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Traveling waves for a diffusive SIR model with delay ☆

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ABSTRACT

In this paper, we study traveling waves connecting the infection-free equilibrium state and the endemic equilibrium state for a diffusive SIR model with delay and saturated incidence rate. Since this system does not enjoy the comparison principle, we will use an iteration process to construct a pair of upper and lower solutions. With the aid of the pair of upper and lower solutions, we can apply the Schauder fixed point theorem to construct a family of solutions of the truncated problems, which, via the limiting argument, can generate the traveling wave. Indeed, we show that there exists $c^* > 0$ such that this system admits a traveling wave solution with speed c iff $c \geq c^*$.

1. Introduction

Traveling waves in a diffusive epidemic model have received attention by several researchers because they can be used to describe the state that a disease propagates spatially with a constant speed. In the biological context, one of the important issues is whether the traveling wave solutions exist. In this paper, we will study this problem for a delayed SIR model

$$S_t(x,t) = d_S S_{xx}(x,t) + \mu(\Lambda - S(x,t)) - F(S(x,t), I(x,t-\tau)), \tag{1.1a}$$

$$I_t(x,t) = d_I I_{xx}(x,t) + F(S(x,t), I(x,t-\tau)) - (\mu+\gamma) I(x,t), \tag{1.1b}$$

$$R_t(x,t) = d_R R_{xx}(x,t) + \gamma I(x,t) - \mu R(x,t), \tag{1.1c}$$

where d_S, d_I, d_R, Λ, μ, γ, and τ are positive constants. Here $S(x,t)$, $I(x,t)$, and $R(x,t)$ stand for the numbers of the susceptible, infected, and removed individuals at position x and time t, respectively, and the parameters d_S, d_I, and d_R are their diffusion coefficients. The constant $\mu\Lambda$ is the recruitment rate of the susceptible population, γ is the recovery rate of the infective population, μ is the natural death rate

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for all the susceptible, the infective, and the removed population, and τ is the latent period of the disease. The constant Λ can be interpreted as a carrying capacity, or maximum possible population size. This model can be used to describe transmission of viral agent diseases such as measles, mumps, and small pox. In practical use, there are various types of the incidence term $F(S,I)$. The common types include bilinear incidence (or mass action incidence) βSI (see, for example, [5,2,12,15]), standard incidence $\frac{\beta SI}{\Lambda}$ (see, for example, [8]), and saturated incidence $\frac{\beta SI}{1+\alpha I}$ (see, for example, [3,8,9,11,13,14,17,16,18]), where β and α are positive constants. In this paper, we focus on the case of saturated incidence and therefore hereafter we assume that $F(S,I) = \frac{\beta SI}{1+\alpha I}$.

Observing that the first two equations of (1.1) form a closed system and the function R can be derived as long as both S and I are solved, from now on we only consider (1.1a)–(1.1b). For convenience, by a change of variables, we rewrite (1.1) in the following form

$$\begin{aligned} S_t(x,t) &= \delta S_{xx}(x,t) + \mu(1 - S(x,t)) - \beta S(x,t) g(I(x,t-\tau)), \\ I_t(x,t) &= I_{xx}(x,t) + \beta S(x,t) g(I(x,t-\tau)) - (\mu+\gamma) I(x,t), \end{aligned} \tag{1.2}$$

where $g(\xi) := \xi/(1+\alpha\xi)$.

Note that the infection-free equilibrium state $(1,0)$ always exists in system (1.2). Besides, when the basic reproduction number $R_0 := \beta/(\gamma+\mu) > 1$, there also exists a positive endemic equilibrium state (s^*, i^*), where

$$s^* := \frac{\alpha\mu + \mu + \gamma}{\beta + \alpha\mu} \text{ and } i^* := \frac{\mu(\beta - \mu - \gamma)}{(\mu+\gamma)(\beta + \alpha\mu)}.$$

Since we are concerned with the spread of disease, we always assume that $R_0 > 1$ throughout this paper.

By a traveling wave solution of system (1.2), we mean a solution of system (1.2) of the form

$$(S(x,t), I(x,t)) = (s(z), i(z)), \quad z = x + ct,$$

with the boundary condition $(s,i)(-\infty) = (1,0)$ and $(s,i)(+\infty) = (s^*, i^*)$. Here the wave speed c is a constant to be determined and the wave profile $(s,i) \in C^2(\mathbb{R}) \times C^2(\mathbb{R})$ is a pair of nonnegative functions. Upon substituting the ansatz on (s,i) into (1.2), we are led to the governing system for (s,i) as follows:

$$\delta s''(z) - cs'(z) + \mu(1 - s(z)) - \beta s(z) g(i(z - c\tau)) = 0, \tag{1.3a}$$

$$i''(z) - ci'(z) + \beta s(z) g(i(z - c\tau)) - (\mu+\gamma) i(z) = 0 \tag{1.3b}$$

on $\mathbb{R}$, together with the boundary conditions

$$(s,i)(-\infty) = (1,0), \quad (s,i)(+\infty) = (s^*, i^*). \tag{1.4}$$

Here the prime indicates differentiation with respect to z.

Linearizing (1.3b) around the point $(1,0)$ leads to the equation

$$i''(z) - ci'(z) + \beta i(z - c\tau) - (\mu+\gamma) i(z) = 0,$$

whose characteristic equation is given by

$$P(\lambda, c) := \lambda^2 - c\lambda + \beta e^{-c\tau\lambda} - \gamma - \mu = 0. \tag{1.5}$$

Since $R_0 > 1$, we obtain that $P(0,c) = \beta - \gamma - \mu > 0$. For each fixed $c > 0$, we note that $P_{\lambda\lambda}(\lambda, c) > 0$ for all $\lambda \in \mathbb{R}$ and $P_\lambda(0,c) < 0$. Since $P(\lambda, 0) > 0$ and $P_c(\lambda, c) < 0$ for all $\lambda > 0$, and, for any fixed $\lambda > 0$, $P(\lambda, c) < 0$ if c is sufficiently large, it follows that

$$c^* := \sup\{c > 0 | P(\lambda, c) > 0, \forall \lambda \in \mathbb{R}\}$$

exists and is positive, and the equation $P(\lambda, c^*) = 0$ has a unique root, denoted by λ^*. One can easy verify that λ^* is positive, and

$$P(\lambda^*, c^*) = 0 \text{ and } P_\lambda(\lambda^*, c^*) = 0. \tag{1.6}$$

In addition, for each $c > c^*$, the equation $P(\lambda, c) = 0$ has two positive roots λ_1 and λ_2, and $P(\lambda, c) < 0$ when $\lambda \in (\lambda_1, \lambda_2)$. *In the sequel, we retain the notations* c^*, λ^*, λ_1, λ_2.

Now we are in a position to state our main result on the existence and non-existence of traveling waves for system (1.2) in the following theorem.

Theorem 1.1. *For* $c \geq c^*$*, system (1.3)–(1.4) admits a nonnegative solution* (s, i) *satisfying the following properties:*

(i) $0 < s < 1$ *and* $0 < i < B := \frac{\beta-\mu-\gamma}{\alpha(\mu+\gamma)}$ *over* $\mathbb{R}$.
(ii) *As* $z \to -\infty$*, we have*

$$i(z) = \begin{cases} \mathcal{O}(e^{\lambda_1 z}), & \text{if } c > c^*, \\ \mathcal{O}(-ze^{\lambda^* z}), & \text{if } c = c^*. \end{cases}$$

However, for $c < c^*$*, there exist no nonnegative solutions of system (1.3)–(1.4).*

Our proof is briefly sketched as follows. To show the existence of traveling waves, our strategy is to construct a solution (s, i) of (1.3) on $\mathbb{R}$ such that it is sandwiched between the upper and lower solutions (s^+, i^+) and (s^-, i^-). Since $(s^+, i^+)(-\infty) = (s^-, i^-)(-\infty) = (1, 0)$, we can infer that (s, i) is a nonnegative solution of (1.3) on $\mathbb{R}$ with $(s, i)(-\infty) = (1, 0)$, which serves as a candidate of a nonnegative solution (s, i) of system (1.3)–(1.4). Then, by constructing a Lyapunov functional and an invariant set, we use the Lyapunov–LaSalle Theorem to show that $(s, i)(+\infty) = (s^*, i^*)$, which confirms that (s, i) is a nonnegative solution of system (1.3)–(1.4).

The remaining parts of this paper are organized as follows. In Section 2, we first establish the non-existence of traveling waves of (1.2). Then, we construct a pair of upper and lower solutions. Finally, we derive the solution of truncated problem of system (1.3). In Section 3, we use a family of solutions of the truncated problems of system (1.3) to obtain the non-critical waves of system (1.2). Section 4 is devoted to the existence of critical waves of system (1.2). Finally, some auxiliary lemmas are given in Appendix A.

2. Preliminary

2.1. Non-existence of traveling waves

The non-existence of traveling waves of (1.2) is stated in the following lemma. Its proof can be easily obtained by following the proof of Theorem 4.3 in [16] and therefore we omit it.

Lemma 2.1. *For* $c < c^*$*, there exists no nonnegative solution of system (1.3)–(1.4).*

2.2. Upper and lower solutions for the case $c > c^*$

In this subsection, we will use an iteration process to construct a pair of upper and lower solutions of (1.3) with $c > c^*$. Therefore, throughout this section, we always assume that $c > c^*$. Specifically, we first construct the s-component of the upper solution s^+, which is immediately employed to construct the i-component of the upper solution i^+. Then i^+ in turn used to generate the s-component of the lower solution s^-. Finally, we use s^- to construct the i-component of the lower solution i^-. The idea of such a construction is motivated by [1]. To begin with, we give the definition of upper and lower solutions of (1.3).

Definition 2.1. (s^+, i^+) and (s^-, i^-) are called a pair of upper and lower solutions of (1.3) if s^+, i^+, s^-, i^- satisfy

$$\begin{aligned}
&\delta(s^+)''(z) - c(s^+)'(z) + \mu(1 - s^+(z)) - \beta s^+(z)g(i^-(z - c\tau)) \le 0,\\
&\delta(s^-)''(z) - c(s^-)'(z) + \mu(1 - s^-(z)) - \beta s^-(z)g(i^+(z - c\tau)) \ge 0,\\
&\big(i^+\big)''(z) - c\big(i^+\big)'(z) + \beta s^+(z)g(i^+(z - c\tau)) - (\gamma + \mu)i^+(z) \le 0,\\
&\big(i^-\big)''(z) - c\big(i^-\big)'(z) + \beta s^-(z)g(i^-(z - c\tau)) - (\gamma + \mu)i^-(z) \ge 0
\end{aligned}$$

except for finitely many points of z in $\mathbb{R}$.

To construct upper and lower solutions, we set $z_0 := \ln B/\lambda_1$, and select $0 < \nu < \min\{c/\delta, \lambda_1\}$ and $0 < \eta < \min\{\nu, \lambda_1, \lambda_2 - \lambda_1\}$ such that

$$c - \delta\nu > 0, \tag{2.1}$$

$\lambda_1 - \nu > 0$, $\eta - \nu < 0$, $\eta - \lambda_1 < 0$, and $P(\lambda_1 + \eta) < 0$. Then we choose

$$M := e^{-\nu z_1} > 1, \tag{2.2}$$

where z_1 is a negative number such that $z_1 < z_0$ and

$$e^{(\lambda_1-\nu)z} \le \frac{\mu}{\beta} \cdot e^{\lambda_1 c\tau}, \forall z \le z_1. \tag{2.3}$$

Finally, we pick

$$L > \max\left\{M, -\frac{\beta(M + \alpha)}{P(\lambda_1 + \eta)}\right\}, \tag{2.4}$$

and set $z_2 = -\ln L/\eta$. Note that $z_2 < z_1 < 0$ since $z_1 = -\ln M/\nu$, $L > M$, and $\eta < \nu$.

Now we define four nonnegative continuous functions s^+, s^-, i^+, and i^- as follows:

$$\begin{aligned}
s^+(z) &:= 1,\\
s^-(z) &:= \begin{cases} 1 - Me^{\nu z}, & z \le z_1,\\ 0, & z > z_1,\end{cases}\\
i^+(z) &:= \begin{cases} e^{\lambda_1 z}, & z \le z_0,\\ B, & z > z_0,\end{cases}\\
i^-(z) &:= \begin{cases} e^{\lambda_1 z} - Le^{(\lambda_1+\eta)z}, & z \le z_2,\\ 0, & z > z_2.\end{cases}
\end{aligned}$$

It is obvious that $s^+(z)$ satisfies the inequality

$$\delta(s^+)''(z) - c(s^+)'(z) + \mu(1 - s^+(z)) - \beta s^+(z)g(i^-(z - c\tau)) \le 0 \tag{2.5}$$

for all $x \in \mathbb{R}$. In the following, we will show that (s^+, i^+) and (s^-, i^-) are a pair of upper and lower solutions of (1.3).

Lemma 2.2. *The function $i^+(z)$ satisfies the equation*

$$\big(i^+\big)''(z) - c\big(i^+\big)'(z) + \beta s^+(z) g(i^+(z - c\tau)) - (\gamma + \mu) i^+(z) \le 0 \tag{2.6}$$

for all $z \ne z_0$.

Proof. For $z > z_0$, since $i^+(z) \equiv B$ in (z_0, ∞) and $i^+(z - c\tau) \le B$, the inequality (2.6) follows from the fact that $\beta g(i^+(z - c\tau)) - (\gamma + \mu) \cdot i^+(z) \le \beta g(B) - (\gamma + \mu)B = 0$. For $z < z_0$,

$$\big(i^+\big)''(z) - c\big(i^+\big)'(z) + \beta i^+(z - c\tau) - (\gamma + \mu) i^+(z) = P(\lambda_1, c) \cdot i^+(z) = 0, \tag{2.7}$$

which, together with $s^+ \equiv 1$ and $g(i^+(z - c\tau)) \le i^+(z - c\tau)$, implies (2.6). □

Lemma 2.3. *The function $s^-(z)$ satisfies the inequality*

$$\delta (s^-)''(z) - c(s^-)'(z) + \mu(1 - s^-(z)) - \beta s^-(z) g(i^+(z - c\tau)) \ge 0 \tag{2.8}$$

for all $z \ne z_1$.

Proof. For $z > z_1$, since $s^-(z) \equiv 0$ in (z_1, ∞), the inequality (2.8) follows. For $z < z_1$, $s^-(z) = 1 - Me^{\nu z}$. By (2.3), we deduce that

$$\mu e^{\nu z} \ge \beta i^+(z - c\tau), \forall z \le z_1. \tag{2.9}$$

Noting that $1 - s^-(z) = Me^{\nu z}$, $s^-(z) \le 1$, and $g(i^+(z - c\tau)) \le i^+(z - c\tau)$, we can use (2.1), (2.2), and (2.9) to deduce that

$$\begin{aligned} &\delta (s^-)''(z) - c(s^-)'(z) + \mu(1 - s^-(z)) - \beta s^-(z) g(i^+(z - c\tau)) \\ &\ge M\nu(c - \delta\nu)e^{\nu z} + \mu M e^{\nu z} - \beta i^+(z - c\tau) \\ &\ge 0. \end{aligned}$$

Hence (2.8) holds. □

Lemma 2.4. *The function $i^-(z)$ satisfies the inequality*

$$(i^-)''(z) - c(i^-)'(z) + \beta s^-(z) g(i^-(z - c\tau)) - (\gamma + \mu) i^-(z) \ge 0 \tag{2.10}$$

for all $z \ne z_2$.

Proof. For $z > z_2$, the inequality (2.10) holds immediately since $i^-(z) \equiv 0$ in (z_2, ∞). For $z < z_2$, $i^-(z) = i^+(z) - Le^{(\lambda_1 + \eta)z}$ and $s^-(z) = 1 - Me^{\nu z}$. A simple computation gives that

$$\begin{aligned} (i^-)'(z) &= (i^+)'(z) - (\lambda_1 + \eta) L e^{(\lambda_1 + \eta)z}, \\ (i^-)''(z) &= (i^+)''(z) - (\lambda_1 + \eta)^2 L e^{(\lambda_1 + \eta)z}, \end{aligned}$$

and

$$
\begin{aligned}
& s^-(z)g(i^-(z-c\tau)) \\
\geq\ & s^-(z)i^-(z-c\tau)[1-\alpha i^-(z-c\tau)] \\
=\ & s^-(z)i^-(z-c\tau) - \alpha s^-(z)i^-(z-c\tau)^2 \\
\geq\ & (1-Me^{\nu z})\left(i^+(z-c\tau) - Le^{(\lambda_1+\eta)(z-c\tau)}\right) - \alpha e^{2\lambda_1(z-c\tau)} \\
& \text{(by definition of } s^- \text{ and } i^- \text{ and the fact that } s^- \leq 1 \text{ and } i^-(z) \leq e^{\lambda_1 z}) \\
\geq\ & i^+(z-c\tau) - Me^{(\nu+\lambda_1)z-\lambda_1 c\tau} - Le^{(\lambda_1+\eta)(z-c\tau)} - \alpha e^{2\lambda_1(z-c\tau)} \\
\geq\ & i^+(z-c\tau) - Me^{(\nu+\lambda_1)z} - Le^{(\lambda_1+\eta)(z-c\tau)} - \alpha e^{2\lambda_1 z}.
\end{aligned}
$$

Together with (2.7) and definition of P, we get

$$
\begin{aligned}
& (i^-)''(z) - c(i^-)'(z) + \beta s^-(z)g(i^-(z-c\tau)) - (\gamma+\mu)i^-(z) \\
\geq\ & e^{(\lambda_1+\eta)z}[-P(\lambda_1+\eta)L - \beta Me^{(\nu-\eta)z} - \beta\alpha e^{(\lambda_1-\eta)z}] \\
\geq\ & 0 \quad (\text{by } e^{(\nu-\eta)z} \leq 1,\ e^{(\lambda_1-\eta)z} \leq 1, \text{ and } (2.4))
\end{aligned}
$$

The proof of this lemma is therefore completed. □

2.3. Truncated problem for the case $c > c^*$

In this subsection, we always assume that $c > c^*$ and we consider the following truncated problem

$$\delta s''(z) - cs'(z) + \mu(1-s(z)) - \beta s(z)g(i(z-c\tau)) = 0 \quad \text{in } I_l := (-l, l), \tag{2.11a}$$

$$i''(z) - ci'(z) + \beta s(z)g(i(z-c\tau)) - (\gamma+\mu)i(z) = 0 \quad \text{in } I_l, \tag{2.11b}$$

$$s(z) = s^-(z),\ i(z) = i^-(z) \quad \text{in } (-\infty, -l] \cup [l, \infty), \tag{2.11c}$$

where $l > \max\{-z_2, |z_0|\}$.

For convenience, we set

$$X := C(\mathbb{R}) \times C(\mathbb{R}) \text{ and } Y := C^1(I_l) \times C^1(I_l).$$

We will apply the Schauder fixed point theorem to show that there exists a pair of functions $(s, i) \in X \cap Y$ satisfying (2.11). To begin with, we introduce the set

$$E := \{(s,i) \in X |\ s^- \leq s \leq s^+ \text{ and } i^- \leq i \leq i^+ \text{ in } \mathbb{R}\},$$

which is a closed convex set in the Banach space X equipped with the norm $\|(f_1, f_2)\|_X = \|f_1\|_{C(\mathbb{R})} + \|f_2\|_{C(\mathbb{R})}$. Next, we define the mapping $\mathcal{F}E \to E$ as follows: given $(s_0, i_0) \in E$,

$$\mathcal{F}(s_0, i_0) := (s, i),$$

where (s, i) is a pair of functions $(s, i) \in X \cap Y$ satisfying

$$\delta s''(z) - cs'(z) + \mu(1-s(z)) - \beta s(z)g(i_0(z-c\tau)) = 0 \quad \text{in } I_l, \tag{2.12a}$$

$$i''(z) - ci'(z) + \beta s_0(z)g(i_0(z-c\tau)) - (\gamma+\mu)i(z) = 0 \quad \text{in } I_l. \tag{2.12b}$$

$$s(z) = s^-(z),\ i(z) = i^-(z) \quad \text{in } (-\infty, -l] \cup [l, \infty). \tag{2.12c}$$

Obviously, any fixed point of $\mathcal{F}$ is a pair of functions $(s, i) \in X \cap Y$ satisfying (2.11). In the following, we will verify that the mapping $\mathcal{F}$ satisfies the conditions of the Schauder fixed point theorem.

Lemma 2.5. *The mapping $\mathcal{F}$ is well-defined; that is, for a given $(s_0, i_0) \in E$, there exists a unique pair of functions $(s, i) \in X \cap Y$ satisfying (2.12). Moreover, $s^- \leq s \leq s^+$ and $i^- \leq i \leq i^+$ in $\mathbb{R}$.*

Proof. Since system (2.12) is not a coupled system and the equations (2.12a) and (2.12b) are inhomogeneous linear equations, the existence and uniqueness of solutions to the boundary value problem (2.12) can be easily obtained by [7, Theorem 3.1 of Chapter 12]. Moreover, since $\delta s''(z) - cs'(z) - [\mu + \beta g(i_0(z - c\tau))]s(z) = -\mu \leq 0$ on I_l and $s(\pm l) = s^-(\pm l) \geq 0$, it follows from the maximum principle that $s > 0$ over I_l. By a similarly way, we also deduce that $i > 0$ over I_l.

Now it remains to show that $s^- \leq s \leq s^+$ in I_l. Using (2.12a) and the fact that $i_0 \leq i^+$, we deduce that

$$\delta s''(z) - cs'(z) + \mu(1 - s(z)) - \beta s(z)g(i^+(z - c\tau)) \leq 0 \quad \text{in } I_l.$$

Together with (2.8), we see that the function $w_1 := s - s^-$ satisfies $\delta w_1''(z) - cw_1'(z) - (\mu + \beta g(i^+(z - c\tau)))w_1(z) \leq 0$ in $(-l, z_1)$. In addition, from (2.12c) and the fact $s(z_1) > 0$ and $s^-(z_1) = 0$, we know that $w_1(z_1) > 0$ and $w_1(l) = 0$. Hence the maximum principle asserts that $w_1 \geq 0$ in $(-l, z_1)$, which implies that $s^- \leq s$ in $(-l, z_1)$. Together with the fact that $s^- \equiv 0 \leq s$ in $[z_1, l)$, we get $s^- \leq s$ in I_l. Next, we show that $s \leq s^+$ in I_l. Since $i_0 \geq i^-$, it follows that

$$\delta s''(z) - cs'(z) + \mu(1 - s(z)) - \beta s(z)g(i^-(z - c\tau)) \geq 0 \quad \text{in } I_l.$$

Note that $s(\pm l) \leq s^+(\pm l)$, we can use (2.5) and the maximum principle to get that $s \leq s^+$ in I_l.

Finally, we claim that $i^- \leq i \leq i^+$ in I_l. Since

$$s^-(z)g(i^-(z - c\tau)) \leq s_0(z)g(i_0(z - c\tau)) \leq s^+(z)g(i^+(z - c\tau)),$$

it follows that

$$i''(z) - ci'(z) + \beta s^-(z)g(i^-(z - c\tau)) - (\gamma + \mu)i(z) \leq 0 \tag{2.13}$$

and

$$i''(z) - ci'(z) + \beta s^+(z)g(i^+(z - c\tau)) - (\gamma + \mu)i(z) \geq 0 \tag{2.14}$$

for all z in I_l. Consider the function $w_2 = i - i^-$. From (2.12c) and the fact $i(z_2) > 0$ and $i^-(z_2) = 0$, we know that $w_2(z_2) > 0$ and $w_2(-l) = 0$. In addition, (2.10) and (2.13) give that $w_2''(z) + cw_2'(z) - (\gamma + \mu)w_2(z) \leq 0$ for all $z \in (-l, z_2)$. Then it follows from the maximum principle that $w_2 \geq 0$ in $(-l, z_2)$. This implies that $i^- \leq i$ in $(-l, z_2)$. Together with the fact that $i^- \equiv 0 \leq i$ in $[z_2, l)$, we get $i^- \leq i$ in I_l. To prove $i \leq i^+$ on I_l, we note that $\bar{i}(z) := B$ satisfies

$$\bar{i}''(z) - c\bar{i}'(z) + \beta s^+(z)g(\bar{i}(z - c\tau)) - (\gamma + \mu)\bar{i} = 0 \text{ in } I_l.$$

Since $s_0(z)g(i_0(z - c\tau)) \leq s^+(z)g(\bar{i}(z - c\tau))$, it follows that

$$i''(z) - ci'(z) + \beta s^+(z)g(\bar{i}(z - c\tau)) - (\gamma + \mu)i(z) \geq 0 \text{ in } I_l.$$

Note that $i(\pm l) \leq B$. Then one can easily use the maximum principle to deduce that $i \leq B$ in I_l. Together with the fact that $i^+(z) = B$ in $[z_0, l)$, we get that $i \leq i^+$ in $[z_0, l)$. To prove $i \leq i^+$ in $(-l, z_0]$, we note that

$i(-l) \le i^+(-l)$ and $i(z_0) \le B = i^+(z_0)$. Together with (2.6), (2.14) and the maximum principle, we deduce that $i \le i^+$ in $(-l, z_0]$. Hence the proof of this lemma is completed. □

Finally, arguing as the proofs of Lemma 4.4 and Lemma 4.5 in [6] and using Lemma A.1 and Lemma A.2, one can verify that the mapping $\mathcal{F}$ is continuous and precompact. Then we can apply the Schauder fixed point theorem to conclude that $\mathcal{F}$ has a fixed point $(s_l, i_l) \in X \cap Y$, which is a pair of functions satisfying (2.11) and $s^- \le s_l \le s^+$ and $i^- \le i_l \le i^+$ on $\mathbb{R}$. From the above discussion, we have the following existence result for the truncated problem (2.11).

Lemma 2.6. *There exists a pair of functions $(s_l, i_l) \in X \cap Y$ satisfying (2.11). Moreover,*

$$0 \le s^- \le s_l \le s^+ \equiv 1 \text{ and } 0 \le i^- \le i_l \le i^+ \le B \tag{2.15}$$

on $\mathbb{R}$.

3. Existence of non-critical waves

In this section, we use the solution (s_l, i_l) of the truncated problem (2.11) and the limiting argument to obtain a solution (s, i) of system (1.3) satisfying $(s, i)(-\infty) = (1, 0)$. The condition that $(s, i)(\infty) = (s^*, i^*)$ will be verified by the Lyapunov–LaSalle Theorem. Now we state the existence result for non-critical waves.

Lemma 3.1. *For $c > c^*$, system (1.3)–(1.4) admits a nonnegative solution (s, i) with $0 < s < 1$ and $0 < i < B$ over $\mathbb{R}$. Moreover, $i(z) = \mathcal{O}(e^{\lambda_1 z})$ as $z \to -\infty$.*

Proof. Let $\{l_n\}_{n \in \mathbb{N}}$ be an increasing sequence in (z_2, ∞) such that $l_1 > \max\{-z_2, |z_0|\}$ and $l_n \to \infty$ as $n \to \infty$, and let (s_n, i_n), $n \in \mathbb{N}$, be a pair of functions in $X \times Y$ satisfying (2.11) with $l = l_n$ and (2.15) on $\mathbb{R}$. For any fixed $N \in \mathbb{N}$, since the sequences

$$\{s_n\}_{n \ge N} \text{ and } \{i_n\}_{n \ge N}$$

are uniformly bounded in $[-l_N, l_N]$, we can use Lemma A.2 to infer that the sequences

$$\{s'_n\}_{n \ge N} \text{ and } \{i'_n\}_{n \ge N}$$

are also uniformly bounded in $[-l_N, l_N]$. Using (2.11), we can express s''_n and i''_n in terms of s_n, i_n, s'_n and i'_n. Differentiating (2.11), we can use the resulting equations to express s'''_n and i'''_n in terms of s_n, i_n, s'_n, i'_n, s''_n and i''_n. Consequently, the sequences

$$\{s''_n\}_{n \ge N}, \{i''_n\}_{n \ge N}, \{s'''_n\}_{n \ge N} \text{ and } \{i'''_n\}_{n \ge N}$$

are uniformly bounded in $[-l_N, l_N]$. With the aid of Arzela–Ascoli theorem, we can use a diagonal process to get a subsequence $\{(s_{n_j}, i_{n_j})\}$ of $\{(s_n, i_n)\}$ such that

$$s_{n_j} \to s, s'_{n_j} \to s', s''_{n_j} \to s'',$$

and

$$i_{n_j} \to i, i'_{n_j} \to i', i''_{n_j} \to i'',$$

uniformly in any compact interval of $\mathbb{R}$ as $n \to \infty$, for some functions s and i in $C^2(\mathbb{R})$. Then it is easy to see that (s, i) is a nonnegative solution of system (1.3) and satisfies

$$s^- \le s \le s^+ = 1 \text{ and } i^- \le i \le i^+ \le B$$

over $\mathbb{R}$. Together with definitions of $s^\pm$ and $i^\pm$, we deduce that $(s,i)(-\infty) = (1,0)$, and $i(z) = \mathcal{O}(e^{\lambda_1 z})$ as $z \to -\infty$.

Furthermore, we claim that $0 < s < 1$ and $0 < i < B$ over $\mathbb{R}$. For contradiction, we assume that $i(\tilde{z_2}) = 0$ for some $\tilde{z_2} \in \mathbb{R}$. Then $i'(\tilde{z_2}) = 0$. Therefore the uniqueness gives that $i \equiv 0$, which contradicts the fact that $i \ge i^- > 0$ on $(-\infty, z_2)$. Hence $i > 0$ over $\mathbb{R}$. To prove $s < 1$ over $\mathbb{R}$, we also use a contradictory argument and assume that $s(\tilde{z_2}) = 1$ for some $\tilde{z_2} \in \mathbb{R}$. In this case, $s'(\tilde{z_2}) = 0$ and $s''(\tilde{z_2}) \le 0$. This contradicts (1.3a) with $z = \tilde{z_2}$. Hence $s < 1$ over $\mathbb{R}$. By a similar way, we also have $i < B$ over $\mathbb{R}$.

Now it remains to verify that $(s,i)(\infty) = (s^*, i^*)$. To this end, we rewrite (1.3) as a system of first-order ODEs:

$$\begin{aligned}
&s'(z) = w(z),\\
&\delta w'(z) = cw(z) + \mu(s(z) - 1) + \beta s(z) g(i(z - c\tau)),\\
&i'(z) = y(z),\\
&y'(z) = cy(z) - \beta s(z) g(i(z - c\tau)) + (\gamma + \mu) i(z).
\end{aligned} \tag{3.16}$$

In the remaining of the proof, we will use the Lyapunov–LaSalle Theorem to show that $(s,i)(\infty) = (s^*, i^*)$. We divide the proof into several steps.

Step 1: We construct the Lyapunov functional.

Motivated by [10], we define the Lyapunov functional $\mathcal{L}$ by

$$\mathcal{L}(s,w,i,y) = \mathcal{L}_1(s,w,i,y) + \mathcal{L}_2(s,w,i,y) + c(\mu+\gamma) i^* \mathcal{L}_3(s,w,i,y), \tag{3.17}$$

where

$$\mathcal{L}_1(s,w,i,y) := -\left(\delta w - cs - \delta s^* \frac{w}{s} + cs^* \ln \frac{s}{s^*}\right),$$

$$\mathcal{L}_2(s,w,i,y) := -\left(y - ci - g(i^*)\frac{y}{g(i)} + cg(i^*) \int_{i^*}^{i} \frac{1}{g(\sigma)} d\sigma\right),$$

and

$$\mathcal{L}_3(s,w,i,y) := \int_0^\tau \frac{g(i(z - c\theta))}{g(i^*)} - 1 - \ln \frac{g(i(z - c\theta))}{g(i^*)} d\theta.$$

Along the solution $\chi(z) := (s(z), w(z), i(z), y(z))$, where $w := s'(z)$ and $y := i'(z)$, we have that

$$\begin{aligned}
\frac{d}{dz}\mathcal{L}_1(\chi(z)) &= -\delta s^* \frac{w(z)^2}{s(z)^2} + [\mu(1 - s(z)) - \beta s(z) g(i(z - c\tau))]\left(1 - \frac{s^*}{s(z)}\right)\\
&= -\delta s^* \frac{w(z)^2}{s(z)^2} + [\mu(s^* - s(z)) + (\mu+\gamma) i^* - \beta s(z) g(i(z - c\tau))]\left(1 - \frac{s^*}{s(z)}\right)\\
&\qquad \text{(using the fact that } \mu = \mu s^* + (\mu+\gamma) i^*\text{)},\\
\frac{d}{dz}\mathcal{L}_2(\chi(z)) &= -g(i^*) g'(i(z)) \frac{y(z)^2}{g(i(z))^2} + [\beta s(z) g(i(z - c\tau)) - (\gamma+\mu) i(z)]\left(1 - \frac{g(i^*)}{g(i(z))}\right),
\end{aligned}$$

and

$$\frac{d}{dz}\mathcal{L}_3(\chi(z)) = \int_0^\tau \frac{d}{dz}\left\{\frac{g(i(z-c\theta))}{g(i^*)} - 1 - \ln\frac{g(i(z-c\theta))}{g(i^*)}\right\} d\theta$$
$$= -\frac{1}{c}\int_0^\tau \frac{d}{d\theta}\left\{\frac{g(i(z-c\theta))}{g(i^*)} - 1 - \ln\frac{g(i(z-c\theta))}{g(i^*)}\right\} d\theta$$
$$= -\frac{1}{c}\left\{\frac{g(i(z-c\tau))}{g(i^*)} - \frac{g(i(z))}{g(i^*)} - \ln\frac{g(i(z-c\tau))}{g(i(z))}\right\}$$
$$= -\frac{1}{c}\left\{\frac{g(i(z-c\tau))}{g(i^*)} - \frac{g(i(z))}{g(i^*)} - \ln\frac{s^*}{s(z)} - \ln\frac{s(z)g(i(z-c\tau))}{s^*g(i(z))}\right\}.$$

Consequently,

$$\begin{aligned}
&\frac{d}{dz}\mathcal{L}(\chi(z)) \\
&= -\delta s^*\frac{w(z)^2}{s(z)^2} - g(i^*)g'(i(z))\frac{y(z)^2}{g(i(z))^2} \\
&\quad + \mu(s^*-s(z))\left(1-\frac{s^*}{s(z)}\right) + (\gamma+\mu)i^*\left(1-\frac{s^*}{s(z)}\right) \\
&\quad + \beta s^* g(i(z-c\tau)) - \beta g(i^*)\frac{s(z)g(i(z-c\tau))}{g(i(z))} - (\gamma+\mu)i(z)\left(1-\frac{g(i^*)}{g(i(z))}\right) \\
&\quad + (\mu+\gamma)i^*\left[-\frac{g(i(z-c\tau))}{g(i^*)} + \frac{g(i(z))}{g(i^*)} + \ln\frac{s^*}{s(z)} + \ln\frac{s(z)g(i(z-c\tau))}{s^*g(i(z))}\right].
\end{aligned}$$

Together with $\beta s^* g(i^*) = (\mu+\gamma)i^*$, we deduce that

$$\begin{aligned}
&\frac{d}{dz}\mathcal{L}(\chi(z)) \\
&= -\delta s^*\frac{w(z)^2}{s(z)^2} - g(i^*)g'(i(z))\frac{y(z)^2}{g(i(z))^2} \\
&\quad + \mu(s^*-s(z))\left(1-\frac{s^*}{s(z)}\right) + (\gamma+\mu)i^*\left(1-\frac{s^*}{s(z)} + \ln\frac{s^*}{s(z)}\right) \\
&\quad - (\gamma+\mu)i(z)\left(1-\frac{g(i^*)}{g(i(z))}\right) + (\mu+\gamma)i^*\frac{g(i(z))}{g(i^*)} - (\mu+\gamma)i^* \\
&\quad + (\mu+\gamma)i^*\left[1-\frac{s(z)g(i(z-c\tau))}{s^*g(i(z))} + \ln\frac{s(z)g(i(z-c\tau))}{s^*g(i(z))}\right] \\
&= -\delta s^*\frac{w(z)^2}{s(z)^2} - g(i^*)g'(i(z))\frac{y(z)^2}{g(i(z))^2} \\
&\quad + \mu(s^*-s(z))\left(1-\frac{s^*}{s(z)}\right) + (\gamma+\mu)i^*\left(1-\frac{s^*}{s(z)} + \ln\frac{s^*}{s(z)}\right) \\
&\quad - (\gamma+\mu)i^*\left(1-\frac{g(i^*)}{g(i(z))}\right)\left(\frac{i(z)}{i^*} - \frac{g(i(z))}{g(i^*)}\right) \\
&\quad + (\mu+\gamma)i^*\left[1-\frac{s(z)g(i(z-c\tau))}{s^*g(i(z))} + \ln\frac{s(z)g(i(z-c\tau))}{s^*g(i(z))}\right].
\end{aligned}$$

Since $\Psi_1(\xi) := g(\xi)/\xi$ is decreasing and $\Psi_2(\xi) := 1 - \xi + \ln \xi$ is non-positive for $\xi > 0$, it follows that

$$\left(1 - \frac{g(i^*)}{g(i(z))}\right)\left(\frac{i(z)}{i^*} - \frac{g(i(z))}{g(i^*)}\right) \geq 0,$$
$$1 - \frac{s^*}{s(z)} + \ln \frac{s^*}{s(z)} \leq 0,$$

and

$$1 - \frac{s(z)g(i(z-c\tau))}{s^* g(i(z))} + \ln \frac{s(z)g(i(z-c\tau))}{s^* g(i(z))} \leq 0.$$

Hence we deduce that

$$\frac{d}{dz}\mathcal{L}(\chi(z)) \leq 0.$$

Step 2: We claim that there exist positive constants L_i, $i = 1, 2, 3, 4$, such that

$$-L_1 s(z) < s'(z) < L_2 s(z) \text{ and } -L_3 g(i(z)) < i'(z) < L_4 g(i(z)) \tag{3.18}$$

for all $z \geq 0$.

(1) We show that $-L_1 s(z) < s'(z)$ for all $z \geq 0$, if L_1 is a positive constant sufficiently large such that $-L_1 s(0) < s'(0)$ and $L_1 \geq 2\beta g(B)/c$.

Let

$$\Phi_1(z) := s'(z) + L_1 s(z).$$

It suffices to show that $\Phi_1(z) > 0$ for all $z \geq 0$. Note that $\Phi_1(0) > 0$. For contradiction, we assume that there exists $\hat{z_1} > 0$ such that $\Phi_1(\hat{z_1}) = 0$ and $\Phi_1'(\hat{z_1}) \leq 0$. Then there are two possibilities: either

$$\Phi_1(z) \leq 0, \forall z \geq \hat{z_1} \tag{3.19}$$

or

$$\Phi_1(\hat{z_2}) = 0 \text{ and } \Phi_1'(\hat{z_2}) \geq 0, \tag{3.20}$$

for some $\hat{z_2} \geq \hat{z_1}$. For the first case, (3.19) and the fact that $L_1 \geq 2\beta g(B)/c$ gives

$$cs'(z) \leq -2\beta g(B)s(z), \forall z \geq \hat{z_1}.$$

Together with the fact that $0 \leq i \leq B$ and $s < 1$, we deduce from (1.3a) that

$$\delta s''(z) = cs'(z) + \beta s(z)g(i(z-c\tau)) + \mu(s(z) - 1) \leq -\beta g(B)s(z) < 0, \forall z \geq \hat{z_1},$$

which implies that s' is decreasing in $[\hat{z_1}, \infty)$. Hence $s'(z) \leq s'(\hat{z_1}) \leq -L_1 s(\hat{z_1}) < 0$ for all $z \geq \hat{z_1}$, which contradicts the boundedness of s. For the second case, (3.20) yields that

$$s'(\hat{z_2}) = -L_1 s(\hat{z_2}) < 0 \tag{3.21}$$

and

$$s''(\hat{z}_2) \geq -L_1 s'(\hat{z}_2) > 0. \tag{3.22}$$

Using (1.3a), we deduce that

$$\begin{aligned} 0 &= \delta s''(\hat{z}_2) - cs'(\hat{z}_2) + \mu(1 - s(\hat{z}_2)) - \beta s(\hat{z}_2) g(i(\hat{z}_2 - c\tau)) \\ &\geq cL_1 s(\hat{z}_2) - \beta s(\hat{z}_2) g(B) \\ &\qquad \text{(by (3.21) and (3.22), and the fact that } 0 < s < 1 \text{ and } 0 < i \leq B) \\ &\geq \beta s(\hat{z}_2) g(B) \quad \text{(by definition of } L_1) \\ &> 0, \end{aligned}$$

a contradiction again.

(2) We show that $s'(z) < L_2 s(z)$ for all $z \geq 0$, if L_2 is a positive constant sufficiently large such that $s'(0) < L_2 s(0)$ and $\delta L_2^2 - cL_2 - \mu - \beta g(B) > 0$.

Let

$$\Phi_2(z) := s'(z) - L_2 s(z).$$

It suffices to show that $\Phi_2(z) < 0$ for all $z \geq 0$. Note that $\Phi_2(0) < 0$. For contradiction, we assume that there exists $\hat{z}_3 \geq 0$ such that $\Phi_2(\hat{z}_3) = 0$ and $\Phi_2'(\hat{z}_3) \geq 0$. Then

$$s'(\hat{z}_3) = L_2 s(\hat{z}_3) \text{ and } s''(\hat{z}_3) \geq L_2 s'(\hat{z}_3) = L_2^2 s(\hat{z}_3).$$

Together with the fact that $i \leq B$, we deduce from (1.3a) that

$$\begin{aligned} 0 &= \delta s''(\hat{z}_3) - cs'(\hat{z}_3) + \mu - \mu s(\hat{z}_3) - \beta s(\hat{z}_3) g(i(z_3 \hat{-} c\tau)) \\ &\geq \left(\delta L_2^2 - cL_2 - \mu - \beta g(B)\right) s(\hat{z}_3) > 0, \end{aligned}$$

a contradiction.

(3) We show that $-L_3 g(i(z)) < i'(z)$ for all $z \geq 0$, where L_3 is a positive number sufficiently large such that $-L_3 g(i(0)) < i'(0)$ and $L_3 \geq (1 + \alpha B)(\mu + \gamma)/c$.

Let

$$\Phi_3(z) := i'(z) + L_3 g(i(z)).$$

It suffices to show that $\Phi_3(z) > 0$ for all $z \geq 0$. Note that $\Phi_3(0) > 0$. For contradiction, we assume that there exists $\hat{z}_4 > 0$ such that $\Phi_3(\hat{z}_4) = 0$ and $\Phi_3'(\hat{z}_4) \leq 0$. Then there are two possibilities: either

$$\Phi_3(z) \leq 0, \forall z \geq \hat{z}_4 \tag{3.23}$$

or

$$\Phi_3(\hat{z}_5) = 0 \text{ and } \Phi_3'(\hat{z}_5) \geq 0 \tag{3.24}$$

for some $\hat{z}_5 \geq \hat{z}_4$. For the first case, (3.23) yields

$$i'(z) \leq -L_3 g(i(z)), \forall z \geq \hat{z}_4.$$

Together with (1.3b) the fact that

$$g(i(z)) \geq i(z)/(1+\alpha B), \tag{3.25}$$

we deduce that

$$i''(z) = ci'(z) - \beta s(z)g(i(z-c\tau)) + (\mu+\gamma)i(z) \leq 0, \forall z \geq \hat{z}_4,$$

which implies that $i'(z)$ is decreasing in $[\hat{z}_4, \infty)$. Hence $i'(z) \leq i'(\hat{z}_4) = -L_3 i(\hat{z}_4) < 0$ for all $z \geq \hat{z}_4$, which contradicts the boundedness of i. For the second case, (3.24) and the fact that $g'(\xi) > 0$ for $\xi > 0$ yields that

$$i'(\hat{z}_5) = -L_3 g(i(\hat{z}_5)) < 0 \tag{3.26}$$

and

$$i''(\hat{z}_5) \geq -L_3 g'(i(\hat{z}_5))i'(\hat{z}_5) > 0. \tag{3.27}$$

Then we deduce from (1.3b) that

$$\begin{aligned} 0 &= i''(\hat{z}_5) - ci'(\hat{z}_5) + \beta s(\hat{z}_5)g(i(\hat{z}_5 - c\tau)) - (\mu+\gamma)i(\hat{z}_5) \\ &\geq cL_3 g(i(\hat{z}_5)) - (\mu+\gamma)i(\hat{z}_5) \\ &\qquad \text{(by (3.26) and (3.27))} \\ &> 0 \quad \text{(by (3.25) and definition of } L_3), \end{aligned}$$

a contradiction again.

(4) We show that $i'(z) < L_4 g(i(z))$ for all $z \geq 0$, if L_4 is a positive constant sufficiently large such that $i'(0) < L_4 g(i(0))$ and $(1+\alpha B)^{-2}L_4^2 - cL_4 - (\mu+\gamma)(1+\alpha B) > 0$.

Let

$$\Phi_4(z) := i'(z) - L_4 g(i(z)).$$

It suffices to show that $\Phi_4(z) < 0$ for all $z \geq 0$. Note that $\Phi_4(0) < 0$. For contradiction, we assume that there exists $\hat{z}_6 \geq 0$ such that $\Phi_4(\hat{z}_6) = 0$ and $\Phi_4'(\hat{z}_6) \geq 0$. Then

$$i'(\hat{z}_6) = L_4 g(i(\hat{z}_6)) \text{ and } i''(\hat{z}_6) \geq L_4 g'(i(\hat{z}_6))i'(\hat{z}_6) = L_4^2 g'(i(\hat{z}_6))g(i(\hat{z}_6)).$$

Together with the fact that $g'(i) \geq (1+\alpha B)^{-2}$, $i \leq (1+\alpha B)g(i)$, and $(1+\alpha B)^{-2}L_4^2 - cL_4 - (\mu+\gamma) > 0$, we deduce from (1.3b) that

$$\begin{aligned} 0 &= i''(\hat{z}_6) - ci'(\hat{z}_6) + \beta s(\hat{z}_6)g(i(\hat{z}_6 - c\tau)) - (\mu+\gamma)i(\hat{z}_6) \\ &\geq L_4^2 g'(i(\hat{z}_6))g(i(\hat{z}_6)) - cL_4 g(i(\hat{z}_6)) - (\mu+\gamma)(1+\alpha B)g(i(\hat{z}_6)) > 0, \end{aligned}$$

a contradiction.

Step 3: We show that $(s,i)(\infty) = (s^*, i^*)$.

Now we construct the open bounded set $\mathcal{D}$ of system (3.16) as follows:

$$\mathcal{D} := \{(s,w,i,y) | 0 < s < 1, 0 < i < B, -L_1 s < w < L_2 s, -L_3 g(i) < y < L_4 g(i)\}.$$

Then (3.18) assert that the solution $\chi(z) = (s(z), w(z), i(z), y(z))$ with $(w, y) = (s', i')$ of system (3.16) is positively invariant in $\mathcal{D}$ for all $z \geq 0$. Recall the orbital derivative of $\mathcal{L}$ along $\chi(z)$ is non-positive. Besides, one can easily see that $\mathcal{L}$ is continuous and bounded below on $\mathcal{D}$. Taken together, it follows from the Lyapunov–LaSalle Theorem that $\chi(z) \to (s^*, 0, i^*, 0)$ as $z \to \infty$, and so $(s, i)(\infty) = (s^*, i^*)$. This completes the proof of Lemma 3.1. $\square$

4. Existence of critical waves

In this section, we establish the existence of traveling waves of system (1.2) with critical speed $c = c^*$. To this end, we need to construct a pair of upper and lower solutions of (1.3) with $c = c^*$. For this, we set $z_0 := -1/\lambda^*$ and $\rho := eB\lambda^*$ and select $0 < \nu < c^*/\delta$ such that

$$c^* - \delta\nu > 0. \tag{4.28}$$

Then we set

$$M := e^{-\nu z_1} > 1, \tag{4.29}$$

where z_1 is a negative number such that $z_1 < z_0$ and

$$e^{(\lambda^*-\nu)z} \leq \frac{\mu}{\beta} \cdot e^{\lambda^* c^* \tau}, \forall z \leq z_1. \tag{4.30}$$

Noting that $\lim_{z\to-\infty} \left[M\rho(-z)^{5/}e^{\nu z} + \alpha\rho^2(-z)^{7/2}e^{\lambda^*(z-c^*\tau)}\right] = 0$, there exists a number $z_2 < \min\{z_1, -c^*\tau, -1/\rho^2\}$ such that

$$M\rho(-z)^{5/2}e^{\nu z} + \alpha\rho^2(-z)^{7/2}e^{\lambda^*(z-c^*\tau)} < \frac{1}{16}(c^*)^2\tau^2, \forall z \leq z_1. \tag{4.31}$$

Set $L := \rho\sqrt{-z_2} > 1$.

Motivated by [4], we define four nonnegative continuous functions s^+, s^-, i^+, and i^- as follows:

$$s^+(z) := 1,$$
$$s^-(z) := \begin{cases} 1 - Me^{\nu z}, & z \leq z_1, \\ 0, & z > z_1, \end{cases}$$
$$i^+(z) := \begin{cases} -\rho z e^{\lambda^* z}, & z \leq z_0, \\ B, & z > z_0, \end{cases}$$
$$i^-(z) := \begin{cases} \left[-\rho z - L(-z)^{1/2}\right] e^{\lambda^* z}, & z \leq z_2, \\ 0, & z > z_2. \end{cases}$$

As before, $s^+(z)$ satisfies the inequality

$$\delta(s^+(z))'' - c^*(s^+(z))' + \mu(1 - s^+(z)) - \beta s^+(z) g(i^-(z - c^*\tau)) \leq 0$$

for all $x \in \mathbb{R}$. In the following, we will show that (s^+, i^+) and (s^-, i^-) are a pair of upper and lower solutions of (1.3) with $c = c^*$.

Lemma 4.1. *The function $i^+(z)$ satisfies the equation*

$$\left(i^+\right)''(z) - c^*\left(i^+\right)'(z) + \beta s^+(z)g(i^+(z - c^*\tau)) - (\gamma + \mu)i^+(z) \le 0 \tag{4.32}$$

for all $z \ne z_0$.

Proof. For $z > z_0$, it is obvious that (4.32) holds. For $z < z_0$, $i^+(z) = -\rho z e^{\lambda^* z}$. Then we have that

$$\begin{aligned}&\left(i^+\right)''(z) - c^*\left(i^+\right)'(z) + \beta i^+(z - c^*\tau) - (\gamma + \mu)i^+(z)\\ &= -\rho e^{\lambda^* z}[P(\lambda^*, c^*)z + P_\lambda(\lambda^*, c^*)] = 0.\end{aligned} \tag{4.33}$$

Together with $s^+ \equiv 1$ and $g(i^+(z - c^*\tau)) \le i^+(z - c^*\tau)$, we get (4.32). □

Lemma 4.2. *The function $s^-(z)$ satisfies the inequality*

$$\delta(s^-)''(z) - c^*(s^-)'(z) + \mu(1 - s^-(z)) - \beta s^-(z)g(i^+(z - c^*\tau)) \ge 0 \tag{4.34}$$

for all $z \ne z_1$.

Proof. For $z > z_1$, since $s^-(z) \equiv 0$ in (z_1, ∞), the inequality (4.34) follows. For $z < z_1$, $s^-(z) = 1 - Me^{\nu z}$. By (4.30), we deduce that

$$\mu e^{\nu z} \ge \beta i^+(z - c^*\tau), \forall z \le z_1. \tag{4.35}$$

Noting that $1 - s^-(z) = Me^{\nu z}$, $s^-(z) \le 1$, and $g(i^+(z - c^*\tau)) \le i^+(z - c^*\tau)$, we can use (4.28), (4.35), and (4.29) to deduce that

$$\begin{aligned}&\delta(s^-(z))'' - c^*(s^-(z))' + \mu(1 - s^-(z)) - \beta s^-(z)g(i^+(z - c^*\tau))\\ &\ge M\nu(c^* - \delta\nu)e^{\nu z} + \mu Me^{\nu z} - \beta i^+(z - c^*\tau)\\ &\ge 0.\end{aligned}$$

Hence (4.34) holds. □

Lemma 4.3. *The function $i^-(z)$ satisfies the inequality*

$$(i^-)''(z) - c^*(i^-)'(z) + \beta s^-(z)g(i^-(z - c^*\tau)) - (\gamma + \mu)i^-(z) \ge 0 \tag{4.36}$$

for all $z \ne z_2$.

Proof. For $z > z_2$, the inequality (4.36) holds immediately since $i^-(z) \equiv 0$ in (z_2, ∞). For $z < z_2$, $i^-(z) = i^+(z) - L(-z)^{1/2}e^{\lambda^* z}$ and $s^-(z) = 1 - Me^{\nu z}$. A simple computation gives that

$$(i^-(z))' = (i^+(z))' + Le^{\lambda^* z}\left[\frac{1}{2}(-z)^{-1/2} - \lambda^*(-z)^{1/2}\right], \tag{4.37}$$

$$\begin{aligned}(i^-(z))'' &= (i^+(z))'' + Le^{\lambda^* z}\left[\frac{1}{4}(-z)^{-3/2} + \lambda^*(-z)^{-1/2} - (\lambda^*)^2(-z)^{1/2}\right]\\ &\ge (i^+(z))'' + Le^{\lambda^* z}\left[\lambda^*(-z)^{-1/2} - (\lambda^*)^2(-z)^{1/2}\right],\end{aligned} \tag{4.38}$$

and

$$\begin{aligned}
&s^-(z)g(i^-(z-c^*\tau)) \\
&\geq s^-(z)i^-(z-c^*\tau)[1-\alpha i^-(z-c^*\tau)] \\
&= s^-(z)i^-(z-c^*\tau) - \alpha s^-(z)i^-(z-c^*\tau)^2 \\
&\geq (1-Me^{\nu z})\left[i^+(z-c^*\tau) - L(-z+c^*\tau)^{1/2}e^{\lambda^*(z-c^*\tau)}\right] - \alpha\rho^2 z^2 e^{2\lambda^*(z-c^*\tau)} \\
&\quad \text{(by definition of } s^- \text{ and } i^-\text{, and the fact } s^- \leq 1 \text{ and } i^-(z) \leq -\rho z e^{\lambda^* z}) \\
&\geq i^+(z-c^*\tau) + M\rho z e^{\nu z + \lambda^*(z-c^*\tau)} - L(-z+c^*\tau)^{1/2}e^{\lambda^*(z-c^*\tau)} - \alpha\rho^2 z^2 e^{2\lambda^*(z-c^*\tau)}.
\end{aligned} \tag{4.39}$$

Applying the Taylor's Theorem, we have that

$$(-z+c^*\tau)^{1/2} \leq (-z)^{1/2} + \frac{1}{2}c^*\tau(-z)^{-1/2} - \frac{1}{8}(c^*)^2\tau^2(-z)^{-3/2} + \frac{1}{16}(c^*)^3\tau^3(-z)^{-5/2}. \tag{4.40}$$

Combining (4.39) and (4.40), we deduce that

$$\begin{aligned}
&s^-(z)g(i^-(z-c^*\tau)) \\
&\geq i^+(z-c^*\tau) + M\rho z e^{\nu z+\lambda^*(z-c^*\tau)} - \alpha\rho^2 z^2 e^{2\lambda^*(z-c^*\tau)} \\
&\quad - Le^{\lambda^*(z-c^*\tau)}\left[(-z)^{1/2} + \frac{1}{2}c^*\tau(-z)^{-1/2} - \frac{1}{8}(c^*)^2\tau^2(-z)^{-3/2} + \frac{1}{16}(c^*)^3\tau^3(-z)^{-5/2}\right].
\end{aligned} \tag{4.41}$$

Now, using (4.37), (4.38), (4.41) and (1.6), we get

$$\begin{aligned}
&(i^-(z))'' - c^*(i^-(z))' + \beta s^-(z)g(i^-(z-c^*\tau)) - (\gamma+\mu)i^-(z) \\
&\geq \beta M\rho z e^{\nu z+\lambda^*(z-c^*\tau)} - \beta\alpha\rho^2 z^2 e^{2\lambda^*(z-c^*\tau)} \\
&\quad + \beta L e^{\lambda^*(z-c^*\tau)}\left[\frac{1}{8}(c^*)^2\tau^2(-z)^{-3/2} - \frac{1}{16}(c^*)^3\tau^3(-z)^{-5/2}\right] \\
&\geq \beta(-z)^{-3/2}e^{\lambda^*(z-c^*\tau)}\left[\frac{1}{16}L(c^*)^2\tau^2 - M\rho(-z)^{5/2}e^{\nu z} - \alpha\rho^2(-z)^{7/2}e^{\lambda^*(z-c^*\tau)}\right] \\
&\quad + \frac{1}{16}\beta L(c^*)^2\tau^2(-z)^{-3/2}e^{\lambda^*(z-c^*\tau)}\left(1+\frac{c^*\tau}{z}\right) \\
&\geq 0 \quad \text{(using the fact } L > 1 \text{ and } z < z_2 < -c^*\tau \text{ and by (4.31))}
\end{aligned}$$

The proof of this lemma is therefore completed. □

Finally, following the proofs of Lemma 2.5, Lemma 2.6, and Lemma 3.1 by replacing c and λ_1 by c^* and λ^* respectively and using the upper and lower solutions established in this section, we establish the existence of traveling waves of system (1.2) with critical speed $c = c^*$.

Lemma 4.4. *For $c = c^*$, system (1.3)–(1.4) admits a nonnegative solution (s, i) with the $0 < s < 1$ and $0 < i < B$ over $\mathbb{R}$. Moreover, $i(z) = \mathcal{O}(-ze^{\lambda^* z})$ as $z \to -\infty$.*

Appendix A

In this appendix, we provide some *a priori* estimates for solutions of the inhomogeneous linear equation

$$w''(z) - Aw'(z) + f(z)w(z) = h(z). \tag{A.1}$$

Lemma A.1. *Let A be a positive constant and let f and h be continuous functions on $[a, b]$. Suppose that $w \in C([a, b]) \cap C^2((a, b))$ satisfies the differential equation (A.1) in (a, b) and $w(a) = w(b) = 0$. If*

$$-C_1 \leq f \leq 0 \text{ and } |h| \leq C_2 \text{ on } [a, b],$$

for some constants C_1, C_2, then there exists a positive constant C_3, depending only on A, C_1, and the length of the interval $[a, b]$, such that

$$\|w\|_{C([a,b])} \leq C_2 C_3.$$

Proof. Note that the function $\tilde{w}(z) := w(-z)$ satisfies

$$\tilde{w}''(z) + A\tilde{w}'(z) + \tilde{f}(z)\tilde{w}(z) = \tilde{h}(z)$$

on $(-b, -a)$, where $\tilde{f}(z) := f(-z)$ and $\tilde{h}(z) := h(-z)$. Then it follows from Lemma 3.2 of [6] that there exists a positive constant C_3, depending only on A, C_1, and the length of the interval $[-b, -a]$, such that

$$\|\tilde{w}\|_{C([-b,-a])} \leq C_2 C_3.$$

This implies that

$$\|w\|_{C([a,b])} \leq C_2 C_3. \qquad \square$$

Lemma A.2. *Let A, f, and h be as in Lemma A.1. Suppose that $w \in C([a, b]) \cap C^2((a, b))$ satisfies (A.1) in (a, b). If $\|w\|_{C([a,b])} \leq C_0$ for some constant C_0, then there exists a positive constant C_4, depending only on A, C_0, C_1, C_2, and the length of the interval $[a, b]$, such that*

$$\|w'\|_{C([a,b])} \leq C_4.$$

Proof. Arguing as the proof of Lemma A.1 and using Lemma 3.3 of [6], one can easily get the proof. $\square$

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