lambda t_j)^m}{m!} E_R\left[\exp(m\theta + \sigma_Y\sqrt{m}Z)1_D\right] \\
&= H(0)\exp\left(-\int_0^{t_j} \delta(s)ds\right) \sum_{m=0}^{\infty} \frac{e^{-\lambda(\beta+1)t_j}(\lambda t_j)^m}{m!} \exp\left(m\left(\theta + \frac{1}{2}\sigma_Y^2\right)\right) X_m \\
&= H(0)\exp\left(-\int_0^{t_j} \delta(s)ds\right) \sum_{m=0}^{\infty} \frac{e^{-\lambda(\beta+1)\,t_j}\left(\lambda(\beta+1)t_j\right)^m}{m!} X_m,
\end{aligned}$$

where Z is a standard normal random variable and $\beta = E(Y_i - 1) = e^{\theta + \frac{\sigma_Y^2}{2}} - 1$. Given $N(t_j) = m$, X_m is defined as follows:

$$\begin{aligned}
X_m &= E_R\left[\exp\left(\sigma_Y\sqrt{m}Z - \frac{1}{2}\sigma_Y^2 m\right) 1_D\right] \\
&= \int_{-\infty}^{\infty} 1_D \exp\left(\sigma_Y\sqrt{m}\,z - \frac{1}{2}\sigma_Y^2 m\right) \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}z^2\right) dz \\
&= \int_{-\infty}^{\infty} 1_D \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}(z - \sigma_Y\sqrt{m})^2\right) dz.
\end{aligned}$$

Let $Z^* = Z - \sigma_Y\sqrt{m}$. Hence, the housing price process, given $N(t_j) = m$, satisfies

$$H(t_j) = H(0)B(t_j)\exp\left\{\int_0^{t_j} \left(-\delta(s) - \lambda\beta + \frac{1}{2}\sigma_H^2(s)\right)ds + (\theta + \sigma_Y^2)m + \int_0^{t_j} \hat{\sigma}_H(s) \cdot dW_R(s) + \sigma_Y\sqrt{m}\,Z^*\right\}$$

and X_m can be rewritten as,

$$\begin{aligned}
X_m &= \Pr\left[\int_0^{t_j} \hat{\sigma}_H(s) \cdot dW_R(s) + \sigma_Y\sqrt{m}Z^* \le -\left(x_j + \frac{1}{2}\int_0^{t_j} \sigma_H^2(s)\,ds + (\theta + \sigma_Y^2)m\right)\right] \\
&= \Phi(-d_{1,j}(m)).
\end{aligned}$$

This completes the proof of Proposition 1.

Appendix 2. Proof of Proposition 2.

By virtue of Equation (6) and (27), $C_L(t_j)$ can be rewritten as follows:

$$C_L(t_j) = E_Q\left[\frac{\left[BAL(t_j) - H(t_j)\right]^+}{B(t_j)}\right] = E_Q\left[\left[BAL(0)\pi(j)e^{\pi_r t_j} - H(0)e^{A(t_j)+\int_0^{t_j}\hat{\sigma}_H(s)\cdot dW(s)+L(t_j)}\right]^+\right].$$

Assuming that $K_j = BAL(0)\pi(j)e^{\pi_r t_j}$ and $H_B(t) = H(t)/B(t)$, we have,

$$\begin{aligned} C_L(t_j) &= E_Q\left[(K_j - H_B(t_j))\,1_{(H_B(t)\le K_j)}\right] = E_Q\left[(K_j - H_B(t_j))(1 - 1_G)\right] \\ &= K_j - E_Q(H_B(t_j)) + E_Q\left[(H_B(t_j) - K_j)^+\right] = K_j - H(0)e^{-\int_0^{t_j}\delta(s)\,ds} + C(t_j, k_j), \end{aligned}$$

where $k_j = \ln K_j$ and $G = \{H_B(t_j) > K_j\}$. Then $C(t_j, k_j)$ can be rewritten as,

$$C(t_j, k_j) \equiv E^Q\left[\left[e^{h(t_j)} - e^{k_j}\right]^+\right] = \int_{k_j}^{\infty}\left(e^{h(t_j)} - e^{k_j}\right)Q\big(h(t_j)\big)dh(t_j); \qquad \text{(B.1)}$$

where $h(t)$ is equal to $\ln H(0) + A(t) + \int_0^t \hat{\sigma}_H(s)\cdot dW(s) + L(t)$ with a risk-neutral probability density $Q_{h(t)}$ and a characteristic function $\phi_{h(t)}(\omega)$ as follows:

$$\begin{aligned} \phi_{h(t)}(\omega) &= E_Q\left(e^{i\omega h(t)}\right) = \exp(i\omega(\ln H(0) + A(t)))E_Q\left(e^{i\omega\int_0^t\hat{\sigma}_H(s)\cdot dW(s)}\right)E_Q\left(e^{i\omega L(t)}\right) \\ &= \exp\left(i\omega(\ln H(0) + A(t)) - \frac{\omega^2}{2}\int_0^t \sigma_H^2(s)ds - t\,\psi(\omega)\right). \end{aligned}$$

To solve $C(t_j, k_j)$, following the Carr & Madan (1999) approach, we define a modified call price as:

$$C_{\text{mod}}(t_j, k_j) \equiv e^{\alpha k_j}C(t_j, k_j) \qquad \text{(B.2)}$$

where $C_{\text{mod}}(t_j, k_j)$ is expected to satisfy the integrability condition

$$\int_{-\infty}^{\infty}\left|C_{\text{mod}}(t_j, k_j)\right|dk_j < \infty,$$

by carefully choosing $\alpha > 0$. Consider a Fourier transform of $C_{\text{mod}}(t_j, k_j)$ as follows:

$$\Lambda_{t_j}(\omega) = \int_{-\infty}^{\infty} e^{i\omega k_j}C_{\text{mod}}(t_j, k_j)\,dk_j.$$

Substituting equations (B.1) and (B.2) and interchanging integrals yields

$$\begin{aligned}\Lambda_{t_j}(\omega) &= \int_{-\infty}^{\infty} e^{i\omega k_j} e^{\alpha k_j} C(t_j, k_j)\, dk_j = \int_{-\infty}^{\infty} e^{i\omega k_j} e^{\alpha k_j} \int_{k_j}^{\infty} \left(e^{x} - e^{k_j}\right) Q_{h(t_j)}(x) dx\, dk_j \\ &= \int_{-\infty}^{\infty} Q_{h(t_j)}(x) \int_{-\infty}^{x} e^{i\omega k_j} \left(e^{x+\alpha k_j} - e^{(1+\alpha)k_j}\right) dk_j dx \\ &= \int_{-\infty}^{\infty} Q_{h(t_j)}(x) \left(\frac{e^{(\alpha+1+i\omega)x}}{\alpha + i\omega} - \frac{e^{(\alpha+1+i\omega)x}}{\alpha + 1 + i\omega}\right) dx = \frac{\phi_{h(t_j)}(\omega - (\alpha+1)i)}{(\alpha + i\omega)(\alpha + 1 + i\omega)}.\end{aligned}$$

Thus, $C(t_j, k_j)$ can be obtained by an inverse Fourier transform of $\Lambda_{t_j}(\omega)$:

$$C\left(t_j, k_j\right) = \frac{e^{-\alpha k_j}}{2\pi} \int_{-\infty}^{\infty} e^{-i\omega k_j} \Lambda_{t_j}(\omega)\, d\omega = \frac{e^{-\alpha k_j}}{2\pi} \int_{-\infty}^{\infty} e^{-i\omega k_j} \frac{\phi_{h(t_j)}(\omega - (\alpha+1)i)}{(\alpha + i\omega)(\alpha + 1 + i\omega)} d\omega.$$