行政院國家科學委員會專題研究計畫 期末報告

一維 Emden-Fowler 型半線性波方程式

計 畫 類 別 ： 個別型
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執 行 期 間 ： 101 年 08 月 01 日至 102 年 07 月 31 日
執 行 單 位 ： 國立政治大學應用數學學系

計畫主持人： 李明融
共同主持人： 謝宗翰
計畫參與人員： 碩士班研究生-兼任助理人員：鄭富元

報 告 附 件 ： 移地研究心得報告

公 開 資 訊 ： 本計畫涉及專利或其他智慧財產權，2 年後可公開查詢

中 華 民 國 102 年 07 月 13 日

中 文 摘 要 ： 於此文中吾人以自有方法證明一維有界域上 Emden-Fowler 型態之半線性波方程式解之局部存在性

中文關鍵詞： Emden-Fowler 型態, 半線性波方程式

英 文 摘 要 ： In this study we use our methode to prove the local existence of solution to Emden-Fowler type semilinear wave equation in bounded domain in 1-space dimension.

英文關鍵詞： Emden-Fowler type, semilinear wave equation.

On the existence of solution to Emden-Fowler type semilinear wave equation in bounded domain in 1-space dimension

$$t^2 u_{tt} - u_{xx} = u^p$$

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1 Introduction

We will consider the existence of solutions for the initial value problem for the Emden-Fowler type semilinear wave equation of the form

$$t^2 u_{tt} - u_{xx} = f(u) \quad in \quad [t_0, T) \times [R_1, R_2], \tag{1.1}$$

where $t_0 > 0$, $f : R \to R$ is a real valued function. The case $f(u) = u^p$, $p \geq 1$ corresponds to the classical Emden-Fowler equation $t^2 u'' = u^p$.

The existence result to the relative equation $u_{tt} - u_{xx} = f(u)$ is proved $[Li3]$ and the positive solution blows-up in finite time under some conditions $[Li2]$, we want to use another method to estimate the life-span of the solution to Emden-Fowler type semilinear wave equation (1.2) and later to study the Emden-Fowler type semilinear wave equation (1.3)

$$t^2 u_{tt} - u_{xx} = u^p \quad in \quad [t_0, t_0 + T) \times [R_1, R_2], \tag{1.2}$$

$$t^2 u_{tt} - u_{xx} = \varepsilon u^p \quad in \quad [t_0, t_0 + T) \times [R_1, R_2], \tag{1.3}$$

$$u(t_0, \cdot) = u_0 \in H^2(R_1, R_2) \cap H_0^1(R_1, R_2),$$

$$u_t(t_0, \cdot) = u_1 \in H_0^1(R_1, R_2),$$

but it is not clearly whether it is true for any $p > 1$? If so, we would want to estimate the blow-up time and the blow-up rate under such a situation in the future.

It is also important to study the asymptotic behavior of the solution u, u_t, the velocity and the rate of the approximation for ε approaches to zero. Such questions are also not easy to answer and the case for the ordinary differential equation

$$t^2 u'' = u^p, \quad u(t_0) = u_0, \quad u'(t_0) = u_1,$$

should be studied. We have studied the blow-up behavior of the solution for semilinear wave equation and got some estimates on blow-up time and blow-up rate $[Li4]$ but it is difficult to find the real blow-up time (life-span). Further literature could be fund in $[S], [R], [W1]$ and $[W2]$.

In this study we hope that our ideals used in $[Li2]$, $[Li4]$, $[Li5]$, $[Li7]$, $[Li8]$, $[LiLinShieh]$, $[ShiehLi]$ and $[SLLLW]$ can do help us dealing such problem (1.1) on our topics.

2 Fundamental Lemma

From the local Lipschitz functions, $p > 1$ the initial-boundary value semilinear wave equation

$$u_{tt} - u_{xx} = u^p \tag{2.1}$$

possesses a unique solution in $H1 := C^1\left(0, T, H_0^1(R_1, R_2)\right) \cap C^2\left(0, T, L^2(R_1, R_2)\right)$ $[Li1]$, but for the equation (1.2) of Emden-Fowler type has to be overcome.

Lemma 2.3: *Suppose that $u \in H1$ is a weakly positive solution of* (2.1) *with $E(0) = 0$ for $p > 1$, then for $a(0) > 0$ we have:*

$E(0) = 0$ *for* $p > 1$, *then for* $a(0) > 0$ *we have*:

(i) $a \in C2(\mathbb{R}^+)$ *and* $E(t) = E(0) \quad \forall t \in [0, T)$.

(ii) $a\prime(t) > 0 \quad \forall t \in [0, T)$, *provided* $a\prime(0) > 0$.

(iii) $a\prime(t) > 0 \quad \forall t \in (0, T)$, *if* $a\prime(0) = 0$.

(iv) For $a'(0) < 0$, *there exists a constant* $t_0 > 0$ *with* $a'(t) > 0 \quad \forall\, t > t_0$ *and* $a\prime(t_0) = 0$.

Note that the conclusions are not always valid for equation (1.2), which assertions remain true? It should be proven in another method.

Lemma 2.4: *Suppose that u is a positive weakly solution in $H1$ of equation* (2.1) *with* $u(0, \cdot) = 0 = u(0, \cdot)$ in $L^2(R_1, R_2)$. *For* $p > 1$, we have $u \equiv 0$ in $H1$.

Note that the result in lemma 2.4 could perhaps not be applied to the case of equation (1.2).

3 Estimates for the Life-Span

Estimates for the Life-Span of the Solutions of (2.1) under Null-Energy

We hav studied the case that $E(0) = 0, \ \ p > 1$ and divide it into two parts

(i) $a(0) > 0, \ a\prime(0) \geq 0$ and (ii) $a(0) > 0, \ a\prime(0) < 0$.

Remark 3.

1) The local existence and uniqueness of solutions of equation (2.1) in $H1$ are known [$Li2$].

2) For $p > 1$ and $E(0) = 0$, the life-span of the positive solution $u \in H1$ of equation (2.1) is bounded.

Estimates for the Life-Span of the Solutions of equation (2.1) under Negativ-Energy

We use the following result and those argumentations of proof are not true for positive energy, so under positive energy we need another method to show the similar results.

Lemma 3: *Suppose that $u \in H1$ is a positive weakly solution of equation (2.1) with $a(0) > 0$ and $E(0) < 0$. Then (i) for $a\prime(0) \geq 0$, we have $a\prime(t) > 0 \forall t > 0$;*(ii) *for $a\prime(0) < 0$, there exists a constant $t_1 > 0$ with $a'(t) > 0$*

$\forall t > t_1, \ a\prime(t_1) = 0$ and

$$t \leq t^* = -\frac{a'(0)}{(p-1)(\delta^2 - E(0))},$$

where δ is the positive root of the equation $2\lambda_{p+1}^{p+1} r^{p+1} - (p+1) r^2 + (p+1) E(0) = 0$.

4 Positive solutions of equation (2.1) near blow-up solutions

In the furture we want to utilize our ideals used in [$Li2$], [$Li4$], [$Li5$], [$Li7$], [$Li8$], [$LiLinShieh$], [$ShiehLi$] and [$SLLLW$] to deal problem (1.2) on our topics under some conditions and want to obtain similar conclusions resulted from (2.1) later:

Theorem 4.1: *Suppose that $u \in H1$ is a weakly positive solution of* (2.1) *with $E(0) \leq 0$ for $p > 1$, then for $a(0) > 0$ we have: the weakly positive solution u of* (2.1) *blows up in finite time.*

Theorem 4.2: *Suppose that* $u \in H1$ *is a weakly positive solution of* (2.1) *with* $E(0) = 0$ *for* $p > 1$ *we have:*

the weakly positive solution u *of* (2.1) *blows up in finite time.*

Theorem 4.3: *Suppose that* $u \in H1$ *is a weakly positive solution of* (2.1)*with* $E(0) > 0$ *for* $p > 1$*, then for* $a(0) > 0$ *we have: the weakly positive solution* u *of* (2.1) *blows up in finite time.*

5 Existence of solutions to the equation (1.2) in $H2$

After some long redundant argumentation using Banch fixed point theory we can obtain the existence of solutions for equation (1.2).

Theorem 5.1 *There exist positive* $T > 0$ *and* u *in* $H2$ *satisfing equation* (1.2)*, where* $H2$ *is the space*

$$H2 := C^1\left(t_0, t_0 + T, H_0^1\left(R_1, R_2\right)\right) \cap C^2\left(t_0, t_0 + T, L^2\left(R_1, R_2\right)\right).$$

Lemma 5.2 [LM, p.95] For $u_0 \in H_0^1\left(R_1, R_2\right), u_1 \in L^2\left(R_1, R_2\right)$ and $h \in W^{1,1}\left(t_0, t_0 + T, L^2\left(R_1, R_2\right)\right)$, the initial-boundary value problem for the linear wave equation of the form

$$u_{tt} - u_{xx} + h\left(t, x\right) = 0, \tag{5.1}$$

$$u\left(t_0, \cdot\right) = u_0,$$

$$u_t\left(t_0, \cdot\right) = u_1,$$

posseses exactly one solution u in $H2$.

Lemma 5.3 Suppose that u is the solution of the linear wave equation (5.1) and $u_0 \in H_0^1\left(R_1, R_2\right), u_1 \in L^2\left(R_1, R_2\right)$ and $h \in W^{1,1}\left(t_0, t_0 + T, L^2\left(R_1, R_2\right)\right)$, then we can have the estimate

$$\|Du\|_2\left(t\right) := \sqrt{\int_{R_1}^{R_2}\left(u_t^2 + u_x^2\right)\left(t, x\right)dx}$$

$$\leq \sqrt{\int_{R_1}^{R_2}\left(u_1\left(x\right)^2 + u_0'\left(x\right)^2\right)dx} + \int_{t_0}^{t}\sqrt{\int_{R_1}^{R_2} h\left(r, x\right)^2 dx}dr.$$

Sketch proof for the Theorem:

Step 1: Take a transformation $t = e^s$, $u(t,x) = v(s,x)$ then $u_t = v_s \cdot t^{-1}$, $t^2 u_{tt} = v_{ss} - v_s$ and the equation (1.2) can be rewritten into the form

$$v_{ss} - v_{xx} = v_s + v^p,$$

$$v(s_0, x) = u_0(t_0, x) = v(\ln t_0, x) := v_0(x),$$

$$v_s(s_0, x) = s_0 t_0 u_1(x),$$

where $s_0 = \ln t_0$.

Step 2: Set $v_0(s,x) = u_0(t_0,x) = v(\ln t_0, x) := v_0(x) \in H_0^1(R_1, R_2)$, $v_1(s,x) := st_0 u_1(x)$. Suppose that $v_2(s,x)$ be the solution for the linear wave equation

$$v_{ss} - v_{xx} = v_{1s} + v_0^p = t_0 u_1 + v_0^p, \tag{5.2}$$

$$v(s_0, x) = v_0(x), v_s(s_0, x) = s_0 t_0 u_1(x).$$

Step 3: Checking $v_{1s} + v_0^p \in W^{1,1}\left(0, T, L^2(R_1, R_2)\right)$, by Lemma 5.2, Lemma 5.3, $v_2 \in W^{1,1}\left(0, T, L^2(R_1, R_2)\right)$ and

$$\|Dv_2\|_2(s)$$

$$\leq \sqrt{\int_{R_1}^{R_2} \left(s_0^2 t_0^2 u_1(x)^2 + u_0'(x)^2\right) dx} + (s - s_0)\sqrt{\int_{R_1}^{R_2} \left(t_0 u_1 + v_0^p\right)(x)^2 dx}.$$

setting v_{n+1} the solution of the linear wave equation

$$v_{ss} - v_{xx} = v_{ns} + v_n^p, \tag{5.3}$$

$$v(s_0, x) = v_0(x), v_s(s_0, x) = s_0 t_0 u_1(x).$$

Step 4. To check $v_{ns} + v_n^p \in W^{1,1}\left(0, T, L^2(R_1, R_2)\right)$, by Lemma 5.2 and Lemma5.3, $v_{n+1} \in W^{1,1}\left(0, T, L^2(R_1, R_2)\right)$ and

$$\|Dv_{n+1}\|_2(s)$$

$$\leq \sqrt{\int_{R_1}^{R_2} \left(s_0^2 t_0^2 u_1(x)^2 + u_0'(x)^2\right) dx} + \int_{s_0}^{s} \sqrt{\int_{R_1}^{R_2} \left(v_{ns} + v_n^p\right)(r,x)^2 dx dr}.$$

Step 5. Under some constructions, prove that v_n is a Chauchy sequence in $H2$. Then the local existence result for the equation (5.1) in $H2$ can be got, and than local existence result for equation (1.2) also can be abtained.

Remark 1: The nonexistence of global solution in time for the equation (1.2) is an interesting problem, in some days later we will study it and estimate life-spans of T_ε of u_ε of (1.3) and T of u of (1.2).

Remark 2: The decade rate of the difference of life-spans of T_ε of u_ε of (1.3) and T of u of (1.2), can not be estimated very well for $\varepsilon \to 0$; thus it will be a good topic on asymptotic behavior near the blow-up solutions.

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二零一三年訪問香港城市大學數學系系主任/講座教授楊彤心得報告

一月二十日(日)　　　　下午七點半抵達城市大學

一月二十一日(一)　　　早上十點 ~下午十四點 楊彤教授有事開會 無暇討論
下午十五點 ~ 十七點 與楊彤教授討論
一維有界域上 Emden-Fowler 型半線波方程
(Emden-Fowler Type Semi-linear Wave Equations)解之存在性問題

一月二十二/三日(二/三) 十點 ~ 十六點 楊彤教授有事開會　無法討論

一月二十四日(四)　　　早上十點 ~十二點　與楊彤教授討論
一維有界域上 Emden-Fowler 型半線波方程
(Emden-Fowler Type Semi-linear Wave Equations)解之存在性問題之細節
下午十四點 ~ 十七點楊彤教授有事開會　無法討論

一月二十五日(五)　　　早上九點 ~十四點　完整繕寫一維有界域上 Emden-Fowler 型半線波方程(Emden-Fowler Type Semi-linear Wave Equations) 解之存在性
下午十四點 ~ 十七點與楊彤教授討論一維有界域上 Emden-Fowler 型半線波方程(Emden-Fowler Type Semi-linear Wave Equations) 解之可能性質

一月二十六日(六)　　　早上九點 ~下午十七點　思考一維有界域上 Emden-Fowler 型半線波方程(Emden-Fowler Type Semi-linear Wave Equations) 解之可能性質

一月二十七日(日)　　　早上九點 ~下午十五點訪問香港中文大學數學研究所副所長辛周平教授

一月二十八日(一)　　　早上九點 ~下午十五點楊彤教授有事開會/指導學生　無暇討論　下午十五點~十七點與楊彤教授討論一維有界域上 Emden-Fowler 型半線波方程(Emden-Fowler Type Semi-linear Wave Equations) 解之可能性質

一月二十九日(二)　　　早上九點 ~下午十四點楊彤教授有事開會/指導學生　無法討論　下午十四點~十七點與楊彤教授繼續討論一維有界域上 Emden-Fowler 型半線波方程(Emden-Fowler Type Semi-linear Wave Equations) 解之可能性質並獲得一些正面結果　與楊彤教授討論訪問政大應數系問題

一月三十日(三)　　　楊彤教授有事到日本　無法討論

一月三十一日(四)　　　準備返台

國科會補助計畫衍生研發成果推廣資料表

日期:2013/07/13

<table>
<tr><td rowspan="3">國科會補助計畫</td><td colspan="2">計畫名稱: 一維 Emden-Fowler 型半線性波方程式</td></tr>
<tr><td colspan="2">計畫主持人: 李明融</td></tr>
<tr><td>計畫編號: 101-2115-M-004-001-</td><td>學門領域: 隨機分析與統計物理</td></tr>
<tr><td colspan="3">無研發成果推廣資料</td></tr>
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101 年度專題研究計畫研究成果彙整表

計畫主持人：李明融　　**計畫編號：**101-2115-M-004-001-

計畫名稱：一維 Emden-Fowler 型半線性波方程式

成果項目			量化			單位	備註（質化說明：如數個計畫共同成果、成果列為該期刊之封面故事...等）
			實際已達成數（被接受或已發表）	預期總達成數(含實際已達成數)	本計畫實際貢獻百分比		
國內	論文著作	期刊論文	0	0	100%	篇	
		研究報告/技術報告	1	0	100%		
		研討會論文	2	0	100%		
		專書	0	0	100%		
	專利	申請中件數	0	0	100%	件	
		已獲得件數	0	0	100%		
	技術移轉	件數	0	0	100%	件	
		權利金	0	0	100%	千元	
	參與計畫人力（本國籍）	碩士生	0	0	100%	人次	
		博士生	0	0	100%		
		博士後研究員	0	0	100%		
		專任助理	0	0	100%		
國外	論文著作	期刊論文	1	0	100%	篇	
		研究報告/技術報告	0	0	100%		
		研討會論文	0	0	100%		
		專書	0	0	100%	章/本	
	專利	申請中件數	0	0	100%	件	
		已獲得件數	0	0	100%		
	技術移轉	件數	0	0	100%	件	
		權利金	0	0	100%	千元	
	參與計畫人力（外國籍）	碩士生	0	0	100%	人次	
		博士生	0	0	100%		
		博士後研究員	0	0	100%		
		專任助理	0	0	100%		

其他成果 (無法以量化表達之成果如辦理學術活動、獲得獎項、重要國際合作、研究成果國際影響力及其他協助產業技術發展之具體效益事項等，請以文字敘述填列。)	無

	成果項目	量化	名稱或內容性質簡述
科教處計畫加填項目	測驗工具(含質性與量性)	0	
	課程/模組	0	
	電腦及網路系統或工具	0	
	教材	0	
	舉辦之活動/競賽	0	
	研討會/工作坊	0	
	電子報、網站	0	
	計畫成果推廣之參與（閱聽）人數	0	

國科會補助專題研究計畫成果報告自評表

請就研究內容與原計畫相符程度、達成預期目標情況、研究成果之學術或應用價值（簡要敘述成果所代表之意義、價值、影響或進一步發展之可能性）、是否適合在學術期刊發表或申請專利、主要發現或其他有關價值等，作一綜合評估。

1. 請就研究內容與原計畫相符程度、達成預期目標情況作一綜合評估

■達成目標

□未達成目標（請說明，以 100 字為限）

□實驗失敗

□因故實驗中斷

□其他原因

說明：

2. 研究成果在學術期刊發表或申請專利等情形：

論文：□已發表 ■未發表之文稿 □撰寫中 □無

專利：□已獲得 □申請中 ■無

技轉：□已技轉 □洽談中 ■無

其他：（以 100 字為限）

3. 請依學術成就、技術創新、社會影響等方面，評估研究成果之學術或應用價值（簡要敘述成果所代表之意義、價值、影響或進一步發展之可能性）（以 500 字為限）