科技部補助專題研究計畫成果報告
期末報告

熱帶曲線上因子之秩的計算

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執 行 期 間 ： 102年08月01日至103年09月30日
執 行 單 位 ： 國立政治大學應用數學學系

計 畫 主 持 人 ： 蔡炎龍

計畫參與人員： 碩士班研究生-兼任助理人員：黃明怡

報 告 附 件 ： 移地研究心得報告

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中 華 民 國 103年12月29日

中 文 摘 要 ： 本報告主要是研究離散型熱帶因子，特別是熱帶曲線上的熱帶因子之秩的計算。我們主要結果可以將一個熱帶因子的秩的所有可能解果降至幾個數字。在許多情況中，我們甚至可以直接得到答案。另外，我們猜測連續型的熱帶因子也會有相同的結果。

中文關鍵詞： 熱帶因子，熱帶幾何，秩

英 文 摘 要 ： In this report, we survey the discrete tropical divisor theory, and focus on the computation of the rank of a tropical divisor on a tropical curve. Our main result can reduced the possible numbers for rank of a divisor into just a few numbers. In many cases, we can even get exact number for the rank of a tropical divisor. We conjecture that

英文關鍵詞： Tropical Divisor, Tropical Geometry, Rank

國科會專題研究成果報告:
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計畫編號: NSC 102-2115-M-004 -001 -
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December 29, 2014

Chapter 1

報告内容

1.1 Introduction

Tropical geometry rise much attention these years. One major reason is that Mikhalkin [?] successfully developed and applied techniques of tropical geometry to calculate the Gromov-Witten invariants in $\mathbb{CP}^2$, which had been proved previously by Kontsevich [?], and Caporaso and Harris [?] with much deep methods of algebraic geometry. We refer to Gathmann's article [?] for basic concepts and applications of tropical geometry.

As in the classical algebraic geometry, we can define divisors for tropical curves. There are at least two types of definitions, one is called the "discrete" version of divisors, and the other is "continues" version of divisors. For simplicity, we only discuss discrete version of divisors of a tropical curve, yet many results can be easily applied to the continuous situation.

The rank of a divisor D for a tropical curve is the tropical counterpart of the dimension of the vector space of meromorphic functions satisfying $\mathrm{Div}(f) + D$ is effective. We have tropical analogous of the Riemann-Roch theorem. Baker and Norine [?] introduced a version of the Riemann-Roch theorem for graphs. Gathmann and Kerber [?], and Mikhalkin and Zharkov [?] used the result to prove the Riemann-Roch theorem for tropical curves. Roughly speaking, they extended Baker and Norine's result to metric graphs. After that, Amini and Caporaso [?] extended the Riemann-Roch theorem to weighted tropical curves.

All versions of the Riemann-Roch theorem for tropical curves of course give us an important tool to calculate the ranks of divisors of tropical curves. Besides that, Hladký, Král, and Norine [?] give an algorithm to calculate the ranks. We will take different approach from their method, which we shall explain in the following sections.

1.2 Basic Definitions and Results

Let Γ be a tropical curve. Γ is naturally corresponding to a finite graph $G = (E, V)$ which we will explain more in details latter. For any graph G, we define (discrete) *divisors* are formal sum of $\mathbb{Z}$-linear combination of the vertices. That is, a divisor D on G (Γ) is of the form:

$$D = \sum_{v \in V} a_v \cdot v,$$

where $a_v \in \mathbb{Z}$. We say a divisor D on the curve Γ is exactly a divisor on the corresponding graph G. The set of all divisors on G (Γ) is denoted by $\mathrm{Div}(G)$ or $\mathrm{Div}(\Gamma)$. The *degree* of a divisor is the sum of all coefficients.

A *meromorphic function* on G is simply a function

$$f \colon V \to \mathbb{Z}.$$

That is, $f \in \mathrm{Hom}(V, \mathbb{Z})$ and we denote the set $\mathrm{Hom}(V, \mathbb{Z})$ by $\mathcal{M}(G)$.

Each $f \in \mathcal{M}(G)$ is corresponding to a divisor

$$D(f) = \sum_{v \in V} \delta_v(f) \cdot v,$$

where

$$\delta_v(f) = \sum_{e = wv \in E_v} (f(v) - f(w)).$$

A divisor of this form is called a *principal divisor*. Two divisors D_1, D_2 are equivalent ($D_1 \sim D_2$) if they are differed by a principal divisor. That is, there is $f \in \mathcal{M}(G)$ such that

$$D_1 - D_2 = D(f).$$

An *effective divisor* E is a divisor that coefficients are all nonnegative, and we use $E \geq 0$ to indicate it is an effective divisor. For a divisor $D \in \mathrm{Div}(G)$, we define the *linear system* associated to D to be the set

$$|D| = \{E \in \mathrm{Div}(G) \mid E \geq 0, E \sim D\}.$$

Finally, we define the *rank* of a given divisor D. The rank of a divisor $D \in \mathrm{Div}(G)$ is defined as the following.

$$\operatorname{rank} D = \max\{s \mid |D - E| \neq \emptyset \text{ for all } E \geq 0 \text{ and } \deg E = s\}.$$

For a graph G, we define the *canonical divisor*

$$K = \sum_{v \in V(G)} (\deg(v) - 2)(v).$$

Baker and Norine [**?**] gave a version of *tropical Riemann-Roch Theorem.*

$$\operatorname{rank}(D) - \operatorname{rank}(K - D) = \deg(D) + g - 1. \tag{1.1}$$

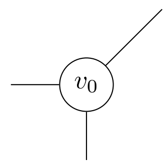
The $\operatorname{rank}(D)$ is what we are primely interested. In the following section, we shall use examples to show the ideas of our project.

1.3 Illustrated Example

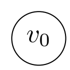
In this section, we give examples that basically explain what we did.

Let $G = (E, V)$ be a graph where E is the collection of edges and V is the collection of vertices.

Let Γ be a tropical line, we emphasize the vertex at center by making it a large point, and denoted the vertex by v_0.



Removing the rays pointing to the infinity, we get exactly one point (the vertex v_0.



We can "verify" the tropical Riemann-Roch theorem. Let D be the divisor

$$D = 3 \cdot (v_0).$$

Since a meromorphic function f on G is simply a function from $V(G) = \{v_0\}$ to $\mathbb{Z}$, there is a $c \in \mathbb{Z}$ such that $f(v_0) = c$. Thus we can find the divisor corresponding to f:

$$(f) = 0 \cdot (v_0).$$

The rank of D is either

$$\max\{n \mid \text{for all } E, \deg(E) = n, E \geq 0, \text{we have } |D - E| \neq \emptyset\},$$

or

$$\min\{m | \text{there is } E \geq 0, \text{such that} \deg(E) = m, |D - E| = \emptyset - 1\}。$$

The only divisor E such that $\deg(E) = 3$ is $E = 3 \cdot v_0$. Therefore, $D - E = 0 \cdot v_0 = (f)$. We conclude that $\operatorname{rank}(D) \geq 3$。

The only divisor E such that $\deg(E) = 4$ is $E = 4 \cdot v_0$. Clearly, $|D - E| = \emptyset$, thus we have

$$\operatorname{rank}(D) = 3.$$

It is easy to check that the canonical divisor is

$$K = -2 \cdot v_0.$$

Then $K - D = -5 \cdot (v_0)$, so $|K - D| = \emptyset$. That is,

$$\operatorname{rank}(K - D) = -1。$$

The left hand side of the tropical Riemann-Roch **??** is

$$r(D) - r(K - D) = 3 - (-1) = 4.$$

Since $\deg(D) = 3$, and the genus $g = |E(G)| - |V(G)| + 1 = 0$, so the right hand side of the tropical Riemann-Roch theorem is

$$\deg(D) - g + 1 = 4.$$

Thus, we verify the tropical Riemann-Roch theorem for a tropical line. In general, calculating the rank would be not so easy. We have to use the tropical Riemann-Roch to find a good estimation.

1.4 Basic Settings

Let Γ be a tropical curve. We remove the rays of the tropical curve. Then we have the corresponding finite graph G. Define

$$\mathrm{Div}(\Gamma) := \mathrm{Div}(G).$$

The graph G is called the graph corresponding to the tropical curve Γ. What we mean by a tropical divisor D is actually a divisor on the graph G.

1.5 Rank Theorem

We state our main theorem here.

Main Theorem. Let Γ be a tropical curve and let D be a divisor on Γ.

(a) If $\deg D < 0$ then $\mathrm{rank}(D) = -1$.

(b) If $\deg D \geq 0$ then $\deg D - g \leq \mathrm{rank}(D) \leq \deg D$.

Proof. Part (a) is easy. Since $\deg D < 0$, by definition $|D - E|$ is empty for all effective divisor E on Γ. Thus, $\mathrm{rank}\, D = -1$.

For part (b), we consider two cases: $|K - D| = \emptyset$ and $|K - D| \neq \emptyset$.

If $|K - D| = \emptyset$, we have $\mathrm{rank}(K - D) = -1$ by definition. By the tropical Riemann-Roch theorem, we obtain

$$\mathrm{rank}(D) - (-1) = \deg D - g + 1,$$

thus $\mathrm{rank}(D) = \deg D - g$. In particular,

$$\deg D - g \leq \mathrm{rank}(D) \leq \deg D$$

holds.

Now, if $|K - D| \neq \emptyset$. Let E be an arbitrary effective divisor on Γ of degree $\deg D + 1$. Then

$$\begin{aligned}\deg(D-E) &= \deg D - \deg E,\\ &= -1\end{aligned}$$

Hence $|D-E| = \emptyset$. Therefore, $\operatorname{rank} D$ is at most $\deg D$.

Note that $|K-D| \neq \emptyset$, so $\operatorname{rank}(K-D) \geq 0$. By the tropical Riemann-Roch theorem, we have

$$\operatorname{rank}(D) \geq \deg D - g + 1.$$

Hence,

$$\deg D - g \leq \operatorname{rank}(D) \leq \deg D.$$

□

Remark 1. Let $D \in \operatorname{Div}(\Gamma)$ such that $\deg D \geq 0$. In the proof of our Main Theorem, we can get an even better inequality for the cases $|K-D| \neq 0$, namely

$$\deg D - g + 1 \leq \operatorname{rank}(D) \leq \deg D.$$

1.6 Applications to Our Theorem

Our main theorem from previous section gives us a range for the rank of a divisor D. Therefore, we only need to check a few possible numbers to see which one is the correct number for $\operatorname{rank}(D)$. Sometimes, we even get the rank immediately such as the following example.

Example 1. Let Γ be a tropical curve of genus 1. Let $D = 3 \cdot v_1 - 2 \cdot v_2 + 5 \cdot v_3 \in \operatorname{Div}(\Gamma)$. We have $\deg D = 3 - 2 + 5 = 6 \geq 0$. By the Remark **??**, we have

$$\deg D - g + 1 \leq \operatorname{rank}(D) \leq \deg D,$$

but

$$\deg D - g + 1 = \deg D - 1 + 1 = \deg D = 6.$$

Therefore, we have $\operatorname{rank}(D) = 6$.

Chapter 2

成果自評

We survey the theory of tropical discrete divisors. The goal is to quickly find the rank of given divisors, and we did find a very good bound for the rank of a divisor. Our main theorem provide a quick way to get the rank of a divisor. Sometime it require to check a few possible numbers, sometimes it reduced to only one possible. We plan to publish these results in the near future.

Finally, we discuss some plans for future research projects. First, what we did here is for discrete tropical divisors, and naturally, we would like to see if our theories can be applied to continuous tropical divisors. We are very confident that is the case.

Moreover, for some interesting geometry objects, like elliptic curves, it is very interesting to see what kind of information the tropical divisor theory can provide. Of course, the first step is to give a “good” definition of tropical version of the objects we are interested in, and then check if they have the same properties as the classical objects.

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國科會專題研究出國報告:
熱帶曲線上因子之秩的計算

計畫編號: NSC 102-2115-M-004 -001 -
蔡炎龍

訪問地點: 韓國首爾誠信女子大學
訪問時間: 2014 年 8 月 8 日至 2014 年 8 月 22 日

非常感謝這次有國科會計畫的支持, 至韓國首爾誠信女子大學 (성신여자대학교, Sungshin Women's University) 訪問尹基憲 (윤기헌, Ki-Heon Yun) 教授, 並且也順道參與在首爾舉辦的數學界的盛會 ICM (International Congress of Mathematicians) 2014, 得到對研究相關許多新的啓發。

尹教授主要是對在 4-manifold 上的 Lefschetz fibrations 拓樸相關性質有深入的瞭解與興趣。我也介紹了 tropical geometry 手法運用在, 尤其是拓樸相關不變量 (如 Gromov-Witten 不變量) 的手法。我們希望有效的把一些手法放在尤其是 monodromy 上, 這是不是可行的方向仍待進一步的評估。

另外在 ICM 中, 對未來研究方向有相當的啓發。其中今年費爾茲獎得主 Manjul Bhargava 他對橢圓曲線的問題做了深入淺出的介紹。他著重當然是在數論方面。但令我好奇的就是:

1. 可不可能在 tropical geometry 中給橢圓曲線給合理的定義?
2. Tropical 橢圓曲線的 Picard group 可以如古典般和橢圓曲線上的點一一對應, 也保持運算, 因此也會給一個群的結構?

這個問題在經過之後的一些研究, 有了一些初步的成果, 並且將列爲明年的研究計畫。

有一些演講在目前看來和自己的研究沒有什麼關係, 卻在方法上有很大的啓發。例如張益唐介紹他近年來非常受到重視「攣生質數」方面的研究成果和手法。他的手法其實很樸實, 甚至想法提出自然到會讓人覺得本來這個問題就該這樣問的感覺。但不容易的地方也就在於這樣: 事實上大家沒有想到, 但他一提出就覺得自然應該這麼想的。

因爲 tropical geometry 的 divisor theory 和圖論有許多關聯, 所以也特別留意圖論相關的演講, 尤其是和代數幾何相關的。Jánon Pach 的 “Geometric Intersection Patterns and the Theory of Topological Graphs” 中, 他主要運用 semialgebraic sets 相關的理論, 去解決圖論用純組合手法很難處理的問題。許多地方和我們研究的方向相關, 但以前對這方面的理論較不熟悉, 會是未來更關注的主題。

Tropical geometry 本身就是定義在 tropical semifield 上的代數幾何。這類「近似」傳統代數結構的應用, 近年來似乎有不少有趣的成果。除了剛剛提到 Pach 等人運用 semi algebraic sets 在圖論方面的研究, 像 Ben Green 也介紹另一個方向的想法, 他和陶哲軒在 “approximate group” 做了不少工作。概念是這樣的: 假設我們要討論的群是 G, G 的子集合 A 如果符合 $A \cdot A^{-1} = A$ 自然就是一個子群了。如果 $A \cdot A^{-1} \neq A$ 但 $|A \cdot A^{-1}|$ 又沒有大得太誇張, 我們就叫 A 是一個 approximate group。精確的定義是說, 對一個常數 K, 如果我們有 $|A \cdot A^{-1}| \leq K|A|$, 我們就說 A 是一個 K-aproximate group。這裡是說如果我們一個子集合沒有「太離譜」, 它也可以考慮成一個群。更進一步的說, Green, Tao 等人證明了這些 approximate group 必然是某些固定的形式。目前不是很清楚這樣子的手法是不是能運用到我們關切的問題, 不過是個挺有意思的主題, 尤其是在和幾何有關的群討論這樣的問題。

另外一些數學應用的主題也相當有趣, 雖然目前可能還沒有和自己的研究相關。比如說 Stanley Osher 的 Level Set Method (LSM) 可以非常精確描述許多東西和現象, 例如畫出火焰及倒入水進一個容器, 這些都可以用數學表示出來。又如 Emmanuel Candes 的 compressive sensing 相關的研究。在幾何上中基本就是研究什麼樣的不變量可以把我們的幾何物件還原。需實務上我們不一定要完整還原, 只要還原到某個可接受的範圍就好。Candes 給了許多有意思的例子, 例如讓 MRI 可以用八倍的速度掃描, 把 (如大數據中) 欠缺的資料補上, 及在影竹片中分離人物和背景等等應用。

科技部補助計畫衍生研發成果推廣資料表

日期:2014/12/01

科技部補助計畫	計畫名稱: 熱帶曲線上因子之秩的計算
	計畫主持人: 蔡炎龍
	計畫編號: 102-2115-M-004-001- 學門領域: 代數幾何

無研發成果推廣資料

102 年度專題研究計畫研究成果彙整表

計畫主持人：蔡炎龍 | 計畫編號：102-2115-M-004-001-

計畫名稱：熱帶曲線上因子之秩的計算

成果項目			量化：實際已達成數（被接受或已發表）	量化：預期總達成數(含實際已達成數)	量化：本計畫實際貢獻百分比	單位	備註（質化說明：如數個計畫共同成果、成果列為該期刊之封面故事...等）
國內	論文著作	期刊論文	0	0	100%	篇	
		研究報告/技術報告	1	1	100%		
		研討會論文	1	1	100%		
		專書	0	0	100%		
	專利	申請中件數	0	0	100%	件	
		已獲得件數	0	0	100%		
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		權利金	0	0	100%	千元	
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	成果項目	量化	名稱或內容性質簡述
科教處計畫加填項目	測驗工具(含質性與量性)	0	
	課程/模組	0	
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	教材	0	
	舉辦之活動/競賽	0	
	研討會/工作坊	0	
	電子報、網站	0	
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科技部補助專題研究計畫成果報告自評表

請就研究內容與原計畫相符程度、達成預期目標情況、研究成果之學術或應用價值（簡要敘述成果所代表之意義、價值、影響或進一步發展之可能性）、是否適合在學術期刊發表或申請專利、主要發現或其他有關價值等，作一綜合評估。

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   其他：（以 100 字為限）

3. 請依學術成就、技術創新、社會影響等方面，評估研究成果之學術或應用價值（簡要敘述成果所代表之意義、價值、影響或進一步發展之可能性）（以 500 字為限）

   本次的研究，我們充份瞭解熱帶幾何中，divisors 的定義及計算方式。我們也計算了一條熱帶曲線，它的 rank 範圍估計。這些計算可以應用在幾何，尤其是試著把古典 divior theory 傳換到熱帶幾何的情況。另外，因為熱帶曲線和圖論有非常強烈的相關，一些圖論的問題也可望多一個熱帶幾何的工具可以處理。